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Positive Definite Distributions and Subspaces of L_{-p} With Applications to Stable Processes

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Abstract. We define embedding of an n-dimensional normed space into L_{-p}, $0 < p < n$ by extending analytically with respect to p the corresponding property of the classical L_p-spaces. The well-known connection between embeddings into L_p and positive definite functions is extended to the case of negative p by showing that a normed space embeds in L_{-p} if and only if $\|x\|^{-p}$ is a positive definite distribution. We show that the technique of embedding in L_{-p} can be applied to stable processes in some situations where standard methods do not work. As an example, we prove inequalities of correlation type for the expectations of norms of stable vectors. In particular, for every $p \in [n-3, n)$, $\mathbb{E}(\max_{i=1,\dots,n} |X_i|^{-p}) \geq \mathbb{E}(\max_{i=1,\dots,n} |Y_i|^{-p})$, where $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are jointly q-stable symmetric random variables, $0 < q \leq 2$, so that, for some $k \in \mathbb{N}$, $1 \leq k < n$, the vectors $(X_1, \dots, X_k)$ and $(X_{k+1}, \dots, X_n)$ have the same distributions as $(Y_1, \dots, Y_k)$ and $(Y_{k+1}, \dots, Y_n)$, respectively, but Y_i and Y_j are independent for every choice of $1 \leq i \leq k$, $k+1 \leq j \leq n$.

1 Introduction

The connections between stable measures, positive definite norm dependent functions and embedding of normed spaces in L_p were discovered by P. Lévy [14] as parts of his theory of stable processes, and, since then, those connections have been under intensive development (see [10], [16] for the most recent surveys). In particular, P. Lévy pointed out that an n-dimensional normed space $B = (\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $p > 0$ if and only if there exists a finite Borel measure γ on the unit sphere Ω in $\mathbb{R}^n$ so that

$$\|x\|^p = \int_S |(x,\xi)|^p \, d\gamma(\xi) \tag{1}$$

for every $x \in \mathbb{R}^n$. On the other hand, for $0 < p \leq 2$, the representation (1) exists if and only if the function $\exp(-\|x\|^p)$ is positive definite and, hence, is the characteristic function of a symmetric stable measure in $\mathbb{R}^n$. We call (1) the Blaschke-Lévy representation of the norm with the exponent p and measure γ (see [11] for the history, generalizations and applications of this representation).

Several applications of the Blaschke-Lévy representation to stable processes depend on the standard procedure of using (1) to estimate the expectation of the norm of a stable vector (we give an example in Section 4). Sometimes, those applications do not use the Banach space structure of the space L_p, and they work equally well for $p \geq 1$ and $p \in (0,1)$. Moreover, when $p < 2$ becomes smaller one can expect more normed spaces to admit the representation (1) with the exponent p, because for $0 < p_1 < p_2 \leq 2$, the space L_{p_2}

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embeds isometrically in L_{p_1} (see [1]). However, the spaces ℓ_∞^n, $n \geq 3$ do not embed in any of the spaces L_p with $p > 0$, and the spaces ℓ_q^n, $n \geq 3$, $q > 2$ do not embed in L_p with $0 < p \leq 2$ (see [15], [7]; note that the latter results solved the 1938 Schoenberg's problems on positive definite functions [18].) These spaces (especially ℓ_∞^n) are particularly important in the theory of stable processes, and it seems to be natural to try to modify the standard technique so that it works for those spaces.

These were the reasons which led the author to an attempt to get more norms involved by generalizing the Blaschke-Lévy representation (and embedding in L_p) to the case of negative p. In Section 2, we define the Blaschke-Lévy representation in $\mathbb{R}^n$ with negative exponents $-p$, $0 < p < n$, and we say that the existence of such a representation for a normed space means that the space embeds in L_{-p}. The definition is "analytic" with respect to p, which might allow us to transfer properties of the spaces L_p in both directions between the positive and negative values of p. We show that the connection between embeddings in L_p and positive definiteness remains in force, namely, a space $B = (\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if $\|x\|^{-p}$ is a positive definite distribution on $\mathbb{R}^n$. Recall that in the positive case the condition is that the distribution $\Gamma(-p/2)\|x\|^p$ must be positive definite outside of the origin (see [8]; p is not an even integer).

