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Generalized Triple Homomorphisms and Derivations

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Abstract. We introduce generalized triple homomorphisms between Jordan–Banach triple systems as a concept that extends the notion of generalized homomorphisms between Banach algebras given by K. Jarosz and B. E. Johnson in 1985 and 1987, respectively. We prove that every generalized triple homomorphism between JB*-triples is automatically continuous. When particularized to C^*-algebras, we rediscover one of the main theorems established by Johnson. We will also consider generalized triple derivations from a Jordan–Banach triple E into a Jordan–Banach triple E-module, proving that every generalized triple derivation from a JB*-triple E into itself or into E^* is automatically continuous.

1 Introduction

During the last seventy years, a multitude of studies have been published proving that a homomorphism T between Banach algebras (*i.e.*, a linear map with $T(ab) = T(a)T(b)$ for all a, b) must be, under some additional conditions, continuous (*cf.* [9], [10] and [28]). For example, it follows from Gelfand's original theory that every homomorphism from a Banach algebra to a commutative, semisimple Banach algebra is automatically continuous. It is well known that every $*$-homomorphism between C^*-algebras is continuous. It is due to B. E. Johnson that if a unital C^*-algebra has no closed cofinite ideals (*e.g.*, $L(H)$, where H is an infinite dimensional Hilbert space), then each homomorphism from it into a Banach algebra is continuous (*cf.* [19]).

Johnson and K. Jarosz considered *generalized homomorphisms* (also called *ε-multiplicative linear maps* or *ε-isomorphisms*) between Banach algebras in [18], [21] and [20]. Let A and B be Banach algebras. A linear mapping $T\colon A \to B$ is a generalized homomorphism if there exists $\varepsilon > 0$ satisfying $\|T(ab) - T(a)T(b)\| \leq \varepsilon \|a\|\,\|b\|$, for every $a, b \in A$. The first result in this line is due to Jarosz, who proved that every generalized homomorphism from a Banach algebra into a unital abelian C^*-algebra is necessarily continuous (*cf.* [18, Proposition 5.5]). Johnson established in [20, Theorem 4] that a generalized homomorphism T between C^*-algebras is continuous if and only if the mapping $a \mapsto T(a^*)^* - T(a)$ is continuous. A *generalized $*$-homomorphism* between Banach $*$-algebras A and B is a generalized homomorphism $T\colon A \to B$ for which the mapping $a \mapsto T(a^*)^* - T(a)$ is continuous.

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Every Banach algebra A can be regarded as an element in the class of Jordan–Banach triples with respect to the product

$$\{a, b, c\} := \frac{1}{2}(abc + cba). \tag{1}$$

JB*-triples constitute a subclass of the Jordan–Banach triples which contains all C^*-algebras and plays a similar role of that played by the latter inside the class of Banach algebras (see definitions in Section 2). However, according to our knowledge, the automatic continuity of triple homomorphisms between Jordan–Banach triples (*i.e.*, linear mappings T satisfying $T(\{a, b, c\}) = \{T(a), T(b), T(c)\}$, for every a, b, c) have not been deeply studied. The forerunners in this line reduce to a work of T. J. Barton, T. Dang, and G. Horn, where these authors prove the automatic continuity of every triple homomorphism between JB*-triples (see [3, Lemma 1]).

In Section 3 we define a *generalized triple homomorphism* between Jordan–Banach triples E and F as a linear mapping $T\colon E \to F$ for which there exists $\varepsilon > 0$ satisfying

$$\|T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\| \le \varepsilon \|a\| \, \|b\| \, \|c\|,$$

for all a, b, c in E. We show that every generalized homomorphism between Banach algebras A and B is a generalized triple homomorphism when A and B are equipped with the product defined in (1). We further prove that every generalized $*$-homomorphism between Banach $*$-algebras A and B is a generalized triple homomorphism when A and B are equipped with the product $\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a)$ (see Proposition 1). In this section we also study the basic properties of the separating space of a generalized triple homomorphism T between Jordan–Banach triples E and F, proving that the separating space $\sigma_F(T)$ is a closed triple ideal of the closed subtriple of F generated by $T(E)$ (compare Proposition 3).

In Section 4 we establish some theorems of automatic continuity of generalized triple homomorphisms between Jordan–Banach triples. One of the main results states that every generalized triple homomorphism between JB*-triples is automatically continuous (see Theorem 8). Since every generalized $*$-homomorphism between C^*-algebras is a generalized triple homomorphism, the aforementioned result of Johnson (see [20, Theorem 4]) follows as a direct consequence. Theorem 14 provides necessary and sufficient conditions, in terms of the quadratic annihilator of the separating space, to characterize when a generalized triple homomorphism from a JB*-triple to a Jordan–Banach triple is continuous. We also prove that every generalized triple homomorphism from a Hilbert space, regarded as a type I Cartan factor, or from a spin factor into an anisotropic Jordan–Banach triple is automatically continuous (*cf.* Lemmas 15 and 16).

In the last section we consider *generalized triple derivations* from a Jordan–Banach triple E to a Jordan–Banach triple E-module X. A conjugate linear mapping $\delta\colon E \to X$ is said to be a *generalized derivation* when there exists $\varepsilon > 0$ satisfying:

$$\|\delta\{a, b, c\} - \{\delta(a), b, c\} - \{a, \delta(b), c\} - \{a, b, \delta(c)\}\| \le \varepsilon \|a\| \, \|b\| \, \|c\|,$$

for every a, b, c in E. In a recent paper, B. Russo and the second author prove that every triple derivation from a real or complex JB*-triple, E, into its dual space E^* (*i.e.*, a conjugate linear map $\delta\colon E \to E^*$ satisfying $\delta\{a,b,c\} = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}$) is automatically continuous (compare [25, Corollary 15]). We complement this result by proving that every generalized triple derivation from a real or complex JB*-triple E into itself or into E^* is automatically continuous (see Theorem 18). When specialized to C^*-algebras, we show that every generalized triple derivation from a C^*-algebra A to a Jordan–Banach triple A-module is automatically continuous (compare Theorem 22). Our results are not mere generalizations of those forerunners due to Johnson [20] and A. M. Peralta and Russo [25], the proofs are completely independent and the theorems presented here are novelties of independent interest even in the category of C^*-algebras.

2 Preliminaries

We recall that a complex (resp., real) *(normed) Jordan triple* is a complex (resp., real) (normed) space E equipped with a continuous triple product

$$E \times E \times E \to E(xyz) \mapsto \{x,y,z\}$$

that is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called "*Jordan Identity*",

$$L(a,b)L(x,y) - L(x,y)L(a,b) = L\big(L(a,b)x,y\big) - L\big(x,L(b,a)y\big),$$

for all a,b,x,y in E, where $L(x,y)z := \{x,y,z\}$. If E is complete with respect to the norm (*i.e.*, if E is a Banach space), then it is called a complex (resp., real) *Jordan–Banach triple*. Every normed Jordan triple can be completed in the usual way to become a Jordan–Banach triple. Unless otherwise stated, the term "normed Jordan triple" (resp., "Jordan–Banach triple") will always mean a real or complex normed Jordan triple (resp., "Jordan–Banach triple").

For each element a in a Jordan triple E, $Q(a)$ will denote the mapping defined by $Q(a)(x) := \{a,x,a\}$.

Given an element a in a Jordan triple E and a natural number n, we denote $a^{[1]} = a$, and $a^{[2n+1]} := Q(a)^n(a)$. The Jordan identity implies that $a^{[5]} = \big\{a,a,a^{[3]}\big\}$, and by induction, $a^{[2n+1]} = L(a,a)^n(a)$ for all $n \in \mathbb{N}$. The element a is called *nilpotent* if $a^{[2n+1]} = 0$ for some n. Jordan triples are power associative, that is, $\big\{a^{[k]},a^{[l]},a^{[m]}\big\} = a^{[k+l+m]}$.

A Jordan triple E for which the vanishing of $\{a,a,a\}$ implies that a itself vanishes is said to be *anisotropic*. It is easy to check that E is anisotropic if and only if zero is the unique nilpotent element in E.

A real (resp., complex) *Jordan algebra* is a (non-necessarily associative) algebra over the real (resp., complex) field whose product $\circ$ is abelian and satisfies $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$. A normed Jordan algebra is a Jordan algebra A equipped with a norm, $\|\cdot\|$, satisfying $\|a \circ b\| \le \|a\|\,\|b\|$, $a,b \in A$. A *Jordan–Banach algebra* is a normed Jordan algebra whose norm is complete.

Every real or complex associative Banach algebra (resp., Jordan Banach algebra) is a real Jordan–Banach triple with respect to the product $\{a,b,c\} = \frac{1}{2}(abc + cba)$ (resp., $\{a,b,c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$).

A JB*-algebra is a complex Jordan–Banach algebra A equipped with an algebra involution $*$ satisfying $\| \{a, a^*, a\} \| = \|2(a \circ a^*) \circ a - a^2 \circ a^*\| = \|a\|^3$, $a \in A$.

A *(complex)* JB**-triple* is a complex Jordan–Banach triple E satisfying the following axioms:

(JB* 1) For each a in E the map $L(a, a)$ is an hermitian operator on E with nonnegative spectrum.

(JB* 2) $\|\{a, a, a\}\| = \|a\|^3$ for all a in A.

We recall that a *real* JB**-triple* is a norm-closed real subtriple of a complex JB*-triple (see [17]).

We also recall that a subspace I of a normed Jordan triple E is a *triple ideal* (resp., a *subtriple*) if $\{E, E, I\} + \{E, I, E\} \subseteq I$ (resp., $\{I, I, I\} \subseteq I$). The quotient of a normed Jordan triple by a closed triple ideal is a normed Jordan triple. It is also known that the quotient of a JB*-triple (resp., a real JB*-triple) by a closed triple ideal is a JB*-triple (resp., a real JB*-triple) (compare [22]).

