Matters for Debate

Why avoid induction?

JOHN BAYLIS

1. *Introduction*

T. Koshy [1] has recently given an interpretation of the Fibonacci numbers using a simple graph, the powers of its adjacency matrix, and the eigenvalues of that matrix. Using these, he gave combinatorial proofs (avoiding induction as far as possible) of several of the rather striking identities satisfied by the Fibonacci numbers and the closely related Lucas numbers. The main combinatorial, induction-avoiding idea used was that of double counting: if you employ a counting technique to derive a formula F for the size of a certain set S, then devise a different technique leading to an alternative formula G for the size of the same set, you will have established an identity, $F = G$.

Two questions arise from these introductory remarks: first, the Fibonacci numbers arise from models far more elementary than the graph-theoretic one, so will these models yield a similar range of identities? Second, what is the virtue of avoiding induction?

Of the elementary models the best known is Fibonacci's rabbits: at the start of month 1 a new born male/female pair of rabbits starts the population. They produce no offspring for two months, but at the start of month 3, and the start of every subsequent month, they produce another male/female pair who behave in exactly the same way. The nth term of the sequence (F_n) is the number of pairs in the population at the start of month n. Even if you are willing to accept the strange biology of Fibonacci's rabbits (each pair stays together for life, they only produce offspring in male/female pairs, and they are immortal), there is still some work to do in order to establish the basic recurrence.

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geqslant 3. \tag{1}$$

Slightly more realistic, but still requiring a distortion and over-simplification of the biology, F_n can be regarded as the number of ancestors of a single drone bee (generation 1), $n - 1$ generations prior to this (see [2]). In this model equation (1) can also be demonstrated, but in neither model is this recurrence and other identities obvious.

Now to address the second of our questions, it must surely be acknowledged that induction is one of the most powerful and versatile methods of proof. Its range is immense: most branches of algebra, some parts of geometry, graph theory, number theory, set theory, logic, even analysis. It is quite a challenge to find any branch of mathematics devoid of an inductive proof. But induction is also deeply unsatisfying. After following an inductive proof one usually has a very firm conviction that the result is indeed true, but a feeling for why it is true is often elusive. This is

https://doi.org/10.1017/mag.2015.37 Published online by Cambridge University Press

especially the case when the result is particularly neat or surprising, and if the inductive proof is the only one available, dissatisfaction increases.

To put the issue into a context, consider the binomial identity, $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. Using only the formula for $\binom{n}{k}$ it is easy but messy to prove this identity either directly or by induction on k. But the proof by 'combinatorial reasoning' straight from the definition of $\binom{n}{k}$ as the number of size k subsets in an n-element set is so much more satisfying; it shows the result emerging from the meaning of $\binom{n}{k}$ rather than as if by magic after playing around with algebraic fractions and factorials.

2. *The staircase climbing model* (see [3])

In this model the Fibonacci recurrence relation is as close to obvious as things get in mathematics: you are standing at level 1 of a staircase and wish to climb to level n ($n > 1$). At each stride you have the option of climbing 1 or 2 steps. In how many ways can you arrive at level n? We'll call this number F_n. You can do it either by landing on level $n - 1$ on your penultimate stride and then climb one more step to complete the job, or by landing on level $n - 2$ and then finishing with a 2-step stride. These two options are clearly mutually exclusive and they cover all possibilities, so $F_n = F_{n-1} + F_{n-2}$.

The only slightly artificial feature (if we want the convenience of $F_1 = 1$) is that 'do nothing' has to be the single option allowed to achieve level 1, but this can't be allowed at any stage in climbing to level n for $n > 1$. This is probably more satisfying as a teaching strategy than 'F_1 is conventionally defined to be 1'.

3. *A sample of combinatorial proofs using the staircase model*

The following results, routinely proved by induction, are chosen to illustrate the greater transparency achieved by combinatorial reasoning.

Result 1: $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$.

Proof: There is only one way of reaching level $n + 2$ entirely with single step strides, so the right-hand side of this identity is the number of ways of reaching this level with at least one 2-step stride.

