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ON GENERALIZED CUMULATIVE ENTROPIES

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In the present paper, we introduce a generalization of the cumulative entropy proposed by Di Crescenzo and Longobardi [8]. This new notion is related to the lower records and the reversed relevation transform. Dynamic version of the newly proposed measure is considered. Several properties including the effect of linear transformations, a two-dimensional version, a normalized version, bounds, stochastic ordering, etc. are studied for the generalized cumulative entropy (GCE). Similar results are obtained for the dynamic GCE. Various relationships with other functions are derived. A class of distributions is introduced and several properties are studied. Finally, empirical GCE is proposed to estimate the newly proposed information measure.

1. INTRODUCTION

In the context of probability theory, entropy describes the amount of uncertainty associated with a random variable. For an information source having n components with associated probabilities of occurrence $p_1, \ldots, p_n$, Shannon [30] showed that the quantity $\mathcal{S} = -\sum_{i=1}^{n} p_i \ln p_i$ is the measure of uncertainty in the outcome of a particular event. Let X denote the random lifetime of a component with absolutely continuous cumulative distribution function $F_X(x)$, probability density function $f_X(x)$ and reliability (survival) function $\bar{F}_X(x)(= 1 - F_X(x))$. Consider the support of X as $(0, +\infty)$. The entropy $\mathcal{S}$ can be generalized for a non-negative random variable X as

$$\mathcal{H}(X) = -\int_0^{+\infty} f_X(x) \ln f_X(x) dx. \tag{1.1}$$

The measure given by (1.1) is known as the differential entropy which may be negative, whereas the Shannon's entropy $\mathcal{S}$ is always positive. For various properties and applications of the differential entropy we refer to Cover and Thomas [5]. A wide variety of competing measures of entropy have been proposed by several authors which can be considered as the generalizations of the differential entropy (1.1). In this direction, we refer to Renyi [26], Varma [32], Kapur [15] and Tsallis [31].

There exist several situations where $\mathcal{H}(X)$ is not appropriate to measure uncertainty of a component when its age has to be considered. One's interest may be in studying the residual lifetime of a component which is still working at time $t > 0$. However, the

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random lifetime is not necessarily related to the future but to the past. Consider a system which is working during a specified time interval and its state is observed only at certain pre-specified inspection times. Suppose the system is inspected for the first time and it is found to be down, then the uncertainty relies in the interval $(0, t)$, it has stopped working. Ebrahimi [11] and Di Crescenzo and Longobardi [7] proposed dynamic versions of (1.1), respectively known as residual and past entropy to deal with these situations. Residual entropy quantifies uncertainty of the random lifetime $X_{(t)} = [X - t|X \geq t]$, whereas past entropy quantifies uncertainty of the random lifetime $X_{[t]} = [t - X|X \leq t]$. Note that $X_{(t)}$ and $X_{[t]}$ are known as residual and inactivity time, respectively. For various properties and applications of residual and past entropies, we refer to Ebrahimi and Pellerey [13], Ebrahimi [11], Ebrahimi and Kirmani [12], Belzunce et al. [3], Nanda and Paul [19], Kundu, Nanda, and Maiti [18] and Sachlas and Papaioannou [27].

Rao et al. [24] pointed out few drawbacks of the differential entropy given by (1.1).

- It is defined when distributions have probability density functions.
- It may be negative. For example, the differential entropy of a uniform random variable in $(0, a)$ is negative when $a < 1$.

They introduced another type of entropy known as cumulative residual entropy (CRE) in order to provide a way to accommodate random variables that do not have a well-defined probability density function. The CRE of a non-negative random variable X is given by

$$\mathcal{E}(X) = -\int_0^{+\infty} \bar{F}_X(x) \ln \bar{F}_X(x) dx. \tag{1.2}$$

It has several important properties. Equation (1.2) is consistent in both discrete and continuous domains. It is always non-negative. The CRE of the uniform distribution in $(0, a)$ is non-negative for $0 < a < +\infty$, though the differential entropy is negative for $a < 1$. For various other properties and applications of (1.2), we refer to Wang et al. [34,35], Rao et al. [24], Rao [23], Asadi and Zohrevand [2], Wang and Vemuri [33] and Navarro, Del Aguila, and Asadi [20]. A dynamic version of (1.2) for residual lifetime has been proposed by Asadi and Zohrevand [2] as

$$\mathcal{E}(X; t) = -\int_t^{+\infty} \frac{\bar{F}_X(x)}{\bar{F}_X(t)} \ln \frac{\bar{F}_X(x)}{\bar{F}_X(t)} dx, \tag{1.3}$$

for $t > 0$ such that $\bar{F}_X(t) > 0$. Motivated by (1.2), a dual concept of the CRE was introduced by Di Crescenzo and Longobardi [8], which is suitable to describe the uncertainty of the problems related to aging properties of reliability theory based on the past and on the inactivity times. For a non-negative random variable X the cumulative entropy (CE) is defined as

$$\mathcal{CE}(X) = -\int_0^{+\infty} F_X(x) \ln F_X(x) dx. \tag{1.4}$$

Motivated by Di Crescenzo and Longobardi [7] and analogous to (1.3), Di Crescenzo and Longobardi [8] proposed a dynamic version of the CE to quantify the uncertainty contained in past lifetime. It is defined as

$$\mathcal{CE}(X;t) = -\int_0^t \frac{F_X(x)}{F_X(t)} \ln \frac{F_X(x)}{F_X(t)} dx, \tag{1.5}$$

for $t > 0$ such that $F_X(t) > 0$. Recently, Psarrakos and Navarro [21] extended the concept of the CRE given by (1.2) as

$$\mathcal{E}_n(X) = \frac{1}{n!} \int_0^{+\infty} \bar{F}_X(x)[\bar{\Lambda}(x)]^n dx, \quad n = 1, 2, \ldots, \tag{1.6}$$

where $\bar{\Lambda}(x)(= -\ln \bar{F}_X(x))$ is known as the cumulative hazard function. They named it generalized cumulative residual entropy (GCRE). For $n = 1$, (1.6) reduces to (1.2), that is, $\mathcal{E}_1(X) = \mathcal{E}(X)$. Also, when $n = 0$, then $\mathcal{E}_0(X) = \int_0^{+\infty} \bar{F}_X(x)dx = E(X)$, provided expectation exists. In analogy to (1.3), a dynamic version of the GCRE was also introduced by the authors. It is given by

$$\mathcal{E}_n(X;t) = \frac{1}{n!} \int_t^{+\infty} \frac{\bar{F}_X(x)}{\bar{F}_X(t)} \Big[-\ln \frac{\bar{F}_X(x)}{\bar{F}_X(t)} \Big]^n dx, \quad n = 1, 2, \ldots, \tag{1.7}$$

for $t > 0$ such that $\bar{F}_X(t) > 0$. It is easy to see that $\mathcal{E}_n(X;0) = \mathcal{E}_n(X)$ and $\mathcal{E}_1(X;t) = \mathcal{E}(X;t)$. Also,

$$\mathcal{E}_0(X;t) = \int_t^{+\infty} \frac{\bar{F}_X(x)}{\bar{F}_X(t)} dx = E(X_{(t)}),$$

known as the mean residual life. Psarrakos and Navarro [21] obtained various results related to characterizations, stochastic ordering and aging classes for $\mathcal{E}_n(X)$ and $\mathcal{E}_n(X;t)$. In this paper, we introduce a new information measure which is dual to GCRE given by (1.6). It is noticed that the proposed measure is related to the lower records of a sequence of independent and identically distributed non-negative random variables and the reversed relevation transform. We also consider its dynamic version for inactivity time. The paper is arranged as follows. In Section 2, we recall some definitions and preliminary results. In Section 3, we obtain various results on GCE. It includes basic properties such as the effect of linear transformations, a two-dimensional representation of it, bounds and stochastic orderings. Further, some relationships with other functions are derived. The dynamic version of the GCE is considered in Section 4. Various results similar to the Section 3 are obtained. A new class of distributions is introduced and studied. Empirical GCE is proposed in Section 5 to estimate the GCE. It is computed for the exponentially distributed random sample.

Throughout the paper we assume that the terms increasing and decreasing are used in non-strict sense. All expectations, conditional random variables and derivatives wherever used are implicitly assumed to exist. By convention, we assume that $\sum_{k=0}^{i} = 0$, for $i < 0$.