A real JB*-algebra is a closed $*$-invariant real subalgebra of a (complex) JB*-algebra. Real C^*-algebras (*i.e.*, closed $*$-invariant real subalgebras of C^*-algebras) equipped with the Jordan product $a\circ b = \frac{1}{2}(ab+ba)$ are examples of real JB*-algebras.

3 The Separating Space of a Generalized Triple Homomorphism

Let $T\colon E \to F$ be a (not necessarily continuous) linear mapping between normed Jordan triples. We define $\check{T}\colon E \times E \times E \to F$ by the rule

$$(a, b, c) \mapsto \check{T}(a, b, c) = T(\{a, b, c\}) - \{T(a), T(b), T(c)\}.$$

The mapping $\check{T}$ is symmetric and linear in the outer variables and conjugate linear in the middle one (trilinear when E is a real Jordan triple). The mapping T is said to be a *generalized triple homomorphism* if $\check{T}$ is (jointly) continuous, equivalently, if there exists $C > 0$ such that

$$\|\check{T}(a, b, c)\| = \|T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\| \le C\|a\| \, \|b\| \, \|c\|.$$

Let A, B be Banach algebras. We have already mentioned that a linear mapping $T\colon A \to B$ is a generalized homomorphism when the bilinear mapping

$$(a, b) \to T(ab) - T(a)T(b)$$

is continuous. Every Banach algebra is a Jordan–Banach triple when endowed with the triple product

$$2\{a, b, c\} = abc + cba. \tag{2}$$

We will refer to this product as the *elemental* (Jordan) triple product of A.

A richer structure on the Banach algebra A provides richer ternary products. For example, when A is a $*$-algebra we can consider the Jordan triple product given by

$$2\{a, b, c\} = ab^*c + cb^*a. \tag{3}$$

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Let A and B be Banach $*$-algebras. A linear mapping $T\colon A \to B$ is said to be a *generalized $*$-homomorphism* if T is a generalized homomorphism and the mapping

$$a \mapsto S(a) = T(a^*)^* - T(a)$$

is continuous. Generalized $*$-homomorphisms were already considered by Johnson in [20, Theorem 4].

Our next result explores the connections between generalized $(*\text{-})$ homomorphisms and generalized triple homomorphisms between Banach $(*\text{-})$algebras.

Proposition 1 *Let A, B be Banach algebras. Every generalized homomorphism $T\colon A \to B$ is a generalized triple homomorphism when A and B are equipped with the elemental triple product $2\{a, b, c\} = abc + cba$.*

When A and B are Banach $*$-algebras and T is a generalized $*$-homomorphism, then T is a generalized triple homomorphism with respect to the triple product $2\{a, b, c\} = ab^*c + cb^*a$.

Proof We start proving the first statement. Let $T\colon A \to B$ be a generalized homomorphism between Banach algebras. We will show that T is a generalized triple homomorphism when A and B are equipped with the triple product (2).

Throughout this proof, $\tilde{T}$ will denote the continuous bilinear mapping from $A \times A$ into B defined by $\tilde{T}(a, b) := T(ab) - T(a)T(b)$.

First, let us see that the (real) trilinear mapping $(a, b, c) \mapsto T(a)\tilde{T}(b, c)$ is continuous. Applying the uniform boundedness principle we see that a trilinear mapping from the cartesian product of three Banach spaces to another Banach space is (jointly) continuous if, and only if, it is continuous whenever we fix two variables. Since $\tilde{T}$ is continuous, the desired statement will follow as soon as we prove that the linear mapping $x \mapsto T(x)\tilde{T}(b, c)$ is continuous whenever we fix b and c in A. Let (x_n) be a norm-null sequence in A, then

$$\begin{aligned}\lim_n T(x_n)\tilde{T}(b, c) &= \lim_n T(x_n)T(bc) - T(x_n)T(b)T(c)\\ &= \lim_n \tilde{T}(x_n, b)T(c) + \tilde{T}(x_n b, c) - \tilde{T}(x_n, bc) = 0,\end{aligned}$$

which proves the desired continuity.

Now, the identity

$$T(abc) - T(a)T(b)T(c) = \tilde{T}(a, bc) + T(a)\tilde{T}(b, c)$$

implies that the assignment $(a, b, c) \mapsto T(abc) - T(a)T(b)T(c)$ defines a (jointly) continuous trilinear mapping. It follows that the assignment

$$\begin{aligned}(a, b, c) &\mapsto T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\\ &= \frac{1}{2}\big(T(abc) + T(cba) - T(a)T(b)T(c) - T(c)T(b)T(a)\big)\\ &= \frac{1}{2}\big(T(abc) - T(a)T(b)T(c)\big) + \frac{1}{2}\big(T(cba) - T(c)T(b)T(a)\big)\end{aligned}$$

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defines a continuous trilinear mapping, which gives the first statement.

Let us suppose now that T is a generalized $*$-homomorphism between Banach $*$-algebras A and B. By the first part of the proof, T is a generalized triple homomorphism when A and B are equipped with the triple product (2). We have actually shown that the mapping

$$(a,b,c) \mapsto T(abc) - T(a)T(b)T(c) \tag{4}$$

is continuous. We will see that T is a generalized triple homomorphism when A and B are endowed with the product defined in (3).

Let us write $S(x) = T(x^*)^* - T(x)$. Fix two elements a, c in A. We claim that the (real) linear mapping

$$x \mapsto T(ax^*c + cx^*a) - T(a)T(x)^*T(c) - T(c)T(x)^*T(a) \tag{5}$$

is continuous. Clearly, it is enough to check that the restriction to A_{sa} is continuous. Let x be a self-adjoint element in A, then

$$\begin{aligned}
&T(axc) - T(a)T(x)^*T(c) \\
&\quad = T(axc) - T(a)T(x)T(c) - T(a)T(x)^*T(c) + T(a)T(x)T(c) \\
&\quad = T(axc) - T(a)T(x)T(c) - T(a)\big(T(x)^* - T(x)\big)T(c),
\end{aligned}$$

and hence

$$\begin{aligned}
&T(axc + cxa) - T(a)T(x)^*T(c) - T(c)T(x)^*T(c) \\
&\quad = \big(T(axc + cxa) - T(a)T(x)T(c) - T(c)T(x)T(a)\big) \\
&\qquad - \big(T(a)S(x)T(c) + T(c)S(x)T(a)\big),
\end{aligned}$$

which proves the claim.

Now, we fix a, b in A and claim that the linear mapping

$$x \mapsto T(ab^*x) - T(a)T(b)^*T(x) \tag{6}$$

is continuous. To this end, let (x_n) be a norm null sequence in A. Then by (4),

$$\begin{aligned}
&\lim_n T(ab^*x_n) - T(a)T(b)^*T(x_n) \\
&\quad = \lim_n \big(T(ab^*x_n) - T(a)T(b^*)T(x_n)\big) \\
&\qquad + \big(T(a)T(b^*)T(x_n) - T(a)T(b)^*T(x_n)\big) \\
&\quad = \lim_n \big(T(ab^*x_n) - T(a)T(b^*)T(x_n)\big) \\
&\qquad + \lim_n T(a)\tilde{T}(x_n^*, b)^* + T(a)T(b)^*S(x_n) \\
&\qquad - T(a)\tilde{T}(b^*, x_n) - T(a)S(b^*x_n) = 0.
\end{aligned}$$

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Similarly, for every b, c in A the linear mapping

$$x \mapsto T(xb^*c) - T(x)T(b)^*T(c) \tag{7}$$

is continuous.

Combining (5), (6), and (7) with the uniform boundedness principle we deduce that the (real) trilinear mapping $(x, y, x) \mapsto T(xy^*z) - T(x)T(y)^*T(z)$ is jointly continuous, and hence, T is a generalized triple homomorphism for the product defined in (3). ■

The separating space of a linear mapping played an important role in many problems of automatic continuity (compare [2, 7, 13, 25, 26, 29], among others). Let $T\colon X \to Y$ be a linear mapping between two normed spaces. We recall that the *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$. It is well known that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$.

When $T\colon A \to B$ is a generalized homomorphism between Banach algebras and $z \in \sigma_Y(T)$ it is clear that $T(a)z$ and $zT(a)$ lie in $\sigma_Y(T)$, for every $a \in A$. This was actually noticed and applied by Johnson to show that the separating space of T is a closed two-sided ideal of the closed subalgebra of B generated by $T(A)$ (compare [20, Lemma 1]).

We are interested in the properties of the separating space of a generalized triple homomorphism T between Jordan–Banach triples E and F. Clearly, the image of a generalized triple homomorphism $T\colon E \to F$ and the image of $\check{T}$ are both contained in the subtriple of F generated by $T(E)$. However, $T(E)$ and $\check{T}(E \times E \times E)$ need not be Jordan subtriples of F. Moreover, it is not so easy to check that the separating space of T is a closed triple ideal of the closed subtriple of F generated by the image of T. The difficulties in the triple setting grow seriously. For this reason, we will require an appropriate description of the subtriple of F generated by a subset.

In the following we need the notion of a *triple monomial* or an *odd triple monomial*. Let $x_1, x_2, \ldots$ be a sequence of indeterminates. Then a triple monomial is a term that can be obtained by the following recursive procedure:

(i) Every indeterminate x_k is a triple monomial of degree 1.

(ii) If V_1, V_2, and V_3 are triple monomials of degrees d_1, d_2, and d_3 respectively, then $V := \{V_1, V_2, V_3\}$ is a triple monomial of degree $d_1 + d_2 + d_3$, where $\{\cdot, \cdot, \cdot\}$ is a "formal triple product" in three variables.