The number of ways in which the steps from k to $k + 2$ is the last 2-step is F_k because from level k the subsequent sequence of step lengths is fixed as 2,1,1,1,1,…,1. This last 2-step can be 1 to 3, 2 to 4, 3 to 5, …, or n to $n + 2$, so by summing from $k = 1$ to $k = n$ the result follows.

Result 2: $F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}$.

Proof: A similar approach works here. The right hand side is the number of ways of reaching level $2n$. The starting level is 1, so classify the methods according to the last odd level visited. If level k is the last there is no further choice for the sequence of subsequent strides; it must be 1,2,2,2,…,2, so F_k is the number of routes in which k is the last odd level visited, and the result follows by summing over the odd k from 1 to $2n - 1$.

Result 3: $1 + F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1}$.

Proof: Classify the routes to level $2n + 1$ by the last even level reached. (The 1 on the left-hand side is from the single route which visits no even level.) The details are left as an exercise.

Corollary of Results 2 *and* 3

By considering the difference between results 2 and 3 an alternating sum is obtained:

$$F_1 - F_2 + F_3 - F_4 + \ldots + (-1)^{n+1}F_n = 1 + (-1)^{n+1}F_{n-1}.$$

Result 4: $F_nF_{n+1} = F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2$.

Proof: To apply combinatorial thinking to this one, consider two climbers A and B. A stops at level n and B at level $n + 1$ both having started at level 1. They can reach their respective summits in F_n and F_{n+1} ways, so F_nF_{n+1} is the number of ways in which A and B can jointly achieve these final levels.

For example, if A climbs to level 20 and B to 21, A's path could be via levels 1, 3, 5, 6, 8, 9, 10, 11, 12, 14, 16, 18, 20 while B's could be via 1, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 21. Level 11 is the highest visited by both climbers. Now classify these pairs of routes according to their highest common level (hcl). Their routes beyond the hcl are uniquely determined: one will consist entirely of 2-steps and the other will have an initial 1-step followed by 2-steps. If this is not immediately obvious think about the possibilities beyond level 11, and explore a couple more examples.

So in general the number of joint routes with hcl k is F_k^2, and since the hcl can be anything from 1 to n the result follows by summing over this range of values.

Result 5: $F_n = F_kF_{n-k+1} + F_{k-1}F_{n-k}$.
(In our model this makes sense for $n > k \geqslant 2$, but 'a conventional $F_0 = 0$' extends the validity to $n \geqslant k \geqslant 1$.)

Proof: The number of routes to level n is the sum of the number of routes visiting k and those which bypass k.

A route via k consists of a route from 1 to k followed by one from k to

n, and the latter is equivalent (in terms of the number of ways of achieving it) to a route from 1 to $n - k + 1$. A route bypassing k consists of one from 1 to $k - 1$, followed by a 2-step, followed by one from $k + 1$ to n, and the last of these is equivalent to a route from 1 to $n - k$.

Hence the result

$$F_n = F_k F_{n-k+1} + F_{k-1} F_{n-k}.$$

A connection with binomial coefficients

Since each step is either 1 or 2, and the height climbed in going from level 1 to level n is $n - 1$, F_n is equal to the number of series with sum $n - 1$ each term of which is 1 or 2. Such a series with k 2's will have $n - 1 - 2k$ 1's, so $n - 1 - k$ terms in all. The 2's can be placed in any k of the $n - 1 - k$ positions, and $\left\lfloor \frac{n-1}{2} \right\rfloor$ is the maximum number of 2's. Hence

$$F_n = \sum_{k=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} \binom{n-1-k}{k}.$$

In terms of Pascal's triangle this is the sum of the circled 'diagonal' entries shown in Figure 1.

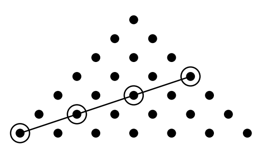

FIGURE 1

Result 6: $F_n < (n - 1)!$ for $n \geqslant 4$.