2. SOME PRELIMINARY RESULTS AND DEFINITIONS

In this section, we review some preliminary results, definitions and well-known notions of aging and stochastic orders. Let $X_1^*, X_2^*, \ldots, X_n^*$, etc. be a sequence of independent and identically distributed non-negative random variables with a common absolutely continuous distribution function $F_X(x)$ and probability density function $f_X(x)$. An observation X_i^* is said to be a lower record if $X_i^* < X_j^*$ for all $j < i$. Assume that X_i^* occurs at time i, then the record time sequence is defined as $L(1) = 1$ and $L(n+1) = \min\{i : X_i^* < X_{L(n)}^*\}$ for $n = 1, 2, \ldots$. The random variables $X_n = X_{L(n)}^*$, $n = 1, 2, \ldots$ are said to be the lower records. Consider a system which is repaired instantaneously after each failure. Assume that once repair is completed, the duration of the next lifetime is stochastically smaller than the previous lifetime. It is due to imperfect repairs. Here, $\{X_n\}_{n\geq 1}$ may be interpreted as the sequence of lifetimes of this system. The marginal distribution of X_n is given by

$$f_n(x) = \frac{[-\ln F_X(x)]^{n-1}}{(n-1)!} f_X(x), \quad x > 0,\ n = 1, 2, \ldots. \tag{2.1}$$

The cumulative distribution function of X_n is

$$\begin{aligned} F_n(t) &= \frac{1}{(n-1)!}\int_0^t [-\ln F_X(x)]^{n-1} f_X(x)dx = \sum_{k=0}^{n-1} \frac{[-\ln F_X(t)]^k}{k!} F_X(t) \\ &= \sum_{k=0}^{n-1} \frac{[\Lambda(t)]^k}{k!} F_X(t), \end{aligned} \tag{2.2}$$

where $\Lambda(t)(= -\ln F_X(t))$ is known as the cumulative reversed hazard function. For details on records, we refer to Arnold, Balakrishnan, and Nagaraja [1]. An alternative interpretation of (2.2) can be given from the notion of reversed relevation transform. For detail we refer to Di Crescenzo and Toomaj [10]. Let X and Y be two non-negative absolutely continuous random variables with cumulative distribution functions $F_X(x)$ and $F_Y(x)$, respectively. Assume X and Y to be independent. Then the reversed relevation transform of X and Y is defined as

$$F_X \tilde{\#} F_Y(x) = F_X(x) + F_Y(x) \int_x^{+\infty} \frac{dF_X(t)}{F_Y(t)}, \quad x > 0. \tag{2.3}$$

Let $F_X(x) = F_Y(x)$ and $F_n(x)$ denotes the cumulative distribution function of the nth failure, $n = 1, 2, \ldots$. Then we have

$$F_n(x) = \begin{cases} F(x), & \text{if } n = 1, \\ F_{n-1}\tilde{\#}F(x), & \text{if } n \geq 2. \end{cases} \tag{2.4}$$

Along the lines of the proof of the theorem (see p. 113, Krakowski [17]), it can be showed that (2.2) and (2.4) are equivalent. The expression given by (2.2) can also be viewed as the cumulative distribution function of a Poisson process with mean $\Lambda(t)$. Raqab and Asadi [25] introduced the notion of the mean residual waiting time of records and present some monotonic and aging properties. In order to prove our main results, hereafter, we obtain results similar to Raqab and Asadi [25] based on the waiting time elapsed of records. The waiting time elapsed of the record model is $[t - X_n | X_n \leq t]$, where $t > 0$ and $n = 1, 2, \ldots$. The mean waiting inactivity time (MWIT) is defined as

$$\mu_n(t) = E(t - X_n | X_n \leq t) = \frac{\int_0^t F_n(x)dx}{F_n(t)} = \frac{\sum_{k=0}^{n-1}\int_0^t \eta_k(x)dx}{\sum_{k=0}^{n-1}\eta_k(t)}, \quad n = 1, 2, \ldots, \tag{2.5}$$

where the last equality is due to (2.2) and $\eta_k(x) = [\Lambda(t)]^k F_X(x)/k!$. For $k = 0, 1, \ldots, n-1$, we define

$$M_k(t) = \int_0^t \left[\frac{\Lambda(x)}{\Lambda(t)}\right]^k \frac{F_X(x)}{F_X(t)} dx \tag{2.6}$$

and

$$p_k(t) = \frac{[\Lambda(t)]^k F_X(t)/k!}{\sum_{j=0}^{n-1}([\Lambda(t)]^j F_X(t)/j!)} = \frac{P(Y_t = k)}{P(Y_t < n)}, \tag{2.7}$$

where Y_t is a Poisson random variable with mean $\Lambda(t)$. Therefore, from (2.5) we obtain

$$\mu_n(t) = \sum_{k=0}^{n-1} p_k(t) M_k(t), \quad n = 1, 2, \ldots. \tag{2.8}$$

Note that when $n = 1$, the MWIT of the first lower record reduces to

$$\mu_1(t) = \int_0^t \frac{F_X(x)}{F_X(t)} dx,$$

which is the mean inactivity time of the parent distribution. Now we consider the following definitions which will be useful in the next sections.

Definition 2.1: *Let X and Y be two non-negative random variables with absolutely continuous cumulative distribution functions $F_X(x)$ and $F_Y(x)$, survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(x)$ and probability density functions $f_X(x)$ and $f_Y(x)$, respectively. Then*

- *X is said to be smaller than Y in likelihood ratio order, denoted by $X \leq_{lr} Y$, if $f_Y(x)/f_X(x)$ is increasing in $x > 0$.*
- *X is said to be smaller than Y in usual stochastic order, denoted by $X \leq_{st} Y$, if $\bar{F}_X(x) \leq \bar{F}_Y(x)$, for all $x > 0$.*
- *X is said to be smaller than Y in the decreasing convex order, denoted by $X \leq_{dcx} Y$, if $E(\phi(X)) \leq E(\phi(Y))$, for all decreasing convex functions ϕ.*

Also we have $X \leq_{st} Y$ if, and only if,

$$E(\phi(X)) \leq E(\phi(Y)), \tag{2.9}$$

for all increasing functions ϕ for which the expectations exist. For details on usual stochastic order and related results, we refer to Shaked and Shanthikumar [29].

3. GENERALIZED CUMULATIVE ENTROPY

In this section, in analogy to (1.6), we propose a generalization of the CE given by (1.4). Let X be a non-negative random variable with absolutely continuous distribution function $F_X(x)$. Then the GCE of X is defined as

$$\mathcal{CE}_n(X) = \frac{1}{n!} \int_0^{+\infty} F_X(x)[\Lambda(x)]^n dx, \quad n = 1, 2, \ldots. \tag{3.1}$$

From (3.1), it follows that $\mathcal{CE}_n(X)$ takes values from the interval $[0, +\infty]$. In particular, for a degenerate random variable it is easy to obtain $\mathcal{CE}_n(X) = 0$. Moreover, when $n = 1$, $\mathcal{CE}_n(X)$ reduces to $\mathcal{CE}(X)$. Also for $n = 0$, $\mathcal{CE}_0(X) = \int_0^{+\infty} F_X(x) dx$, which may be divergent. The following remark is for the random variables with finite support.

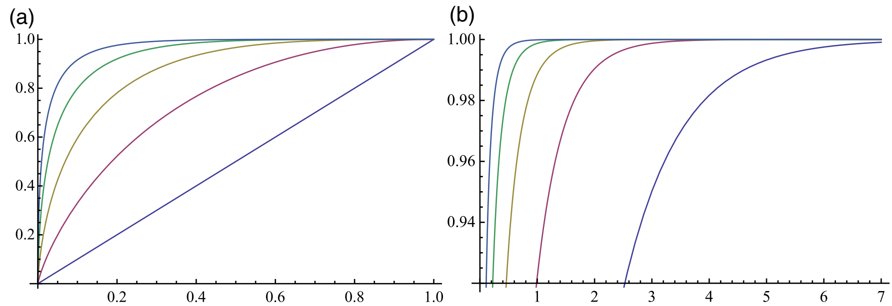

FIGURE 1. (a) Plot of $F_n(x)$ for uniform distribution for $n = 1, 2, 3, 4, 5$ (from below). (b) Plot of $F_n(x)$ for exponential distribution for $n = 1, 2, 3, 4, 5$ (from below). Here, $\mathcal{CE}_n(X)$ corresponds to the areas between these functions for $n = 1, 2, 3, 4$.