Notice that this procedure is neither commutative nor associative in general, and the degrees of triple monomials are always odd numbers. If the triple monomial V does not contain any indeterminate x_j with $j > n$, we also write $V = V(x_1, \ldots, x_n)$. In that case, for every JB*-triple E and every $a = (a_1, \ldots, a_n) \in E^n$ the element $V(a) = V(a_1, \ldots, a_n) \in E$ is well defined—just specialize every x_k to a_k and the "formal triple product" to the concrete triple product of E. In this sense V induces a polynomial map $E^n \to E$ which is denoted by the same symbol (or by V_E to avoid confusion). Now, for each fixed odd integer $n \geq 1$, denote by $\mathcal{OP}^n$ the set of all triple monomials V of degree n in which every x_k with $1 \leq k \leq n$ occurs precisely once.

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Then $V = V(x_1, \ldots, x_n)$ and the induced map $V_E \colon E^n \to E$ is multilinear for every JB*-triple E.

The symbol $\mathcal{OP}^{2m+1}(E)$ will stand for the set of all multilinear mappings of the form V_E, where V runs in $\mathcal{OP}^{2m+1}$, while $\mathcal{OP}(E)$ will denote the set of all odd triple monomials of any degree on E. It should be noted here that when F is another Jordan triple, each triple monomial V in $\mathcal{OP}^{2m+1}$ induces an element V_F in $\mathcal{OP}^{2m+1}(F)$ by just replacing the triple product of E in the definition of V with the corresponding triple product on F.

Lemma 2 *Let $T \colon E \to F$ be a generalized triple homomorphism between normed Jordan triples and m a natural number. Let V be an odd triple monomial of degree $2m+1$, which can be regarded as an element in $\mathcal{OP}^{2m+1}(E)$ or in $\mathcal{OP}^{2m+1}(F)$ indistinctly. Suppose V of the form $V = \{\,\cdot\,, W, P\}$ (resp.,$V = \{W, \,\cdot\,, P\}$), and let $j = \deg(W)$. Then*

$$\lim_{n\to\infty} V\big(T(x_n), T(a_1), \ldots, T(a_{2m})\big) - T\big(V(x_n, a_1, \ldots, a_{2m})\big) = 0,$$

$$\text{(resp., } \lim_{n\to\infty} V\big(T(a_1), \ldots, T(a_j), T(x_n), T(a_{j+1}), \ldots, T(a_{2m})\big)$$
$$- T\big(V(a_1, \ldots, a_j, x_n, a_{j+1}, \ldots, a_{2m})\big) = 0\text{)},$$

for every norm-null sequence (x_n) and $a_1, \ldots, a_{2m}$ in E.

Proof We will proceed by induction on m. Since T is a generalized triple homomorphism, the statement trivially holds for every odd triple monomial of degree 3. Now, let us suppose that the statement is true for odd triple monomials of degree less or equal than $2m-1$.

Let V be an odd triple monomial of degree $2m+1$. We will assume $V = \{\,\cdot\,, W, P\}$, the case $V = \{W, \,\cdot\,, P\}$ follows similarly. Pick a norm-null sequence (x_n) and $a_1, \ldots, a_{2m}$ in E. The odd triple monomials W and P can be written in the form $W = \{W_1, W_2, W_3\}$ and $P = \{P_1, P_2, P_3\}$ for some odd triple monomials P_i, W_i, $i = 1, 2, 3$. Clearly $1 \le \deg(W_i), \deg(P_i) < 2m-1$.

Applying the Jordan identity we have

(8)
$$\begin{aligned}
&V(T(x_n), T(a_1), \ldots, T(a_{2m})) = \{T(x_n), W(T(a_i)), P(T(a_j))\}\\
&= \big\{ T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, \{P_1(T(a_{j_1})), P_2(T(a_{j_2})), P_3(T(a_{j_3}))\}\big\}\\
&= \big\{ \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\}\\
&- \big\{ P_1(T(a_{j_1})), \{\{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, T(x_n), P_2(T(a_{j_2}))\}, P_3(T(a_{j_3}))\big\}\\
&+ \big\{ P_1(T(a_{j_1})), P_2(T(a_{j_2})), \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_3(T(a_{j_3}))\}\big\}.
\end{aligned}$$

We will treat the summands in the right-hand side independently. We claim that

(9)
$$\begin{aligned}
&\lim_n \big\{ \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\}\\
&\qquad - T\big(\{\{x_n, \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_1(a_{j_1})\}, P_2(a_{j_2}), P_3(a_{j_3})\}\big) = 0.
\end{aligned}$$

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Indeed, consider the monomial $Q = \big\{ \big\{ \cdot, \{W_1, W_2, W_3\}, P_1 \big\}, P_2, P_3 \big\}$. It is clear that $\deg(Q) \le 2m-1$, and

$$(10)\qquad \big\{ T(x_n), \big\{ W_1\big(T(a_{i_1})\big), W_2\big(T(a_{i_2})\big), W_3\big(T(a_{i_3})\big) \big\}, P_1 T\big((a_{j_1})\big) \big\} \\ = Q\big(T(x_n), T(a_{i_1}), T(a_{i_2}), T(a_{i_3}), T(a_{j_1})\big).$$

Taking limits in $n \to \infty$ and applying the induction hypothesis we get

$$(11)\quad \lim_n Q\big(T(x_n), T(a_{i_1}), T(a_{i_2}), T(a_{i_3}), T(a_{j_1})\big) - T\big(Q(x_n, a_{i_1}, a_{i_2}, a_{i_3}, a_{j_1})\big) = 0.$$

Let $z_n := Q(x_n, a_{i_1}, a_{i_2}, a_{i_3}, a_{j_1})$. It follows from the continuity of the triple product that (z_n) is a norm-null sequence in E.

Consider now the monomial $Q' = \{\cdot, P_2, P_3\}$. Since $\deg(Q') \le 2m-1$ we can apply the induction hypothesis to prove

$$(12)\qquad \lim_n \big\{ T(z_n), P_2\big(T(a_{j_2})\big), P_3\big(T(a_{j_3})\big) \big\} - T\big(\{z_n, P_2(a_{j_2}), P_3(a_{j_3})\}\big) \\ = \lim_n Q'\Big(T(z_n), T\big((a_{j_2}), T(a_{j_3})\big)\Big) - T\big(Q'(z_n, a_{j_2}, a_{j_3})\big) = 0.$$

Combining (10), (11), and (12) we have

$$\lim_n \big\{ \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\} \\ - T\big(\{\{x_n, \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_1(a_{j_1})\}, P_2(a_{j_2}), P_3(a_{j_3})\}\big) \\ = \lim_n \big\{ Q(T(x_n), T(a_{i_1}), T(a_{i_2}), T(a_{i_3}), T(a_{j_1})), P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\} \\ - T\big(\{Q(x_n, a_{i_1}, a_{i_2}, a_{i_3}, a_{j_1}), P_2(a_{j_2}), P_3(a_{j_3})\}\big) \\ = \lim_n \{T(z_n), P_2(T(a_{j_1})), P_3(T(a_{j_2}))\} - T\big(\{z_n, P_2(a_{j_1}), P_3(a_{j_2})\}\big) = 0,$$

which proves the claim (9).

We can similarly prove that

$$(13)\quad \lim_n \{P_1(T(a_{j_1})), \{\{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, T(x_n), P_2(T(a_{j_2}))\}, P_3(T(a_{j_3}))\} \\ - T\big(\{P_1(a_{j_1}), \{\{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, T(x_n), P_2(a_{j_2})\}, P_3(a_{j_3})\}\big) = 0$$

and

$$(14)\quad \lim_n \{P_1(T(a_{j_1})), P_2(T(a_{j_2})), \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_3(T(a_{j_3}))\}\} \\ - T\big(\{P_1(a_{j_1}), P_2(a_{j_2}), \{T(x_n)\{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_3(a_{j_3})\}\}\big) = 0.$$

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Finally, from (8), (9), (13), and (14) we obtain

$$\begin{aligned}
&\lim_n V\big(T(x_n), T(a_1), \ldots, T(a_{2m})\big) - T\big(V(x_n, a_1, \ldots, a_{2m})\big) = \text{(from the Jordan identity)}\\
&\lim_n \{\{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\}\\
&- T(\{\{x_n, \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_1(a_{j_1})\}, P_2(a_{j_2}), P_3(a_{j_3})\})\\
&- \{P_1(T(a_{j_1})), \{\{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, T(x_n), P_2(T(a_{j_2}))\}, P_3(T(a_{j_3}))\}\\
&+ T(\{P_1(a_{j_1}), \{\{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, T(x_n), P_2(a_{j_2})\}, P_3(a_{j_3})\})\\
&+ \{P_1(T(a_{j_1})), P_2(T(a_{j_2})), \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_3(T(a_{j_3}))\}\}\\
&- \lim_n T(\{P_1(a_{j_1}), P_2(a_{j_2}), \{T(x_n), \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_3(a_{j_3})\}\}) = 0,
\end{aligned}$$

as we desired. ■

We recall that two elements a and b in a Jordan–Banach triple E are said to be *orthogonal* (written $a \perp b$) if $L(a,b) = L(b,a) = 0$. A direct application of the Jordan identity yields that, for each c in E,

$$a \perp \{b, c, b\} \quad \text{whenever} \quad a \perp b.$$

When E is anisotropic, $a \perp b$ if and only if $L(a,b) = 0$. In case E is a real or complex JB*-triple, the relation of being orthogonal admits several equivalent reformulations (*cf.* [6, Lemma 1]).

Given a subset M of a Jordan–Banach triple, E, we write $M^{\perp}_{E}$ for the *(orthogonal) annihilator of* M, defined by

$$M^{\perp}_{E} := \{y \in E : y \perp x, \forall x \in M\}.$$

When no confusion arises, we will write $M^{\perp}$ instead of $M^{\perp}_{E}$.

