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CATALAN'S TRAPEZOIDS

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Named after the French–Belgian mathematician Eugène Charles Catalan, Catalan's numbers arise in various combinatorial problems [12]. Catalan's triangle, a triangular array of numbers somewhat similar to Pascal's triangle, extends the combinatorial meaning of Catalan's numbers and generalizes them [1,5,11]. A need for a generalization of Catalan's triangle itself arose while conducting a probabilistic analysis of the Asymmetric Simple Inclusion Process (ASIP) — a model for a tandem array of queues with unlimited batch service [7–10]. In this paper, we introduce *Catalan's trapezoids*, a countable set of trapezoids whose first element is Catalan's triangle. An iterative scheme for the construction of these trapezoids is presented, and a closed-form formula for the calculation of their entries is derived. We further discuss the combinatorial interpretations and applications of Catalan's trapezoids.

1. CATALAN'S NUMBERS AND CATALAN'S TRIANGLE

Consider a string of numbers composed of n $(+1)$'s and n (-1)'s, arranged in a row from left to right, such that the sum over every initial substring is non-negative. What is the total number of different such strings? Consider equivalently a path that: (i) starts at the origin of a two dimensional lattice; (ii) consists of n right $(\rightarrow)$ steps and n up $(\uparrow)$ steps; (iii) does not go above the line $y = x$. What is the total number of different such paths? As it turns out, the solution to these combinatorial problems is given by the n^{th} Catalan number (Thomas) [12]:

$$C(n) = \binom{2n}{n} - \binom{2n}{n-1} \qquad \textbf{(1.1)}$$

$(n = 1, 2, 3, \ldots)$, with $C(0) = 1$ by definition. Specifically, the first Catalan numbers are given by: $1, 1, 2, 5, 14, 42, 132, 429$.

One can generalize the combinatorial problem mentioned above by considering strings of n $(+1)$'s and k (-1)'s or, alternatively, paths of n right steps and k up steps. In this case,

TABLE 1. Some entries of Catalan's triangle.

n/k	0	1	2	3	4	5	6	7
0	1							
1	**1**	1						
2	**1**	2	2					
3	**1**	3	5	5				
4	**1**	4	9	14	14			
5	**1**	5	14	28	42	42		
6	**1**	6	20	48	90	132	132	
7	**1**	7	27	75	165	297	429	429

the number of different strings for which the sum over every initial substring is non-negative is given by:

$$C(n,k) = \begin{cases} 1 & k = 0, \\ \binom{n+k}{k} - \binom{n+k}{k-1} & 1 \le k \le n, \\ 0 & k > n, \end{cases} \tag{1.2}$$

$(n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots)$, and the same is true for the number of paths that start at the origin of a two dimensional lattice and do not go above the line $y = x$.

The numbers $C(n, k)$ are referred to in combinatorial mathematics as the entries of *Catalan's triangle* (Thomas, Bailey [1,12]). These entries facilitate the solution to Bertrand's famous ballot problem (Feller [4]): "In an election where candidate A receives n votes and candidate B receives k votes, what is the probability that A will not trail behind B throughout the entire count of votes?". Indeed, the answer to this version of Bertrand's problem is given by the ratio $C(n, k)/\binom{n+k}{k}$.

Catalan's triangle, illustrated in Table 1, has the following iterative construction. By definition, all entries that are positioned on the left boundary of the triangle $(k = 0)$ are given by the boundary condition $C(n, 0) = 1$. In Table 1, these entries are highlighted in bold. Entries positioned to the right of the main diagonal $k = n$ are zero. In Table 1, these entries are indicated by empty squares. All the other entries of Catalan's triangle follow the recursion

$$C(n, k) = C(n-1, k) + C(n, k-1), \tag{1.3}$$

i.e., each entry is a sum of the entry above it and the entry to its left. In Table 1, a specific example, $9 + 5 = 14$, is highlighted in magenta. Entries on the diagonal of Catalan's triangle $(k = n)$ are the Catalan numbers: $C(n, n) = C(n)$. In Table 1, these entries are highlighted in blue.