This is a rather crude upper bound for F_n. It is trivial, but is included here because a combinatoric proof is just as easy as the inductive proof given in [4].

Proof: We make use of the fact that $(n - 1)!$ is the number of permutations of $n - 1$ objects; any route to level n can be specified by the list of levels visited, in order, followed by the list of levels stepped over, again in increasing order. (We omit level 1 since this is the starting point of all routes.) For example, a route to level 10 which goes

$$1 \to 2 \to 4 \to 5 \to 7 \to 8 \to 9 \to 10$$

would be associated with the permutation of {2, 3, 4, 5, 6, 7, 8, 9, 10} 2, 4, 5, 7, 8, 9, 10, 3, 6.

In general this sets up a bijection between all routes from level 1 to level n and a subset of the permutations of $\{2, 3, 4, \ldots, n\}$. Also, since $n \geqslant 4$, the subset is a strict one because no permutation representing a legal route can begin with 4.

https://doi.org/10.1017/mag.2015.37 Published online by Cambridge University Press

A theorem of Cayley

This result, proved in 1876 and referred to in [5], is about certain types of partition, now called compositions. They are expressions of the natural number n ($n \geqslant 1$) as a sum of natural numbers in which sums having the same parts but in different orders are counted as different. For example, the compositions of 4 are 4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2 and 1+1+1+1.

Cayley's theorem says that the number of compositions of n in which there are no 1's is F_{n-1} $(n \geqslant 2)$. We'll call such compositions 'specials' and use S_n for their number. Clearly $S_1 = 0, S_2 = S_3 = 1$, so the proof is completed by showing that $S_{n+2} = S_{n+1} + S_n$ for all $n \geqslant 1$.

Think about a special for $n + 2$. Its last part is either 2 or a number greater than 2. Those in the first set can be thought of as a special for n with an extra +2 tagged on at the end. Those in the second are obtained by increasing the last part of a special for $n + 1$ by 1.

This construction has set up a bijection between the union of these two disjoint sets and the set of specials for $n + 2$, from which the result follows.

Martin Griffiths [6] has used a similar approach in his note on compositions, and in [7] on coin-tossing sequences.

The odd-numbered Fibonacci terms

Rajesh and Leversha [8] have given some identities concerning the odd-numbered terms of the Fibonacci sequence, starting from their lemma, which when expressed in our notation is $F_{2k+3} = 3F_{2k+1} - F_{2k-1}$. They prove this by repeated use of the basic recurrence for F_i. An alternative is just one use of the recurrence combined with a combinatorial argument.

Any route to $2k + 3$ falls into just one of the following types,

$$\left.\begin{array}{l} \ldots,\ 2k+1,\ 2k+2,\ 2k+3, \\ \ldots,\ 2k+1,\ 2k+3 \end{array}\right\} \text{those routes which visit } 2k+1$$

$$\ldots,\ 2k,\ 2k+2,\ 2k+3 \quad \text{those routes which step over } 2k+1.$$

There are F_{2k+1} of each of the first two types, and F_{2k} of the third. Hence $F_{2k+3} = 2F_{2k+1} + F_{2k}$ and then replacing F_{2k} by $F_{2k+1} - F_{2k-1}$ gives the lemma.

They use induction to prove, as a consequence of this lemma, that $F_{2k-1}^2 + F_{2k+1}^2 + 1 = 3F_{2k-1}F_{2k+1}$.

With a view to finding a combinatorial proof of this, the following rearrangement seems more promising:

$$F_{2k-1}^2 + F_{2k+1}^2 - 2F_{2k-1}F_{2k+1} = F_{2k-1}F_{2k+1} - 1$$

i.e. $(F_{2k+1} - F_{2k-1})^2 = F_{2k-1}F_{2k+1} - 1$, which can then be rewritten by the basic recurrence as

$$F_{2k}^2 = F_{2k-1}F_{2k+1} - 1.$$

Now let S_i be the set of all sequences of levels visited in paths from level 1 to level i. A typical member of S_i is a sequence 1, … , i in which the jump from each term to the next is 1 or 2.

