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Congruences concerning truncated hypergeometric series

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We employ some formulae on hypergeometric series and p-adic Gamma function to establish several congruences.

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1. Introduction

Following [1], we define the hypergeometric series by

$$ {}_{r+1}F_s\left[\begin{matrix} a_0 & a_1 & \cdots & a_r & \\ & b_1 & b_2 & \cdots & b_s \end{matrix}; z\right] := \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_s)_k} z^k, $$

where $(z)_n$ is given by

$$ (z)_0 = 1, (z)_n = z(z+1)\cdots(z+n-1). $$

The truncated hypergeometric series are defined by

$$ {}_{r+1}F_s\left[\begin{matrix} a_0 & a_1 & \cdots & a_r & \\ & b_1 & b_2 & \cdots & b_s \end{matrix}; z\right]_n := \sum_{k=0}^{n} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k!(b_1)_k \cdots (b_s)_k} z^k, $$

i.e. the truncation of the series after the z^n term. Thus, each truncated hypergeometric series is a rational function in a_i, b_i and z if it is well defined.

Recall that the function $\Gamma(x)$ is a meromorphic function in $\mathbb{C}$ defined by [1]

$$ \frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n}, $$

where γ is the Euler constant defined as

$$ \gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right). $$

One of the most important properties of $\Gamma(z)$ is the Euler reflection formula,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}.$$

Let p be an odd prime and n a positive integer. We define the p-adic Gamma function as

$$\Gamma_p(n):=(-1)^n\prod_{\substack{j<n\\ p\nmid j}} j.$$

Then we extend this to all $x\in\mathbb{Z}_p$ (where $\mathbb{Z}_p$ denotes the set of all rational numbers a/b with $a\in\mathbb{Z}$, $b\in\mathbb{Z}_{>0}$, $\gcd(a,b)=1$ and $p\nmid b$) by setting

$$\Gamma_p(x)=\lim_{n\to x}\Gamma_p(n),$$

where n runs through any sequence of positive integers p-adically approaching x and $\Gamma_p(0)=1$. The fact that the above limit exists is independent of how n approaches x and determines a continuous function on $\mathbb{Z}_p$.

The topic of congruences is related to the p-adic Gamma function, Gauss and Jacobi sums, hypergeometric series [5], modular forms, Calabi–Yau manifolds and some sophisticated combinatorial identities involving harmonic numbers (see, for example, [8]). Actually, many congruences have been obtained by using the Wilf–Zeilberger method (see [3,16]). Various supercongruences were conjectured by many mathematicians including van Hamme [17,18], Zudilin [19], Chan *et al*. [2], Z.-W. Sun [13–15] and Z.-H. Sun [9–12]. In particular, van Hamme [18, (C.2)] conjectured the following congruence.

Conjecture 1.1. *If p is an odd prime, then*

$$\sum_{k=0}^{(p-1)/2}(4k+1)\binom{-\frac{1}{2}}{k}^4\equiv p\pmod{p^3}. \tag{1.1}$$

Here and below, we use the notation $A\equiv B\pmod{p^l}$ to denote that, for $A,B\in\mathbb{Q}$, $(A-B)/p^l$ is a p-integer, where p-integers are rational numbers of the form a/b with $a\in\mathbb{Z}$, $b\in\mathbb{Z}_{>0}$, $\gcd(a,b)=1$ and $p\nmid b$. We shall give a new proof of congruence (1.1) by using a formula for basic hypergeometric series (see [1, § 10.9] for the definition of basic hypergeometric series).

Our main results are the following congruences.

Theorem 1.2. *Let $p\geqslant 5$ be a prime. Then*

$${}_4F_3\left[\begin{matrix}\frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ & \frac{1}{4} & 1 & 1\end{matrix};1\right]_{p-1}\equiv\begin{cases}0 & \text{if } p\equiv 1\pmod 4\\ -16\Gamma_p(\frac{3}{4})^4 & \text{if } p\equiv 3\pmod 4\end{cases}\pmod{p^2}, \tag{1.2}$$

$${}_4F_3\left[\begin{matrix}\frac{7}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ & \frac{1}{6} & 1 & 1\end{matrix};1\right]_{p-1}\equiv\begin{cases}p\Gamma_p(\frac{2}{3})^3 & \text{if } p\equiv 1\pmod 6\\ -6\Gamma_p(\frac{2}{3})^3 & \text{if } p\equiv 5\pmod 6\end{cases}\pmod{p^2}, \tag{1.3}$$

$$ {}_4F_3\left[\begin{matrix}\frac{9}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4};1\\ & \frac{1}{8} & 1 & 1\end{matrix}\right]_{p-1} \equiv \begin{cases} p\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 & \text{if } p\equiv 1,7 \pmod 8\\ 0 & \text{if } p\equiv 3 \pmod 8\\ -8\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 & \text{if } p\equiv 5 \pmod 8\end{cases} \pmod{p^2} \tag{1.4}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{64^k} \equiv \begin{cases}\frac{1}{4}\Gamma_p(\frac{1}{4})^4 & \text{if } p\equiv 1 \pmod 4\\ -4\Gamma_p(\frac{3}{4})^4 & \text{if } p\equiv 3 \pmod 4\end{cases} \pmod{p^2}, \tag{1.5}$$

$$\sum_{k=1}^{p-1}\frac{(\frac{1}{3})_k^3}{(k-1)!k!^2} \equiv \begin{cases}\frac{1}{6}(p\Gamma_p(\frac{2}{3})^3 - \Gamma_p(\frac{1}{3})^6) & \text{if } p\equiv 1 \pmod 6\\ -\Gamma_p(\frac{2}{3})^3 & \text{if } p\equiv 5 \pmod 6\end{cases} \pmod{p^2}. \tag{1.6}$$

THEOREM 1.3. *Let $p \geqslant 5$ be a prime. Then*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4};1\\ & 1 & 1\end{matrix}\right]_{p-1} \equiv \begin{cases}-\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 1 \pmod 8\\ \frac{1}{8}p\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 3,5 \pmod 8\\ 0 & \text{if } p\equiv 7 \pmod 8\end{cases} \pmod{p^2} \tag{1.7}$$

and

$$\sum_{k=1}^{p-1}\frac{(\frac{1}{4})_k^3}{(k-1)!k!^2}$$
$$\equiv \begin{cases}\frac{1}{8}(p\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 + \Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8})) & \text{if } p\equiv 1 \pmod 8\\ -\frac{1}{64}p\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 3 \pmod 8\\ -\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 - \frac{1}{64}p\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 5 \pmod 8\\ \frac{1}{8}p\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 & \text{if } p\equiv 7 \pmod 8\end{cases} \pmod{p^2}. \tag{1.8}$$