The combinatorial meaning of Eq. (1.3) and its validity for $1 \le k \le n$ become immediately clear after conducting a binary partition of all valid strings according to their last digit $+1$ or -1. Indeed, since $k \le n$ the sum over a string of n $(+1)$'s and k (-1)'s is non-negative. Moreover, if the string ends with $+1$ there are exactly $C(n-1, k)$ ways to choose the order of the first $n-1$ $(+1)$'s and k (-1)'s such that the sum over every initial substring is non-negative. Similarly, if the string ends with a -1 there are exactly $C(n, k-1)$ ways to choose the order of the first n $(+1)$'s and $k-1$ (-1)'s such that the sum over every initial substring is non-negative. Equation (1.3) readily follows.

TABLE 2. Some entries of Catalan's trapezoid of order $m = 2$ (top) and $m = 3$ (bottom). Entries on the left and upper boundaries are marked in bold. Null entries positioned to the right of the diagonal $k = n + m - 1$ are indicated by empty squares. All other entries follow the recursive rule given in Eq. (2.1). Two specific examples, $429 + 572 = 1001$ and $117 + 83 = 200$, are highlighted in magenta.

n/k	0	1	2	3	4	5	6	7	8
0	**1**	**1**							
1	**1**	2	2						
2	**1**	3	5	5					
3	**1**	4	9	14	14				
4	**1**	5	14	28	42	42			
5	**1**	6	20	48	90	132	132		
6	**1**	7	27	75	165	297	429	429	
7	**1**	8	35	110	275	572	1001	1430	1430

n/k	0	1	2	3	4	5	6	7	8	9
0	**1**	**1**	**1**							
1	**1**	2	3	3						
2	**1**	3	6	9	9					
3	**1**	4	10	19	28	28				
4	**1**	5	15	34	62	90	90			
5	**1**	6	21	55	117	207	297	297		
6	**1**	7	28	83	200	407	704	1001	1001	
7	**1**	8	36	119	319	726	1430	2431	3432	3432

2. CATALAN'S TRAPEZOIDS

The need for a generalization of Catalan's triangle naturally arose while conducting a probabilistic analysis of the Asymmetric Simple Inclusion Process (ASIP) — a model for a tandem array of queues with unlimited batch service (Reuveni, Eliazar and Yechiali [7–9]). Analyzing the ASIP, it so turned out that steady state occupation probabilities in the model can be written in terms of entries taken from trapezoid number arrays whose iterative construction is identical to that of Catalan's triangle, albeit a small change in boundary conditions (Reuveni et al. [10]). Hence, we set out to construct the family of *Catalan's trapezoids*.

Let $C_m(n,k)$ denote the (n,k) entry of the Catalan's trapezoid of order m ($m = 1, 2, 3, \ldots$). Defining Catalan's trapezoid of order $m = 1$ to be Catalan's triangle we have $C_1(n,k) = C(n,k)$. The iterative construction of higher order trapezoids is similar to that of Catalan's triangle. All elements on the left boundary ($k = 0$) of the trapezoid are given by the boundary condition $C_m(n,0) = 1$, all elements on the upper boundary of the trapezoid ($n = 0; 0 \le k \le m-1$) are given by the boundary condition $C_m(0,k) = 1$, and all elements positioned to the right of the diagonal $k = n + m - 1$ are set to zero. The rest of the elements in the trapezoid follow a recursive rule similar to the one given in Eq. (1.3), albeit replacing the numbers $C(n,k)$ by the numbers $C_m(n,k)$:

$$C_m(n,k) = C_m(n-1,k) + C_m(n,k-1), \tag{2.1}$$

i.e., each entry is a sum of the entry above it and the entry to its left. Some entries of Catalan's trapezoid of order $m = 2$ and of order $m = 3$ are given in Table 2.

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A closed form expression for $C_m(n,k)$ is given by

$$C_m(n,k) = \begin{cases} \binom{n+k}{k} & 0 \le k < m, \\ \binom{n+k}{k} - \binom{n+k}{k-m} & m \le k \le n+m-1, \\ 0 & k > n+m-1, \end{cases} \tag{2.2}$$

$(n = 0,1,2,\ldots; k = 0,1,2,\ldots; m = 1,2,3,\ldots)$. Indeed, substituting Eq. (2.2) into (2.1) and making use of the well-known Pascal's rule [4] one can easily verify that the recursion rule in Eq. (2.1) holds. The validity of the trapezoid boundary conditions can be easily verified as well.