In Section 3, we give an example of how the standard technique of the theory of stable processes can be modified by using embeddings in L_{-p}. For $B = (\mathbb{R}^n, \|\cdot\|)$, $p \in \mathbb{R}$, we consider the problem of optimization of the expectation $\mathbb{E}(\|X\|^p)$ of the norm of a symmetric q-stable random vector X in $\mathbb{R}^n$ in the following sense. Let $1 \leq k < n$, $0 < q \leq 2$ and $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be jointly q-stable symmetric random variables, so that the vectors $(X_1, \dots, X_k)$ and $(X_{k+1}, \dots, X_n)$ have the same distributions as $(Y_1, \dots, Y_k)$ and $(Y_{k+1}, \dots, Y_n)$, respectively, but Y_i and Y_j are independent for every choice of $1 \leq i \leq k$, $k+1 \leq j \leq n$. We compare the expectations $\mathbb{E}(\|X\|^p)$ and $\mathbb{E}(\|Y\|^p)$. First, we apply the standard methods to the case where $p > 0$ and B is a subspace of L_p, and we prove that $\mathbb{E}(\|X\|^p) \leq \mathbb{E}(\|Y\|^p)$ for each $p < q$. Then, we show that the technique of embedding in L_{-p} leads to similar results for a larger class of spaces B. In particular, for every $p \in [n-3, n)$,

$$\mathbb{E}(\max_{i=1,\dots,k} |X_i|^{-p}) \geq \mathbb{E}(\max_{i=1,\dots,k} |Y_i|^{-p}).$$

The question of what happens to the latter inequality when the exponent $-p$ is replaced by 1 is open, and, in the Gaussian case, this question is the matter of the weak version of the well-known Gaussian correlation problem (see [17] for the most recent developments).

2 Positive Definite Distributions and Embeddings in L_{-p}

The main tool of this paper is the Fourier transform of distributions. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle$ for every test function ϕ. A distribution is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f(x), \phi(x/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle$ for every test function ϕ and every $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. If $p > -1$ and p is not an even integer, then the Fourier transform of the function $h(z) = |z|^p$, $z \in \mathbb{R}$ is equal to $(|z|^p)^\wedge(t) = c_p |t|^{-1-p}$ (see [4, p. 173]), where $c_p = \frac{2^{p+1}\sqrt{\pi}\,\Gamma((p+1)/2)}{\Gamma(-p/2)}$.

The well-known connection between the Radon transform and the Fourier transform is that, for every $\xi \in \Omega$, the function $t \to \hat{\phi}(t\xi)$ is the Fourier transform of the function $z \to R\phi(\xi; z) = \int_{(x,\xi)=z} \phi(x)\,dx$ (R stands for the Radon transform). A distribution f is called positive definite if, for every test function ϕ, $\langle f, \phi * \overline{\phi(-x)}\rangle \geq 0$. A distribution is positive definite if and only if it is the Fourier transform of a tempered measure in $\mathbb{R}^n$ ([5, p. 152]). Recall that a (non-negative, not necessarily finite) measure μ is called tempered if

$$\int_{\mathbb{R}^n} (1 + \|x\|_2)^{-\beta}\, d\mu(x) < \infty$$

for some $\beta > 0$. Every positive distribution (in the sense that $\langle f, \phi\rangle \geq 0$ for every non-negative test function ϕ) is a tempered measure [5, p. 147].

Throughout the paper $\|x\|$ stands for a homogeneous of degree 1, continuous, positive outside of the origin function on $\mathbb{R}^n$. We say that $B = (\mathbb{R}^n, \|\cdot\|)$ is a homogeneous n-dimensional space. Clearly, the class of homogeneous spaces contains all finite dimensional normed and quasi-normed spaces. It is easily seen that every functional $\|x\|$ is equivalent to the Euclidean norm in the sense that, for every $x \in \mathbb{R}^n$, $K_1\|x\|_2 \leq \|x\| \leq K_2\|x\|_2$ for some positive constants K_1, K_2. Hence, $\|x\|^{-p}$ is a locally integrable function on $\mathbb{R}^n$ for every $p \in (0, n)$.