REMARK 3.1: *Let X be a random variable with support $[0, b]$, where b is finite. Then $\mathcal{CE}_n(X) = (1/n!) \int_0^b F_X(x)[\Lambda(x)]^n dx,\ n = 1, 2, \ldots$. In particular, $\mathcal{CE}_0(X) = b - E(X)$.*

REMARK 3.2: *Making use of (2.2), the GCE given by (3.1) can be written as $\mathcal{CE}_n(X) = \int_0^{+\infty} [F_{n+1}(x) - F_n(x)]dx$, $n = 1, 2, \ldots$. This implies that for $n = 1, 2, \ldots$, $\mathcal{CE}_n(X)$ can be interpreted as the area between the cumulative distribution functions $F_{n+1}(x)$ and $F_n(x)$. In particular, for $n = 0$, $\mathcal{CE}_0(X)$ represents the area under $F_1(x) = F_X(x)$.*

Figure 1 represents the plot of the areas under $F_n(x)$ for uniform and exponential distributions with cumulative distribution functions $F_X(x) = x,\ 0 < x < 1$ and $F_X(x) = 1 - \exp\{-x\},\ x > 0$, respectively. The area under $F_1(x) = F_X(x)$ in Figure 1a is $E(X) = 0.5$.

REMARK 3.3: *Let X be a symmetric random variable with respect to the finite mean $m = E(X)$, that is, $F_X(x + m) = 1 - F_X(m - x)$, for all $x \in \mathcal{R}$. Then*

$$\mathcal{CE}_n(X) = \mathcal{E}_n(X), \quad n = 1, 2, \ldots.$$

Making use of (2.1) in (3.1) we obtain for $n = 1, 2, \ldots$,

$$\begin{aligned} \mathcal{CE}_n(X) &= \frac{1}{n!} \int_0^{+\infty} \frac{1}{r_X(x)} [\Lambda(x)]^n f_X(x)dx \\ &= \int_0^{+\infty} \frac{1}{r_X(x)} f_{n+1}(x)dx = E\Big(\frac{1}{r_X(X_{n+1})}\Big), \end{aligned} \quad \textbf{(3.2)}$$

where $r_X(x) = f_X(x)/F_X(x)$ is the reversed failure rate function and X_{n+1} is the $(n+1)$th record with cumulative distribution function $F_{n+1}(x)$ and probability density function $f_{n+1}(x)$. From (2.1) we have

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{\Lambda(x)}{n}, \quad n = 1, 2, \ldots,$$

which is a decreasing function in $x > 0$, where $f_{n+1}(x)$ is the probability density function of the $(n+1)$th record. Thus, we have $X_{n+1} \le_{lr} X_n$. It implies $X_{n+1} \le_{st} X_n$, that is, $F_{n+1}(x) \ge F_n(x)$. From (2.9) and (3.2) we obtain the following proposition. The proof is simple. Hence, we omit it.

Proposition 3.4: *Let the reversed failure rate function $r_X(x)$ be decreasing (increasing) in $x > 0$. Then for $n = 0, 1, 2, \ldots,$*

$$\mathcal{CE}_n(X) \geq (\leq) \mathcal{CE}_{n+1}(X). \quad \textbf{(3.3)}$$

Note that the assumption $F_X(x)$ is decreasing reversed failure rate can be replaced by $F_X(x)$ is log-concave in Proposition 3.4 as they are equivalent. For detail on log-concave distributions we refer to Sengupta and Nanda [28]. There are many distributions for which the reversed failure rate function is decreasing. For example, two-parameter Weibull, gamma, Pareto and log-normal distributions belong to this class. But, there does not exist any life distribution which has increasing reversed failure rate over the domain $[0, +\infty)$ (see Block, Savits, and Singh [4]). Analogous to the normalized CE due to Di Crescenzo and Longobardi [8], hereafter, we introduce a normalized version of (3.1) for log-concave cumulative distribution functions. From the recurrence relation given by (3.3) we notice that for any non-degenerate random variable with log-concave distribution function, $\mathcal{CE}_n(X)$ is decreasing in n, that is,

$$\mathcal{CE}_0(X) \geq \mathcal{CE}_1(X) \geq \mathcal{CE}_2(X) \geq \cdots \geq \mathcal{CE}_n(X) \geq \cdots. \quad \textbf{(3.4)}$$

Thus, for a non-negative random variable X with non-zero $\mathcal{CE}_0(X)$, the normalized GCE is defined as

$$\mathcal{NCE}_n(X) = \frac{\mathcal{CE}_n(X)}{\mathcal{CE}_0(X)} = \frac{1}{n!\, \mathcal{CE}_0(X)} \int_0^{+\infty} F_X(x)[\Lambda(x)]^n dx, \quad n = 1, 2, \ldots. \quad \textbf{(3.5)}$$

Using (3.4) in (3.5) it is easy to see that the normalized GCE takes values from the interval $[0, 1]$. In the following, we discuss the effect of the linear transformations on the GCE.

Proposition 3.5: *Let $Y = aX + b$ be the linear transformation with $a > 0$ and $b \geq 0$. Then*

$$\mathcal{CE}_n(Y) = a\, \mathcal{CE}_n(X), \quad n = 1, 2, \ldots. \quad \textbf{(3.6)}$$

Proof: The proof follows from $F_{aX+b}(x) = F_X((x - b)/a)$, for all $x \in \mathcal{R}$. Hence, omitted. ■

Remark 3.6: *From (3.6) we observe that the right-hand side does not depend on b. This implies that $\mathcal{CE}_n(X)$ is a shift-independent measure. Moreover, it can be easily showed that this property also holds for the generalized CRE given by (1.6).*

In this part of the paper, we introduce two-dimensional version of $\mathcal{CE}_n(X)$. Let X and Y be two non-negative random variables with joint cumulative distribution function $F_{XY}(x, y) = P(X \leq x, Y \leq y)$ and the joint probability density function $f_{XY}(x, y)$. Here, X and Y may be thought of as the lifetimes of two components of a system. The marginal cumulative distribution functions of X and Y are denoted by $F_X(x)$ and $F_Y(y)$, respectively. There are several ways in extending a one dimension (univariate) concept to higher dimensions. A natural extension of the GCE given by (3.1) to the bivariate setup can be obtained by substituting the joint cumulative distribution function $F_{XY}(x, y)$ in place of $F_X(x)$. The bivariate GCE is defined as

$$\mathcal{CE}_n(X, Y) = \frac{1}{n!} \int_0^{+\infty} \int_0^{+\infty} F_{XY}(x, y)[-\ln F_{XY}(x, y)]^n dx dy, \quad n = 1, 2, \ldots. \quad \textbf{(3.7)}$$

The measure (3.7) reduces to the two-dimensional analog of (1.4) for $n = 1$. Assume X and Y are independent, that is, $F_{XY}(x, y) = F_X(x)F_Y(y)$. Then using binomial theorem in (3.7) we obtain the following proposition.

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PROPOSITION 3.7: *The bivariate GCE of two independent random variables X and Y can be expressed as*

$$\mathcal{CE}_n(X,Y) = \sum_{k=0}^{n} \mathcal{CE}_{n-k}(X)\ \mathcal{CE}_k(Y). \tag{3.8}$$

From (3.8) we get the following particular relation:

$$\mathcal{CE}_1(X,Y) = \left[\int_0^{+\infty} F_X(x)dx\right]\mathcal{CE}(Y) + \left[\int_0^{+\infty} F_Y(y)dy\right]\mathcal{CE}(X), \tag{3.9}$$

which is obtained by Di Crescenzo and Longobardi [8]. Note that the integrals inside the square brackets of (3.9) may be divergent. Thus we have the following remark by restricting our attention to random variables with finite supports.