We will now show that $C_m(n,k)$ is the number of different strings of n $(+1)$'s and k (-1)'s for which the sum over every initial substring is larger than, or equal to, a threshold level $1-m$ $(m = 1,2,3,\ldots)$. Setting $m = 1$, we note that this combinatorial interpretation generalizes the combinatorial interpretation given for the entries of Catalan's triangle.

In order to prove that our combinatorial interpretation is correct we will consider an equivalent path counting problem. In the non-negative quadrant of a two-dimensional lattice $\{(x,y)—x,y = 0,1,2,3,\ldots\}$, what is the total number of paths that: (i) start at the origin $(0,0)$; (ii) are composed out of n right steps $(\rightarrow)$ and k up steps $(\uparrow)$; (iii) do not go above the line $y = x+m-1$ $(m = 1,2,3,\ldots)$? The formulation of this path counting problem asserts that if a path meets the above-mentioned requirements then at any point along the path the number of right steps minus the number of up steps is larger than or equal to $1-m$. Noting the one-to-one correspondence between $(+1)$'s and right steps and (-1)'s and up steps, it is clear that the path counting problem we have introduced is equivalent to the string counting problem used to combinatorialy interpret the entries of Catalan's trapezoid of order m. Our proof will be concluded by showing that Eq. (2.2) is the answer to the path counting problem presented above.

Firstly, consider the case $0 \le k < m$. In this case, paths cannot go above the line $y = x + m - 1$, so every path is legitimate and the total number of paths is $\binom{n+k}{k}$. Secondly, consider the case $k > n+m-1$. In this case, all paths will end at a point, which is positioned above the line $y = x+m-1$, thus yielding no legitimate paths. Thirdly, note that when $m \le k \le n+m-1$ some paths will go above the line $y = x+m-1$ (illegitimate paths) while others will not (legitimate paths). Clearly, the number of legitimate paths is given by the total number of paths minus the number of illegitimate paths. In order to count the number of illegitimate paths we apply a reflection principle.

An illegitimate path connecting the origin with the point (n,k) is illustrated in Figure 1. Every illegitimate path must hit the line $y = x+m$ at least once and we denote the first (leftmost) hitting point by P. The point P divides the illegitimate path into two segments. The path segment positioned to the left of P connects it with the origin (the dashed blue segment in Figure 1). The path segment positioned to the right of P connects it with the point (n,k) (the solid magenta segment in Figure 1). Reflecting the blue segment with respect to a mirror plane placed along the line $y = x+m$ results in a new path segment that connects the point $(-m,m)$ with the point P (the dashed red segment in Figure 1). Concatenating the red segment with the magenta segment results in a semi-reflected path that connects the point $(-m,m)$ with the point (n,k) via P. Since $k \le n+m-1$, the point (n,k) lies below the line $y = x+m$ and hence every path that

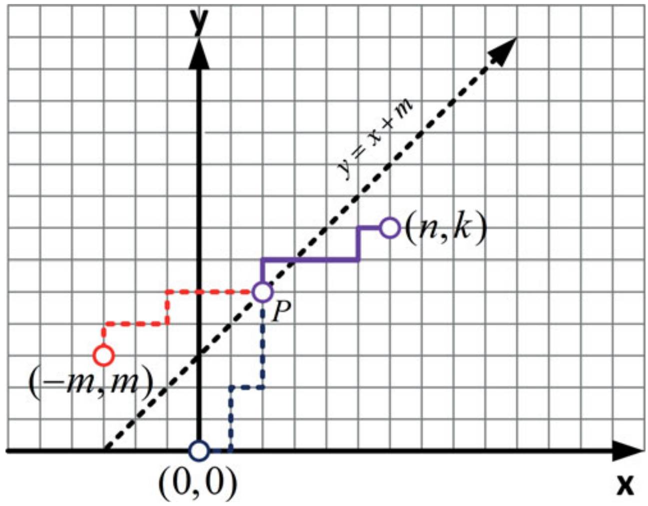

Figure 1. (Color online) An illustration of the reflection principle ($m = 3$).

starts at $(-m, m)$ and ends at (n, k) must cross this line at least once. Denoting the first (leftmost) crossing point by P asserts a one-to-one correspondence between illegitimate paths and paths that: (i) start at $(-m, m)$; (ii) are composed of $n + m$ right steps and $k - m$ up steps. The number of illegitimate paths is thus given by $\binom{n+k}{k-m}$. In turn, since the total number of paths is $\binom{n+k}{k}$, we conclude that the number of legitimate paths is given by $\binom{n+k}{k} - \binom{n+k}{k-m}$.