Let E be a Jordan–Banach triple and $S \subseteq E$. The norm-closed Jordan subtriple of E generated by S is the smallest norm-closed subtriple of E containing S and will be denoted by E_S. Clearly, E_S coincides with the norm-closure of the linear span of the set

$$\mathcal{OP}_E(S) := \{V(a_1, \ldots, a_{2m+1}) : m \in \mathbb{N}, V \in \mathcal{OP}^{2m+1}(E), a_1, \ldots, a_{2m+1} \in S\}.$$

When a is an element in E, we write E_a instead of $E_{\{a\}}$.

Proposition 3 *Let $T\colon E \to F$ be a generalized triple homomorphism between two Jordan–Banach triples. Let I and $\widetilde{F}$ denote $\sigma_F(T)$ and the norm-closed subtriple of F generated by $T(E)$, respectively. Then we have the following:*

(i) *I is a (closed) triple ideal of $\widetilde{F}$.*
(ii) *$I^{\perp}_{\widetilde{F}}$ contains all the elements of the form $\check{T}(a,b,c)$.*

Further, if J is a closed triple ideal of $\widetilde{F}$ containing $I^{\perp}_{\widetilde{F}}$, then $\pi \circ T$ is a triple homomorphism, where π is the quotient map $\widetilde{F} \to \widetilde{F}/J \cap \widetilde{F}$.

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Proof (i) Since I is a closed linear subspace of F, we only have to prove that $\{\widetilde{F},\widetilde{F},I\}+\{\widetilde{F},I,\widetilde{F}\}\subseteq I$. Since $\mathcal{OP}_F\big(T(E)\big)$ is dense in $\widetilde{F}$, it is enough to show that

$$V\big(I,T(E),\dots,T(E)\big)+V'\big(T(E),\dots,T(E),I,T(E),\dots,T(E)\big)\subseteq I,$$

where V and V' are arbitrary odd triple monomials of the form $\{W,\cdot,P\}$ and $\{\cdot,W',P'\}$, respectively.

Let z be an element in I, then there exists a norm-null sequence (z_n) in E such that $z=\lim_n T(z_n)$. Now let $V=\{W,\cdot,P\}$ and $V'=\{\cdot,W',P'\}$ be odd triple monomials of degree $2m_1+1$ and $2m_2+1$, respectively, with $j=\deg(W)$. Let us fix $a_1,\dots,a_{2m_1},b_1,\dots,b_{2m_2}$ in E. By Lemma 2,

$$\begin{aligned}V'\big(z,T(a_1),\dots,T(a_{2m_1})\big)&=\lim_n V'\big(T(z_n),T(a_1),\dots,a_{2m_1}\big)\\&=\lim_n T\big(V'(z_n,a_1,\dots,a_{2m_1})\big),\end{aligned}$$

and

$$\begin{aligned}&V\big(T(b_1),\dots,T(b_j),z,T(b_{j+1}),\dots,T(b_{2m_2})\big)\\&\quad=\lim_n V\big(T(b_1),\dots,T(b_j),T(z_n),T(b_{j+1}),\dots,T(b_{2m_2})\big)\\&\quad=\lim_n T\big(V(b_1,\dots,b_j,z_n,b_{j+1},\dots,b_{2m_2})\big).\end{aligned}$$

By the continuity of the Jordan triple product $x_n=V'(z_n,a_1,\dots,a_{2m_1})$ and $y_n=V(b_1,\dots,b_j,z_n,b_{j+1},\dots,b_{2m_2})$ are norm-null sequences in E, and thus

$$V'\big(z,T(a_1),\dots,T(a_{2m_1})\big)=\lim_n T(x_n)\in I$$

and

$$V\big(T(b_1),\dots,T(b_j),z,T(b_{j+1}),\dots,T(b_{2m_2})\big)=\lim_n T(y_n)\in I.$$

(ii) In order to see that $I^{\perp}_{\widetilde{F}}\supseteq\check{T}(E,E,E)$, we will show that

$$L\big(I,\check{T}(a,b,c)\big)\big|_{\widetilde{F}}=L\big(\check{T}(a,b,c),I\big)\big|_{\widetilde{F}}=0,\quad\forall a,b,c\in E.$$

Let $z=\lim T(z_n)$ in I, where (z_n) is a norm-null sequence in E, V and odd triple monomial of degree $2m+1$ and $a,b,c,a_1,\dots,a_{2m+1}$ in E. Then

$$\begin{aligned}&L\big(z,\check{T}(a,b,c)\big)\Big(V\big(T(a_1),\dots,T(a_{2m+1})\big)\Big)\\&\quad=\lim_n\big\{T(z_n),\check{T}(a,b,c),V\big(T(a_1),\dots,T(a_{2m+1})\big)\big\}\\&\quad=\lim_n\big\{T(z_n),T(\{a,b,c\}),V\big(T(a_1),\dots,T(a_{2m+1})\big)\big\}\\&\qquad\quad-\big\{T(z_n),\{T(a),T(b),T(c)\},V\big(T(a_1),\dots,T(a_{2m+1})\big)\big\}=(\text{by Lemma 2})\\&\quad=\lim_n T\big(\big\{z_n,\{a,b,c\},V(a_1,\dots,a_{2m+1})\big\}\big)\\&\qquad\quad-T\big(\big\{z_n,\{a,b,c\},V(a_1,\dots,a_{2m+1})\big\}\big)=0.\end{aligned}$$

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We can similarly show that $L\big(\check{T}(a,b,c),z\big)\,\big(V(a_1,\ldots,a_{2m+1})\big)=0$. Therefore, it follows from the density of $\mathcal{OP}_F\big(T(E)\big)$ in $\widetilde{F}$ and the continuity of the triple product that $L\big(I,\check{T}(a,b,c)\big)\big|_{\widetilde{F}}=L\big(\check{T}(a,b,c),I\big)\big|_{\widetilde{F}}=0$, which proves (ii).

Finally, to see the last statement we observe that, since $I_{\widetilde{F}}^{\perp}$ contains all the elements of the form $\check{T}(a,b,c)$, we have

$$\begin{aligned}0=\pi\big(\check{T}(a,b,c)\big)&=\pi\big(T(\{a,b,c\})-\{T(a),T(b),T(c)\}\big)\\&=\pi\big(T(\{a,b,c\})\big)-\pi\big(\{T(a),T(b),T(c)\}\big),\quad\forall a,b,c\in E,\end{aligned}$$

so $\pi\circ T$ is a triple homomorphism. ■

Let us suppose that, in the hypothesis of Proposition 3 above, F is assumed to be a JB*-triple. In this setting two elements a,b in F are orthogonal if and only if $\{a,a,b\}=0$ (*cf.* [6, Lemma 1]). Under these assumptions, let z be an element in I and pick arbitrary a,b,c in E. Since there exists a null sequence (z_n) in E such that $z=\lim_n T(z_n)$, by Lemma 2 and the uniform boundedness principle, we have

$$\begin{aligned}\{z,z,\check{T}(a,b,c)\}=\lim_n&\big\{T(z_n),T(z_n),T(\{a,b,c\}\big\}\\&-\big\{T(z_n),T(z_n),\{T(a),T(b),T(c)\}\big\}=0,\end{aligned}$$

which implies $I_F^{\perp}\supseteq I_{\widetilde{F}}^{\perp}\supseteq\check{T}(E,E,E)$.

4 Automatic Continuity

4.1 Generalized Triple Homomorphisms Between Jordan–Banach Triples

A celebrated result of J. Cuntz states that a linear mapping $T\colon A\to X$ from a C^*-algebra to a Banach space is continuous if and only if its restriction to any C^*-subalgebra of A generated by a single hermitian element is continuous (*cf.* [8]). Some years before A. M. Sinclair [27] established that a similar automatic continuity result holds for homomorphism from a C^*-algebra to a Banach algebra. At this point, the reader should be tempted to ask if a similar statement holds for linear mappings whose domain is a JB*-triple (by replacing C^*-subalgebras generated by a single hermitian element by JB*-subtriples generated by a single element). Unfortunately, we will see next that the answer to this question is negative.

Example 4 A complex Hilbert space H becomes a JB*-triple when endowed with the triple product defined by $\{a,b,c\}=\frac{1}{2}\big((a|b)c+(c|b)a\big)$, where $(\cdot|\cdot)$ denotes the inner product of H. It can be easily seen that every norm-one element e in E is tripotent (*i.e.*, $\{e,e,e\}=e$). Therefore, the JB*-subtriple of E generated by a single element a coincides with $\mathbb{C}a$. This implies that, for each Banach space X, the restriction of any linear mapping $T\colon H\to X$ to any JB*-subtriple of H generated by a single element is continuous. When H is infinite-dimensional, we can easily find a discontinuous linear mapping from H into a Banach space. We can similarly consider a JB*-triple E of infinite dimension with finite rank (*e.g.*, all E_a have finite dimensions, see [4, Section 3]).

The above example shows that a simple translation to the setting of JB*-triples of the hypotheses assumed by Cuntz in [8] is not enough to guarantee that a linear mapping from a JB*-triple to a Banach space is automatically continuous. Finding an assumption to avoid the previous counterexample, we will replace the subtriple generated by a single element by the norm-closed inner ideal generated by a single element. We recall that a subspace J of a JB*-triple E is said to be an *inner ideal* if $\{J, E, J\}$ is contained in J. Let a be an element in E and let $E(a)$ denote the norm closure of $\{a, E, a\}$ in E. It is known that $E(a)$ coincides with the norm-closed inner ideal of E generated by a (*cf.* [5, pp. 19–20]). Let us notice that in the previous Example 4, $H(a) = H$ for every norm-one element $a \in H$.

