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MODERATE DEVIATION PRINCIPLE OF SAMPLE QUANTILES AND ORDER STATISTICS

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In this paper, we mainly study the moderate deviation principle of sample quantiles and order statistics for stationary m-dependent random variables. The results obtained in this paper extend the corresponding ones for an independent and identically distributed sequence to a stationary m-dependent sequence.

Keywords: moderate deviation, order statistics, p-quantile, sample quantiles, stationary m-dependent random variables

1. INTRODUCTION

A quantile has no restrictions on moment conditions, which enables it to be widely used in various problems in finance, for instance, quantile-hedging, optimal portfolio allocation, risk management, and so forth. In practice, the large sample theory, which can give the asymptotic properties of sample estimator, is a significant approach to analyze statistical problems.

To present our main results, firstly let us recall the concept of m-dependence. A sequence $\{X_n, n \geq 1\}$ of random variables is called m-dependent if for a given fixed m, $\{X_i, i \in A\}$ and $\{X_j, j \in B\}$ are independent whenever $\rho(A, B) > m$ for $A, B \subset \mathbb{N}$, where

$$\rho(A, B) \triangleq \inf\{|i - j| : i \in A, j \in B\}.$$

If we take $m = 0$, then m-dependence is equivalent to independence, so the concept of m-dependence is an extension of independence. There are some results based on m-dependent sequence, one can refer to Hoeffding and Robbins [8], Sen [17], Schönfeld [16], Romano and Wolf [15] for instance.

The random sequence $\{X_n, n \geq 1\}$ is called stationary, if for any positive integers $t_1 < t_2 < \cdots < t_l$ and all positive integer k, $(X_{t_1}, X_{t_2}, \ldots, X_{t_l})$ has the same distribution with $(X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_l+k})$.

In this paper, suppose that we have a stationary m-dependent sample of size n from a distribution function $F(x)$ with a continuous probability density function $f(x)$. For $0 <$

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$p < 1$, the p-quantile of F is defined as

$$\xi_p = \inf\{x : F(x) \geq p\}.$$

As is known, there are two important estimators to estimate the p-quantile, one of which is the sample p-quantile, which is denoted as

$$\hat{\xi}_{np} = \inf\{x : F_n(x) \geq p\}, \quad 0 < p < 1,$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x), x \in \mathbb{R}$ is the empirical distribution function. The other estimator of the p-quantile is order statistics. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics of the sample $\{X_1, X_2, \ldots, X_n\}$ of observations on $F(x)$.

There are many results for the deviation between the sample quantiles and the quantile. Bahadur [4] introduced a representation for the sample quantiles based on independent and identically distributed (i.i.d.) random variables. Subsequently the result, called Bahadur representation, was extended to numerous dependent sequences, for instance, Ling [10] and Xing and Yang [20] studied the Bahadur representation for sample quantiles under negatively associated (NA) sequence; Wendler [25] established the Bahadur representation for U-quantiles of strongly mixing random variables and functionals of absolutely regular sequences; Li et al. [9] further investigated the Bahadur representation for sample quantiles under negatively orthant dependent (NOD) sequence; Yang et al. [28] obtained the Bahadur representation for sample quantiles under widely orthant dependent (WOD) sequence, which is weaker than NOD sequence, and thus weaker than NA sequence. For weakly dependent sequences, one can refer to Babu and Gingh [3], Sen [18], Wang, Hu, and Yang [23], Xing et al. [21], and so on. There are also some results on Berry–Esséen bounds of sample quantiles, we refer the readers to Yang et al. [26,27], Liu et al. [11] among others. Meanwhile, some results were established for order statistics, one can refer to David [6], Park [13,14], Childs et al. [5], Adler [1,2], Wang, Zhuang, and Hu [24] for the details.

Recently, Xu and Miao [22] established the following asymptotic properties of the moderate deviation between the sample quantiles $\hat{\xi}_{np}$ and the quantile ξ_p.