Now we are ready to define the Blaschke-Lévy representation with negative exponents p. Indeed, the formula (1) does not make sense if $p < -1$. However, let us start with positive p and apply functions in both sides (1) to a test function ϕ :

$$\begin{aligned}
\int_{\mathbb{R}^n} \|x\|^p \phi(x)\,dx &= \int_\Omega d\gamma(\xi) \int_{\mathbb{R}^n} |(x,\xi)|^p \phi(x)\,dx \\
&= \int_\Omega d\gamma(\xi) \int_{\mathbb{R}} |z|^p \left(\int_{(x,\xi)=z} \phi(x)\,dx \right) dz \\
&= \int_\Omega \langle |z|^p, R\phi(\xi; z)\rangle\, d\gamma(\xi) \\
&= c_p \int_\Omega \langle |t|^{-1-p}, \hat{\phi}(t\xi)\rangle\, d\gamma(\xi).
\end{aligned}$$

If p is negative the function $|t|^{-1-p}$ is locally integrable, which allows to write $\langle |t|^{-1-p}, \hat{\phi}(t\xi)\rangle$ as an integral, and this is how we extend the Blaschke-Lévy representation:

Definition Let $B = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional homogeneous space, $p \in (0, n)$. We say that the norm of B admits the Blaschhke-Lévy representation with the exponent $-p$, if there exists a finite symmetric measure γ on the sphere Ω so that, for every test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x)\,dx = \int_\Omega d\gamma(\xi) \int_{\mathbb{R}} |t|^{p-1} \hat{\phi}(t\xi)\,dt. \tag{2}$$

If the norm of B satisfies (2) with a measure γ, we also say that the space B embeds in L_{-p}.

It is easy to show the uniqueness of the representation (2). In fact, consider the test functions ϕ of the form

$$\phi(x) = h(t)u(\xi), \quad x = t\xi, \quad t \in \mathbb{R}, \quad t > 0, \quad \xi \in \Omega, \tag{3}$$

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where h is a non-negative test function on $\mathbb{R}$, and u is an infinitely differentiable even function on the sphere Ω. If a norm admits the representation (2) with two measures γ_1 and γ_2, then applying (2) to the test functions whose Fourier transforms have the form (3), we get that, for every u,

$$\int_\Omega u(\xi)\, d\gamma_1(\xi) = \int_\Omega u(\xi)\, d\gamma_2(\xi),$$

which implies $\gamma_1 = \gamma_2$.

We need the following simple fact.

Lemma 1 *Let μ be a tempered measure on $\mathbb{R}^n$ which is, at the same time, a homogeneous distribution of degree $-n + p$, $p \in (0, n)$. Then there exists a finite Borel measure γ on the sphere Ω so that, for every test function ϕ,*

$$\langle \mu, \phi \rangle = \int_\Omega d\gamma(\xi) \int_{\mathbb{R}} |t|^{p-1} \phi(t\xi)\, dt.$$

Proof Let us first show that μ can not have an atom at the origin. In fact, suppose that $\mu = \mu_1 + a\delta$, where $\mu_1(\{0\}) = 0$, and δ is the unit mass at the origin. Since μ is homogeneous of degree $-n + p$, for every non-negative test function ϕ with $\phi(0) > 0$ and every $t > 0$, we have $\langle \mu, \phi(x/t) \rangle = t^p \langle \mu, \phi \rangle \to 0$ as $t \to 0$. On the other hand, $\langle \mu, \phi(x/t) \rangle = \langle \mu_1, \phi(x/t) \rangle + a\phi(0)$, so $a = 0$.

For every Borel subset $A \subset \Omega$ and interval $(a, b] \in [0, \infty)$ denote by $A \times (a, b] = \{x \in R^n : x = t\theta, t \in (a, b], \theta \in A\}$, and let $\chi_{A\times(a,b]}$ be the indicator of this set.

By the definition of a homogeneous distribution, we have $\langle \mu, \phi(x/t) \rangle = t^p \langle \mu, \phi \rangle$ for every test function ϕ and $t > 0$. Using the dominated convergence theorem to extend the latter equality to non-smooth functions, we get

$$\mu\bigl(A \times [0, k]\bigr) = \langle \mu, \chi_{A\times[0,1]}(x/k) \rangle = k^p \mu\bigl(A \times [0, 1]\bigr).$$

Now, for every Borel subset $A \subset \Omega$ and every $0 \le a < b$ we have $\mu(A \times (a, b]) = (b^p - a^p)\mu(A \times [0, 1])$.