3. APPLICATIONS

In this section, we consider the application of Catalan's trapezoids to the analysis of three different problems.

3.1. A Generalized Ballot Problem

Consider a generalized ballot problem in which candidate A begins the race $m - 1$ votes ahead of candidate B ($m = 1, 2, 3, \ldots$), and collects n more votes for a total of $n + m - 1$ to B's k votes ($n = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots$). What is the probability that candidate A will not trail behind candidate B throughout the entire count of votes? We note in passing that an equivalent problem is one in which the voting starts off with no head start, and the probability that candidate A will not trail behind candidate B, by more than $m - 1$ votes, is the one of interest.

Catalan's trapezoids facilitate a solution to the above-mentioned generalization of the ballot problem. Indeed, this can be seen by shifting Figure 1's path and reflection line $m - 1$ units to the right, for a path from $(m - 1, 0)$ to $(n + m - 1, k)$ and a semi-reflected path starting at $(-1, m)$. The reflecting boundary line is now given by $y = x + 1$ and its crossing deems a path illegitimate from the stance of B's tally exceeding A's at the point of reflection (even with A's initial vote lead). It is thus clear that the solution to the problem is precisely $C_m(n, k)/\binom{n+k}{k}$.

3.2. Random-Walk Hitting Probabilities

The problems described in this subsection and in the consecutive one, naturally arose in the course of the probabilistic analysis of the ASIP model, and are in a sense two sides of the same coin. Readers who are interested in the ASIP model [7–9] and its connections to Catalan's trapezoids are referred to [10] for a more elaborate discussion.

Consider a random walk in the non-negative quadrant of a two dimensional lattice $\{(x,y)|—x,y=0,1,2,3,\ldots\}$. The point (x,y) will be called a boundary point of the non-negative quadrant if $x=0$ and/or $y=0$. Assume that the walk starts at an non-boundary point (k,m) $(k=1,2,3,\ldots;m=1,2,\ldots)$, and that at every time step the walker chooses between a down step ($\downarrow$) and an up-left step ($\nwarrow$) with equal probability. What is the probability, $P_{hit}^{k,m}(k',m')$, that the random walker hits a specific boundary point (k',m') before it hits any other boundary point?

The answer to the above-mentioned question is given in terms of the entries of the Catalan's trapezoids. Indeed,

$$P_{hit}^{k,m}(k',0)=\left(\frac{1}{2}\right)^{2k+m-2k'}C_1(k+m-k'-1,k-k') \tag{3.1}$$

$(k'=1,2,\ldots,k)$,

$$P_{hit}^{k,m}(0,m')=\left(\frac{1}{2}\right)^{2k+m-m'}C_m(k-1,m+k-m') \tag{3.2}$$

$(m'=2,3,\ldots,k+m)$, and $P_{hit}^{k,m}(k',m')=0$ otherwise.

In order to prove Eqs. (3.1) and (3.2) we must count the number of paths that: (i) are composed out of down and up-left steps only; (ii) connect the point (k,m) with a specific boundary point (k',m'); (iii) do not pass through any other boundary point. Paths that comply with conditions (i), (ii) and (iii) will be called legitimate paths. We note that the number of up-left steps in a legitimate path is given by $k-k'$ and that the number of down steps is given by $k-k'+m-m'$. Thus, the total number of steps equals $2k-2k'+m-m'$. Hence, the probability for realizing a specific path is given by $\left(\frac{1}{2}\right)^{2k-2k'+m-m'}$ and it follows that $P_{hit}^{k,m}(k',m')$ is given by $\left(\frac{1}{2}\right)^{2k-2k'+m-m'}$ times the number of legitimate paths connecting (k,m) with (k',m'). This idea is further illustrated in Figure 2.