THEOREM 1.1: *Let $\{X_1, X_2, \ldots, X_n\}$ be independent and identically distributed random variables with a continuous distribution $F(x)$, and let ξ_p be a p-quantile of F for $0 < p < 1$. Corresponding to the sample $\{X_1, X_2, \ldots, X_n\}$, the sample p-quantile which is denoted by $\hat{\xi}_{np}$ is defined as the p-quantile of the empirical distribution function $F_n(x)$. Assume that $F(x)$ has a continuous density function $f(x)$ in the neighborhood of ξ_p and $f(\xi_p) > 0$. In addition, let $\{b_n\}$ be a positive sequence satisfying*

$$b_n \to \infty \text{ and } \frac{b_n}{\sqrt{n}} \to 0, \quad \text{as} \quad n \to \infty.$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}|\hat{\xi}_{np} - \xi_p| \geq r\right) = -\frac{f^2(\xi_p)r^2}{2p(1-p)}.$$

Miao et al. [12] obtained the asymptotic properties of the moderate deviation between order statistics and the quantile ξ_p.

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THEOREM 1.2: *Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of $\{X_1, X_2, \ldots, X_n\}$ which is a sequence of independent and identically distributed random variables with a continuous distribution $F(x)$, and let ξ_p be a p-quantile of F for $0 < p < 1$. Assume that $F(x)$ has a continuous density function $f(x)$ in the neighborhood of ξ_p and $f(\xi_p) > 0$. In addition, let $\{b_n\}$ be a positive sequence satisfying*

$$b_n \to \infty \quad \text{and} \quad \frac{b_n}{\sqrt{n}} \to 0, \quad \text{as} \quad n \to \infty.$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}|X_{(k)} - \xi_p| \geq r\right) = -\frac{f^2(\xi_p)r^2}{2p(1-p)},$$

where $k = np + o(b_n\sqrt{n})$.

The aim of this paper is to further study the asymptotic properties of the moderate deviation of sample quantiles and order statistics under the sequence of m-dependent random variables. The results obtained in this paper extend the corresponding ones of Theorems 1.1 and 1.2 for i.i.d. setting to stationary m-dependent setting. We should point out that the methods used in the paper are different from those in Theorems 1.1 and 1.2.

The layout of our work is as follows. Main results are presented in Section 2. The proofs of the main results are provided in Section 3. Throughout the paper, let $\log x = \ln \max(x, e)$, $\lfloor x \rfloor$ denotes the integer part of x and $I(A)$ represents the indicator function of the set A.

2. MAIN RESULTS

First, we present the result of moderate deviation principle between the sample quantiles $\hat{\xi}_{np}$ and the quantile ξ_p.

THEOREM 2.1: *Let $\{X_1, X_2, \ldots, X_n\}$ be stationary m-dependent random variables with a continuous distribution $F(x)$. Let $\hat{\xi}_{np}$ and ξ_p be the sample p-quantile and the p-quantile, respectively. Assume that $F(x)$ has a continuous density function $f(x)$ in the neighborhood of ξ_p and $f(\xi_p) > 0$. In addition, let $\{b_n\}$ be a positive sequence satisfying*

$$b_n \to \infty \quad \text{and} \quad \frac{b_n}{\sqrt{n}} \to 0, \quad \text{as} \quad n \to \infty.$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}|\hat{\xi}_{np} - \xi_p| \geq r\right) = -\frac{f^2(\xi_p)r^2}{2\delta^*},$$

where $\delta^* = E\eta_1^2 + 2\sum_{i=1}^{m} E\eta_1\eta_{i+1} > 0$ with $\eta_i = I(X_i \geq \xi_p) - EI(X_i \geq \xi_p)$.

The result of moderate deviation principle between order statistics and the quantile ξ_p is given as follows.