Define a measure μ_0 on Ω by $\mu_0(A) = p\mu(A \times [0, 1])$ for every Borel set $A \subset \Omega$. Clearly,

$$\int_\Omega d\mu_0(\theta) \int_{\mathbb{R}} |t|^{p-1} \chi_{A\times(a,b]}(t\theta)\, dt = (b^p - a^p)\mu_0(A).$$

Therefore, we get the equality (2) with $\phi = \chi_{A\times(a,b]}$ and the result follows since A, a, b are arbitrary. ■

Similar to the positive case, embedding into L_{-p} is closely related to positive definiteness. The following fact will serve as a tool for checking whether certain spaces embed in L_{-p}.

Theorem 1 *An n-dimensional homogeneous space $B = (\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p}, $p \in (0, n)$ if and only if $\|x\|^{-p}$ is a positive definite distribution.*

Proof Suppose that B embeds in L_{-p}. For every non-negative test function ϕ, using (2) and the fact that $(\hat{\phi})^\wedge(x) = (2\pi)^n \phi(-x)$, we get

$$\langle (\|x\|^{-p})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x)\, dx = (2\pi)^n \int_\Omega d\gamma(\xi) \int_{\mathbb{R}} |t|^{p-1} \phi(t\xi)\, dt \ge 0,$$

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which shows that $(\|x\|^{-p})^\wedge$ is a positive distribution over $\mathcal{S}(\mathbb{R}^n,)$ and, hence, the distribution $\|x\|^{-p}$ is positive definite.

Conversely, since the function $\|x\|^{-p}$ is a positive definite distribution and it is homogeneous of degree $-p$, the Fourier transform $(\|x\|^{-p})^\wedge$ is a tempered measure μ on $\mathbb{R}^n$, which is a homogeneous distribution of degree $-n+p$. By Lemma 1, there exists a measure γ on the sphere Ω so that, for every test function ϕ,

$$\langle (\|x\|^{-p})^\wedge, \phi\rangle = \langle \mu, \phi\rangle \int_\Omega d\gamma(\xi) \int_{\mathbb{R}} |t|^{p-1}\phi(t\xi)\, dt.$$

The result follows. ■

In order to prove that every homogeneous space embeds in every L_{-p} with $p \in [n-1, n)$, we use the following simple facts taken from [11], [12].

Lemma 2 *Let $p \in (n-1, n)$ and let f be an even homogeneous function of degree $-p$ on $\mathbb{R}^n \setminus \{0\}$ such that $f|_\Omega \in L_1(\Omega)$. Then for every $\xi \in \mathbb{R}^n$*

$$\hat{f}(\xi) = \frac{\pi}{c}\int_\Omega |(\theta, \xi)|^{-n+p} f(\theta)\, d\theta.$$

where $c = 2^{-n+p+1}\sqrt{\pi}\Gamma\big((-n+p+1)/2\big)/\Gamma\big((n-p)/2\big) > 0$. In particular, $\hat{f}|_\Omega \in L_1(\Omega)$.

Lemma 3 *Let f be an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$ so that $f|_\Omega \in L_1(\Omega)$. Then, for every $\xi \in \Omega$,*

$$\hat{f}(\xi) = \pi \int_{\Omega\cap\{(\theta,\xi)=0\}} f(\theta)\, d\theta.$$

If $B = (\mathbb{R}^n, \|\cdot\|)$ is a homogeneous space and $p \in [n-1, n)$, the function $f(x) = \|x\|^{-p}$ satisfies the conditions of Lemma 2 or Lemma 3. Therefore, the Fourier transform $(\|x\|^{-p})^\wedge$ is a homogeneous of degree $-n+p$, positive, locally integrable in $\mathbb{R}^n$ function, and, hence, it is a positive distribution. By Theorem 1,

Corollary 1 *Every n-dimensional homogeneous space embeds in L_{-p} for every $p \in [n-1, n)$.*

Let us show that each of the spaces L_{-p} is large enough to contain all finite dimensional subspaces of L_q, $0 < q \le 2$.