THEOREM 1.4. *Let $p \geqslant 5$ be a prime. Then we have the following.*

(i) *If $p \equiv 1 \pmod 3$, then*

$$p\cdot {}_3F_2\left[\begin{matrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3};1\\ & \frac{7}{6} & \frac{2}{3}\end{matrix}\right]_{(p-1)/3} \equiv \Gamma_p(\tfrac{1}{3})^6\Gamma_p(\tfrac{2}{3})^3 \pmod{p^3}. \tag{1.9}$$

(ii) *If $p \equiv 2 \pmod 3$, then*

$$p\cdot {}_3F_2\left[\begin{matrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3};1\\ & \frac{7}{6} & \frac{2}{3}\end{matrix}\right]_{(2p-1)/3} \equiv \tfrac{1}{2}p\Gamma_p(\tfrac{1}{3})^6\Gamma_p(\tfrac{2}{3})^3 \pmod{p^2}. \tag{1.10}$$

We shall provide some auxiliary results in the next section. A new proof of congruence (1.1) will be given in § 3. Section 4 is devoted to our proof of theorems 1.2–1.4.

2. Some auxiliary results

In order to prove theorems 1.2–1.4, we need some auxiliary results. We first mention the following result, which is crucial in the derivation of theorem 1.4.

Let $\zeta = \mathrm{e}^{2\pi\mathrm{i}/3}$ for $a \neq 0$ and $j \in \{0, 1, 2\}$. Then, from the fact that

$$\begin{aligned}(a + b\zeta^j p)_k &= (a + b\zeta^j p)(a + b\zeta^j p + 1) \cdots (a + b\zeta^j p + k - 1) \\ &= (a)_k (1 + b\zeta^j p A(k) + b^2 \zeta^{2j} p^2 B(k)) \pmod{p^3},\end{aligned}$$

where

$$A(k) = \sum_{l=1}^{k} \frac{1}{a + l - 1}$$

and

$$B(k) = \sum_{1 \leqslant l < m \leqslant k} \frac{1}{(a + l - 1)(a + m - 1)},$$

we have, for any $a, b \in \mathbb{R}$,

$$(a + bp)_k (a + b\zeta p)_k (a + b\zeta^2 p)_k = (a)_k^3 \pmod{p^3}. \tag{2.1}$$

We recall some basic properties of the Morita p-adic Gamma function.

Lemma 2.1 (Cohen [4, § 11.6]). *Let p be an odd prime and $x \in \mathbb{Z}_p$. Then*

$$\Gamma_p(1) = -1, \tag{2.2}$$

$$\frac{\Gamma_p(x + 1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } x \notin p\mathbb{Z}_p, \\ -1 & \text{if } x \in p\mathbb{Z}_p, \end{cases} \tag{2.3}$$

$$\Gamma_p(x)\Gamma_p(1 - x) = (-1)^{a_0(x)}, \tag{2.4}$$

where $a_0(x) \in \{1, 2, \ldots, p\}$ such that $a_0(x) \equiv x \pmod{p}$.

Actually, (2.3) is a p-adic analogue of the well-known property $\Gamma(x+1) = x\Gamma(x)$, and (2.4) is a p-adic analogue of Euler's reflection formula.

We now mention an important result, which follows readily from the definition of $\Gamma_p(x)$, and which Long and Ramakrishna [7] used but did not state explicitly.

Lemma 2.2 (Long and Ramakrishna [7, lemma 18]). *Let p be an odd prime, $m \geqslant 3$ be an integer and let ζ be the mth primitive root of unity. Suppose $a \in \mathbb{Z}_p[\zeta]$ and $n \in \mathbb{N}$ such that $a + k \notin p\mathbb{Z}_p[\zeta]$ for all $k \in \{0, 1, \ldots, n - 1\}$. Then*

$$(a)_n = (-1)^n \frac{\Gamma_p(a + n)}{\Gamma_p(a)}.$$

The following result on the (p-adic) expansion of p-adic Gamma function is also very important in the proof of theorems 1.2–1.4.

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LEMMA 2.3 (Long and Ramakrishna [7, theorem 15]). *For $p \geqslant 5$, $r \in \mathbb{N}$, $a \in \mathbb{Z}_p$ and $m \in \mathbb{C}_p$ satisfying $v_p(m) \geqslant 0$ and $t \in \{0,1,2\}$, we have*

$$\frac{\Gamma_p(a+mp^r)}{\Gamma_p(a)} \equiv \sum_{k=0}^{t} \frac{G_k(a)}{k!}(mp^r)^k \pmod{p^{(t+1)r}},$$

where

$$G_k(a) = \frac{\Gamma_p^{(k)}(a)}{\Gamma_p(a)} \in \mathbb{Z}_p$$

and $\Gamma_p^{(k)}(x)$ is the kth derivative of $\Gamma_p(x)$.

3. A new proof of (1.1)

Recall the following identity for basic hypergeometric series (see [6, theorem 11.3]):

$$\frac{(\alpha^2 q^2, \alpha^2 ab/q^2; q^2)_m}{(\alpha^2 a, \alpha^2 b; q^2)_m} {}_5\phi_4\left[\begin{matrix} q^{-2m} & q^2/a & q^2/b & \lambda & q\lambda \\ & q\alpha & q^2\alpha & \lambda^2 & q^4/\alpha^2 abq^{2m} \end{matrix}; q^2, q^2\right]$$
$$= \sum_{n=0}^{m} \frac{(1+\alpha q^{2n})(q^{-2m}, \alpha^2, q^2/a, q^2/b; q^2)_n(-q, q\alpha/\lambda; q)_n(\alpha^2\lambda abq^{2m-2})^n}{(1+\alpha)(q^2, \alpha^2 q^{2m+2}, \alpha^2 a, \alpha^2 b; q^2)_n(\alpha, -\lambda; q)_n},$$

where m is a non-negative integer,

$${}_5\phi_4\left[\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ & b_1 & b_2 & b_3 & b_4 \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k(a_3;q)_k(a_4;q)_k(a_5;q)_k}{(q;q)_k(b_1;q)_k(b_2;q)_k(b_3;q)_k(b_4;q)_k} z^k,$$

$$(a;q)_0 = 1, (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{for } n \geqslant 1, \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