THEOREM 2.2: *Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of $\{X_1, X_2, \ldots, X_n\}$ which is a sequence of stationary m-dependent random variables with a continuous distribution $F(x)$, and let ξ_p be a p-quantile of F for $0 < p < 1$. Assume that $F(x)$ has a*

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continuous density function $f(x)$ in the neighborhood of ξ_p and $f(\xi_p) > 0$. In addition, let $\{b_n\}$ be a positive sequence satisfying

$$b_n \to \infty \quad \text{and} \quad \frac{b_n}{\sqrt{n}} \to 0, \quad \text{as} \quad n \to \infty.$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}|X_{(k)} - \xi_p| \geq r\right) = -\frac{f^2(\xi_p)r^2}{2\delta^*},$$

where $k = np + o(b_n\sqrt{n})$, and δ^* is defined in Theorem 2.1.

Remark 1: As is known, m-dependence is equivalent to independence if we take $m = 0$. Thus, we have $\delta^* = E\eta_1^2 = p(1-p)$ if $m = 0$, which is obtained in Theorems 1.1 and 1.2. Hence, the results of Theorems 2.1 and 2.2 extend the corresponding ones of Theorems 1.1 and 1.2 for i.i.d. case to stationary m-dependent case, respectively. We should point out that the methods used in the proofs of Theorems 2.1 and 2.2 are somewhat different from those in Theorems 1.1 and 1.2. The proofs of Theorems 2.1 and 2.2 are more difficult than the corresponding ones of Theorems 1.1 and 1.2.

3. PROOFS OF MAIN RESULTS

To prove the main results of the paper, we need the following two lemmas. The first one is a basic property for distribution function, which can be found in Serfling [19], for instance.

LEMMA 3.1: *Let $F(x)$ be a right-continuous distribution function. The inverse function $F^{-1}(t),\ 0 < t < 1$, is non-decreasing and left-continuous, and satisfies*

$$F^{-1}(F(x)) \leq x, \quad x \in (-\infty, \infty),$$

and

$$F(F^{-1}(t)) \geq t, \quad t \in (0, 1).$$

Then we have

$$F(x) \geq t \quad \text{if and only if} \quad x \geq F^{-1}(t).$$

The following lemma is also essential to prove our main results, whose proof is placed in Section 4.

LEMMA 3.2: *Let $\{X_n, n \geq 1\}$ be a sequence of stationary m-dependent random variables with a continuous distribution $F(x)$, and $\{c_n, n \geq 1\}$ be a sequence of positive numbers such that $c_n \to 0$ as $n \to \infty$. Denote*

$$\Phi_n = \sum_{i=1}^{n} (I(X_i \geq a + c_n) - EI(X_i \geq a + c_n)),$$

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$$\Theta_n = \sum_{i=1}^{n}(I(X_i \le a - c_n) - EI(X_i \le a - c_n))$$

and

$$\phi_i = I(X_i \ge a) - EI(X_i \ge a), \quad \theta_i = I(X_i \le a) - EI(X_i \le a),$$

where $a \in \mathbb{R}$ *is a constant. Then*

$$\lim_{n\to\infty} \frac{E\Phi_n^2}{n} = \beta_1, \quad \lim_{n\to\infty} \frac{E\Theta_n^2}{n} = \beta_2,$$

where $\beta_1 = E\phi_1^2 + 2\sum_{i=1}^{m} E\phi_1\phi_{i+1}$ *and* $\beta_2 = E\theta_1^2 + 2\sum_{i=1}^{m} E\theta_1\theta_{i+1}$*. Moreover,* $\phi_i = -\theta_i$ *a.s. for each* $i \ge 1$*, and thus,* $\beta_1 = \beta_2 \triangleq \beta^* > 0$*.*

Proof of Theorem 2.1: It can be easily obtained for any $r > 0$ that

$$P\left(\frac{\sqrt{n}}{b_n}|\hat{\xi}_{np} - \xi_p| \ge r\right) = P\left(\hat{\xi}_{np} \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right) + P\left(\hat{\xi}_{np} \le \xi_p - \frac{b_n r}{\sqrt{n}}\right). \qquad \textbf{(3.1)}$$

