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L^P ESTIMATES FOR THE HOMOGENIZATION OF STOKES PROBLEM IN A PERFORATED DOMAIN

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Abstract In this paper, we consider the Stokes equations in a perforated domain. When the number of holes increases while their radius tends to 0, it is proven in Desvillettes *et al.* [*J. Stat. Phys.* **131** (2008) 941–967], under suitable dilution assumptions, that the solution is well approximated asymptotically by solving a Stokes–Brinkman equation. We provide here quantitative estimates in L^p-norms of this convergence.

Keywords: Stokes equations; homogenization; suspension flows

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1. Introduction

Let Ω be a connected smooth bounded domain in $\mathbb{R}^3$. Given $N \in \mathbb{N}$, we consider $(B_i^N)_{i\in\{1,\dots,N\}}$ a family of N balls in $\mathbb{R}^3$ such that:

$$B_i^N := B\left(x_i^N, \frac{r_i^N}{N}\right) \subset \Omega, \quad \text{for all } i \in \{1,\dots,N\}.$$

Defining the perforated set $\mathcal{F}^N$ by

$$\mathcal{F}^N = \Omega \Big\backslash \bigcup_{i=1}^N \overline{B_i^N},$$

we denote $(u^N, \pi^N) \in H^1(\mathcal{F}^N) \times L_0^2(\mathcal{F}^N)$ (here the subscript 0 fixes that π^N has mean 0 on $\mathcal{F}^N$) the unique solution to the Stokes problem:

$$\begin{cases} -\Delta u^N + \nabla \pi^N = 0, \\ \operatorname{div} u^N = 0, \end{cases} \quad \text{on } \mathcal{F}^N, \tag{1}$$

completed with boundary conditions:

$$\begin{cases} u^N(x) = V_i^N, & \text{on } \partial B_i^N, \\ u^N(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $(V_i^N)_{i=1,\dots,N} \in (\mathbb{R}^3)^N$ are given. In [5], the authors show that, if $r_i^N = 1$ uniformly, if the holes are sufficiently dilute and the empirical measures associated to the distributions of $(x_i^N, V_i^N)_{i=1,\dots,N}$ converge to a sufficiently smooth particle distribution function $f(x,v)\,dx\,dv$, then the associated sequence of velocity fields $(u^N)_{N\in\mathbb{N}}$ converges weakly to the velocity field $\bar{u}$ of the unique solution $(\bar{u},\bar{\pi}) \in H^1(\Omega)\times L^2_0(\Omega)$ to the Stokes–Brinkman problem:

$$\begin{cases} -\Delta\bar{u} + \nabla\bar{\pi} = (j - \rho\bar{u}), \\ \operatorname{div}\bar{u} = 0, \end{cases} \quad \text{on } \Omega, \tag{3}$$

completed with boundary condition:

$$\bar{u} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

In (3), the flux j and density ρ are computed respectively to the given particle distribution function f by:

$$j(x) = 6\pi\int_{\mathbb{R}^3} vf(x,v)\,dv \quad \rho(x) = 6\pi\int_{\mathbb{R}^3} f(x,v)\,dv, \quad \forall x\in\Omega.$$

We emphasize that here and below (in the definition of discrete densities and fluxes), we include the factor 6π in the formulas. This factor is reminiscent of the Stokes law for the resistance force applied by a viscous fluid on an immersed moving sphere (see next section). Via a standard compact-embedding argument, it entails from [5] that we have also strong convergence of the u^N to $\bar{u}$ in L^p spaces (for $p<6$) up to the extraction of a subsequence. We are interested herein in providing a quantitative estimate of the convergence of u^N to $\bar{u}$.