Let $T\colon E \to F$ be a generalized triple homomorphism between Jordan–Banach triples and suppose that T is continuous when restricted to any norm-closed inner ideal generated by a single (norm-one) element. Let z be an element in $\sigma_F(T)$. Then there exists a norm-null sequence (z_n) in E such that $z = \lim_n T(z_n)$. Pick a norm-one element a in E. Then

$$\begin{aligned}\{T(a), z, T(a)\} &= \lim_n \{T(a), T(z_n), T(a)\} = \lim_n T(\{a, z_n, a\}) - \check{T}(a, z_n, a)\\ &= \lim_n T_{|E(a)}(\{a, z_n, a\}) - \check{T}(a, z_n, a) = 0,\end{aligned}$$

since $\{a, z_n, a\}$ is a norm-null sequence in $E(a)$ and $\check{T}$ and $T_{|E(a)}$ are continuous by hypothesis. Therefore $\{T(z_n), z, T(z_n)\} = 0$, for every natural n, and hence $z^{[3]} = \lim_n \{T(z_n), z, T(z_n)\} = 0$, which affirms that all elements in $\sigma_F(T)$ are nilpotents.

Definition 5 A Jordan–Banach triple E has *Cohen's factorization property* (CFP) if given a norm-null sequence (a_n) in E there exist a norm-null sequence (b_n) and two elements x, y in E such that $a_n = \{x, b_n, y\}$, $\forall n \in \mathbb{N}$.

Every Jordan–Banach algebra with a bounded approximate identity has Cohen's factorisation property (compare [1]). In particular, JB and JB*-algebras have Cohen factorisation property (see [16, Proposition 3.5.4]). It follows from [5, pp. 19–20] (see also [12, Lemma 3.2]) that for every norm-one element a in a JB*-triple E, $E(a)$ satisfies CFP.

Our next result is an extension of Sinclair's result [27, Corollary 4.3].

Theorem 6 *Let $T\colon E \to F$ be a linear mapping between two Jordan–Banach triples and suppose that one of the following statements holds:*

(i) *T is a generalized triple homomorphism and F is anisotropic;*
(ii) *E has Cohen's factorisation property.*

If the restriction of T to any closed inner ideal generated by a single element is continuous, then T is continuous.

Proof The proof under hypothesis (i) was already given in the paragraph preceding Definition 5. Suppose E satisfies CFP. Let (y_n) be a norm-null sequence in E and let $a \in E$. Since $T|_{E(a)}$ is continuous, we have $\lim_n T\{a, y_n, a\} = 0$. Since a was arbitrarily chosen, we deduce that

$$\lim_n T(\{a, y_n, b\}) = 0, \tag{15}$$

for every $a, b \in E$.

Let us pick $z \in \sigma_F(T)$ and a norm-null sequence (z_n) in E satisfying $T(z_n) \to z$. By hypothesis, there exist a, b in E and a norm-null sequence $(y_n) \subseteq E$ such that $z_n = \{a, y_n, b\}$. In such a case, by (15),

$$z = \lim_n T(z_n) = \lim_n T(\{a, y_n, b\}) = 0. \qquad \blacksquare$$

Remark 7 Let $T\colon E \to F$ be a linear mapping between Banach spaces. A useful property of the separating space $\sigma_F(T)$ asserts that for every bounded linear map R from F to another Banach space Z, the composition RT is continuous if and only if $\sigma_F(T) \subseteq \ker(R)$. It is also known that $\sigma(RT) = \overline{R(\sigma(T))}^{\|\cdot\|}$ (see [28, Lemma 1.3]).

Based on the Commutative Gelfand Theory established by W. Kaup (*cf.* [22]), T. J. Barton, T. Dang, and G. Horn proved the automatic continuity of triple homomorphisms between JB*-triples (see [3, Lemma 1]). The natural extension of this automatic continuity property to the setting of generalized triple homomorphisms is contained in our next result.

Theorem 8 *Every generalized triple homomorphism between* JB**-triples is continuous.*

Proof Let $T\colon E \to F$ be a generalized triple homomorphism between JB*-triples. The norm closed subtriple of F generated by $T(E)$ will be again denoted by $\widetilde{F}$, while the symbol I will stand for the separating space $\sigma_F(T)$. Since $\widetilde{F}$ is a norm-closed subtriple of F, then $\widetilde{F}$ is a JB*-triple itself. Proposition 3 (i) assures that I is a closed ideal of $\widetilde{F}$, and by [25, Lemma 4] $I_{\widetilde{F}}^{\perp}$ is a norm-closed triple ideal of $\widetilde{F}$.

The final statement in Proposition 3 guarantees that the linear mapping $\pi \circ T\colon E \to \widetilde{F}/I_{\widetilde{F}}^{\perp}$ is a triple homomorphism. Since the quotient $\widetilde{F}/I_{\widetilde{F}}^{\perp}$ is a JB*-triple, the triple homomorphism $\pi \circ T$ is continuous (*cf.* [3, Lemma 1]). By Remark 7, we have $I = \sigma_F(T) \subseteq \ker(\pi) = I_{\widetilde{F}}^{\perp}$, and the latter implies that $I = \sigma_F(T) = 0$. $\blacksquare$

Since every C^*-algebra, endowed with the triple product given in (3), is a JB*-triple, Theorem 8, together with Proposition 1, allows us to rediscover the following result, which is originally due to Johnson [20, Theorem 4].

Corollary 9 ([20, Theorem 4]) *Every generalized $*$-homomorphism between C^*-algebras is continuous.* $\blacksquare$

Our next goal is to explore the automatic continuity of a generalized triple homomorphism from a JB*-triple to a Jordan–Banach triple. To this end we will require some additional concepts and tools.

Let E be a real or complex Jordan–Banach triple system. We will say that E is *algebraic* if all singly generated (norm-closed) subtriples of E are finite-dimensional. If in fact there exists $m \in \mathbb{N}$ such that single-generated subtriples of E have dimension $\leq m$, then E is said to be of *bounded degree*, and the minimum of such an m will be called the degree or the rank of E. For real and complex JB*-triples algebraic and bounded degree are the same (*cf.* [4, Section 3]).

Our next result owes much to the proof given in [25, Proposition 12] by Russo and the second author.

Theorem 10 *Let $T\colon E \to X$ be a linear mapping from a* JB**-triple to a Banach space. Let $J_T := \{a \in E : T \circ Q(a), T \circ L(a,a) \text{ are continuous}\}$. Suppose that J_T has the following properties:*

(i) $J_T + J_T \subseteq J_T$.

(ii) $\{E, E, J_T\} + \{E, J_T, E\} \subseteq J_T$.

(iii) *If I is a norm-closed triple ideal containing J_T, then E/I is algebraic of bounded degree.*

Then T is continuous if and only if J_T is norm-closed.

Proof When T is continuous, J_T coincides with E and nothing has to be proved. Suppose now that J_T is norm-closed. It follows from (i) and (iii) that J_T is a norm-closed triple ideal of E. We claim that the restriction of T to J_T is continuous. Indeed, the assignment $(a, b, c) \mapsto W(a, b, c) = T(\{a, b, c\})$ defines a (real) trilinear mapping $W\colon J_T \times J_T \times J_T \to F$. From (i) and the definition of J_T, W is separately continuous whenever we fix two variables. An application of the uniform boundedness principle implies that W is jointly continuous. Therefore, there exists a positive constant M such that $\|T\{a, b, c\}\| \leq M\|a\|\,\|b\|\,\|c\|$, for every a, b, c in J_T. Since J_T is a JB*-subtriple of E, for each a in J_T there exists b in J_T such that $b^{[3]} = a$. In this case

$$\|T(a)\| = \|T(\{b, b, b\})\| \leq M\|b\|^3 = M\|\{b, b, b\}\| = M\|a\|,$$

which shows that $T|_{J_T}$ is continuous.

Finally, let us prove that $J_T = E$. By hypothesis (iii), E/J_T is algebraic of bounded degree m. Thus, for each element $a + J_T$ in E/J_T there exist mutually orthogonal minimal tripotents $e_1 + J_T, \ldots, e_k + J_T$ in E/J_T and $0 < \lambda_1 \leq \cdots \leq \lambda_k$ with $k \leq m$ such that $a + J_T = \sum_{j=1}^{k} \lambda_k e_k + J_T$. We will show that $e_1, \ldots, e_k \in J_T$, and hence, $a \in J_T$, which proves $E = J_T$.

Let $e + J_T$ be a minimal tripotent in E/J_T. Henceforth, $\pi\colon E \to E/J_T$ will denote the canonical projection. Take an arbitrary norm-null sequence (a_n) in E. For each natural n, there exists a scalar $\mu_n \in \mathbb{C}$ such that $\pi\big(Q(e)(a_n)\big) = \mu_n(e + J_T)$. The continuity of π and the Peirce projection $P_2(e + J_T)$ assure that $\mu_n \to 0$. It follows that $Q(e)(a_n) - \mu_n e$ lies in J_T and tends to zero in norm. Since, by hypothesis, $T|_{J_T}$ is continuous we have

$$T\big(Q(e)(a_n)\big) = T\big(Q(e)(a_n) - \mu_n e\big) + \mu_n T(e) \to 0.$$

The arbitrarity of (a_n) guarantees that $T \circ Q(e)$ is continuous, or equivalently, e lies in J_T. ■

The following auxiliary lemmas will be needed later.

Lemma 11 *Let E be a real* JB**-triple and J a subset of E satisfying that whenever we have two sequences $(x_n), (y_n)$ in E such that $Q(y_n)Q(x_n) = Q(x_n)$ and $Q(y_n)Q(x_m) = 0$ for $n \neq m$, then the x_n lie in J except (perhaps) for finitely many n. Suppose I is a norm-closed triple ideal of E containing J then E/I is algebraic of bounded degree.*

Proof Since I contains J then I also has the property assumed in the hypothesis.

Let us write $F = E/I$. As noticed in the proof of Corollary 8 in [25] for $\overline{a} = a + I$ we have $F_{\overline{a}} = E_a/(E_a \cap I)$.