It follows from Lemma 3.1 that

$$\begin{aligned}
P\left(\hat{\xi}_{np} \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right) &= P\left(p \ge F_n\left(\xi_p + \frac{b_n r}{\sqrt{n}}\right)\right) \\
&= P\left(\frac{1}{n}\sum_{i=1}^{n} I(X_i \le \xi_p + \frac{b_n r}{\sqrt{n}}) \le p\right) \\
&= P\left(\sum_{i=1}^{n} I(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}) \ge n(1-p)\right) \\
&= P\left(\sum_{i=1}^{n}\left(I\left(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right) - EI\left(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right)\right)\right. \\
&\qquad \left.\ge n(1-p) - nEI\left(X_1 \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right)\right) \\
&= P\left(\sum_{i=1}^{n} Y_{ni} \ge b_n\sqrt{n}\sigma_1\right), \qquad \textbf{(3.2)}
\end{aligned}$$

where

$$Y_{ni} = I\left(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right) - EI\left(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right),$$

and

$$\sigma_1 = \frac{n(1-p) - nEI(X_1 \ge \xi_p + (b_n r/\sqrt{n}))}{b_n\sqrt{n}}.$$

It follows from Taylor's theorem that

$$\begin{aligned}
EI\left(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right) &= P\left(X_i \ge \xi_p + \frac{b_n r}{\sqrt{n}}\right) = 1 - F\left(\xi_p + \frac{b_n r}{\sqrt{n}}\right) \\
&= 1 - \left(F(\xi_p) + F'(\xi_p)\frac{b_n r}{\sqrt{n}} + o\left(\frac{b_n}{\sqrt{n}}\right)\right) \\
&= 1 - p - f(\xi_p)\frac{b_n r}{\sqrt{n}} + o\left(\frac{b_n}{\sqrt{n}}\right).
\end{aligned}$$

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Thus, we have that

$$\sigma_1 = \frac{n(1-p) - n(1-p) + f(\xi_p) r b_n\sqrt{n} + o(b_n\sqrt{n})}{b_n\sqrt{n}} = f(\xi_p)r + o(1).$$

Denote $k = \lfloor n/(m+a_n) \rfloor$, where $\{a_n, n \geq 1\}$ is a sequence of positive integers such that $a_n \to \infty$ and $a_n b_n/\sqrt{n} \to 0$. Hence, we can see that $a_n > m+1$ for all n large enough. Denote for each $j = 0, 1, \ldots, k-1$ that

$$S_j = \sum_{i=1}^{a_n} Y_{n,(a_n+m)j+i}, \text{ and } T_j = \sum_{i=1}^{m} Y_{n,(a_n+m)j+a_n+i}.$$

Then

$$\sum_{i=1}^{n} Y_{ni} = \sum_{j=0}^{k-1} S_j + \sum_{j=0}^{k-1} T_j + \sum_{i=(m+a_n)k+1}^{n}, Y_{ni} \triangleq \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Now we give the Cramér function of the random variable $\sum_{i=1}^{n} Y_{ni}$. Let $\{d_n, n \geq 1\}$ be a sequence of positive constants such that $d_n \to \infty$ and $d_n = o(a_n)$. Then for any $\lambda \in \mathbb{R}$, we obtain by Hölder's inequality that for all n large enough,

$$\begin{aligned} \frac{1}{b_n^2} \log E \exp\left\{\frac{\lambda b_n}{\sqrt{n}} \sum_{i=1}^{n} Y_{ni}\right\} &\leq \frac{d_n - 2}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}(d_n-2)} \Sigma_1\right\} \\ &\quad + \frac{1}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}} \Sigma_2\right\} \\ &\quad + \frac{1}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}} \Sigma_3\right\} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.3}$$