and if n is finite or infinite and l is a positive integer, we use the following notation:

$$(a_1, a_2, \ldots, a_l; q)_n := \prod_{k=1}^{l}(a_k;q)_n.$$

Making the substitutions $\alpha \to q^\alpha$, $a \to q^a$, $b \to q^b$, $\lambda \to q^\lambda$ and letting $q \to 1$, we have

$$\frac{(1+\alpha)_m(\alpha+\frac{1}{2}a+\frac{1}{2}b-1)_m}{(\alpha+\frac{1}{2}a)_m(\alpha+\frac{1}{2}b)_m}$$
$$\times {}_5F_4\left[\begin{matrix} -m & 1-\frac{1}{2}a & 1-\frac{1}{2}b & \frac{1}{2}\lambda & \frac{1}{2}(1+\lambda) \\ & \frac{1}{2}(1+\alpha) & 1+\frac{1}{2}\alpha & \lambda & 2-\alpha-\frac{1}{2}a-\frac{1}{2}b-m \end{matrix}; 1\right]$$
$$= \sum_{n=0}^{m} \frac{(-m)_n(1-\frac{1}{2}a)_n(1-\frac{1}{2}b)_n(1+\alpha-\lambda)_n}{n!(\alpha+m+1)_n(\alpha+\frac{1}{2}a)_n(\alpha+\frac{1}{2}b)_n}. \quad (3.1)$$

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Let $\zeta = \mathrm{e}^{2\pi\mathrm{i}/3}$. Setting $\alpha = \frac{1}{2}$, $a = 1+\zeta p$, $b = 1+\zeta^2 p$, $m = \frac{1}{2}(p-1)$ and $\lambda = 1$ in (3.1), we get

$$
{}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}(1-p) & \frac{1}{2}(1-\zeta p) & \frac{1}{2}(1-\zeta^2 p)\\ & 1+\frac{1}{2}p & 1+\frac{1}{2}\zeta p & 1+\frac{1}{2}\zeta^2 p\end{matrix};1\right]
= \frac{(\frac{3}{2})_{(p-1)/2}(\frac{1}{2}(1-p))_{(p-1)/2}}{(1+\frac{1}{2}\zeta p)_{(p-1)/2}(1+\frac{1}{2}\zeta^2 p)_{(p-1)/2}}
\times {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}(1-p) & \frac{1}{2}(1-\zeta p) & \frac{1}{2}(1-\zeta^2 p)\\ & \frac{3}{4} & \frac{5}{4} & 1\end{matrix};1\right]. \quad (3.2)
$$

By (2.3) and lemma 2.2,

$$
(\tfrac{3}{2})_{(p-1)/2} = p(\tfrac{1}{2})_{(p-1)/2} = (-1)^{(p-1)/2} p\frac{\Gamma_p(\frac{1}{2}p)}{\Gamma_p(\frac{1}{2})} = (-1)^{(p+1)/2} p\frac{\Gamma_p(1+\frac{1}{2}p)}{\Gamma_p(\frac{1}{2})},
$$

$$
(\tfrac{1}{2}(1-p))_{(p-1)/2} = (-1)^{(p-1)/2}\frac{\Gamma_p(0)}{\Gamma_p(\frac{1}{2}(1-p))} = (-1)^{(p-1)/2}\frac{1}{\Gamma_p(\frac{1}{2}(1-p))},
$$

$$
(1+\tfrac{1}{2}\zeta p)_{(p-1)/2} = (-1)^{(p-1)/2}\frac{\Gamma_p(\frac{1}{2}(1+p+\zeta p))}{\Gamma_p(1+\frac{1}{2}\zeta p)} = (-1)^{(p-1)/2}\frac{\Gamma_p(\frac{1}{2}(1-\zeta^2 p))}{\Gamma_p(1+\frac{1}{2}\zeta p)},
$$

$$
(1+\tfrac{1}{2}\zeta^2 p)_{(p-1)/2} = (-1)^{(p-1)/2}\frac{\Gamma_p(\frac{1}{2}(1+p+\zeta^2 p))}{\Gamma_p(1+\frac{1}{2}\zeta^2 p)} = (-1)^{(p-1)/2}\frac{\Gamma_p(\frac{1}{2}(1-\zeta p))}{\Gamma_p(1+\frac{1}{2}\zeta^2 p)}.
$$

Then

$$
\frac{(\frac{3}{2})_{(p-1)/2}(\frac{1}{2}(1-p))_{(p-1)/2}}{(1+\frac{1}{2}\zeta p)_{(p-1)/2}(1+\frac{1}{2}\zeta^2 p)_{(p-1)/2}}
= -p\frac{\Gamma_p(1+\frac{1}{2}p)\Gamma_p(1+\frac{1}{2}\zeta p)\Gamma_p(1+\frac{1}{2}\zeta^2 p)}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{2}(1-p))\Gamma_p(\frac{1}{2}(1-\zeta^2 p))\Gamma_p(\frac{1}{2}(1-\zeta p))}.
$$

By lemma 2.3,

$$
\Gamma_p(1+\tfrac{1}{2}\zeta^j p) \equiv \Gamma_p(1)(1+G_1(1)\tfrac{1}{2}\zeta^j p + G_2(1)\tfrac{1}{8}(\zeta^{2j}p^2)) \pmod{p^3},
$$
$$
\Gamma_p(\tfrac{1}{2}(1+\zeta^j p)) \equiv \Gamma_p(\tfrac{1}{2})(1+G_1(\tfrac{1}{2})\tfrac{1}{2}\zeta^j p + G_2(\tfrac{1}{2})\tfrac{1}{8}(\zeta^{2j}p^2)) \pmod{p^3},
$$

for $j \in \{0,1,2\}$. Hence,

$$
\Gamma_p(1+\tfrac{1}{2}p)\Gamma_p(1+\tfrac{1}{2}\zeta p)\Gamma_p(1+\tfrac{1}{2}\zeta^2 p) \equiv \Gamma_p(1)^3 \pmod{p^3},
$$
$$
\Gamma_p(\tfrac{1}{2}(1-p))\Gamma_p(\tfrac{1}{2}(1-\zeta^2 p))\Gamma_p(\tfrac{1}{2}(1-\zeta p)) \equiv \Gamma_p(\tfrac{1}{2})^3 \pmod{p^3},
$$

and so by (2.2) and (2.4),

$$
\frac{(\frac{3}{2})_{(p-1)/2}(\frac{1}{2}(1-p))_{(p-1)/2}}{(1+\frac{1}{2}\zeta p)_{(p-1)/2}(1+\frac{1}{2}\zeta^2 p)_{(p-1)/2}} \equiv -p\frac{\Gamma_p(1)^3}{\Gamma_p(\frac{1}{2})^4} = p \pmod{p^3}. \quad (3.3)
$$