By the trick used in the proof of Result 4, $F_{2k}^2 = |S_{2k} \times S_{2k}|$ and $F_{2k-1}F_{2k+1} = |S_{2k-1} \times S_{2k+1}|$ so the proof will be achieved if we can find a bijection between $S_{2k-1} \times S_{2k+1}$ with one member missing and $S_{2k} \times S_{2k}$.

Here is one method which I have to admit came to me after several ideas which turned out not to work! We change the model slightly and think of a member of $S_{2k} \times S_{2k}$ as a pair of towers consisting of bricks each of height 1 or 2 units, and manipulate this pair of towers into a pair corresponding to a member of $S_{2k-1} \times S_{2k+1}$. Figure 2 shows a member of $S_{10} \times S_{10}$ and its transformation to a member of $S_9 \times S_{11}$.

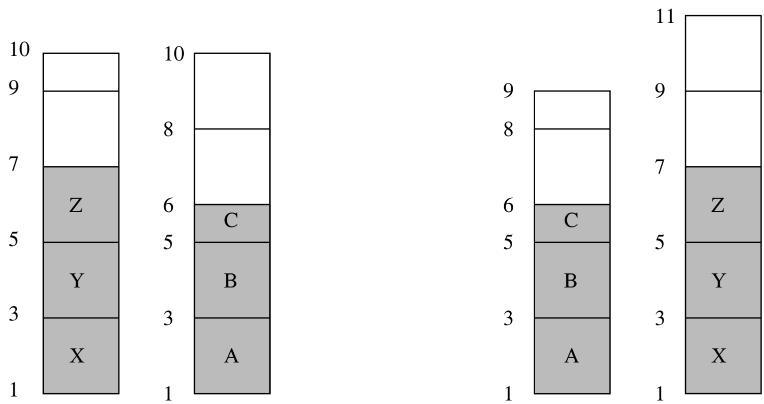

FIGURE 2

Scan the twin towers from the bottom to locate the first level f at which the left tower is one unit higher than a level of the right hand one. In Figure 2 this is $f = 7$. Then interchange the bricks up to level f on the left with those up to level $f - 1$ on the right (bricks X, Y, Z with bricks A, B, C in Figure 2). The result is the twin towers shown on the right of Figure 2, which in general will be a member of $S_{2k-1} \times S_{2k+1}$.

What remains is to show that this map is an injection, and exactly one member of $S_{2k-1} \times S_{2k+1}$ is not the image of any member of $S_{2k} \times S_{2k}$.

First, since the towers of $S_{2k} \times S_{2k}$ rise from level 1 to an even level neither of them can consist entirely of height 2 bricks.

Figure 3 shows the two types of $S_{2k} \times S_{2k}$ and their images in $S_{2k-1} \times S_{2k+1}$.