It follows from $|Y_{ni}| \leq 1$ that $|S_0| \leq a_n$ and thus $|\lambda b_n d_n S_0/[\sqrt{n}(d_n-2)]| \leq \lambda a_n b_n d_n/[\sqrt{n}(d_n-2)] \to 0$. Hence, for any $\vartheta \in (0,1)$,

$$\begin{aligned} \left| E \exp\left\{\frac{\vartheta \lambda b_n d_n}{\sqrt{n}(d_n-2)} S_0\right\} \frac{\left(\frac{\lambda b_n d_n}{\sqrt{n}(d_n-2)} S_0\right)^3}{3!} \right| &\leq C \frac{\lambda a_n b_n d_n}{\sqrt{n}(d_n-2)} E\left(\frac{\lambda b_n d_n}{\sqrt{n}(d_n-2)} S_0\right)^2 \\ &= o(1) E\left(\frac{\lambda b_n d_n}{\sqrt{n}(d_n-2)} S_0\right)^2. \end{aligned}$$

Note that S_j are independent for $j = 0, 1, \ldots, k-1$, we have by Taylor's theorem and Lemma 3.2 that

$$\begin{aligned} \lim_{n\to\infty} I_1 &= \lim_{n\to\infty} \frac{(d_n-2)k}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}(d_n-2)} S_0\right\} \\ &= \lim_{n\to\infty} \frac{(d_n-2)k}{b_n^2 d_n} \log\left\{1 + \frac{(1+o(1))\lambda^2 b_n^2 d_n^2}{2n(d_n-2)^2}\left(a_n E Y_{n1}^2 + 2\sum_{i=1}^{m}(a_n - i) E Y_{n1} Y_{n,i+1}\right)\right\} \\ &= \lim_{n\to\infty} \frac{(d_n-2)k}{b_n^2 d_n} \cdot \frac{(1+o(1))\lambda^2 b_n^2 d_n^2}{2n(d_n-2)^2}\left(a_n E Y_{n1}^2 + 2\sum_{i=1}^{m}(a_n - i) E Y_{n1} Y_{n,i+1}\right) \\ &= \frac{\lambda^2 \delta^*}{2}. \end{aligned} \tag{3.4}$$

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Similarly, T_j are also independent for $j = 0, 1, \ldots, k-1$; thus, it follows from the stationarity and $d_n = o(a_n)$ that

$$\begin{aligned}\lim_{n\to\infty} I_2 &= \lim_{n\to\infty} \frac{k}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}} T_0\right\} \\ &= \lim_{n\to\infty} \frac{k}{b_n^2 d_n} \log\left\{1 + \frac{(1+o(1))\lambda^2 b_n^2 d_n^2}{2n}\left(mEY_{n1}^2 + 2\sum_{i=2}^{m}(m-i+1)EY_{n1}Y_{ni}\right)\right\} \\ &= \lim_{n\to\infty} \frac{(1+o(1))\lambda^2 k d_n}{2n}\left(mEY_{n1}^2 + 2\sum_{i=2}^{m}(m-i+1)EY_{n1}Y_{ni}\right) = 0. \end{aligned} \tag{3.5}$$

Since $|Y_{ni}| \le 1$, we can easily obtain that

$$\begin{aligned}\limsup_{n\to\infty} I_3 &\le \limsup_{n\to\infty} \frac{1}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}}(m + a_n)\right\} \\ &= \limsup_{n\to\infty} \frac{1}{b_n^2 d_n} \log \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}}(m + a_n)\right\} \\ &= \limsup_{n\to\infty} \frac{(m+a_n)\lambda}{b_n\sqrt{n}} = 0. \end{aligned} \tag{3.6}$$

Combining (3.3)–(3.6), we can see that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log E \exp\left\{\frac{\lambda b_n}{\sqrt{n}} \sum_{i=1}^{n} Y_{ni}\right\} \le \frac{\lambda^2 \delta^*}{2}. \tag{3.7}$$