Theorem 2 *Every n-dimensional subspace of L_q with $0 < q \le 2$ embeds in L_{-p} for each $p \in (0, n)$.*

Proof By a well-known result of P. Lévy [14], for every n-dimensional subspace $B = (\mathbb{R}^n, \|\cdot\|)$ of L_q with $0 < q \le 2$, the function $\exp(-\|x\|^q)$ is the Fourier transform of a q-stable symmetric measure μ on $\mathbb{R}^n$. We have

$$\|x\|^{-p} = \frac{q}{\Gamma(p/q)}\int_0^\infty t^{p-1}\exp(-t^q\|x\|^q)\, dt.$$

For every non-negative test function ϕ,

$$\begin{aligned}\langle (\|x\|^{-p})^\wedge, \phi\rangle &= \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x)\,dx \\ &= \frac{q}{\Gamma(p/q)} \int_0^\infty t^{p-1}\,dt \int_{\mathbb{R}^n} \hat{\phi}(x) \exp(-t^q \|x\|^q)\,dx \\ &= \frac{q}{\Gamma(p/q)} \int_0^\infty t^{p-1}\,dt \int_{\mathbb{R}^n} \phi(tx)\,d\mu(x) \geq 0.\end{aligned}$$

Therefore, $(\|x\|^{-p})^\wedge$ is a positive distribution. ∎

A detailed proof of our next result will appear in a later paper [13]. We denote by $\|x\|_q$ the norm of the space ℓ_q^n, $2 < q \leq \infty$, where $\|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q}$ if $2 < q < \infty$ and $\|x\|_\infty = \max(|x_1|, \ldots, |x_n|)$.

Theorem 3 *Let $2 < q \leq \infty$, $0 < p < n$, $n \geq 3$. The function $\|x\|_q^{-p}$ is a positive definite distribution if $p \in [n-3, n)$, and it is not positive definite if $p \in (0, n-3)$. Therefore, the space ℓ_q^n embeds in L_{-p} if and only if $p \in [n-3, n)$.*

3 Inequalities of Correlation Type for the Expectations of Norms of Stable Vectors

For $0 < q \leq 2$, let $X = (X_1, \ldots, X_n)$ be a symmetric q-stable random vector which means that the characteristic functional of the vector X has the form

$$\phi(\xi) = \exp\Big(-\Big\|\sum_{i=1}^n \xi_i s_i\Big\|_q^q\Big), \quad \xi \in \mathbb{R}^n, \tag{4}$$

where $s_1, \ldots, s_n \in L_q([0,1])$. In this section, we use the notation $\|\cdot\|_q$ for the norm of the space $L_q([0,1])$.

Fix an integer k, $1 \leq k < n$, and consider the set $\mathcal{A}(X,k)$ of all n-dimensional symmetric q-stable random vectors whose first k coordinates have the same joint distribution as $X_1, \ldots, X_k$, and whose last $n-k$ coordinates have the same joint distribution as $X_{k+1}, \ldots, X_n$. We denote by $Y = (Y_1, \ldots, Y_n)$ the vector from $\mathcal{A}(X,k)$ for which every Y_i and Y_j with $1 \leq i \leq k$, $k+1 \leq j \leq n$ are independent. Then, the characteristic functional of Y is equal to

$$\phi_0(\xi) = \exp\Big(-\Big\|\sum_{i=1}^k \xi_i s_i\Big\|_q^q - \Big\|\sum_{i=k+1}^n \xi_i s_i\Big\|_q^q\Big).$$

Given an n-dimensional homogeneous space $B = (\mathbb{R}^n, \|\cdot\|)$ and a real number p, we are interested in conditions on B and p under which the independent case is extremal in the sense that the expectation $\mathbb{E}(\|Y\|^p)$ is the minimal or maximal value of $\mathbb{E}(\|Z\|^p)$, $Z \in \mathcal{A}(X,k)$.

First, we consider the case where $B = (\mathbb{R}^n, \|\cdot\|)$ is an n-dimensional subspace of L_p, $p > 0$ satisfying the following symmetry condition: for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$,

$$\|(u,v)\| = \|(u,-v)\|. \tag{$*$}$$

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We use the representation (1) and a standard argument from the theory of stable processes to show that, if $0 < p \le q$ then

$$\mathbb{E}(\|Y\|^p) = \max\{\mathbb{E}(\|Z\|^p) : Z \in \mathcal{A}(X,k)\}.$$

As it was mentioned in the Introduction, the condition that B is a subspace of L_p is restricting, for example, the most interesting case of $B = \ell_\infty^n$ is not covered. However, we replace the standard argument by the technique of embedding in L_{-p}, which allows to get more spaces involved. We prove that if B embeds into L_{-p}, $p \in (0,n)$ and has the symmetry $(*)$ then

$$\mathbb{E}(\|Y\|^{-p}) = \min\{\mathbb{E}(\|Z\|^{-p}) : Z \in \mathcal{A}(X,k)\}.$$

Let us start with the standard technique. If B is a subspace of L_p with $p > 0$, then one can use the well-known formula for the expectations of the scalar products of q-stable vectors with fixed vectors to reduce the estimation of $\mathbb{E}(\|X\|^p)$ to simple properties of the L_q-norms.