This problem is related to the homogenization of Stokes problem in perforated domains with homogeneous boundary conditions and a forcing term. In this case, previous studies prove convergence of the sequence of N-hole solutions to the solution of the Stokes–Brinkman problem (or other ones depending on the dilution regime of the holes) in the periodic as in the random setting [1, 2, 13]. These results extend to the Stokes problem previous analysis for the Laplace equations [4]. The problem with nonhomogeneous boundary conditions that we consider herein is introduced by [5] in a tentative to justify a Vlasov–Navier–Stokes or Vlasov–Stokes problem that is applied in spray theory [3, 7]. The strategy here is to couple the Stokes problem (1)–(2) by prescribing that the holes are particles whose position/velocity $(x_i^N, V_i^N)_{i=1,\dots,N}$ evolve according to Newton laws:

$$\frac{d}{dt}x_i^N = V_i^N, \tag{5}$$

$$m\frac{d}{dt}V_i^N = -\int_{\partial B_i^N}(\nabla u + \nabla u^\top - p\mathbb{I}_3)n\,d\sigma. \tag{6}$$

Here we denote by m the mass of the particles and n the normal to ∂B_i^N directed toward B_i^N. Note that, contrary to the stationary problem we are studying in this paper, in this target system the holes/particles are moving. As classical in these ‘many-particle systems’, one crucial issue to complete a rigorous derivation is to control the distance between the particles. Partial improvements have been obtained in this direction either by increasing the family of datas for which transition from the N-hole stationary Stokes problem to the Stokes–Brinkman problem holds [9] or by completing successfully the kinetic program for the odes (5)–(6) with singular forcing terms [8]. In this paper, we do not tackle this issue on the distance between particles. Keeping in mind that, in the full problem, one wants to couple the dynamical equations for the particles with the pde governing the fluid problem, we infer that a quantitative description of the convergence of the N-hole solutions to the Stokes–Brinkman problem is necessary. The main motivation of this paper is then to discuss in which norms such quantitative estimates may be computed. It turns out that our proof simplifies the approach of [5] and encompasses the case of a polydispersed cloud made of spheres.

We make precise now the main assumptions that are in force throughout the paper:

- the balls are sufficiently spaced:

$$\exists\, C_0 > 0 \text{ independent of } i \neq j, N \text{ s.t. } \mathrm{dist}(B_i^N, B_j^N) \geqslant \frac{C_0}{N^{1/3}}, \quad \mathrm{dist}(B_i^N, \partial\Omega) \geqslant \frac{C_0}{N^{1/3}}; \tag{H1}$$

- the normalized radii $r_i^N > 0$ are uniformly bounded:

$$\exists\, R_0 > 0 \text{ independent of } i, N \text{ s.t. } r_i^N \leqslant R_0; \tag{H2}$$

- the kinetic energies of the data are uniformly bounded:

$$\exists\, E_0 > 0 \text{ independent of } N \text{ such that } \frac{1}{N}\sum_{i=1}^{N} |V_i^N|^2 \leqslant |E_0|^2. \tag{H3}$$

Then, following [5] and [9] we introduce empirical measures to describe the asymptotic behavior of the distribution $(x_i^N, V_i^N, r_i^N)_{i=1,\dots,N}$:

$$S_N(x, v, r) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^N, V_i^N, r_i^N}(x, v, r) \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3 \times]0, \infty[).$$

We denote then by ρ^N and j^N its two first momentums:

$$\rho^N := 6\pi \int_{\mathbb{R}^3 \times]0,\infty[} r S_N(dv\, dr), \quad j^N := 6\pi \int_{\mathbb{R}^3 \times]0,\infty[} r v S_N(dv\, dr). \tag{7}$$

The sequence of densities ρ^N (respectively fluxes j^N) are then measures (respectively vectorial measures) on $\mathbb{R}^3$ with support in Ω. Compared to [5], the main new assumptions are that the radii of the holes may depend on i, N.

With the above assumptions, for arbitrary $N \in \mathbb{N}$, the domain $\mathcal{F}^N$ has a smooth boundary and there exists a solution to (1)–(2) (see [6, § IV]). We have thus at hand a sequence $(u^N, \pi^N) \in H^1(\mathcal{F}^N) \times L_0^2(\mathcal{F}^N)$. Under assumption (H1)–(H2)–(H3) one may prove that up to the extraction of a subsequence ρ^N (respectively j^N) converges to some density $\rho \in L^\infty(\Omega)$ (respectively flux $j \in L^2(\Omega)$). We have then a unique solution $(\bar{u}, \bar{\pi})$ to the Stokes–Brinkman problem (3)–(4) for this density/flux pair (see next section for more details). In order to compute the distance between u^N and $\bar{u}$ we extend u^N to the whole Ω by setting:

$$E_\Omega[u^N] := \begin{cases} u^N, & \text{on } \mathcal{F}^N, \\ V_j^N, & \text{on } B_j^N. \end{cases}$$

Because of boundary conditions (2), these extended velocity fields satisfy $E_\Omega[u^N] \in H_0^1(\Omega)$. With these notations, we state now our two results on the convergence of the sequence $(E_\Omega[u^N])_{N\in\mathbb{N}}$ toward $\bar{u}$.