The commutative JB*-triple E_a is triple isomorphic to some $C_0(L)$ (*cf.* [22, Section 1]). We will identify E_a with $C_0(L)$. It is known that $F_{a+I} \cong C_0(\Gamma)$ where

$$\Gamma = \{t \in L : b(t) = 0, \forall b \in E_a \cap I\}.$$

We claim that Γ is finite. Otherwise, there exists an infinite sequence (t_n) in Γ and a sequence of open disjoint sets $\{U_n\}_n$. By local compactness we can find open sets V_n, W_n with $\overline{V_n}$ and $\overline{W_n}$ compact, such that $t_n \in V_n \subseteq \overline{V_n} \subseteq W_n \subseteq \overline{W_n} \subseteq U_n$.

By Urysohn's lemma, for each natural n, we can find $f_n \in C_0(L)$ with $t_n \in \operatorname{supp}(f_n) \subseteq W_n$ and $g_n \in C_0(L)$ such that $g_n \equiv 1$ in $\overline{W_n}$ and vanishing outside U_n. Since $f_n(t_n), g_n(t_n) \neq 0, \forall n \in \mathbb{N}$, then $f_n, g_n \notin I, \forall n \in \mathbb{N}$. In this case the sequences $(f_n), (g_n)$ verify that $Q(g_n)Q(f_n) = Q(f_n)$ and $Q(g_n)Q(f_m) = 0$ for $n \neq m$, and they do not lie in I, which is a contradiction.

It follows that Γ is finite and therefore F_{a+I} is finite dimensional. Since $a + I$ was arbitrary chosen, the statement of the lemma follows from [4, Theorem 3.8]. ■

Lemma 12 *Let $T \colon E \to F$ be a generalized triple homomorphism between real Jordan–Banach triples, and let $(x_n), (y_n)$ be sequences of elements in E such that $Q(y_n)Q(x_n) = Q(x_n)$ and $Q(y_n)Q(x_m) = 0$ for $n \neq m$. Then $Q(T(x_n))T$ and $TQ(x_n)$ are continuous for all but a finite number of n.*

Proof Let us suppose that $Q\big(T(x_n)\big)\,T$ is discontinuous for infinitely many n in $\mathbb{N}$. By passing to a subsequence if necessary, we can assume that $Q\big(T(x_n)\big)\,T$ is discontinuous for all n in $\mathbb{N}$. We observe that, since T is a generalized triple homomorphism the identity

$$\{T(x_n), T(b), T(x_n)\} = T(\{x_n, b, x_n\}) - \check{T}(x_n, b, x_n),$$

holds for every $b \in E$ and $n \in \mathbb{N}$. It is then clear that $Q\big(T(x_n)\big)\,T$ is continuous if and only if $TQ(x_n)$ is. So, we may assume that $TQ(x_n)$ is discontinuous for all n in $\mathbb{N}$. Choose (a_n) in E such that $\|a_n\| \leq 2^{-n}\|x_n\|^{-2}$ and

$$\|TQ(x_n)(a_n)\| \geq 2^n\big(1 + \|T(y_n)\|^2\big) + \|\check{T}\|\,\|y_n\|^2.$$

Let $a = \sum_{m\geq 1}\{x_m, a_m, x_m\}$. Since $\{y_n, a, y_n\} = \{x_n, a_n, x_n\}$ we have

$$\begin{aligned} 2^n\big(1 + \|T(y_n)\|^2\big) + \|\check{T}\|\,\|y_n\|^2 &\leq \|TQ(x_n)(a_n)\| \\ &= \|TQ(y_n)(a)\| = \big\|\,Q\big(T(y_n)\big)\,\big(T(a)\big) + \check{T}(y_n, a, y_n)\big\| \\ &\leq \|T(y_n)\|^2\,\|T(a)\| + \|\check{T}\|\,\|y_n\|^2\,\|a\| \leq \big(1 + \|T(y_n)\|^2\big)\,\|T(a)\| + \|\check{T}\|\,\|y_n\|^2. \end{aligned}$$

So we have that $\|T(a)\| \geq 2^n, \forall n \in \mathbb{N}$, which is impossible. ■

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Let $T: E \to F$ be a generalized triple homomorphism between Jordan–Banach triples. Following the notation employed in Proposition 3, the symbol $\widetilde{F}$ will denote the norm-closed subtriple of F generated by $T(E)$.

According to the notation defined in [25], for each subset B of a Jordan–Banach triple F, we define its *quadratic annihilator*, $\mathrm{Ann}_F(B)$, as the set

$$\{a \in F : Q(a)(B) = \{a, B, a\} = 0\}.$$

The quadratic annihilator will be used later in a more general setting.

If we set $J := T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$, it not hard to see, from the basic properties of the separating space, that J coincides with the set $\big\{a \in E : Q\big(T(a)\big)T$ is continuous$\big\}$ (compare Remark 7), and since T is a generalized triple homomorphism, the latter equals $\{a \in E : TQ(a)$ is continuous$\}$ (compare the proof of Lemma 12). The following result follows straightforwardly from Lemmas 12 and 11 and the above comments.

Proposition 13 *Let $T: E \to F$ be a generalized triple homomorphism from a real* JB**-triple to a Jordan–Banach triple. The following statements hold:*

(i) *If I is a norm-closed triple ideal containing $T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$, then E/I is algebraic of bounded degree.*

(ii) *Let K be a triple ideal of E. The linear mapping*

$$x \in E \mapsto \{T(a), T(x), T(a)\}$$

is continuous for all a in K if, and only if, K is contained in $T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$.

We can establish now the main result of this section.

Theorem 14 *Let $T: E \to F$ be a generalized triple homomorphism from a* JB**-triple to a Jordan–Banach triple and let $J = T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$. The following statements are equivalent:*

(i) *J is a norm-closed triple ideal of E and*

$$\big\{\mathrm{Ann}_F\big(\sigma_F(T)\big), \mathrm{Ann}_F\big(\sigma_F(T)\big), \sigma_F(T)\big\} = 0.$$

(ii) *T is continuous.*

Proof The implication (ii) $\Rightarrow$ (i) is clear. We will prove (i) $\Rightarrow$ (ii). We already know, by Proposition 13 (ii), that for each element a in J, the linear mapping

$$x \in E \mapsto \{T(a), T(x), T(a)\}$$

is continuous. Let us fix a, b in J. Since J is a linear subspace of E, then $a + b$ also lies in J, that is, the mapping $x \mapsto \{T(a+b), T(x), T(a+b)\}$ is continuous. The identity

$$\begin{aligned} 2\{T(a), T(x), T(b)\} = \{&T(a+b), T(x), T(a+b)\} \\ &- \{T(a), T(x), T(a)\} - \{T(b), T(x), T(b)\}, \end{aligned}$$

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guarantees that the mapping $x \mapsto \{T(a), T(x), T(b)\}$ is continuous, or equivalently (because T is a generalized triple homomorphism), $TQ(a,b)$ is continuous.

Since $\left\{ \mathrm{Ann}_F\big(\sigma_F(T)\big), \mathrm{Ann}_F\big(\sigma_F(T)\big), \sigma_F(T) \right\} = 0$, the linear mapping

$$x \in E \mapsto \{T(a), T(b), T(x)\}$$

is continuous for every $a, b \in J$. Applying that T is a generalized triple homomorphism, we deduce that the linear mapping $x \in E \mapsto T(\{a,b,x\})$ also is continuous for every $a, b \in J$. This shows that the trilinear mapping $W\colon E \times E \times E$, given by $(a,b,c) \mapsto W(a,b,c) = T(\{a,b,c\})$ is continuous whenever we fix two variables in J. An application of the uniform boundedness principle proves that $W|_{J\times J\times J}$ is jointly continuous. Following the argument given in the proof of Theorem 10, we show that $T|J\colon J \to F$ is continuous.

Proposition 13 (i) implies that E/J is algebraic of bounded degree. The proof concludes applying the argument given in the final part of the proof of Theorem 10. ■

The above Theorem 14 admits a more detailed statement in the particular setting of some Cartan factors. We recall that a complex Hilbert space H can be regarded as a *type I Cartan factor* with its natural norm and the product given by

$$2\,\{a,b,c\} := (a|b)c + (c|b)a, \quad (a,b,c \in H),$$

where $(\,\cdot\,|\,\cdot\,)$ denotes the inner product of H.

Lemma 15 *Let H be a complex Hilbert space regarded as a type I Cartan factor, F an anisotropic Jordan–Banach triple and $T\colon H \to F$ a generalized triple homomorphism. Then T is continuous.*

Proof Let $\widetilde{F}$ denote the norm-closed subtriple of F generated by $T(E)$. It is enough to prove that $T\colon H \to \widetilde{F}$ is continuous. Replacing F with $\widetilde{F}$, we may assume, by Proposition 3, that $\sigma_F(T)$ is a norm-closed triple ideal of F and F is generated by $T(E)$. It follows from our hypothesis that the mapping

$$\check{T}(a,b,c) = \frac{1}{2}\big((a|b)T(c) + (c|b)T(a)\big) - \{T(a), T(b), T(c)\}\,, (a,b,c \in H),$$

is continuous. Let z be an element in $\sigma_F(T)$, there exists a norm-null sequence $(x_n) \subset H$ such that $T(x_n) \to z$. If we fix two arbitrary elements a, c in H, by the continuity of $\check{T}$ and the triple product of F we have

$$0 = \lim_n \frac{1}{2}\big((a|x_n)T(c) + (c|x_n)T(a)\big) - \{T(a), T(x_n), T(c)\} = -\{T(a), z, T(c)\}\,.$$

It follows from the arbitrariness of a and c that $\{T(E), \sigma_F(T), T(E)\} = 0$. Similarly, let V and W be odd triple monomials of degree $2m_1 + 1$ and $2m_2 + 1$, respectively,