We need a few simple inequalities for the L_q-norms which follow from Clarkson's inequality (see [2]). For the reader's convenience we include the proof.

Lemma 4 *Let $x, y \in L_q([0,1])$, $0 < q \le 2$. Then*

$$\exp(-\|x+y\|_q^q) + \exp(-\|x-y\|_q^q) \ge 2\exp(-\|x\|_q^q - \|y\|_q^q). \tag{5}$$

Also for every $0 < p \le q$

$$\|x+y\|_q^p + \|x-y\|_q^p \le 2(\|x\|_q^q + \|y\|_q^q)^{p/q}. \tag{6}$$

Finally, for $q = 2$ and $p > 2$ the inequality (6) reverses.

Proof First, note that for any $0 < q \le 2$

$$\|x+y\|_q^q + \|x-y\|_q^q \le 2(\|x\|_q^q + \|y\|_q^q), \tag{7}$$

and this is a simple consequence of the same inequality for real numbers. Now to get (5) apply the relation between the arithmetic and geometric means and then use (7). The inequality (6) also follows from (7):

$$\left(\frac{\|x+y\|_q^p + \|x-y\|_q^p}{2}\right)^{1/p} \le \left(\frac{\|x+y\|_q^q + \|x-y\|_q^q}{2}\right)^{1/q} \le (\|x\|_q^q + \|y\|_q^q)^{1/q}.$$

Finally, if $q = 2$ the latter calculation works for $p > 2$ where the first inequality goes in the opposite direction, and the second inequality turns into an equality. ∎

Proposition 1 *Let q, k, X, Y be as in the beginning of this section, $0 < p \le q$. Let $B = (\mathbb{R}^n, \|\cdot\|)$ be a subspace of L_p satisfying the condition (*). Then*

$$\mathbb{E}(\|Y\|^p) = \max\{\mathbb{E}(\|Z\|^p) : Z \in \mathcal{A}(X,k)\}.$$

Also, if $q = 2$ and $p > 2$ then $\mathbb{E}\,(\|Y\|^p)$ is the minimal value.

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Proof A basic property of the stable vector with the characteristic function (4) is that, for any vector $\xi \in \mathbb{R}^n$, the random variable (X, ξ) has the same distribution as $\| \sum_{i=1}^n \xi_i s_i \|_q U$, where U is the standard one-dimensional q-stable random variable. Therefore, if $p < q$ then

$$\mathbb{E}|(X,\xi)|^p = c_{p,q} \Big\| \sum_{i=1}^n \xi_i s_i \Big\|_q^p, \tag{8}$$

where $c_{p,q}$ is the p-th moment of U (which exists only for $p < q$ if $q < 2$, and it exists for every $p > 0$ if $q = 2$; see [19] for a formula for $c_{p,q}$). Similarly, we get

$$\mathbb{E}|(X_-,\xi)|^p = c_{p,q} \Big\| \sum_{i=1}^k \xi_i s_i - \sum_{i=k+1}^n \xi_i s_i \Big\|_q^p,$$

where $X_- = (X_1, \ldots, X_k, -X_{k+1}, \ldots, -X_n)$. Also,

$$\mathbb{E}|(Y,\xi)|^p = c_{p,q} \Big(\Big\| \sum_{i=1}^k \xi_i s_i \Big\|_q^q + \Big\| \sum_{i=k+1}^n \xi_i s_i \Big\|_q^q \Big)^{p/q}.$$

Since $(\mathbb{R}^n, \|\cdot\|)$ is a subspace of $L_p([0,1])$, we can use the Blaschke-Lévy representation (1) and after that the formula (8) to get

$$\mathbb{E}(\|X\|^p) = \int_S \mathbb{E}\big(|(X,\xi)|^p\big)\, d\gamma(\xi) = c_{p,q} \int_S \Big\| \sum_{i=1}^n \xi_i s_i \Big\|_q^p \, d\gamma(\xi). \tag{9}$$