Using (2.1), (3.3) and the fact that

$$\binom{x}{k} = (-1)^k \frac{(-x)_k}{k!}$$

in (3.2), we obtain

$$p \cdot {}_4F_3\begin{bmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ & \frac34 & \frac54 & 1 \end{bmatrix}; 1\Big]_{(p-1)/2} \equiv {}_4F_3\begin{bmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ & 1 & 1 & 1 \end{bmatrix}; 1\Big]_{(p-1)/2}$$
$$= \sum_{k=0}^{(p-1)/2} \binom{-\frac12}{k}^4 \pmod{p^3}. \tag{3.4}$$

Taking $\alpha = \frac12$, $a = 1 + \zeta p$, $b = 1 + \zeta^2 p$, $m = \frac12(p-1)$ and $\lambda = 0$ in (3.1) and noting that

$$\lim_{\lambda \to 0} \frac{(\frac12\lambda)_k}{(\lambda)_k} = \begin{cases} 1 & \text{if } k = 0, \\ \frac12 & \text{if } k \geqslant 1, \end{cases}$$

we attain

$${}_4F_3\left[\begin{matrix} \frac32 & \frac12(1-p) & \frac12(1-\zeta p) & \frac12(1-\zeta^2 p) \\ & 1+\frac12 p & 1+\frac12\zeta p & 1+\frac12\zeta^2 p \end{matrix}; 1\right]$$
$$= \frac{(\frac32)_{(p-1)/2}(\frac12(1-p))_{(p-1)/2}}{(1+\frac12\zeta p)_{(p-1)/2}(1+\frac12\zeta^2 p)_{(p-1)/2}}$$
$$\times \left(\frac12 + \frac12 {}_4F_3\left[\begin{matrix} \frac12 & \frac12(1-p) & \frac12(1-\zeta p) & \frac12(1-\zeta^2 p) \\ & \frac34 & \frac54 & 1 \end{matrix}; 1\right]\right).$$

Employing (2.1), (3.3) and the facts that

$$\binom{x}{k} = (-1)^k \frac{(-x)_k}{k!} \quad \text{and} \quad \left(\frac32\right)_k = (2k+1)\left(\frac12\right)_k$$

in the above identity, we are led to

$$\sum_{k=0}^{(p-1)/2} (2k+1)\binom{-\frac12}{k}^4 = {}_4F_3\left[\begin{matrix} \frac32 & \frac12 & \frac12 & \frac12 \\ & 1 & 1 & 1 \end{matrix}; 1\right]_{(p-1)/2}$$
$$\equiv p\left(\frac12 + \frac12 {}_4F_3\left[\begin{matrix} \frac12 & \frac12 & \frac12 & \frac12 \\ & \frac34 & \frac54 & 1 \end{matrix}; 1\right]_{(p-1)/2}\right) \pmod{p^3}. \tag{3.5}$$

In view of (3.4) and (3.5), we deduce that

$$\sum_{k=0}^{(p-1)/2} (4k+1)\binom{-\frac12}{k}^4 \equiv p \pmod{p^3}.$$

This concludes the proof.

4. Proof of theorems 1.2–1.4

Proof of theorem 1.2. We recall the following identity on hypergeometric series (see [1, exercise 25(a), p. 182]):

$$
{}_4F_3\left[\begin{matrix} a & \frac{1}{2}a+1 & b & c \\ & \frac{1}{2}a & a-b+1 & a-c+1 \end{matrix};1\right]
= \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(a+1)\Gamma(a-b-c+1)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}. \quad (4.1)
$$

Letting $a=\frac{1}{2}$, $b=\frac{1}{2}(1-p)$, $c=\frac{1}{2}(1+p)$ in (4.1) yields

$$
{}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2}(1-p) & \frac{1}{2}(1+p) \\ & \frac{1}{4} & 1+\frac{1}{2}p & 1-\frac{1}{2}p \end{matrix};1\right] = \frac{\Gamma(1+\frac{1}{2}p)\Gamma(1-\frac{1}{2}p)\Gamma(\frac{3}{4})\Gamma(-\frac{1}{4})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{1}{2}p)\Gamma(\frac{1}{4}-\frac{1}{2}p)}. \quad (4.2)
$$

From Euler's reflection formula and the fact that $\Gamma(x+1)=x\Gamma(x)$ it is easily seen that

$$
\frac{\Gamma(1+\frac{1}{2}p)\Gamma(1-\frac{1}{2}p)}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} = p\frac{\Gamma(\frac{1}{2}p)\Gamma(1-\frac{1}{2}p)}{\Gamma(\frac{1}{2})^2} = p(-1)^{(p-1)/2}. \quad (4.3)
$$

When $p\equiv 1 \pmod 4$, $(\frac{3}{4})_{(p-1)/2}$ has no multiples of p but $(\frac{1}{4}-\frac{1}{2}p)_{(p-1)/2}$ has a multiple of p, which is $-\frac{1}{4}p$, while when $p\equiv 3 \pmod 4$, $(\frac{3}{4})_{(p-1)/2}$ has a multiple of p, which is $\frac{1}{4}p$ but $(\frac{1}{4}-\frac{1}{2}p)_{(p-1)/2}$ has no multiples of p. Then, by lemma 2.2 and the definition of $\Gamma_p(x)$,

$$
\frac{\Gamma(\frac{1}{4}+\frac{1}{2}p)}{\Gamma(\frac{3}{4})} = \left(\frac{3}{4}\right)_{(p-1)/2} = \begin{cases} \dfrac{\Gamma_p(\frac{1}{4}+\frac{1}{2}p)}{\Gamma_p(\frac{3}{4})} & \text{if } p\equiv 1 \pmod 4, \\[2ex] -\dfrac{p}{4}\dfrac{\Gamma_p(\frac{1}{4}+\frac{1}{2}p)}{\Gamma_p(\frac{3}{4})} & \text{if } p\equiv 3 \pmod 4, \end{cases}
$$

and

$$
\frac{\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4}-\frac{1}{2}p)} = \left(\frac{1}{4}-\frac{p}{2}\right)_{(p-1)/2} = \begin{cases} -\dfrac{p}{4}\dfrac{\Gamma_p(-\frac{1}{4})}{\Gamma_p(\frac{1}{4}-\frac{1}{2}p)} & \text{if } p\equiv 1 \pmod 4, \\[2ex] -\dfrac{\Gamma_p(-\frac{1}{4})}{\Gamma_p(\frac{1}{4}-\frac{1}{2}p)} & \text{if } p\equiv 3 \pmod 4. \end{cases}
$$