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and let us fix $a_1, \ldots, a_{2m_1}, b_1, \ldots, b_{2m_2}$ in H. By Lemma 2,

$$\begin{aligned}
&\big\{V\big(T(a_1), \ldots, T(a_{2m_1+1})\big), z, W\big(T(b_1), \ldots, T(b_{2m_2+1})\big)\big\} \\
&\quad = \lim_n \big\{V\big(T(a_1), \ldots, T(a_{2m_1+1})\big), T(x_n), W\big(T(b_1), \ldots, T(b_{2m_2+1})\big)\big\} \\
&\quad = \lim_n T\big(\{V(a_1, \ldots, a_{2m_1+1}), x_n, W(b_1, \ldots, b_{2m_2+1})\}\big) \\
&\quad = \lim_n \frac{1}{2}\big(V(a_1, \ldots, a_{2m_1+1})|x_n\big)\, T\big(W(b_1, \ldots, b_{2m_2+1})\big) \\
&\qquad\quad + \frac{1}{2}\big(W(b_1, \ldots, b_{2m_2+1})|x_n\big)\, T\big(V(a_1, \ldots, a_{2m_1+1})\big) = 0.
\end{aligned}$$

Since we have assumed that F is the Jordan–Banach triple generated by $T(E)$, it follows by linearity and from the continuity of the product of F that $\{F, \sigma_F(T), F\} = 0$. Finally, F being anisotropic implies that $\sigma_F(T) = 0$ and hence T is continuous. ■

A *(complex) spin factor* is a complex Hilbert space S provided with a conjugation (*i.e.*, a conjugate linear isometry of period 2) $x \mapsto \overline{x}$, triple product

$$\{a, b, c\} = \frac{1}{2}\big((a|b)c + (c|b)a - (a|\bar{c})\bar{b}\big),$$

and norm given by $\|a\|^2 = \frac{1}{2}(a|a) + \frac{1}{2}\sqrt{(a|a)^2 - |(a|\overline{a})|^2}$, for every $a, b, c \in S$.

Lemma 16 *Let S be a (complex) spin factor, F an anisotropic Jordan–Banach triple and $T\colon S \to F$ a generalized triple homomorphism. Then T is continuous.*

Proof Let S be a spin factor. The corollary in [11, p. 313] and the proof of the proposition on p. 312 in the just-quoted paper assure that S is the norm closed linear span of a "*spin grid*" $\{u_i, v_i, u_0\}_{i\in\Gamma}$, where $(u_i|u_j) = (v_i|v_j) = (u_i|v_j) = (u_i|v_i) = (u_0|u_i) = (u_0|v_i) = 0$, $\|u_i\| = 1$, $\|v_i\| = 1$, $\|u_0\| = 1$ or 0, $\overline{u_i} = v_i$, and $\overline{u_0} = u_0$, for every $i \neq j$ in Γ. Let S_1 (resp., S_2) denote the norm-closed subspace of S generated by $\{u_i : i \in \Gamma\}$ (resp., $\{v_i : i \in \Gamma\}$). Clearly $S = S_1 \oplus S_2 \oplus \mathbb{C}u_0$. It is easy to see that S_1 and S_2 are norm-closed subtriples of S (*i.e.*, $\{S_i, S_i, S_i\} \subset S_i$) and $\{a, b, c\} = \frac{1}{2}\big((a|b)c + (c|b)a\big)$, for every a, b, c in S_i ($i = 1, 2$). Therefore S_1 and S_2 are Hilbert spaces equipped with structure of type I Cartan factors. Lemma 15 shows that $T|_{S_i}\colon S_i \to F$ is continuous for every $i = 1, 2$. Finally, the continuity of the natural projections of S onto S_1, S_2 and $\mathbb{C}u_0$ assures that T is continuous. ■

According to the comments given before Proposition 17 in [25], the proof of Theorem 10 (and hence the proof of Theorem 14) is only valid for complex JB*-triples, the reason being that, in the real setting, a minimal tripotent e in a real JB*-triple E need not satisfy that $E_2(e) = \mathbb{R}e$. Actually, there exist examples of minimal tripotents e for which $E_2(e)$ is infinite dimensional. The extension of Theorem 14 to the real setting is not a trivial consequence of the result proved in the complex case and constitutes a result of independent interest which remains open in this paper. However, there exists a subclass of real JB*-triples for which the statements of Theorems 10 and 14 remain true. A real JB*-triple E is called *reduced* whenever

$E_2(e) = \mathbb{R}e$ (equivalently, $E^{-1}(e) = 0$) for every minimal tripotent $e \in E$. Reduced real JB*-triples were considered in [24], [23], [14] and [25]. We note that the proof of Theorem 14 is valid for reduced real JB*-triples.

4.2 Generalized Triple Derivations from a JB*-Triple

Russo and the second author carried out in [25] a pioneer study on automatic continuity of ternary derivations from a JB*-triple E into a Jordan–Banach triple E-module. The concept of Jordan–Banach triple module is introduced in the just-quoted paper, where it is also established that every triple derivation from a real or complex JB*-triple into its dual space or into itself is automatically continuous. It seems natural, at this stage, to consider generalized triple derivations in the context of JB*-triples, studying the automatic continuity of these mappings.

Jordan triple modules over Jordan triples were introduced as appropriate extensions of bimodules over associative algebras and Jordan modules over Jordan algebras (*cf.* [25]). The concrete definition reads as follows: Let E be a complex (resp., real) Jordan triple, a *Jordan triple E-module* (also called a *triple E-module*) is a vector space X equipped with three mappings

$$\{\cdot,\cdot,\cdot\}_1 \colon X\times E\times E\to X,\quad \{\cdot,\cdot,\cdot\}_2 \colon E\times X\times E\to X,$$
$$\text{and}\quad \{\cdot,\cdot,\cdot\}_3 \colon E\times E\times X\to X$$

satisfying the following axioms:

(JTM1) $\{x,a,b\}_1$ is linear in a and x and conjugate linear in b (resp., trilinear), $\{a,b,x\}_3$ is linear in b and x and conjugate linear in a (resp., trilinear) and $\{a,x,b\}_2$ is conjugate linear in a,b,x (resp., trilinear).

(JTM2) $\{x,b,a\}_1 = \{a,b,x\}_3$, and $\{a,x,b\}_2 = \{b,x,a\}_2$ for every $a,b \in E$ and $x \in X$.

(JTM3) Denoting by $\{\cdot,\cdot,\cdot\}$ any of the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$, and $\{\cdot,\cdot,\cdot\}_3$, the identity

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\},$$

holds whenever one of the elements a,b,c,d,e is in X and the rest are in E.

When E is a Jordan–Banach triple and X is a triple E-module which is also a Banach space, we will say that X is a *Banach (Jordan) triple E-module* when the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ are (jointly) continuous. From now on, the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ will be simply denoted by $\{\cdot,\cdot,\cdot\}$.

Every real or complex associative algebra A (resp., Jordan algebra J) is a real Jordan triple with respect to $\{a,b,c\} := \frac{1}{2}(abc+cba)$, $a,b,c\in A$ (resp., $\{a,b,c\} = (a\circ b)\circ c + (c\circ b)\circ a - (a\circ c)\circ b$, $a,b,c\in J$). It is not hard to see that every A-bimodule X is a real triple A-module with respect to the products $\{a,b,x\}_3 := \frac{1}{2}(abx+xba)$ and $\{a,x,b\}_2 = \frac{1}{2}(axb+bxa)$, and that every Jordan module X over a Jordan algebra J is a real triple J-module with respect to the products

$$\{a,b,x\}_3 := (a\circ b)\circ x + (x\circ b)\circ a - (a\circ x)\circ b \quad \text{and}$$
$$\{a,x,b\}_2 := (a\circ x)\circ b + (b\circ x)\circ a - (a\circ b)\circ x.$$

The dual space, E^*, of a complex (resp., real) Jordan–Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}},$$

$\forall \varphi \in E^*$, $a, b, x \in E$ (*cf.* [25]).

Given a triple E-module X over a Jordan triple E, the space $E \oplus X$ can be equipped with a structure of real Jordan triple with respect to the product

$$\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} = \{a_1, a_2, a_3\} + \{x_1, a_2, a_3\} + \{a_1, x_2, a_3\} + \{a_1, a_2, x_3\}.$$

The Jordan triple $E \oplus X$ will be called the *triple split null extension* of E and X.

Let X be a Jordan triple E-module over a Jordan triple E. A *triple derivation* from E to X is a conjugate linear map $\delta\colon E \to X$ satisfying $\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}$.

Let E be a real (resp., complex) Jordan–Banach triple and let X be a Jordan–Banach triple E-module. A (conjugate) linear mapping $\delta\colon E \to X$ is said to be a *generalized derivation* when the mapping $\check{\delta}\colon E \times E \times E \to X$,

$$(a, b, c) \mapsto \check{\delta}(a, b, c) := \delta\{a, b, c\} - \{\delta(a), b, c\} - \{a, \delta(b), c\} - \{a, b, \delta(c)\}$$

is (jointly) continuous.

Arguing as in [25], we will associate with each generalized derivation from a JB*-triple E into a Jordan–Banach triple E-module a generalized triple homomorphism, in such a a way that the continuity of these two mappings is mutually determined.