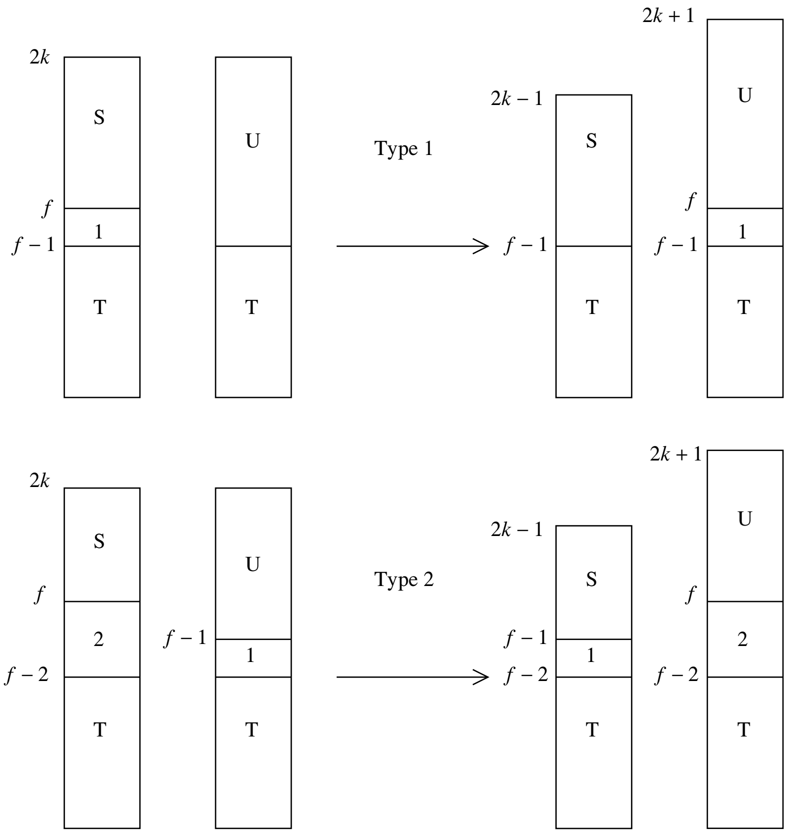

FIGURE 3

In type 1 both towers of $S_{2k} \times S_{2k}$ have only 2-bricks in block T, then the left tower has a 1-brick. In type 2 the left tower has only 2-bricks up to level f, whereas the right has 2-bricks up to level $f - 2$ then a 1-brick. In both cases block T could be empty.

In both types at least one of the towers of $S_{2k-1} \times S_{2k+1}$ must contain a 1-brick, so that the all 2-brick member of $S_{2k-1} \times S_{2k+1}$ is left out.

Since the two types of $S_{2k-1} \times S_{2k+1}$ shown in Figure 3 cover all possibilities except the all 2-brick case, and the maps shown are reversible, then the all 2-brick is indeed the only twin tower which is not an image, and the proof is complete.

Remark

The result just proved is half of a more general property: $F_n^2 = F_{n-1}F_{n+1} + (-1)^{n+1}$, sometimes known as Cassini's identity, and the case for odd n is left to the reader, with the comment that the analogous mapping will leave out a member of the equal height twin towers.

4. *Conclusion*

It now has to be admitted that the combinatorial proofs are getting harder than those employing induction as the main tool. However, I still maintain that the former are worth seeking for the gain in insight. There is a tendency for an automatic 'induction reflex' to take over when faced with any result involving natural numbers as parameters. Succumbing to this reflex may be the only reasonable strategy in very abstract areas such as general ring theory, but in areas with a visual or intuitive content alternatives are worth a try.

To end on a personal note, I recall many years ago coping with consequences of relativity by simply letting the formal mathematics do the work for me, but how I envied those relativists and geometers who were led to their results by an ability to visualise 4-dimensional Minkowski space, or even the strange curved space-times of general relativity.

Any comments?

References

1. T. Koshy, Graph-theoretic models for the Fibonacci family, *Math. Gaz.* **98** (July 2014) pp. 256-265.
2. David A. Cushman, Drone parent numbers (2004), available at www.dave-cushman.net/bee/fibonacci.html
3. A. M. Narasimhamurthy, The steps problem and the Fibonacci sequence, Applied Probability Trust (2000), available at ms.appliedprobability.mg/data/files/Articles%2033/33-1-5.pdf
4. Surajit Rajagopal and Martin Griffiths, On Fibonacci numbers that are factorials, *Math. Gaz.* **98** (March 2014) pp 104-107.
5. George E. Andrews, *Combinatorics: ancient and modern* (Robin Wilson and John J. Watkins, eds) Oxford University Press (2013) p. 223.
6. Martin Griffiths, Further thoughts on ordered partitions, *Math. Gaz.* **96** (March 2012) pp. 152-153.
7. Martin Griffiths, Yet another coin-tossing scenario, *Math. Gaz.* **96** (November 2012) pp. 543-546.
8. V. Rajesh and Gerry Leversha, Some properties of the odd terms of the Fibonacci sequence, *Math. Gaz.* **88** (March 2004) pp. 85-86.

doi:10.1017/mag.2015.37

JOHN BAYLIS

Yr Ystablau, Wiston, Haverfordwest SA62 4PN