Hence, by (4.3),

$$
\frac{\Gamma(1+\frac{1}{2}p)\Gamma(1-\frac{1}{2}p)\Gamma(\frac{3}{4})\Gamma(-\frac{1}{4})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{1}{2}p)\Gamma(\frac{1}{4}-\frac{1}{2}p)}
= \begin{cases} -\dfrac{p^2}{4}\dfrac{\Gamma_p(\frac{3}{4})\Gamma_p(-\frac{1}{4})}{\Gamma_p(\frac{1}{4}+\frac{1}{2}p)\Gamma_p(\frac{1}{4}-\frac{1}{2}p)} & \text{if } p\equiv 1 \pmod 4, \\[2ex] -4\dfrac{\Gamma_p(\frac{3}{4})\Gamma_p(-\frac{1}{4})}{\Gamma_p(\frac{1}{4}+\frac{1}{2}p)\Gamma_p(\frac{1}{4}-\frac{1}{2}p)} & \text{if } p\equiv 3 \pmod 4. \end{cases} \quad (4.4)
$$

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It follows easily from lemma 2.3 that

$$\Gamma_p(\tfrac{1}{4}+\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{4}-\tfrac{1}{2}p) \equiv \Gamma_p(\tfrac{1}{4})^2 \pmod{p^2}.$$

Using the above congruence in (4.4) and then employing (2.3) and (2.4), we get

$$\frac{\Gamma(1+\frac{1}{2}p)\Gamma(1-\frac{1}{2}p)\Gamma(\frac{3}{4})\Gamma(-\frac{1}{4})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{1}{2}p)\Gamma(\frac{1}{4}-\frac{1}{2}p)} \equiv \begin{cases} 0 & \text{if } p\equiv 1 \pmod 4 \\ -16\Gamma_p(\frac{3}{4})^4 & \text{if } p\equiv 3 \pmod 4 \end{cases} \pmod{p^2}. \tag{4.5}$$

By the fact that $(a+bp)_k(a-bp)_k \equiv (a)_k^2 \pmod{p^2}$, we get

$${}_4F_3\begin{bmatrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2}(1-p) & \frac{1}{2}(1+p) \\ & \frac{1}{4} & 1+\frac{1}{2}p & 1-\frac{1}{2}p \end{bmatrix};1 \equiv {}_4F_3\begin{bmatrix} \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{bmatrix};1_{(p-1)/2} \pmod{p^2}. \tag{4.6}$$

In view of (4.2), (4.5) and (4.6), we obtain

$${}_4F_3\begin{bmatrix} \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{bmatrix};1_{(p-1)/2} \equiv \begin{cases} 0 & \text{if } p\equiv 1 \pmod 4 \\ -16\Gamma_p(\frac{3}{4})^4 & \text{if } p\equiv 3 \pmod 4 \end{cases} \pmod{p^2}.$$

This proves (1.2), since $(\frac{1}{2})_k \equiv 0 \pmod p$ for $\frac{1}{2}p < k < p$.

Similarly, taking $a=\frac{1}{3}$, $b=\frac{1}{3}(1-\varepsilon p)$, $c=\frac{1}{3}(1+\varepsilon p)$ and $a=\frac{1}{4}$, $b=\frac{1}{4}(1-\eta p)$, $c=\frac{1}{4}(1+\eta p)$, where

$$\varepsilon = \begin{cases} 1 & \text{if } p\equiv 1 \pmod 3, \\ 2 & \text{if } p\equiv 2 \pmod 3, \end{cases} \quad \text{and} \quad \eta = \begin{cases} 1 & \text{if } p\equiv 1 \pmod 4, \\ 3 & \text{if } p\equiv 3 \pmod 4, \end{cases}$$

in (4.1), we obtain (1.3) and (1.4), respectively.

By (1.2) and the fact that $(\frac{1}{2})_k/k! = \binom{2k}{k}/4^k$,

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 0 & \text{if } p\equiv 1 \pmod 4 \\ -16\Gamma_p(\frac{3}{4})^4 & \text{if } p\equiv 3 \pmod 4 \end{cases} \pmod{p^2}. \tag{4.7}$$

According to [18, (H.2)], we have

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 & \text{if } p\equiv 1 \pmod 4 \\ 0 & \text{if } p\equiv 3 \pmod 4 \end{cases} \pmod{p^2}. \tag{4.8}$$

Then (1.5) follows easily from (4.7), (4.8) and the fact that $\binom{2k}{k} \equiv 0 \pmod p$ for $\frac{1}{2}p < k < p$.

It follows from (1.3) that

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv \begin{cases} p\Gamma_p(\frac{2}{3})^3 & \text{if } p\equiv 1 \pmod 6 \\ -6\Gamma_p(\frac{2}{3})^3 & \text{if } p\equiv 5 \pmod 6 \end{cases} \pmod{p^2}. \tag{4.9}$$

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According to [7, corollary 26], we attain

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{3})_k^3}{k!^3}\equiv\begin{cases}\Gamma_p(\frac{1}{3})^6 & \text{if } p\equiv 1 \pmod 6\\ 0 & \text{if } p\equiv 5 \pmod 6\end{cases} \pmod{p^2}. \tag{4.10}$$

Then (1.6) can be obtained from congruences (4.9) and (4.10). This completes the proof of theorem 1.2. □

Proof of theorem 1.3. We recall from [1, theorem 3.4.1] the following formula for hypergeometric series:

$$ {}_3F_2\begin{bmatrix} a & -b & -c \\ & 1+a+b & 1+a+c \end{bmatrix}; 1\Bigg] = \frac{\Gamma(\frac{1}{2}a+1)\Gamma(a+b+1)\Gamma(a+c+1)\Gamma(\frac{1}{2}a+b+c+1)}{\Gamma(a+1)\Gamma(\frac{1}{2}a+b+1)\Gamma(\frac{1}{2}a+c+1)\Gamma(a+b+c+1)}.$$