In counting the number of paths we note that in every legitimate path that connects the point (k,m) with the point $(k',0)$ $(k'=1,2,3,\ldots,k)$ the last step is always a down step. The remaining, $k-k'$ up-left and $k+m-k'-1$ down, steps must be ordered to form a path that connects the point (k,m) with the point $(k',1)$ without going below the line $y=1$. The total number of these orderings is clearly given by $C_m(k-k',k+m-k'-1)$. However, $C_m(k-k',k+m-k'-1)=C_1(k+m-k'-1,k-k')$, this fact follows by counting the reversed paths in which $k-k'$ down-right ($\searrow$) steps and $k+m-k'-1$ up ($\uparrow$) steps must be ordered to form a path that connects the point $(k',1)$ with the point (k,m) without going below the line $y=1$.

Similarly, in every legitimate path that connects the point (k,m) with the point $(0,m')$ $(m'=2,3,\ldots,k+m)$ the last step is always an up-left step. The remaining, $k-1$ up-left and $k+m-m'$ down, steps must be ordered to form a path that connects the point (k,m) with the point $(1,m'-1)$ without going below the line $y=1$ first. The total number of these orderings is given by $C_m(k-1,k+m-m')$.

Finally, we note that in all other cases there are no legitimate paths that connect the point (k,m) with the point (k',m'). Eqs. (3.1) and (3.2) now follow.

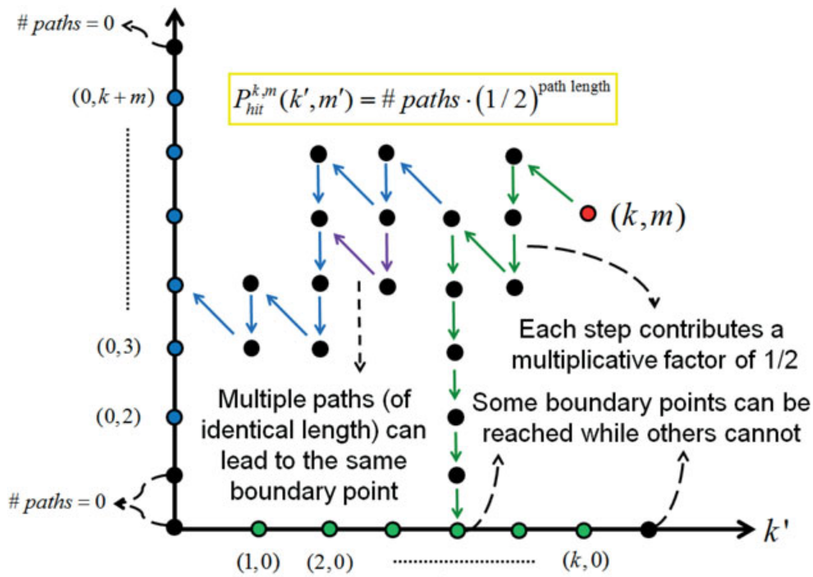

FIGURE 2. (Color online) Computing the hitting probability $P_{hit}^{k,m}(k',m')$ requires the solution of a path counting problem. The answer is given in terms of the entries of Catalan's trapezoids.

3.3. Nested Sums

In this section, we will demonstrate the applicability of Catalan's trapezoids to the algebraic simplification of nested sums. Consider the integers m and n ($m = 1, 2, 3, \ldots; n = 1, 2, 3, \ldots$). Consider also the n indexes $j_1, j_2, \ldots, j_n$ such that j_1 runs from 1 to m and j_{k+1} runs from 1 to $j_k + 1$ ($k = 1, 2, 3, \ldots, n-1$). Let f be a function from the integers and consider the sum

$$S = \sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-1}=1}^{j_{n-2}+1} \sum_{j_n=1}^{j_{n-1}+1} f(j_n). \qquad (3.3)$$

We will hereby show that S can be written as

$$S = \sum_{j=1}^{m+n-1} C_m(n-1, m+n-1-j) f(j), \qquad (3.4)$$

a form which is considerably simpler than the one given by Eq. (3.3). Direct observation implies that the running index j_n in Eq. (3.3) runs from 1 to $m + n - 1$. In order to prove Eq. (3.4), we need to show that $C_m(n-1, m+n-1-j)$ is the exact number of times that j_n receives the value j ($j = 1, 2, 3, \ldots, m+n-1$). That is, we need to show that

$$C_m(n-1, m+n-1-j) = \sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-1}=1}^{j_{n-2}+1} \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j), \qquad (3.5)$$

where $\delta(x, y)$ is the Kronecker delta function.