REMARK 3.8: *Let X and Y be two non-negative, independent random variables having bounded supports $[0, a_1]$ and $[0, a_2]$, respectively. Here a_1 and a_2 are finite. Then for $n = 1$, (3.8) reduces to Proposition 2.2 of Di Crescenzo and Longobardi [8].*

The generalized conditional CE is given by

$$\mathcal{CE}_n(X|Y) = \frac{1}{n!}\int_0^{+\infty} F_{XY}(x|y)[-\ln F_{XY}(x|y)]^n dx, \quad n = 1, 2, \ldots, \tag{3.10}$$

where $F_{XY}(x|y) = F_{XY}(x,y)/F_Y(y)$ is the conditional distribution function of $X|Y$. Note that when X and Y are independent, then from (3.10) it is easy to obtain

$$\mathcal{CE}_n(X|Y) = \mathcal{CE}_n(X), \quad n = 1, 2, \ldots.$$

In the following proposition, we discuss the effect of the linear transformations on the generalized joint CE. It shows that the generalized joint CE is a shift-independent measure.

PROPOSITION 3.9: *Let Y_1 and Y_2 be two non-negative random variables with joint cumulative distribution function $F_{Y_1Y_2}(y_1, y_2)$. Assume $Y_i = b_i X_i + c_i$, with $b_i > 0$ and $c_i \geq 0$, $i = 1, 2$. Then*

$$\mathcal{CE}_n(Y_1, Y_2) = b_1 b_2\ \mathcal{CE}_n(X_1, X_2). \tag{3.11}$$

PROOF: Proof follows from (3.7) and $F_{Y_1Y_2}(x_1,x_2) = F_{X_1X_2}\left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}\right)$. ■

PROPOSITION 3.10: *Let Y_1 and Y_2 be two non-negative random variables as described in Proposition 3.9 Also, let $Y_i = \phi_i(X_i)$, $i = 1, 2$ be one-to-one transformations with $\phi_i(x_i)$'s are differentiable functions. Then*

$$\mathcal{CE}_n(Y_1, Y_2) = \frac{1}{n!}\int_0^{+\infty}\int_0^{+\infty} F_{X_1X_2}(x_1,x_2)[-\ln F_{X_1X_2}(x_1,x_2)]^n |J| dx_1 dx_2, \quad n = 1, 2, \ldots, \tag{3.12}$$

where J is the Jacobian of the transformations.

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Table 1. Numerical values of $\mathcal{CE}_n(X_\eta)$ and $\mathcal{CE}_n(\eta X)$ for uniform distribution in $(0, 2)$.

η	n	$\mathcal{CE}_n(X_\eta)$	$\mathcal{CE}_n(\eta X)$	η	n	$\mathcal{CE}_n(X_\eta)$	$\mathcal{CE}_n(\eta X)$
0.5	1	0.444432	0.249988	1.5	1	0.479963	0.749963
	2	0.148148	0.125000		2	0.288000	0.375000
	3	0.049382	0.062500		3	0.172800	0.187500
	4	0.016460	0.031250		4	0.103680	0.093750

Proposition 3.10 shows that the generalized joint CE is not invariant under non-singular transformations.

Hereafter, we consider two non-negative random variables X and X_η with cumulative distribution functions $F_X(x)$ and $F_{X_\eta}(x)$, respectively. Assume that X and X_η are related to the following relation:

$$F_{X_\eta}(x) = [F_X(x)]^\eta, \quad x > 0, \tag{3.13}$$

where η is a positive real number. The relation in (3.13) implies that X and X_η satisfy the proportional reversed hazard rate model. In this direction, we refer to Di Crescenzo [6] and Gupta and Gupta [14]. Note that (3.13) holds for the entire real line. Since we are interested about the lifetime distributions, we restrict our attention to $x > 0$. From (3.1) and (3.6) we have

$$\mathcal{CE}_n(\eta X) = \frac{\eta}{n!}\int_0^{+\infty} F_X(x)[\Lambda(x)]^n dx, \quad n = 1, 2, \ldots. \tag{3.14}$$

Moreover, from (3.1) and (3.13) we obtain

$$\mathcal{CE}_n(X_\eta) = \frac{\eta^n}{n!}\int_0^{+\infty} F_X^\eta(x)[\Lambda(x)]^n dx, \quad n = 1, 2, \ldots. \tag{3.15}$$

Di Crescenzo and Longobardi [8] proved a theoretical result dealing with comparison between CEs of X and X_η which is not true for the newly proposed information measure as shown in the following table. In Table 1, we present numerical values of $\mathcal{CE}_n(X_\eta)$ and $\mathcal{CE}_n(\eta X)$ for $n = 1, 2, 3, 4$, where X follows uniform distribution with cumulative distribution function $F_X(x) = x/2,\ 0 < x < 2$.

In the way of finding the relation between the GCEs of a pair of stochastically ordered random variables with finite support we consider the following example which shows that the usual stochastic order does not imply the ordering of the GCEs.

Example 3.11: Let X and Y be two random variables with a common support $[0, b]$, with b finite. The cumulative distribution functions of X and Y are $F_X(x) = \exp\{c(1 - b^2/x^2)\},\ 0 < x \le b$ and $F_Y(x) = \exp\{a(1 - b^2/x^2)\},\ 0 < x \le b$, respectively, where $c > 0,\ a > 0$ and $a \le c$. It is easy to show that $X \ge_{st} Y$, when $a \le c$, but $\mathcal{CE}_n(X) \ngeq \mathcal{CE}_n(Y)$ for all $n = 1, 2, 3, 4$ as shown in Table 2.

Thus, naturally the following question arises: under which condition $\mathcal{CE}_n(X)$ is smaller than $\mathcal{CE}_n(Y)$? We find the answer of this question in the next theorem. First, we prove the following lemma (an extension of Proposition 3.4 of Di Crescenso and Longobardi [8]) which is useful in this direction.

TABLE 2. Numerical values of $\mathcal{CE}_n(X)$ and $\mathcal{CE}_n(Y)$ for the random variables X and Y, as described in Example 3.11 for $b = 2$.

c	a	n	$\mathcal{CE}_n(X)$	$\mathcal{CE}_n(Y)$	c	a	n	$\mathcal{CE}_n(X)$	$\mathcal{CE}_n(Y)$
0.5	0.2	1	0.311359	0.314432	1.7	1.2	1	0.228912	0.259407
		2	0.172160	0.145103			2	0.160573	0.170593
		3	0.109280	0.083840			3	0.119012	0.120822
		4	0.076160	0.055463			4	0.092046	0.090555

LEMMA 3.12: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$, probability density function $f_X(x)$ and reversed failure rate function $r_X(x)$ such that $\mathcal{CE}_n(X) < +\infty$. Then*

$$\mathcal{CE}_n(X) = E[R_n^{(2)}(X)], \quad n = 1, 2, \ldots, \tag{3.16}$$

where

$$R_n^{(2)}(x) = \frac{1}{n!}\int_x^{+\infty} [\Lambda(z)]^n dz = \frac{1}{(n-1)!}\int_x^{+\infty}\int_z^{+\infty} r_X(t)[\Lambda(t)]^{n-1} dt dz. \tag{3.17}$$

PROOF: Using $F_X(x) = \int_0^x f_X(t)dt$ in (3.1), we obtain by Fubini's theorem

$$\mathcal{CE}_n(X) = \frac{1}{n!}\int_0^{+\infty}\left[\int_0^x f_X(t)dt\right][\Lambda(x)]^n dx = \int_0^{+\infty} f_X(t)\left[\int_t^{+\infty}\frac{[\Lambda(x)]^n}{n!}dx\right]dt. \tag{3.18}$$

Hence, the result follows. ■

Note that $R_n^{(2)}(x)$ given by (3.17) is a decreasing convex function in x. Then the following theorem immediately follows from (3.16) and (3.17).

THEOREM 3.13: *Let X and Y be two non-negative random variables such that $X \leq_{dcx} Y$. Then*

$$\mathcal{CE}_n(X) \leq \mathcal{CE}_n(Y), \quad n = 1, 2, \ldots. \tag{3.19}$$

It is noted that the Theorem 3.13 is an extension of Proposition 4.8 of Di Crescenso and Longobardi [8]. In the following, we obtain bounds of the GCE. The proof is simple, hence omitted.