Let

$$\varepsilon=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4,\\ 3 & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Applying the above identity with $a=\frac{1}{4}$, $b=\frac{1}{4}(\varepsilon p-1)$, $c=\frac{1}{4}(-1-\varepsilon p)$ gives

$$ {}_3F_2\begin{bmatrix}\frac{1}{4} & \frac{1}{4}(1-\varepsilon p) & \frac{1}{4}(1+\varepsilon p) \\ & 1+\frac{1}{4}\varepsilon p & 1-\frac{1}{4}\varepsilon p\end{bmatrix}; 1\Bigg] = \frac{\Gamma(\frac{9}{8})\Gamma(\frac{5}{8})\Gamma(1+\frac{1}{4}\varepsilon p)\Gamma(1-\frac{1}{4}\varepsilon p)}{\Gamma(\frac{7}{8}+\frac{1}{4}\varepsilon p)\Gamma(\frac{7}{8}-\frac{1}{4}\varepsilon p)\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})}. \tag{4.11}$$

From Euler's reflection formula and the fact that $\Gamma(x+1)=x\Gamma(x)$ it is easy to see that

$$\frac{\Gamma(1+\frac{1}{4}\varepsilon p)\Gamma(1-\frac{1}{4}\varepsilon p)}{\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})}=\varepsilon p\frac{\Gamma(\frac{1}{4}\varepsilon p)\Gamma(1-\frac{1}{4}\varepsilon p)}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}=\varepsilon p(-1)^{(\varepsilon p-1)/4} \tag{4.12}$$

When $p\equiv 1 \pmod 8$ (or $p\equiv 3 \pmod 8$), $(\frac{9}{8})_{(\varepsilon p-1)/4}$ has exactly a multiple of p, which is $\frac{1}{8}p$ (or $\frac{3}{8}p$), while when $p\equiv 5 \pmod 8$ (or $p\equiv 7 \pmod 8$), $(\frac{9}{8})_{(\varepsilon p-1)/4}$ has no multiples of p. Then, by lemma 2.2 and the definition of $\Gamma_p(x)$,

$$\frac{\Gamma(\frac{7}{8}+\frac{1}{4}\varepsilon p)}{\Gamma(\frac{9}{8})}=\left(\frac{9}{8}\right)_{(\varepsilon p-1)/4}=\begin{cases}\dfrac{p}{8}\dfrac{\Gamma_p(\frac{7}{8}+\frac{1}{4}p)}{\Gamma_p(\frac{9}{8})} & \text{if } p\equiv 1 \pmod 8,\\[2ex] \dfrac{3p}{8}\dfrac{\Gamma_p(\frac{7}{8}+\frac{3}{4}p)}{\Gamma_p(\frac{9}{8})} & \text{if } p\equiv 3 \pmod 8,\\[2ex] -\dfrac{\Gamma_p(\frac{7}{8}+\frac{1}{4}p)}{\Gamma_p(\frac{9}{8})} & \text{if } p\equiv 5 \pmod 8,\\[2ex] -\dfrac{\Gamma_p(\frac{7}{8}+\frac{3}{4}p)}{\Gamma_p(\frac{9}{8})} & \text{if } p\equiv 7 \pmod 8.\end{cases} \tag{4.13}$$

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Similarly,

$$\frac{\Gamma(\frac{5}{8})}{\Gamma(\frac{7}{8}-\frac{1}{4}\varepsilon p)}=\left(\frac{7}{8}-\frac{\varepsilon p}{4}\right)_{(\varepsilon p-1)/4}=\begin{cases}\frac{\Gamma_p(\frac{5}{8})}{\Gamma_p(\frac{7}{8}-\frac{1}{4}p)} & \text{if } p\equiv 1 \pmod 8,\\ -\dfrac{p}{8}\dfrac{\Gamma_p(\frac{5}{8})}{\Gamma_p(\frac{7}{8}-\frac{3}{4}p)} & \text{if } p\equiv 3 \pmod 8,\\ -\frac{\Gamma_p(\frac{5}{8})}{\Gamma_p(\frac{7}{8}-\frac{1}{4}p)} & \text{if } p\equiv 5 \pmod 8,\\ \dfrac{5p}{8}\dfrac{\Gamma_p(\frac{5}{8})}{\Gamma_p(\frac{7}{8}-\frac{3}{4}p)} & \text{if } p\equiv 7 \pmod 8.\end{cases} \tag{4.14}$$

By (4.12)–(4.14),

$$\frac{\Gamma(\frac{9}{8})\Gamma(\frac{5}{8})\Gamma(1+\frac{1}{4}\varepsilon p)\Gamma(1-\frac{1}{4}\varepsilon p)}{\Gamma(\frac{7}{8}+\frac{1}{4}\varepsilon p)\Gamma(\frac{7}{8}-\frac{1}{4}\varepsilon p)\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})} = \begin{cases} 8\frac{\Gamma_p(\frac{5}{8})\Gamma_p(\frac{9}{8})}{\Gamma_p(\frac{7}{8}-\frac{1}{4}p)\Gamma_p(\frac{7}{8}+\frac{1}{4}p)} & \text{if } p\equiv 1 \pmod 8,\\ -p\frac{\Gamma_p(\frac{5}{8})\Gamma_p(\frac{9}{8})}{\Gamma_p(\frac{7}{8}-\frac{3}{4}p)\Gamma_p(\frac{7}{8}+\frac{3}{4}p)} & \text{if } p\equiv 3 \pmod 8,\\ -p\frac{\Gamma_p(\frac{5}{8})\Gamma_p(\frac{9}{8})}{\Gamma_p(\frac{7}{8}-\frac{1}{4}p)\Gamma_p(\frac{7}{8}+\frac{1}{4}p)} & \text{if } p\equiv 5 \pmod 8,\\ \dfrac{15p^2}{8}\dfrac{\Gamma_p(\frac{5}{8})\Gamma_p(\frac{9}{8})}{\Gamma_p(\frac{7}{8}-\frac{3}{4}p)\Gamma_p(\frac{7}{8}+\frac{3}{4}p)} & \text{if } p\equiv 7 \pmod 8.\end{cases} \tag{4.15}$$

We can easily deduce from lemma 2.3 that

$$\Gamma_p(\tfrac{7}{8}-\tfrac{1}{4}p)\Gamma_p(\tfrac{7}{8}+\tfrac{1}{4}p)\equiv\Gamma_p(\tfrac{7}{8}-\tfrac{3}{4}p)\Gamma_p(\tfrac{7}{8}+\tfrac{3}{4}p)\equiv\Gamma_p(\tfrac{7}{8})^2 \pmod{p^2}.$$