On the other hand, note that $\Sigma_1 = \sum_{i=1}^{n} Y_{ni} - \Sigma_2 - \Sigma_3$, we have by Hölder's inequality again that for all n large enough,

$$\begin{aligned}\frac{1}{b_n^2} \log E \exp\left\{\frac{\lambda b_n}{\sqrt{n}} \Sigma_1\right\} &\le \frac{d_n - 2}{b_n^2 d_n} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}(d_n - 2)} \sum_{i=1}^{n} Y_{ni}\right\} \\ &\quad + \frac{1}{b_n^2 d_n} \log E \exp\left\{-\frac{\lambda b_n d_n}{\sqrt{n}} \Sigma_2\right\} + \frac{1}{b_n^2 d_n} \log E \exp\left\{-\frac{\lambda b_n d_n}{\sqrt{n}} \Sigma_3\right\},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{1}{b_n^2} \log E \exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}(d_n - 2)} \sum_{i=1}^{n} Y_{ni}\right\} &\ge \frac{d_n}{b_n^2 (d_n - 2)} \log E \exp\left\{\frac{\lambda b_n}{\sqrt{n}} \Sigma_1\right\} \\ &\quad - \frac{1}{b_n^2 (d_n - 2)} \log E \exp\left\{-\frac{\lambda b_n d_n}{\sqrt{n}} \Sigma_2\right\} \\ &\quad - \frac{1}{b_n^2 (d_n - 2)} \log E \exp\left\{-\frac{\lambda b_n d_n}{\sqrt{n}} \Sigma_3\right\} \\ &=: J_1 - J_2 - J_3. \end{aligned} \tag{3.8}$$

Repeating the steps of (3.4)–(3.6), we can easily obtain the following conclusions:

$$\lim_{n\to\infty} J_1 = \frac{\lambda^2 \delta^*}{2}, \quad \lim_{n\to\infty} J_2 = 0, \quad \text{and} \quad \limsup_{n\to\infty} J_3 \le 0,$$

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which together with (3.8) yields that

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{b_n^2}\log E\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^n Y_{ni}\right\}\\ =\liminf_{n\to\infty}\frac{1}{b_n^2}\log E\exp\left\{\frac{\lambda b_n d_n}{\sqrt{n}(d_n-2)}\sum_{i=1}^n Y_{ni}\right\}\geq\frac{\lambda^2\delta^*}{2}.\end{aligned}\tag{3.9}$$

Combining (3.7) and (3.9), we have

$$\Delta(\lambda)=\lim_{n\to\infty}\frac{1}{b_n^2}\log E\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^n Y_{ni}\right\}=\frac{\lambda^2\delta^*}{2}.\tag{3.10}$$

Thus, the Fenchel–Legendre transform of $\Delta(\lambda)$ is

$$\Delta^*(x)=\sup_{\lambda\in\mathbb{R}}\left(\lambda x-\frac{\lambda^2\delta^*}{2}\right)=\frac{x^2}{2\delta^*},$$

which together with Gärtner–Ellis theorem (see Dembo and Zeitouni [7]) yields that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{b_n^2}\log P\left(\frac{\sqrt{n}}{b_n}(\hat{\xi}_{np}-\xi_p)\geq r\right)&=\lim_{n\to\infty}\frac{1}{b_n^2}\log P\left(\sum_{i=1}^n Y_{ni}\geq b_n\sqrt{n}\sigma_1\right)\\ &=-\inf_{x\geq f(\xi_p)r}\Delta^*(x)=-\frac{f^2(\xi_p)r^2}{2\delta^*}.\end{aligned}\tag{3.11}$$

On the other hand, for any $r>0$, we have by Lemma 3.1 again that

$$\begin{aligned}P\left(\hat{\xi}_{np}\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right)&=P\left(\frac{1}{n}\sum_{i=1}^n I\left(X_i\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right)\geq p\right)\\ &=P\left(\sum_{i=1}^n\left(I\left(X_i\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right)-EI\left(X_i\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right)\right)\right.\\ &\qquad\left.\geq np-nEI\left(X_1\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right)\right)\\ &=P\left(\sum_{i=1}^n Z_{ni}\geq b_n\sqrt{n}\sigma_2\right),\end{aligned}\tag{3.12}$$

where

$$Z_{ni}=I\left(X_i\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right)-EI\left(X_i\leq\xi_p-\frac{b_n r}{\sqrt{n}}\right),$$

and

$$\sigma_2=\frac{np-nEI(X_i\leq\xi_p-(b_n r/\sqrt{n}))}{b_n\sqrt{n}}.$$

We can obtain by Taylor's theorem again that

$$\sigma_2=\frac{np-np+f(\xi_p)rb_n\sqrt{n}+o(b_n\sqrt{n})}{b_n\sqrt{n}}=f(\xi_p)r+o(1).$$