PROPOSITION 3.14: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then for $n = 1, 2, \ldots,$*

- $\mathcal{CE}_n(X) \geq \sum_{i=0}^n \frac{(-1)^i}{i!(n-i)!}\int_0^{+\infty} F_X^{i+1}(x)dx.$
- $\mathcal{CE}_n(X) \geq R_n^{(2)}(\mu)$, *where $R_n^{(2)}(x)$ is given by (3.17) and $E(X) = \mu < +\infty$.*
- $\mathcal{CE}_n(X) \leq \frac{1}{n!}\int_0^{+\infty}[\Lambda(x)]^n dx$, *provided $\int_0^{+\infty}[\Lambda(x)]^n dx$ exists.*

In the following theorem, we obtain a relation between GCE and MWIT. First we prove the following lemma.

LEMMA 3.15: *Let X be a random variable as described in Lemma 3.12 such that $\mathcal{CE}_n(X) < +\infty$. Then*

$$\mathcal{CE}_n(X) = \frac{1}{n!}\int_0^{+\infty} r_X(z)\left[\int_0^z [\Lambda(x)]^{n-1}F_X(x)dx\right]dz, \quad n = 1, 2, \ldots. \tag{3.20}$$

PROOF: From (3.1) we have

$$\begin{aligned}\mathcal{CE}_n(X) &= \frac{1}{n!}\int_0^{+\infty} F_X(x)[\Lambda(x)]^{n-1}\Lambda(x)dx \\ &= \frac{1}{n!}\int_0^{+\infty} F_X(x)[\Lambda(x)]^{n-1}\Big[\int_x^{+\infty} r_X(z)dz\Big]dx. \end{aligned} \tag{3.21}$$

Applying Fubini's theorem in (3.21) we get the desired result. ■

THEOREM 3.16: *Let X be a random variable as described in Lemma 3.12 and X_{k+1} is the $(k+1)$th record with probability density function $f_{k+1}(x)$. Then for $n = 1, 2, \ldots,$ the following relation holds:*

$$\mathcal{CE}_n(X) = \frac{1}{n}\left[\sum_{k=0}^{n-1} E(\mu_n(X_{k+1})) - k\ \mathcal{CE}_k(X)\right]. \tag{3.22}$$

PROOF: From (3.20) we have

$$k\ \mathcal{CE}_k(X) = \frac{1}{(k-1)!}\int_0^{+\infty} r_X(z)\left[\int_0^z [\Lambda(x)]^{k-1}F_X(x)dx\right]dz. \tag{3.23}$$

Taking summation both sides with respect to k from 1 to n we get

$$\begin{aligned}\sum_{k=1}^{n} k\ \mathcal{CE}_k(X) &= \int_0^{+\infty} r_X(z)\sum_{k=1}^{n}\frac{1}{(k-1)!}\int_0^z [\Lambda(x)]^{k-1}F_X(x)dxdz \\ &= \int_0^{+\infty} r_X(z)\sum_{k=0}^{n-1}\frac{1}{k!}\int_0^z [\Lambda(x)]^{k}F_X(x)dxdz \\ &= \int_0^{+\infty} r_X(z)\mu_n(z)\sum_{k=0}^{n-1}\frac{[\Lambda(z)]^k}{k!}F_X(z)dz \quad (\text{from}(2.3)) \\ &= \int_0^{+\infty} \mu_n(z)\sum_{k=0}^{n-1}\frac{[\Lambda(z)]^k}{k!}f_X(z)dz \\ &= \sum_{k=0}^{n-1}\int_0^{+\infty}\mu_n(z)f_{k+1}(z)dz \\ &= \sum_{k=0}^{n-1}E(\mu_n(X_{k+1})). \end{aligned} \tag{3.24}$$

Using (3.24) and the relation $\sum_{k=1}^{n} k\ \mathcal{CE}_k(X) = \sum_{k=0}^{n-1} k\ \mathcal{CE}_k(X) + n\ \mathcal{CE}_n(X)$, we get the desired result. ■

REMARK 3.17: *In particular, when $n = 1$, from Theorem 3.16 we get $\mathcal{CE}_1(X) = E(\mu_1(X_1))$, where $\mu_1(x) = \int_0^x F_X(t)dt/F_X(x)$.*

PROPOSITION 3.18: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then for $n = 1, 2, \ldots,$ we have*

$$\mathcal{CE}_n(X) = \frac{1}{n}\left[\sum_{k=0}^{n-1} \frac{1}{k!}\ E([\Lambda(X)]^k \mu_n(X)) - \sum_{k=0}^{n-2} \frac{1}{k!}\ E([\Lambda(X)]^k \mu_{n-1}(X))\right]. \tag{3.25}$$

PROOF: From (2.2) we have

$$F_n(t) - F_{n-1}(t) = \frac{[\Lambda(t)]^{n-1}}{(n-1)!} F_X(t), \quad n = 1, 2, \ldots. \tag{3.26}$$

Using (3.26) in (3.20) we obtain

$$\begin{aligned}
\mathcal{CE}_n(X) &= \frac{1}{n}\int_0^{+\infty} r_X(z)\left[\int_0^z \{F_n(x) - F_{n-1}(x)\}dx\right]dz \\
&= \frac{1}{n}\int_0^{+\infty} r_X(z)[F_n(z)\mu_n(z) - F_{n-1}(z)\mu_{n-1}(z)]dz \quad (\text{from}(2.3)) \\
&= \frac{1}{n}\int_0^{+\infty} f_X(z)\left[\sum_{k=0}^{n-1}\frac{[\Lambda(z)]^k}{k!}\mu_n(z) - \sum_{k=0}^{n-2}\frac{[\Lambda(z)]^k}{k!}\mu_{n-1}(z)\right]dz \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k!}\int_0^{+\infty} f_X(z)[\Lambda(z)]^k\mu_n(z)dz - \frac{1}{n}\sum_{k=0}^{n-2}\frac{1}{k!}\int_0^{+\infty} f_X(z)[\Lambda(z)]^k\mu_{n-1}(z)dz \\
&= \frac{1}{n}\left[\sum_{k=0}^{n-1}\frac{1}{k!}E([\Lambda(X)]^k\mu_n(X)) - \sum_{k=0}^{n-2}\frac{1}{k!}E([\Lambda(X)]^k\mu_{n-1}(X))\right].
\end{aligned} \tag{3.27}$$

This completes the proof. ■

The GCE can be represented in another form which follows from (3.27) and hence the proof is omitted.

PROPOSITION 3.19: *Let X be a non-negative random variable as described in Lemma 3.12 and X_{k+1} be the $(k+1)$th record with probability density function $f_{k+1}(x)$. Then for $n = 1, 2, \ldots,$ we have*

$$\mathcal{CE}_n(X) = \frac{1}{n}\left[\sum_{k=0}^{n-1} E(\mu_n(X_{k+1})) - \sum_{k=0}^{n-2} E(\mu_{n-1}(X_{k+1}))\right]. \tag{3.28}$$

4. DYNAMIC GCE

A dynamic version of the GCRE was proposed by Psarrakos and Navarro [21]. They considered GCRE for residual lifetime, which deals with the uncertainty of future lifetime. There are many situations where uncertainty is related to past. Assume that at time t, a system which is observed only at certain preassigned inspection times, is found to be down. Then

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the uncertainty of the system life relies on the past, that is, on which instant in $(0,t)$ it has failed. Based on this idea Di Crescenzo and Longobardi [7] studied the past entropy over $(0,t)$. In the following, we introduce a new dynamic generalized information measure known as dynamic generalized cumulative entropy (DGCE). It is given by

$$\mathcal{CE}_n(X;t)=\mathcal{CE}_n(X_{[t]})=\frac{1}{n!}\int_0^t \frac{F_X(x)}{F_X(t)}\left[-\ln\frac{F_X(x)}{F_X(t)}\right]^n dx, \quad n=1,2,\ldots, \tag{4.1}$$

for $t>0$ such that $F_X(t)>0$. Note that $\mathcal{CE}_n(X;+\infty)=\mathcal{CE}_n(X)$ and $\mathcal{CE}_0(X;t)=\mu_1(t)$, where

$$\mu_1(t)=E(X_{[t]})=\int_0^t \frac{F_X(x)}{F_X(t)}dx.$$

Also for $n=1$, (4.1) reduces to (1.5). The following proposition which extends the result given in Remark 5.1 of Di Crescenso and Longobardi [8] shows that the DGCE can be expressed in terms of expectation.