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We start by showing that Eq. (3.5) holds for boundary entries of the Catalan's trapezoid of order m. Indeed, when $n = 1$ we have

$$C_m(0, m-j) = \sum_{j_1=1}^{m} \delta(j_1, j) = 1 \tag{3.6}$$

$(m = 1, 2, 3, \ldots; j = 1, \ldots, m)$. In addition, when $j = m + n - 1$, then $\delta(j_n, j) = 1$ if and only if $\{j_1 = m;\ j_2 = m + 1; \ldots;\ j_{n-1} = m + n - 2;\ j_n = m + n - 1\}$. Since this specific configuration is unique, we have

$$C_m(n-1, 0) = \sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-1}=1}^{j_{n-2}+1} \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, m+n-1) = 1 \tag{3.7}$$

$(m = 1, 2, 3, \ldots; n = 1, 2, 3, \ldots)$.

In order to complete our proof we now show that Eq. (3.5) also holds for off-boundary entries of the Catalan's trapezoid of order m. Recall that Eq. (2.1) asserts that the following recursion relation holds

$$C_m(n-1, m+n-1-j) = C_m(n-2, m+n-1-j) + C_m(n-1, m+n-2-j), \tag{3.8}$$

$(n > 1, 1 \leq j < m + n - 1)$. The Catalan's trapezoid of order m is uniquely determined by Eq. (3.8) and the boundary conditions specified in Eqs. (3.6) and (3.7). Substituting Eq. (3.5) into (3.8) we have

$$\begin{aligned} &\sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-1}=1}^{j_{n-2}+1} \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j) \\ &\quad = \sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-2}=1}^{j_{n-3}+1} \sum_{j_{n-1}=1}^{j_{n-2}+1} \delta(j_{n-1}, j-1) \\ &\qquad + \sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-1}=1}^{j_{n-2}+1} \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j+1). \end{aligned} \tag{3.9}$$

Proving that the equality in Eq. (3.9) holds will conclude our proof. Indeed the right-hand side of Eq. (3.9) immediately gives

$$\sum_{j_1=1}^{m} \sum_{j_2=1}^{j_1+1} \sum_{j_3=1}^{j_2+1} \cdots \sum_{j_{n-1}=1}^{j_{n-2}+1} \left[\delta(j_{n-1}, j-1) + \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j+1) \right] \tag{3.10}$$

and it is easy to check that

$$\sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j) = \delta(j_{n-1}, j-1) + \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j+1), \tag{3.11}$$

$(1 \leq j \leq j_{n-1} + 1)$. Indeed, when $1 \leq j \leq j_{n-1}$ we have $\sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j) = \sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j+1) = 1$ and $\delta(j_{n-1}, j-1) = 0$. In addition, when $j = j_{n-1} + 1$, we have $\sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j) = \delta(j_{n-1}, j-1) = 1$ and $\sum_{j_n=1}^{j_{n-1}+1} \delta(j_n, j+1) = 0$. Substituting Eq. (3.11) into (3.10) proves the validity of Eq. (3.9).

4. CONCLUSIONS

Catalan's numbers and their generalizations have found numerous applications in various problems [1–3,5,11,12]. In this paper, we introduced Catalan's trapezoids — a novel combinatorial construct that naturally arose while computing the steady state occupation probabilities of the Asymmetric Simple Inclusion Process [10]. Catalan's trapezoids were shown applicable to the analysis of the generalized ballot problem, the computation of random walk hitting probabilities, and the evaluation of certain nested sums. Our work is a generalization of Bailey's generalization of Catalan's numbers and joins [6] in that regard.

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