Theorem 1.1. *Assume that $j \in L^q(\Omega)$ for some $q > 3$ and $p \in]1, \frac{3}{2}[$. If R_0/C_0^3 is sufficiently small, there exists a constant $K > 0$ depending only on R_0, C_0, p, q, Ω for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leqslant K\left[\|j^N - j\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{\|j\|_{L^q(\Omega)} + E_0}{N^{1/3}}\right],$$

for $N \geqslant (4R_0/C_0)^{3/2}$.

Theorem 1.2. *Given $p \in]1, \frac{3}{2}[$ there exists $K > 0$ depending only on $R_0, C_0, p, \|\rho\|_{L^\infty(\Omega)}$, Ω for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leqslant K\left[\|j - j^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}}\right)^{1/3} E_0\right],$$

for $N \geqslant (4R_0/C_0)^{3/2}$.

The two previous theorems give a quantitative estimate of the weak convergence obtained in [5]. They link the convergence of the sequence $(u^N)_{N\in\mathbb{N}}$ to $\bar{u}$ to the convergence of the fluxes and densities in the so-called bounded-Lipschitz or Fortet–Mourier distance (see [14, § 6]). As the $(\rho^N)_{N\in\mathbb{N}}$ are positive measures on Ω with the same finite mass, we may relate the bounded-Lipschitz distance $\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*}$ to the Wasserstein distance between ρ^N and ρ thanks to the Kantorovich–Rubinstein formula [14, Theorem 5.10]. The definition of the $(\mathcal{C}^{0,1}(\bar{\Omega}))^*$ norm and its relation with Wasserstein distance is explained below. The restriction on the values N is irrelevant as our aim is to describe the asymptotics $N \to \infty$ of u^N. It is due to the fact that our method requires that $B(x_j^N, r_j^N/N) \subset B(x_j^N, C_0/4N^{1/3})$ for arbitrary $j \in \{1, \ldots, N\}$.

The results we state are complementary one to the other. The first one is limited to sufficiently small ratios R_0/C_0^3. This can be interpreted as configurations for which the holes are sufficiently small compared to their relative distances. In this case, the convergence estimate is linear with respect to the convergence of the data ρ^N and j^N. The second result is valid for arbitrary data. The counterpart is that the convergence estimate is now sublinear with respect to the convergence of the densities ρ^N. These results can be extended in several directions. First, we may interpolate these convergences with crude uniform bounds on $E_\Omega[u^N]$ in $L^6(\Omega)$ to extend the convergence to L^p spaces with $p \geqslant 3/2$. But we can also generalize the result by considering convergence of the empirical measures in more general dual spaces. Finally, we state our main results in the dilution regime of [5] but it extends to the dilution regime of [9]. We comment at the end of the paper on the estimates we can attain with this method.

The outline of the paper is as follows. In the next section, we state and prove some technical lemmas on the resolution of the Stokes problem and Stokes–Brinkman problem. In particular, we state a regularity lemma in negative Sobolev spaces which is at the heart of our computations. Section 3 is devoted to the proof of our main results and we provide a discussion on the possible extensions of our results in a closing section.

We list below some possible nonstandard notations that we use during the proofs. First, we use extensively localizing procedures around the balls B_j^N so that we use repeatedly the shortcut $A(x, r_{\text{int}}, r_{\text{ext}})$ for the annulus with center x and internal (respectively external) radius r_{int} (respectively r_{ext}). We also use the notations $\fint_A u$ for the mean of u on the set of positive measure A:

$$\fint_A u(x)\,dx = \frac{1}{|A|}\int_A u(x)\,dx.$$

We denote classically $L^p(\Omega)$ (respectively $W^{m,p}(\Omega)$ or $H^m(\Omega)$) Lebesgue spaces (respectively Sobolev spaces) on Ω. The index zero specifies zero mean when added to Lebesgue spaces and vanishing boundary values when added to Sobolev spaces. For instance, we denote:

$$L^2_0(\Omega) := \left\{v \in L^2(\Omega), \fint_\Omega v = 0\right\}, \quad D_0(\Omega) := \{v \in [H^1_0(\Omega)]^3, \operatorname{div} v = 0\}.$$