Proposition 4.1: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then*

$$\mathcal{CE}_n(X;t)=E[R_n^{(2)}(X;t)|X\le t], \quad n=1,2,\ldots, \tag{4.2}$$

where

$$R_n^{(2)}(x;t)=\frac{1}{n!}\int_x^t \left[-\ln\frac{F_X(z)}{F_X(t)}\right]^n dz. \tag{4.3}$$

Proof: Using $F_X(x)=\int_0^x f_X(z)dz$ in (4.1) and on the application of Fubini's theorem we get the desired result. ■

Remark 4.2: *From (4.1) and Proposition 4.1, the following observations can be made.*

- *$\mathcal{CE}_n(X;t)$ is always non-negative for all $t>0$.*
- $\lim_{t\to 0+}\mathcal{CE}_n(X;t)=0$.
- $\lim_{t\to +\infty}\mathcal{CE}_n(X;t)=\mathcal{CE}_n(X)$.

In the following proposition, we obtain relation between DGCE and the dynamic generalized cumulative residual entropy (DGCRE) for a symmetric distribution. Proof follows along the lines of that of the Theorem 5.1 of Di Crescenzo and Longobardi [8], and hence it is omitted.

Proposition 4.3: *Let X be a random variable with support $[0,b]$ with b finite. Also assume that X is symmetric with respect to $b/2$. Then*

$$\mathcal{CE}_n(X;t)=\mathcal{E}_n(X;b-t), \quad 0<t<b. \tag{4.4}$$

In our next result, in analogy to Proposition 3.14, we obtain bounds of the DGCE.

Proposition 4.4: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then*

- $\mathcal{CE}_n(X;t) \geq \sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} \int_0^t \left[\frac{F_X(x)}{F_X(t)}\right]^{i+1} dx, \quad n = 1, 2, \ldots.$
- $\mathcal{CE}_n(X;t) \geq R_n^{(2)}(\bar{\mu}(t);t)$, *where* $R_n^{(2)}(x;t)$ *is given by* (4.3) *and the mean inactivity time* $E(X|X \leq t) = \bar{\mu}(t) < +\infty$.
- $\mathcal{CE}_n(X;t) \leq \frac{1}{n!} \int_0^t [-\ln \frac{F_X(x)}{F_X(t)}]^n dx.$

Hereafter, we obtain the effect of the linear transformations on the DGCE.

LEMMA 4.5: *Let* $Y = aX + b$, *where* $a > 0$ *and* $b \geq 0$ *are constants. Then for* $t > b$ *we have*

$$\mathcal{CE}_n(Y;t) = a\, \mathcal{CE}_n\left(X; \frac{t-b}{a}\right). \tag{4.5}$$

PROOF: The proof follows from $F_Y(x) = F_X(x - b/a)$ for $x \in \mathcal{R}$, hence omitted. ■

In this part of the paper, we introduce two-dimensional version of (4.1) as

$$\mathcal{CE}_n(X,Y;t_1,t_2) = \frac{1}{n!} \int_0^{t_2} \int_0^{t_1} \frac{F_{XY}(x,y)}{F_{XY}(t_1,t_2)} \left[-\ln \frac{F_{XY}(x,y)}{F_{XY}(t_1,t_2)}\right]^n dxdy, \quad n = 1, 2, \ldots. \tag{4.6}$$

In particular, for $n = 1$, (4.6) reduces to the two-dimensional analog of (1.5). Using binomial expansion, (4.6) can be written as

$$\mathcal{CE}_n(X,Y;t_1,t_2) = \sum_{k=0}^{n} \mathcal{CE}_{n-k}(X;t_1)\, \mathcal{CE}_k(Y;t_2). \tag{4.7}$$

For $n = 1$, we have from (4.7)

$$\mathcal{CE}_1(X,Y;t_1,t_2) = \bar{\mu}_Y(t_2)\mathcal{CE}(X;t_1) + \bar{\mu}_X(t_1)\mathcal{CE}(Y;t_2), \tag{4.8}$$

where

$$\bar{\mu}_X(t_1) = \int_0^{t_1} \frac{F_X(x)}{F_X(t_1)} dx$$

and

$$\bar{\mu}_Y(t_2) = \int_0^{t_2} \frac{F_Y(x)}{F_Y(t_2)} dx$$

are mean inactivity times of X and Y, respectively. In analogy to Propositions 3.9 and 3.10, we have the following results. We omit the proofs as these follow from (4.6).

PROPOSITION 4.6: *Let* Y_1 *and* Y_2 *be two non-negative random variables with joint cumulative distribution function* $F_{Y_1Y_2}(y_1, y_2)$. *Assume* $Y_i = b_iX_i + c_i$, *with* $b_i > 0$ *and* $c_i \geq 0$, $i = 1, 2$. *Then*

$$\mathcal{CE}_n(Y_1,Y_2;t_1,t_2) = b_1b_2\, \mathcal{CE}_n\left(X_1, X_2; \frac{t_1-b_1}{c_1}, \frac{t_2-b_2}{c_2}\right), \quad n = 1, 2, \ldots, \tag{4.9}$$

where $t_1 > b_1$ *and* $t_2 > b_2$.

PROPOSITION 4.7: *Let Y_1 and Y_2 be two non-negative random variables as described in Proposition 4.6 Also, let $Y_i = \phi_i(X_i)$, $i = 1, 2$ be one-to-one transformations with $\phi_i(x_i)$'s are differentiable functions. Then for $n = 1, 2, \ldots,$*

$$\mathcal{CE}_n(Y_1, Y_2; t_1, t_2) = \frac{1}{n!} \int_0^{t_2} \int_0^{t_1} \frac{F_{X_1X_2}(x_1, x_2)}{F_{X_1X_2}(t_1, t_2)} \left[-\ln \frac{F_{X_1X_2}(x_1, x_2)}{F_{X_1X_2}(t_1, t_2)} \right]^n |J| dx_1 dx_2, \quad \textbf{(4.10)}$$

where J is the Jacobian of the transformations. It shows that the bivariate DGCE is not invariant under non-singular transformations.

Using binomial expansion on (4.1), we get the following proposition which is useful to obtain our next results.

PROPOSITION 4.8: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$ such that $\mathcal{CE}_n(X; t) < +\infty$. Then*

$$\mathcal{CE}_n(X; t) = \frac{1}{F_X(t)} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} [\Lambda(t)]^{n-k} \int_0^t F_X(x) [\Lambda(x)]^k dx, \quad n = 1, 2, \ldots. \quad \textbf{(4.11)}$$

Hereafter, we discuss monotonicity property of the DGCE. In this purpose, we prove the following theorem.

THEOREM 4.9: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then*

$$\mathcal{CE}'_n(X; t) = r_X(t) [\mathcal{CE}_{n-1}(X; t) - \mathcal{CE}_n(X; t)], \quad \textbf{(4.12)}$$

for $n = 1, 2, \ldots,$ where $'$ denotes the derivative with respect to t.

PROOF: From (4.11) we have

$$\mathcal{CE}_n(X; t) F_X(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} [\Lambda(t)]^{n-k} \int_0^t F_X(x) [\Lambda(x)]^k dx. \quad \textbf{(4.13)}$$

Differentiating (4.13) with respect to t we get

$$\mathcal{CE}'_n(X; t) F_X(t) + \mathcal{CE}_n(X; t) f_X(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} \left[[\Lambda(t)]^n F_X(t) \right.$$
$$\left. -(n-k) [\Lambda(t)]^{n-k-1} r_X(t) \int_0^t F_X(x) [\Lambda(x)]^k dx \right]. \quad \textbf{(4.14)}$$

Again,

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} \left[[\Lambda(t)]^n F_X(t) \right] = 0. \quad \textbf{(4.15)}$$

Now using (4.15) in (4.14) we obtain

$$\mathcal{CE}'_n(X;t)F_X(t) + \mathcal{CE}_n(X;t)f_X(t) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{k!(n-k-1)!}[\Lambda(t)]^{n-k-1} r_X(t) \times \int_0^t F_X(x)[\Lambda(x)]^k dx = r_X(t)F_X(t)\mathcal{CE}_{n-1}(X;t).$$

Hence, the result follows. ■

In particular, for $n = 1$ we get from (4.12)

$$\mathcal{CE}'_1(X;t) = r_X(t)[\mu(t) - \mathcal{CE}_1(X;t)]. \quad \textbf{(4.16)}$$

The reversed failure rate of the inactivity time $X_{[t]} = [t - X | X \leq t]$ is $r_{X_{[t]}}(t) = f(t-x)/(F(t) - F(t-x))$ for $x > 0$. It can be shown that if $X_{[t]}$ is decreasing (increasing) reversed failure rate, then from (3.3) and (4.1) we obtain

$$\mathcal{CE}_n(X;t) \geq (\leq)\ \mathcal{CE}_{n+1}(X;t), \quad n = 0, 1, \ldots, \quad \textbf{(4.17)}$$

for all $t > 0$. Thus in analogy to (3.5), the normalized version of (4.1) can be defined when $X_{[t]}$ is decreasing reversed failure rate. It is given by

$$\mathcal{NCE}_n(X;t) = \frac{\mathcal{CE}_n(X;t)}{\mathcal{CE}_0(X;t)}, \quad \textbf{(4.18)}$$

for $t > 0$ such that $F_X(t) > 0$. It is easy to see that the normalized DGCE lies in the interval $[0, 1]$.