Let $\delta\colon E \to X$ be a generalized derivation. The symbol $E \oplus X$ will stand for the triple split null extension of E and X equipped with the ℓ_1-norm. We define the mapping

$$\Theta_\delta\colon E \to E \oplus X,$$
$$a \mapsto a + \delta(a).$$

It is clear that δ is continuous if and only if Θ_δ is continuous. Furthermore, the identity

$$\begin{aligned}\check{\delta}(a, b, c) &= \delta\{a, b, c\} - \{\delta(a), b, c\} - \{a, \delta(b), c\} - \{a, b, \delta(c)\}\\ &= \Theta_\delta\{a, b, c\} - \{\Theta_\delta(a), \Theta_\delta(b), \Theta_\delta(c)\} = \check{\Theta}_\delta(a, b, c),\end{aligned}$$

shows that Θ_δ is a generalized triple homomorphism. According to this notation, we set $\Delta := \Theta_\delta(E) = \{a + \delta(a) : a \in E\}$. Let $(E \oplus X)_\Delta$ be the norm closed subtriple of $E \oplus X$ generated by Δ. Since Θ_δ is a generalized triple homomorphism, by Lemma 3,

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the separating space $\sigma_{E\oplus X}(\Theta_\delta)$ is a triple ideal of $(E \oplus X)_\Delta$. It is not hard to see that $\sigma_{E\oplus X}(\Theta_\delta)$ coincides with $\{0\} \times \sigma_X(\delta)$.

A subspace S of a triple E-module X is said to be a *Jordan triple submodule* or a *triple submodule* if $\{E, E, S\} \subseteq S$ and $\{E, S, E\} \subseteq S$. Every triple ideal J of E is a Jordan triple E-submodule of E.

Let $a + x$ and $b + y$ be elements in $(E \oplus X)_\Delta$ and $z \in \{0\} \times \sigma_X(\delta) = \sigma_{E\oplus X}(\Theta_\delta)$. By the definition of the triple product in $E \oplus X$ and the just-quoted fact that $\sigma_{E\oplus X}(\Theta_\delta)$ is a triple ideal of $(E \oplus X)_\Delta$ we have

$$\{a, b, z\} = \{a + x, b + y, z\} \tag{16}$$

and

$$\{a, z, b\} = \{a + x, z, b + y\} \tag{17}$$

Since $(E \oplus X)_\Delta$ contains Δ, it follows from (16) and (17) that $\{E, E, \sigma_X(\delta)\} \subseteq \sigma_X(\delta)$ and $\{E, \sigma_X(\delta), E\} \subseteq \sigma_X(\delta)$. Since $\sigma_X(\delta)$ is always a linear subspace, it is also a triple E-submodule of X.

For each subset A of a triple E-module X, we define its *quadratic annihilator*, $\mathrm{Ann}_E(A)$, as the set $\{a \in E : Q(a)(A) = \{a, A, a\} = 0\}$.

We will also make use of the following equality:

$$\mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big) \oplus X.$$

Remark 17 The quadratic annihilator of a submodule S of a triple module X need not be, in general, a linear subspace (*cf.* [25]). However, it is known that when E is a JB*-triple and $X = E$ or $X = E^*$ then, for each submodule S of X, $\mathrm{Ann}_E(S)$ is a linear subspace, and hence a norm-closed triple ideal of E (see Lemma 1 and Proposition 2 in [25]). Further, Proposition 2 (or Remark 3) in [25] shows that, in this case, $\{\mathrm{Ann}_E(S), \mathrm{Ann}_E(S), S\} = 0$ in the triple split null extension $E \oplus X$.

From now on, we assume that E is a JB*-triple and X denotes E or E^*. In this case, Remark 17 and the fact that $\sigma_X(\delta)$ is a triple E-submodule of X prove that $\mathrm{Ann}_E\big(\sigma_X(\delta)\big)$ is a norm-closed triple ideal of E.

The strategy for obtaining results on automatic continuity for generalized triple derivations will consist in applying Theorem 14 to the generalized triple homomorphism Θ_δ. In order to do this, we will first check that

$$J := \Theta_\delta^{-1}\Big(\mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big)\Big)$$

is a norm-closed triple ideal of E. It is not hard to see that $\mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big) \oplus X$ and

$$\Theta_\delta^{-1}\Big(\mathrm{Ann}_E\big(\sigma_X(\delta)\big) \oplus X\Big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big).$$

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This proves that J is a norm-closed triple ideal of E (see Remark 17). On the other hand,

$$\left\{ \mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big), \mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big), \sigma_{E\oplus X}(\Theta_\delta)\right\} = \left\{ \mathrm{Ann}_E\big(\sigma_X(\delta)\big), \mathrm{Ann}_E\big(\sigma_X(\delta)\big), \sigma_X(\delta)\right\} = 0$$

(compare the final statement in Remark 17). Theorem 14 proves the continuity of Θ_δ and hence the continuity of δ.

Theorem 18 *Let E be a real or complex* JB**-triple and $\delta\colon E \to X$ a generalized triple derivation, where $X = E$ or E^*. Then δ is continuous.* ■

The statement concerning real JB*-triples can be derived from the complex case applying Remark 14 in [25].

Since every triple derivation is a generalized triple derivation we get the following.

Corollary 19 ([25, Corollary 15]) *Let E be a real or complex* JB**-triple and let $\delta\colon E \to X$ be a triple derivation, where $X = E$ or E^*. Then δ is continuous.* ■

4.3 Generalized Triple Derivations Whose Domain is a C^*-algebra

We have already mentioned that every C^*-algebra belongs to the class of JB*-triples. We will conclude this paper by applying some of the previous results to C^*-algebras. The results obtained this way are interesting by themselves.

Lemma 20 *Let $T\colon A_{sa} \to X$ be a linear mapping from the self-adjoint part, A_{sa}, of an abelian C^*-algebra, A, to a Banach space. Suppose that $J_T := \{a \in A_{sa} : TQ(a)$ is continuous $\}$ is a norm-closed subset of A_{sa} with $\{a, A_{sa}, a\} \in J_T$, for every $a \in J_T$. Then J_T is a triple ideal of A_{sa}.*

Proof It is easy to see that every norm-closed inner ideal of the selfadjoint part of an abelian C^*-algebra A is a triple ideal in A_{sa} (norm-closed by assumption). Therefore, we only have to prove that J_T is a linear subspace. To this end, it is enough to show that $a + b \in J_T$ whenever $a, b \in J_T$.

Let a and b be two elements in A_{sa}. First we observe that, since A_{sa} is abelian, $L(a + b) = Q(a + b)$. Obviously, the linear mapping $L_b\colon A_{sa} \to A_{sa}, c \mapsto cb = bc$ is continuous. Since A_{sa} is abelian we have $L(a^2, b) = Q(a)L_b = L_bQ(a)$. Therefore $TL(a^2, b) = TQ(a)L_b$ is continuous for every $a \in J_T, b \in A_{sa}$.

Let us pick $a \in J_T$. We write a in the form $a = a_1 - a_2$ where a_1, a_2 are orthogonal positive elements in A_{sa}. Since $Q(a)A_{sa} \in J_T$, a_1^3 lies in J_T, and hence $a_1^6 A_{sa} = Q(a_1^3)A_{sa} \subseteq J_T$. This implies that J_T contains the norm-closed ideal of A_{sa} generated by a_1^6, which guarantees that J_T contains a_1 and $a_1^{\frac{1}{2}}$. Similarly, we show that J_T contains a_2 and $a_2^{\frac{1}{2}}$. Now

$$TL(a, b) = TL(a_1, b) - TL(a_2, b) = TL\big((a_1^{\frac{1}{2}})^2, b\big) - TL\big((a_2^{\frac{1}{2}})^2, b\big),$$

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and thus $TL(a,b)$ is continuous for every $b \in A_{sa}$. Finally, the equality

$$TQ(a+b) = TL(a+b) = TL(a,a) + TL(b,b) + 2TL(a,b)$$

shows that $TQ(a+b)$ is continuous for every $a, b \in J_T$. ∎

Proposition 21 *Let $\delta\colon A \to X$ be a generalized derivation from an abelian C^*-algebra to a Jordan–Banach triple A-module. Then δ is continuous.*

Proof We will only prove that $\delta_{|A_{sa}}$ is continuous. Let $\Theta_{\delta_0}\colon A_{sa} \to A_{sa} \oplus X$ be the generalized triple homomorphism associated to $\delta_0 := \delta|_{A_{sa}}$. We have already shown that $J = \Theta_{\delta_0}^{-1}\big(\operatorname{Ann}_{A_{sa}\oplus X}\big(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big)\big)$ coincides with $\operatorname{Ann}_{A_{sa}}\big(\sigma_X(\delta_0)\big)$ (see the comments prior to Theorem 18). Therefore, J is the quadratic annihilator of a closed submodule of X, and hence J is norm closed and satisfies $\{a, A_{sa}, a\} \in J$, for every $a \in J$ (*cf.* [25, Section 2.3]).

It is easy to see that J coincides with $\{a \in A_{sa} : \Theta_{\delta_0}Q(a) \text{ is continuous}\}$. Now, Lemma 20 proves that $J = \Theta_{\delta_0}^{-1}\big(\operatorname{Ann}_{A_{sa}\oplus X}\big(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big)\big)$ is a norm-closed triple ideal of A_{sa}, and since A is abelian,

$$\begin{aligned}\big\{\operatorname{Ann}_{A_{sa}\oplus X}\big(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big), \operatorname{Ann}_{A_{sa}\oplus X}\big(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big), \sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big\}\\ = \big\{\operatorname{Ann}_{A_{sa}}\big(\sigma_{A_{sa}}(\delta_0)\big), \operatorname{Ann}_{A_{sa}}\big(\sigma_{A_{sa}}(\delta_0)\big), \sigma_{A_{sa}}(\delta_0)\big\} = 0.\end{aligned}$$

Having in mind that A_{sa} is a reduced real JB*-triple and the validity of Theorem 14 for reduced real JB*-triples, we conclude that $\delta|_{A_{sa}}$ is continuous. ∎

A celebrated result of J. Cuntz (see [8]) establishes that a linear mapping from a C^*-algebra A to a Banach space is continuous if and only if its restriction to each subalgebra of A generated by a single hermitian element is continuous. We finish this note with a consequence of Cuntz' theorem and Proposition 21.

Theorem 22 *Every generalized triple derivation from a real or complex C^*-algebra A to a Jordan–Banach triple A-module is continuous.* ∎

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