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Denote for each $j = 0, 1, \ldots, k-1$ and $k = \lfloor n/(m + a_n) \rfloor$ that

$$S_j^* = \sum_{i=1}^{a_n} Z_{n,(a_n+m)j+i}, \quad \text{and} \quad T_j^* = \sum_{i=1}^{m} Z_{n,(a_n+m)j+a_n+i}.$$

Analogous to (3.3)–(3.10), we have that

$$\Delta(\lambda) = \lim_{n\to\infty} \frac{1}{b_n^2} \log E \exp\left\{ \frac{\lambda b_n}{\sqrt{n}} \sum_{i=1}^{n} Z_{ni} \right\} = \frac{\lambda^2 \delta^*}{2}.$$

Therefore we have by the Gärtner–Ellis theorem again that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n} (\hat{\xi}_{np} - \xi_p) \le -r \right) &= \lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\sum_{i=1}^{n} Z_{ni} \ge b_n \sqrt{n} \sigma_2 \right) \\ &= -\inf_{x \ge f(\xi_p) r} \Delta^*(x) = -\frac{f^2(\xi_p) r^2}{2\delta^*}. \end{aligned} \tag{3.13}$$

The desired result follows immediately from (3.1), (3.11), and (3.13). The proof is completed. ∎

Proof of Theorem 2.2: Utilizing the approach in the proof of Theorem 2.1, for any $r > 0$ we have that

$$P\left(\frac{\sqrt{n}}{b_n} |X_{(k)} - \xi_p| \ge r \right) = P\left(X_{(k)} \ge \xi_p + \frac{b_n r}{\sqrt{n}} \right) + P\left(X_{(k)} \le \xi_p - \frac{b_n r}{\sqrt{n}} \right). \tag{3.14}$$

It follows from Lemma 3.1 that

$$\begin{aligned} P\left(X_{(k)} \ge \xi_p + \frac{b_n r}{\sqrt{n}} \right) &= P\left(k \ge \sum_{i=1}^{n} I\left(X_i \le \frac{b_n r}{\sqrt{n}} + \xi_p \right) \right) \\ &= P\left(\sum_{i=1}^{n} I\left(X_i \ge \frac{b_n r}{\sqrt{n}} + \xi_p \right) \ge n - k + 1 \right) \\ &= P\left(\sum_{i=1}^{n} U_{ni} \ge b_n \sqrt{n} \sigma_3 \right), \end{aligned}$$

where

$$U_{ni} = I\left(X_i \ge \frac{b_n r}{\sqrt{n}} + \xi_p \right) - EI\left(X_i \ge \frac{b_n r}{\sqrt{n}} + \xi_p \right),$$

and

$$\sigma_3 = \frac{n - k + 1 - nEI(X_i \ge (b_n r/\sqrt{n}) + \xi_p)}{b_n \sqrt{n}} = f(\xi_p) r + o(1).$$

On the other hand,

$$\begin{aligned} P\left(X_{(k)} \le \xi_p - \frac{b_n r}{\sqrt{n}} \right) &= P\left(\sum_{i=1}^{n} I\left(X_i \le \xi_p - \frac{b_n r}{\sqrt{n}} \right) \ge k \right) \\ &= P\left(\sum_{i=1}^{n} V_{ni} \ge b_n \sqrt{n} \sigma_4 \right), \end{aligned}$$

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where

$$V_{ni} = I\left(X_i \le \xi_p - \frac{b_n r}{\sqrt{n}}\right) - EI\left(X_i \le \xi_p - \frac{b_n r}{\sqrt{n}}\right),$$

and

$$\sigma_4 = \frac{k - nEI(X_i \le \xi_p - (b_n r/\sqrt{n}))}{b_n\sqrt{n}} = f(\xi_p)r + o(1).$$