THEOREM 4.10: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then for $n = 1, 2, \ldots,$ we have*

$$\mathcal{CE}_n(X) = \frac{1}{n}E\Big(\mathcal{CE}_{n-1}(X;X)\Big) + \frac{1}{n!}\sum_{k=0}^{n-2}\binom{n-1}{k}(-1)^{n-k} \times \int_0^{+\infty}\int_0^z r_X(z)F_X(x)[\Lambda(z)]^{n-k-1}[\Lambda(x)]^k dx dz. \quad \textbf{(4.19)}$$

PROOF: From (4.11) and after some simplification we get

$$\int_0^t F_X(x)[\Lambda(x)]^n dx = n! F_X(t)\mathcal{CE}_n(X;t) - \sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}[\Lambda(t)]^{n-k}\int_0^t F_X(x)[\Lambda(x)]^k dx. \quad \textbf{(4.20)}$$

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Substituting the expression from (4.20) into (3.20) we obtain

$$\begin{aligned}\mathcal{CE}_n(X) &= \frac{1}{n!}\int_0^{+\infty} r_X(z)[(n-1)!F_X(z)\mathcal{CE}_{n-1}(X;z)\\ &\quad + \sum_{k=0}^{n-2}\binom{n-1}{k}(-1)^{n-k}[\Lambda(z)]^{n-k-1}\int_0^z F_X(x)[\Lambda(x)]^k dx]dz\\ &= \frac{1}{n}\int_0^{+\infty}\mathcal{CE}_{n-1}(X;z)f_X(z)dz + \frac{1}{n!}\sum_{k=0}^{n-2}\binom{n-1}{k}(-1)^{n-k}\\ &\quad \times \int_0^{+\infty}\int_0^z r_X(z)F_X(x)[\Lambda(z)]^{n-k-1}[\Lambda(x)]^k dxdz. \end{aligned} \tag{4.21}$$

Hence, the desired result follows. ■

Below, we obtain relationship between $\mu_n(t)$ and $\mathcal{CE}_n(X;t)$. First we prove the following lemma.

Lemma 4.11: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then $M_k(t)$ defined in (2.6) can be expressed as*

$$M_k(t) = \sum_{j=0}^{k}\frac{k!}{(k-j)!}\frac{1}{[\Lambda(t)]^j}\ \mathcal{CE}_j(X;t).$$

Proof: From (2.6) we have

$$\begin{aligned} M_k(t) &= \int_0^t \left[\frac{-\ln(F_X(x)/F_X(t))}{-\ln F_X(t)}+1\right]^k \frac{F_X(x)}{F_X(t)}dx\\ &= \int_0^t \sum_{j=0}^{k}\binom{k}{j}\left[\frac{-\ln(F_X(x)/F_X(t))}{-\ln F_X(t)}\right]^j \frac{F_X(x)}{F_X(t)}dx\\ &= \sum_{j=0}^{k}\binom{k}{j}\frac{1}{[\Lambda(t)]^j}\int_0^t\left[-\ln\frac{F_X(x)}{F_X(t)}\right]^j\frac{F_X(x)}{F_X(t)}dx\\ &= \sum_{j=0}^{k}\frac{k!}{(k-j)!}\frac{1}{[\Lambda(t)]^j}\int_0^t\frac{[-\ln(F_X(x)/F_X(t))]^j}{j!}\frac{F_X(x)}{F_X(t)}dx\\ &= \sum_{j=0}^{k}\frac{k!}{(k-j)!}\frac{1}{[\Lambda(t)]^j}\mathcal{CE}_j(X;t). \end{aligned}$$

This completes the proof. ■

THEOREM 4.12: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Then the MWIT of X can be expressed as follows:*

$$\mu_n(t) = \sum_{j=0}^{n-1} \mathcal{CE}_j(X;t)\ \zeta_j(t),$$

where

$$\zeta_j(t) = \frac{\sum_{k=0}^{n-j-1} [\Lambda(t)]^k/k!}{\sum_{i=0}^{n-1} [\Lambda(t)]^i/i!}, \quad j = 0, 1, 2, \ldots .$$

PROOF: From (2.8) we have

$$\begin{aligned} \mu_n(t) &= \sum_{k=0}^{n-1} M_k(t) p_k(t) \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{k!}{(k-j)!} \frac{1}{[\Lambda(t)]^j} \mathcal{CE}_j(X;t) p_k(t) \quad \text{(from Lemma 4.11)}. \end{aligned} \tag{4.22}$$

After changing the orders of the sums we get from (4.22)

$$\begin{aligned} \mu_n(t) &= \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} \frac{1}{[\Lambda(t)]^j}\ \mathcal{CE}_j(X;t) p_k(t) \\ &= \sum_{j=0}^{n-1} \frac{1}{[\Lambda(t)]^j} \mathcal{CE}_j(X;t) \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} \frac{[\Lambda(t)]^k/k!}{\sum_{i=0}^{n-1} [\Lambda(t)]^i/i!} \\ &= \sum_{j=0}^{n-1} \mathcal{CE}_j(X;t) \frac{\sum_{k=j}^{n-1} [\Lambda(t)]^{k-j}/(k-j)!}{\sum_{i=0}^{n-1} [\Lambda(t)]^i/i!} \\ &= \sum_{j=0}^{n-1} \mathcal{CE}_j(X;t) \frac{\sum_{k=0}^{n-j-1} [\Lambda(t)]^k/k!}{\sum_{i=0}^{n-1} [\Lambda(t)]^i/i!}. \end{aligned}$$

Hence, the desired result. ■

4.1. Class of Lifetime Distributions

In this section, we introduce a class of distributions. Di Crescenzo and Longobardi [8] pointed out that the dynamic CE cannot be decreasing in t. From Remark 4.2, we also observe the same behavior of the DGCE. Consider the following definition.

DEFINITION 4.13: *The cumulative distribution function $F_X(x)$ is said to be increasing dynamic generalized cumulative entropy (IDGCE), if for $n = 1, 2, \ldots$, $\mathcal{CE}_n(X;t)$ is an increasing function of $t > 0$.*

There are many distributions which belong to this class. For example, uniform and exponential distributions. In Figure 2, we plot $\mathcal{CE}_n(X;t)$ for exponential distribution with cumulative distribution function $F_X(x) = 1 - \exp\{-x\}$, $x > 0$ for different values of n. In the next result we obtain necessary and sufficient conditions for $\mathcal{CE}_n(X;t)$ to be IDGCE.

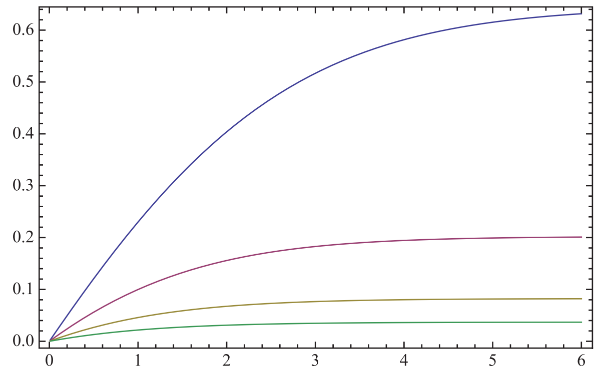

FIGURE 2. Plot of $\mathcal{CE}_n(X;t)$ for exponential distribution for $n = 1, 2, 3, 4$ (from above).