Similarly,

$$\mathbb{E}(\|Y\|^p) = c_{p,q} \int_S \Big(\Big\| \sum_{i=1}^k \xi_i s_i \Big\|_q^q + \Big\| \sum_{i=k+1}^n \xi_i s_i \Big\|_q^q \Big)^{p/q} d\gamma(\xi), \tag{10}$$

$$\mathbb{E}(\|X_-\|^p) = c_{p,q} \int_S \Big\| \sum_{i=1}^k \xi_i s_i - \sum_{i=k+1}^n \xi_i s_i \Big\|_q^p \, d\gamma(\xi). \tag{11}$$

Since $0 < p \le q$, the equalities (9), (10), (11) in conjunction with (6) imply $\mathbb{E}(\|X\|^p) + \mathbb{E}(\|X_-\|^p) \le 2\mathbb{E}(\|Y\|^p)$, and now the result follows from the property of the norm that $\|X\| = \|X_-\|$. In the case $q = 2$, $p > 2$ we use the corresponding part of Lemma 4. ■

Remark For $p > q$, $q < 2$ the expectation of $\|X\|^p$ does not exist so the statement of Proposition 1 does not make sense in that case.

Theorem 4 *Let q, k, X, Y be as in Proposition 1, and suppose that $0 < p < n$ and $B = (\mathbb{R}^n, \|\cdot\|)$ is a homogeneous space which embeds in L_{-p} and whose norm satisfies the symmetry condition (*). Then $\mathbb{E}(\|X\|^{-p}) \ge \mathbb{E}(\|Y\|^{-p})$.*

Proof By Theorem 1, the function $\|x\|^{-p}$ is a positive definite distribution, and by L. Schwartz's generalization of Bochner's theorem [5, p. 152], this function is the Fourier transform of a tempered measure μ on $\mathbb{R}^n$.

Let P_X be the q-stable measure in $\mathbb{R}^n$ according to which the random vector X is distributed. Applying the Parseval equality and formula (4) for the characteristic function of X we get

$$\begin{aligned}\mathbb{E}(\|X\|^{-p}) &= \int_{\mathbb{R}^n} \|x\|^{-p}\, dP_X(x) = \int_{\mathbb{R}^n} \widehat{P_X}(\xi)\, d\mu(\xi)\\ &= \int_{\mathbb{R}^n} \exp\Big(-\Big\|\sum_{i=1}^n \xi_i s_i\Big\|_q^q\Big)\, d\mu(\xi).\end{aligned}$$

Note that the function $\|x\|^{-p}$ is locally integrable in $\mathbb{R}^n$ because $0 < p < n$. Similarly,

$$\mathbb{E}(\|X_-\|^{-p}) = \int_{\mathbb{R}^n} \exp\Big(-\Big\|\sum_{i=1}^k \xi_i s_i - \sum_{i=k+1}^n \xi_i s_i\Big\|_q^q\Big)\, d\mu(\xi),$$

where $X_- = (X_1, \dots, X_k, -X_{k+1}, \dots, -X_n)$, and

$$\mathbb{E}(\|Y\|^{-p}) = \int_{\mathbb{R}^n} \exp\Big(-\Big\|\sum_{i=1}^k \xi_i s_i\Big\|_q^q - \Big\|\sum_{i=k+1}^n \xi_i s_i\Big\|_q^q\Big)\, d\mu(\xi).$$

Now by the inequality (5) from Lemma 4 and taking in account that μ is a positive measure, we get

$$\mathbb{E}(\|X\|^{-p}) + \mathbb{E}(\|X_-\|^{-p}) \geq 2\mathbb{E}(\|Y\|^{-p}),$$

and the result follows from the property (*). ■

The following is an immediate consequence of Theorem 4 in conjunction with Theorems 2, 3 and Corollary 1.

Corollary 2 *Let $B = (\mathbb{R}^n, \|\cdot\|)$ be a homogeneous space, $0 < p < n$, and q, k, X, Y as above. Then the inequality*

$$\mathbb{E}(\|X\|^{-p}) \geq \mathbb{E}(\|Y\|^{-p})$$

holds in each of the following cases:

(i) B is any n-dimensional homogeneous space satisfying the condition (*) and $p \in [n-1, n)$;
(ii) B is an n-dimensional subspace of L_r with $0 < r \leq 2$ satisfying the condition (*) and p is any number from $(0, n)$;
(iii) $B = \ell_q^n$, $n \geq 3$, $2 < q \leq \infty$ and $p \in [n-3, n)$.

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