Since $U_{ni} = Y_{ni}$, $\sigma_3 = \sigma_1$, and $V_{ni} = Z_{ni}$, $\sigma_4 = \sigma_2$, the reminder of the proof follows by repeating (3.3)–(3.13) of Theorem 2.1. Here we omit the details. ■

4. PROOF OF LEMMA 3.2.

We only prove $\lim_{n\to\infty} E\Phi_n^2/n = \beta_1$, and the proof of $\lim_{n\to\infty} E\Theta_n^2/n = \beta_2$ is completely analogous. Denote $\varphi_i = I(X_i \ge a + c_n) - EI(X_i \ge a + c_n)$. Noting that $\{\phi_i, i \ge 1\}$ and $\{\varphi_i, i \ge 1\}$ are both stationary m-dependent sequences, we have that for all $n \ge m+1$,

$$\frac{E\Phi_n^2}{n} = \frac{E\left(\sum_{i=1}^n \varphi_i\right)^2}{n} = E\varphi_1^2 + 2\sum_{i=1}^m \frac{n-i}{n} E\varphi_1\varphi_{i+1},$$

and

$$\frac{E\left(\sum_{i=1}^n \phi_i\right)^2}{n} = E\phi_1^2 + 2\sum_{i=1}^m \frac{n-i}{n} E\phi_1\phi_{i+1}.$$

It follows from the Taylor's theorem that

$$\begin{aligned}
|E\varphi_1^2 - E\phi_1^2| &= |P(X_i \ge a + c_n) - P^2(X_i \ge a + c_n) - P(X_i \ge a) + P^2(X_i \ge a)| \\
&= |(P(X_i \ge a + c_n) - P(X_i \ge a))(1 - P(X_i \ge a + c_n) - P(X_i \ge a))| \\
&\le 3|P(X_i \ge a + c_n) - P(X_i \ge a)| \\
&= 3|F(a + c_n) - F(a)| \\
&= 3f(a)c_n + o(c_n) = o(1),
\end{aligned}$$

and

$$\begin{aligned}
&|E\varphi_1\varphi_{i+1} - E\phi_1\phi_{i+1}| \\
&\quad = |EI(X_1 \ge a + c_n)I(X_{i+1} \ge a + c_n) - P(X_1 \ge a + c_n)P(X_{i+1} \ge a + c_n) \\
&\qquad - EI(X_1 \ge a)I(X_{i+1} \ge a) - P(X_1 \ge a)P(X_{i+11} \ge a)| \\
&\quad \le |EI(X_1 \ge a + c_n)I(X_{i+1} \ge a + c_n) - EI(X_1 \ge a + c_n)I(X_{i+1} \ge a)| \\
&\qquad + |EI(X_1 \ge a + c_n)I(X_{i+1} \ge a) - EI(X_1 \ge a)I(X_{i+1} \ge a)| \\
&\qquad + |P(X_1 \ge a + c_n)P(X_{i+1} \ge a + c_n) - P(X_1 \ge a + c_n)P(X_{i+1} \ge a)| \\
&\qquad + |P(X_1 \ge a + c_n)P(X_{i+1} \ge a) - P(X_1 \ge a)P(X_{i+1} \ge a)| \\
&\quad \le 4|F(a + c_n) - F(a)| = 4f(a)c_n + o(c_n) = o(1).
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{E\Phi_n^2}{n} = E\phi_1^2 + 2\lim_{n\to\infty}\sum_{i=1}^m \frac{n-i}{n} E\phi_1\phi_{i+1} = \beta_1.$$

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Moreover, since the distribution function of X_i is continuous, we have

$$\phi_i + \theta_i = I(X_i = a) - P(X_i = a) = 0 \text{ a.s.},$$

which implies that $\phi_i = -\theta_i$ a.s. for each $i \geq 1$, and thus, $\beta_1 = \beta_2$. The proof is completed. ■

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