THEOREM 4.14: *The cumulative distribution function $F_X(x)$ is IDGCE if, and only if, for $t > 0$.*

$$\mathcal{CE}_{n-1}(X;t) \geq \mathcal{CE}_n(X;t), \quad n = 1, 2, \ldots . \tag{4.23}$$

PROOF: The proof follows from (4.12) and hence omitted. ■

Hereafter, we consider variational behavior of IDGCE class under increasing linear transformations.

THEOREM 4.15: *Let X be a non-negative random variable with absolutely continuous cumulative distribution function $F_X(x)$. Also let $Y = aX + b$, $a > 0$, $b \geq 0$. Then Y is IDGCE if X is IDGCE.*

PROOF: The proof follows from the Lemma 4.11 and the Definition 4.13 and hence omitted. ■

REMARK 4.16: *From the Theorem 4.15, it is clear that IDGCE class is closed under the positive linear transformations.*

As a consequence of the Theorem 4.15 we get the following corollary.

COROLLARY 4.17: *For a non-negative random variable X we have*
(i) aX is IDGCE if X is IDGCE, where $a > 0$; and
(ii) $X + b$ is IDGCE if X is IDGCE, where $b \geq 0$.

5. EMPIRICAL GCE

Di Crescenzo and Longobardi [8] addressed the problem of estimating the CE by means of the empirical CE. The empirical CE may be used as a tool to measure information of neural firing data (see Di Crescenzo and Longobardi [9]). In this section, we consider the empirical GCE which can be used as an estimator of the GCE. Let $X_1, X_2, \ldots, X_m$ be a

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random sample of size m from a lifetime distribution with absolutely continuous cumulative distribution function $F_X(x)$. From (3.1), the empirical GCE is defined as

$$\mathcal{CE}_n(\widehat{F}_m) = \frac{1}{n!}\int_0^{+\infty} \widehat{F}_m(x)[-\ln \widehat{F}_m(x)]^n dx, \quad n = 1, 2, \ldots, \tag{5.1}$$

where

$$\widehat{F}_m(x) = \frac{1}{m}\sum_{i=1}^{m} I_{(X_i \le x)}, \quad x \in \mathcal{R} \tag{5.2}$$

is the empirical distribution of the sample and I is the indicator function. Denote $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$ as the order statistics of the sample. Thus, (5.1) can be written as

$$\mathcal{CE}_n(\widehat{F}_m) = \sum_{j=1}^{m-1} \frac{1}{n!}\int_{X_{(j)}}^{X_{(j+1)}} \widehat{F}_m(x)[-\ln \widehat{F}_m(x)]^n dx, \quad n = 1, 2, \ldots. \tag{5.3}$$

Moreover,

$$\widehat{F}_m(x) = \begin{cases} 0, & x < X_{(1)}, \\ \dfrac{j}{m}, & X_{(j)} \le x < X_{(j+1)}, \quad j = 1, 2, \ldots, n-1, \\ 1, & x \ge X_{(j+1)}. \end{cases}$$

Hence, (5.3) can be written as

$$\mathcal{CE}_n(\widehat{F}_m) = \sum_{j=1}^{m-1} \frac{1}{n!} U_{j+1} \frac{j}{m}\Big(-\ln \frac{j}{m}\Big)^n, \quad n = 1, 2, \ldots, \tag{5.4}$$

where $U_i = X_{(i)} - X_{(i-1)}$, $i = 1, 2, \ldots, n$ and $X_{(0)} = 0$. In the following, we study the empirical GCE for exponentially distributed random samples.

Example 5.1: Let $X_1, X_2, \ldots, X_m$ be a random sample drawn from exponential distribution with parameter θ. Here the sample spacings are independent and U_{j+1} is exponentially distributed with parameter $\theta(m-j)$. For detail we refer to Pyke [22]. Then from (5.4) we obtain

$$E(\mathcal{CE}_n(\widehat{F}_m)) = \frac{1}{\theta}\sum_{j=1}^{m-1} \frac{1}{n!(m-j)} \frac{j}{m}\Big(-\ln \frac{j}{m}\Big)^n, \quad n = 1, 2, \ldots \tag{5.5}$$

and

$$Var(\mathcal{CE}_n(\widehat{F}_m)) = \frac{1}{\theta^2}\sum_{j=1}^{m-1} \frac{1}{n!(m-j)^2} \Big(\frac{j}{m}\Big)^2\Big(-\ln \frac{j}{m}\Big)^{2n}, \quad n = 1, 2, \ldots. \tag{5.6}$$

Below, we present the numerical values of the empirical GCE of the random samples drawn from exponential distribution. We consider different sample sizes ($m = 10$, 20, 30) and different values of the parameter ($\theta = 0.2, 0.5, 1.5$). Assume $n = 1, 2, 3, 4, 5$. From Tables 3 and 4, the following observations are made:

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TABLE 3. Numerical values of $E(\mathcal{CE}_n(\widehat{F}_m))$ for exponential distribution.

θ	m	$E(\mathcal{CE}_1(\widehat{F}_m))$	$E(\mathcal{CE}_2(\widehat{F}_m))$	$E(\mathcal{CE}_3(\widehat{F}_m))$	$E(\mathcal{CE}_4(\widehat{F}_m))$	$E(\mathcal{CE}_5(\widehat{F}_m))$
0.2	10	2.95875	0.971355	0.362776	0.136520	0.048475
	20	3.09498	0.997255	0.392034	0.161878	0.065533
	30	3.13907	1.003530	0.400491	0.170552	0.072464
0.5	10	1.18350	0.388542	0.145111	0.054608	0.019390
	20	1.23799	0.398902	0.156814	0.064751	0.026213
	30	1.25563	0.401412	0.160196	0.068221	0.028985
1.5	10	0.39450	0.129514	0.048370	0.018203	0.006464
	20	0.41267	0.132967	0.052271	0.021584	0.008738
	30	0.41854	0.133804	0.053399	0.022740	0.009662

TABLE 4. Numerical values of $\mathrm{Var}(\mathcal{CE}_n(\widehat{F}_m))$ for exponential distribution.

θ	m	$\mathrm{Var}(\mathcal{CE}_1(\widehat{F}_m))$	$\mathrm{Var}(\mathcal{CE}_2(\widehat{F}_m))$	$\mathrm{Var}(\mathcal{CE}_3(\widehat{F}_m))$	$\mathrm{Var}(\mathcal{CE}_4(\widehat{F}_m))$	$\mathrm{Var}(\mathcal{CE}_5(\widehat{F}_m))$
0.2	10	1.080700	0.124214	0.027173	0.005931	0.001049
	20	0.571024	0.063199	0.015301	0.004226	0.001059
	30	0.387547	0.042257	0.010482	0.003118	0.000891
0.5	10	0.172912	0.019874	0.004347	0.000949	0.000168
	20	0.091364	0.010112	0.002448	0.000676	0.000169
	30	0.062008	0.006761	0.001677	0.000499	0.000143
1.5	10	0.019212	0.002208	0.000483	0.000105	0.000019
	20	0.010152	0.001124	0.000272	0.000075	0.000019
	30	0.006889	0.000751	0.000186	0.000055	0.000016

- The mean of the empirical GCE is increasing (decreasing) in $m(n)$.
- Variance of the empirical GCE is decreasing in m and n.

In the following, we consider example with real dataset taken from Kass, Ventura, and Cai [16].

Example 5.2: The dataset presented below with $m = 29$ is due to a case-study based on a dataset of 242 spike times observed in eight trials on a single neuron.

$$136.842, 145.965, 155.088, 175.439, 184.561, 199.298, 221.053, 231.579, 246.316, 263.158,$$
$$274.386, 282.105, 317.193, 329.123, 347.368, 360.702, 368.421, 389.474, 392.982, 432.281,$$
$$449.123, 463.86, 503.86, 538.947, 586.667, 596.491, 658.246, 668.772, 684.912.$$

Based on these data we get $\mathcal{CE}_1(\widehat{F}_{29}) = 131.223$, $\mathcal{CE}_2(\widehat{F}_{29}) = 55.2174$, $\mathcal{CE}_3(\widehat{F}_{29}) = 24.482$, $\mathcal{CE}_4(\widehat{F}_{29}) = 10.6918$ and $\mathcal{CE}_5(\widehat{F}_{29}) = 4.46299$.

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