Using the above congruences in (4.15) and then employing (2.3) and (2.4) yields

$$\frac{\Gamma(\frac{9}{8})\Gamma(\frac{5}{8})\Gamma(1+\frac{1}{4}\varepsilon p)\Gamma(1-\frac{1}{4}\varepsilon p)}{\Gamma(\frac{7}{8}+\frac{1}{4}\varepsilon p)\Gamma(\frac{7}{8}-\frac{1}{4}\varepsilon p)\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})} \equiv \begin{cases} -\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 1 \pmod 8\\ \frac{1}{8}p\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 3,5 \pmod 8\\ 0 & \text{if } p\equiv 7 \pmod 8\end{cases} \pmod{p^2}. \tag{4.16}$$

By the facts that $(a+bp)_k(a-bp)_k\equiv(a)_k^2 \pmod{p^2}$ and $(\frac{1}{4})_{(\varepsilon p-1)/4}\equiv 0 \pmod p$ for $\frac{1}{4}\varepsilon p<k<p$, we have

$$\begin{aligned} {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4}(1-\varepsilon p) & \frac{1}{4}(1+\varepsilon p)\\ & 1+\frac{1}{4}\varepsilon p & 1-\frac{1}{4}\varepsilon p\end{matrix};1\right] &\equiv {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ & 1 & 1\end{matrix};1\right]_{(\varepsilon p-1)/4}\\ &\equiv {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ & 1 & 1\end{matrix};1\right]_{p-1} \pmod{p^2}. \end{aligned} \tag{4.17}$$

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It follows from (4.11), (4.16) and (4.17) that

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ & 1 & 1\end{matrix};1\right]_{p-1} \equiv \begin{cases} -\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 1 \pmod 8 \\ \frac{1}{8}p\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 3,5 \pmod 8 \\ 0 & \text{if } p\equiv 7 \pmod 8 \end{cases} \pmod{p^2}. $$

This proves (1.7).

By (1.4) and (1.7),

$$ \sum_{k=0}^{p-1}(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \equiv \begin{cases} p\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 & \text{if } p\equiv 1,7 \pmod 8 \\ 0 & \text{if } p\equiv 3 \pmod 8 \\ -8\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 & \text{if } p\equiv 5 \pmod 8 \end{cases} \pmod{p^2} \quad (4.18) $$

and

$$ \sum_{k=0}^{p-1}\frac{(\frac{1}{4})_k^3}{k!^3} \equiv \begin{cases} -\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 1 \pmod 8 \\ \frac{1}{8}p\Gamma_p(\frac{1}{8})^3\Gamma_p(\frac{5}{8}) & \text{if } p\equiv 3,5 \pmod 8 \\ 0 & \text{if } p\equiv 7 \pmod 8 \end{cases} \pmod{p^2}. \quad (4.19) $$

Then (1.8) follows from (4.18) and (4.19). This completes the proof of theorem 1.3. □

Proof of theorem 1.4. Let

$$ \varepsilon = \begin{cases} 1 & \text{if } p\equiv 1 \pmod 3, \\ 2 & \text{if } p\equiv 2 \pmod 3, \end{cases} $$

and $\zeta = \mathrm{e}^{2\pi\mathrm{i}/3}$. Taking $\alpha=\lambda=\frac{1}{3}$, $a=\frac{1}{3}(4+2\varepsilon\zeta p)$, $b=\frac{1}{3}(4+2\varepsilon\zeta^2 p)$, $m=\frac{1}{3}(\varepsilon p-1)$ in (3.1), we get

$$ \begin{aligned} {}_4F_3&\left[\begin{matrix}1 & \frac{1}{3}(1-\varepsilon p) & \frac{1}{3}(1-\varepsilon\zeta p) & \frac{1}{3}(1-\varepsilon\zeta^2 p) \\ & 1+\frac{1}{3}\varepsilon p & 1+\frac{1}{3}\varepsilon\zeta p & 1+\frac{1}{3}\varepsilon\zeta^2 p\end{matrix};1\right] \\ &= \frac{(\frac{4}{3})_{(\varepsilon p-1)/3}(\frac{1}{3}(2-\varepsilon p))_{(\varepsilon p-1)/3}}{(1+\frac{1}{3}\varepsilon\zeta p)_{(\varepsilon p-1)/3}(1+\frac{1}{3}\varepsilon\zeta^2 p)_{(\varepsilon p-1)/3}} \\ &\quad\times {}_4F_3\left[\begin{matrix}\frac{1}{6} & \frac{1}{3}(1-\varepsilon p) & \frac{1}{3}(1-\varepsilon\zeta p) & \frac{1}{3}(1-\varepsilon\zeta^2 p) \\ & \frac{7}{6} & \frac{2}{3} & \frac{1}{3}\end{matrix};1\right]. \end{aligned} \quad (4.20) $$

By lemma 2.2 and (2.3),

$$ \begin{aligned} (\tfrac{4}{3})_{(\varepsilon p-1)/3} &= \varepsilon p(\tfrac{1}{3})_{(\varepsilon p-1)/3} \\ &= (-1)^{(\varepsilon p-1)/3}\varepsilon p\frac{\Gamma_p(\frac{1}{3}\varepsilon p)}{\Gamma_p(\frac{1}{3})} \\ &= (-1)^{1+(\varepsilon p-1)/3}\varepsilon p\frac{\Gamma_p(1+\frac{1}{3}\varepsilon p)}{\Gamma_p(\frac{1}{3})} \end{aligned} \quad (4.21) $$

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$$(1+\tfrac{1}{3}\varepsilon\zeta p)_{(\varepsilon p-1)/3} = (-1)^{(\varepsilon p-1)/3}\frac{\Gamma_p(\frac{1}{3}(2-\varepsilon\zeta^2 p))}{\Gamma_p(1+\frac{1}{3}\varepsilon\zeta p)}, \tag{4.22}$$

$$(1+\tfrac{1}{3}\varepsilon\zeta^2 p)_{(\varepsilon p-1)/3} = (-1)^{(\varepsilon p-1)/3}\frac{\Gamma_p(\frac{1}{3}(2-\varepsilon\zeta p))}{\Gamma_p(1+\frac{1}{3}\varepsilon\zeta^2 p)}. \tag{4.23}$$

When $p\equiv 1 \pmod 3$,

$$(\tfrac{1}{3}(2-p))_{(p-1)/3} = (-1)^{(p-1)/3}\frac{\Gamma_p(\frac{1}{3})}{\Gamma_p(\frac{1}{3}(2-p))}. \tag{4.24}$$

By (4.21)–(4.24),

$$\frac{(\frac{4}{3})_{(p-1)/3}(\frac{1}{3}(2-p))_{(p-1)/3}}{(1+\frac{1}{3}\zeta p)_{(p-1)/3}(1+\frac{1}{3}\zeta^2 p)_{(p-1)/3}} = -p\frac{\Gamma_p(1+\frac{1}{3}p)\Gamma_p(1+\frac{1}{3}\zeta p)\Gamma_p(1+\frac{1}{3}\zeta^2 p)}{\Gamma_p(\frac{1}{3}(2-p))\Gamma_p(\frac{1}{3}(2-\zeta p))\Gamma_p(\frac{1}{3}(2-\zeta^2 p))}. \tag{4.25}$$

By lemma 2.3,

$$\begin{aligned}\Gamma_p(1+\tfrac{1}{3}p) &\equiv \Gamma_p(1)(1+\tfrac{1}{3}pG_1(1)+\tfrac{1}{6}p^2G_2(1)) \pmod{p^3},\\ \Gamma_p(1+\tfrac{1}{3}\zeta p) &\equiv \Gamma_p(1)(1+\tfrac{1}{3}\zeta pG_1(1)+\tfrac{1}{18}\zeta^2p^2G_2(1)) \pmod{p^3},\\ \Gamma_p(1+\tfrac{1}{3}\zeta^2 p) &\equiv \Gamma_p(1)(1+\tfrac{1}{3}\zeta^2 pG_1(1)+\tfrac{1}{18}\zeta p^2G_2(1)) \pmod{p^3}.\end{aligned}$$

Then

$$\Gamma_p(1+\tfrac{1}{3}p)\Gamma_p(1+\tfrac{1}{3}\zeta p)\Gamma_p(1+\tfrac{1}{3}\zeta^2 p) \equiv \Gamma_p(1)^3 \pmod{p^3}.$$

Similarly,

$$\Gamma_p(\tfrac{1}{3}(2-p))\Gamma_p(\tfrac{1}{3}(2-\zeta))\Gamma_p(\tfrac{1}{3}(2-\zeta^2 p)) \equiv \Gamma_p(\tfrac{2}{3})^3 \pmod{p^3}.$$

Using the above two congruences in (4.25) and noticing (2.2) gives

$$\frac{(\frac{4}{3})_{(p-1)/3}(\frac{1}{3}(2-p))_{(p-1)/3}}{(1+\frac{1}{3}\zeta p)_{(p-1)/3}(1+\frac{1}{3}\zeta^2 p)_{(p-1)/3}} \equiv -p\frac{\Gamma_p(1)^3}{\Gamma_p(\frac{2}{3})^3} = \frac{p}{\Gamma_p(\frac{2}{3})^3} \pmod{p^3}. \tag{4.26}$$

It follows from (2.1), (4.20) and (4.26) that

$${}_3F_2\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3};1\\ & 1 & 1\end{bmatrix}_{(p-1)/3} \equiv \frac{p}{\Gamma_p(\frac{2}{3})^3}\,{}_3F_2\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3};1\\ & \frac{7}{6} & \frac{2}{3}\end{bmatrix}_{(p-1)/3} \pmod{p^3}.$$

Therefore, by [7, corollary 26] and the fact that $(\frac{1}{3})_k \equiv 0 \pmod p$ for $\frac{1}{3}p<k<p$,

$$p\cdot {}_3F_2\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3};1\\ & \frac{7}{6} & \frac{2}{3}\end{bmatrix}_{(p-1)/3} \equiv \Gamma_p(\tfrac{1}{3})^6\Gamma_p(\tfrac{2}{3})^3 \pmod{p^3}.$$

This proves (1.9).

When $p\equiv 2 \pmod 3$, $(\frac{1}{3}(2-2p))_{(2p-1)/3}$ has exactly a multiple of p, which is $-\frac{1}{3}p$. Using lemma 2.2 and noting the definition of $\Gamma_p(x)$, we obtain

$$(\tfrac{1}{3}(2-2p))_{(2p-1)/3} = (-1)^{1+(2p-1)/3}\frac{p}{3}\frac{\Gamma_p(\frac{1}{3})}{\Gamma_p(\frac{1}{3}(2-2p))}. \tag{4.27}$$

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Then by (4.21)–(4.23) and (4.27),

$$\begin{aligned}\frac{(\frac{4}{3})_{(p-1)/3}(\frac{1}{3}(2-p))_{(p-1)/3}}{(1+\frac{1}{3}\zeta p)_{(p-1)/3}(1+\frac{1}{3}\zeta^2 p)_{(p-1)/3}} &= \frac{2p^2}{3}\frac{\Gamma_p(1+\frac{1}{3}p)\Gamma_p(1+\frac{1}{3}\zeta p)\Gamma_p(1+\frac{1}{3}\zeta^2 p)}{\Gamma_p(\frac{1}{3}(2-p))\Gamma_p(\frac{1}{3}(2-\zeta p))\Gamma_p(\frac{1}{3}(2-\zeta^2 p))} \\ &\equiv \frac{2p^2}{3}\frac{\Gamma_p(1)^3}{\Gamma_p(\frac{2}{3})^3} = -\frac{2p^2}{3\Gamma_p(\frac{2}{3})^3} \pmod{p^3}. \end{aligned} \tag{4.28}$$

Then from (2.1), (4.20) and (4.28) we are led to

$$ {}_3F_2\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & 1 & 1\end{bmatrix}_{(2p-1)/3} \equiv -\frac{2p^2}{3\Gamma_p(\frac{2}{3})^3}\,{}_3F_2\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\ & \frac{7}{6} & \frac{2}{3}\end{bmatrix}_{(2p-1)/3} \pmod{p^3}.$$

Hence, by [7, corollary 26] and the fact that $(\frac{1}{3})_k \equiv 0 \pmod{p}$ for $\frac{2}{3}p < k < p$,

$$p\cdot {}_3F_2\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3};1 \\ & \frac{7}{6} & \frac{2}{3}\end{bmatrix}_{(2p-1)/3} \equiv \tfrac{1}{2}p\Gamma_p(\tfrac{1}{3})^6\Gamma_p(\tfrac{2}{3})^3 \pmod{p^2},$$

which proves (1.10). The proof of theorem 1.4 is complete. □

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