When there is no ambiguity concerning the definition domain, we only use exponents to denote norms:

$$\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}, \quad \|\cdot\|_{m,q} := \|\cdot\|_{W^{m,q}(\Omega)}.$$

In this paper, we consider empirical measures as particular continuous linear forms on the set of bounded continuous functions $C^0(\overline{\Omega})$. The associated duality pairing is denoted with brackets. As classical, this enables to define convergence of empirical measures in different topologies. We work mostly herein with the bounded-Lipschitz or Fortet–Mourier distance i.e. endow empirical measures with the topology of $\mathcal{C}^{0,1}(\overline{\Omega})^*$. We define then the distance between μ and ρ by:

$$\|\mu - \rho\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} := \sup\{|\langle \mu, \phi\rangle - \langle \rho, \phi\rangle|, \phi \in \mathcal{C}^{0,1}(\bar{\Omega}), \|\phi\|_{\mathcal{C}^{0,1}(\bar{\Omega})} = 1\}.$$

We note that, in the case of densities (the $(\rho^N)_{N\in\mathbb{N}}$ introduced in (7)) our empirical measures are moreover probability measures on $\bar{\Omega}$. Since Ω is bounded, the Kantorovich–Rubinstein formula shows that the Fortet–Mourier distance is equivalent to the 1-Wasserstein distance:

$$W_1(\rho, \mu) := \sup\{|\langle \rho, \phi\rangle - \langle \mu, \phi\rangle|, \phi \in \operatorname{Lip}(\overline{\Omega}), \operatorname{Lip}(\phi) = 1\}$$

with $\operatorname{Lip}(\overline{\Omega})$ the set of Lipschitz functions on $\overline{\Omega}$ and $\operatorname{Lip}(\phi)$ the Lipschitz constant of ϕ.

Given an arbitrary smooth domain $\mathcal{O}$ and $q \in (1, \infty)$, we denote $\mathfrak{B} : L^q_0(\mathcal{O}) \to W^{1,q}_0(\mathcal{O})$ the so-called Bogovskii operator (see [6, § III.3]). It is a continuous linear map which, given $f \in L^q_0(\mathcal{O})$ provides a solution u to the problem:

$$\begin{cases} \operatorname{div} u = f, & \text{on } \mathcal{O}, \\ u = 0, & \text{on } \partial\mathcal{O}. \end{cases}$$

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If $\mathcal{O} = A(x_0, r_{\rm int}, r_{\rm ext})$, we specify the Bogovskii operator by indices: $\mathfrak{B}_{x_0,r_{\rm int},r_{\rm ext}}$. Such operators have been extensively studied in [1]. The main results we apply here are summarized in [9, Appendix A].

Finally, in the whole paper we use the symbol $\lesssim$ to express an inequality with a multiplicative constant depending on irrelevant parameters.

2. Preliminary results on the Stokes and Stokes–Brinkman equations

In this section, we prove some lemmas concerning the resolution of the Stokes and Stokes–Brinkman equations that will help in the proofs of our main results.

2.1. Analysis of the Stokes–Brinkman equation in a bounded domain

In this whole part Ω is a fixed smooth bounded domain. Given a boundary condition $u^* \in H^{1/2}(\Omega)$ and $\rho \in L^\infty(\Omega)$, we consider the Stokes–Brinkman problem:

$$\begin{cases} \rho u - \Delta u + \nabla \pi = j, \\ \operatorname{div} u = 0, \end{cases} \quad \text{on } \Omega, \tag{8}$$

completed with boundary condition:

$$u = u^* \quad \text{on } \Omega. \tag{9}$$

We assume below that $\rho \geqslant 0$ including possibly $\rho = 0$. In this latter case, the Stokes–Brinkman equations degenerate into the Stokes equations. We refer the reader to [6, § IV] for a comprehensive study of Stokes equations. Herein, we also apply the variational characterization of solutions that is provided in [9, § 2]. It is straightforward to extend the existence theory of these references to the Stokes–Brinkman equations with an arbitrary bounded weight $\rho \geqslant 0$ yielding the following theorem:

Theorem 2.1. *Let $j \in L^{6/5}(\Omega; \mathbb{R}^3)$ and $\rho \in L^\infty(\Omega)$ such that $\rho \geqslant 0$. Given $u^* \in H^{1/2}(\Omega)$ satisfying:*

$$\int_{\partial\Omega} u^* \cdot n \, d\sigma = 0,$$

the following equivalent statements hold true and furnish a solution to (8)–(9):

(i) *there exists a unique pair $(u, \pi) \in H^1(\Omega) \times L^2_0(\Omega)$ satisfying (8) in the sense of $\mathcal{D}'(\Omega)$ and (9) in the sense of traces;*

(ii) *there exists a unique divergence-free $u \in H^1(\Omega)$ satisfying (9) in the sense of traces and:*

$$\int_\Omega \nabla u : \nabla v = \int_\Omega (j - \rho u) \cdot v, \quad \textit{for all } v \in D_0(\Omega); \tag{10}$$

(iii) *if we assume furthermore that $j = 0$, there exists a unique solution to the minimization problem:*

$$\inf\left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 + \rho |v|^2, \, v \in [H^1(\Omega)]^3, \operatorname{div} v = 0, \, v = u^* \ \textit{on } \partial\Omega \right\}. \tag{11}$$

The proof of this theorem is a straightforward extension of [6, § IV] and [9, § 2] and is left to the reader.

As stated in [6, Theorem IV.6.1], in the case $\rho = 0$ and $u^* = 0$ we have also that, if $j \in W^{m,p}(\Omega)$ for some $m \in \mathbb{N}$ and $p \in (1, \infty)$ then the solution u satisfies $u \in W^{m+2,p}(\Omega)$. We may extend this regularity statement to our Stokes–Brinkman problem:

Theorem 2.2. *Let $\rho \in L^\infty(\Omega)$ such that $\rho \geqslant 0$ and assume that $u^* = 0$, $j \in L^q(\Omega)$, for some $q \in [6/5, \infty)$. Then, there exists a unique pair $(u, \pi) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ satisfying (8)–(9). Moreover, there exists $C = C(\Omega, q, \|\rho\|_\infty) > 0$ such that:*

$$\|u\|_{2,q} \leqslant C\|j\|_q.$$

Proof. Because Ω is bounded and $q \geqslant 6/5$ we have that $j \in L^{6/5}(\Omega)$. Theorem 2.1 yields the existence and uniqueness of the solution $(u, \pi) \in H^1(\Omega) \times L^2_0(\Omega)$. We recall that we focus on homogeneous boundary conditions. In this case $u \in H^1_0(\Omega)$ so that Poincaré inequality entails that $\|u\|_{1,2} \lesssim \|\nabla u\|_2$.

At first, let assume further that $q \leqslant 6$. Because $H^1_0(\Omega) \subset L^q(\Omega)$, we remark that (u, π) satisfies the Stokes equation with data $f = j - \rho\, u \in L^q(\Omega)$. The regularity theorem for Stokes equations implies that $(u, \pi) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ with:

$$\|u\|_{2,q} \leqslant C\|j - \rho u\|_q \leqslant C(\|j\|_q + \|\rho\|_\infty \|u\|_q)$$

for some positive constant $C > 0$ depending only on Ω and q. Thus, we want to bound $\|u\|_q$ by $\|j\|_q$. To this end, we apply the weak formulation of Stokes–Brinkman problem (10) with $v = u \in D_0(\Omega)$ to get that:

$$\begin{aligned}\int_\Omega |\nabla u|^2 &= \int_\Omega j \cdot u - \int_\Omega \rho |u|^2 \\ &\leqslant \|j\|_q \|u\|_{q'} \\ &\lesssim \|j\|_q \|\nabla u\|_2\end{aligned}$$

where we applied again the embedding $H^1_0(\Omega) \subset L^{q'}(\Omega)$ since $q \geqslant 6/5$. This entails that $\|u\|_q \lesssim \|u\|_{1,2} \lesssim \|j\|_2$ and concludes the proof.

If we assume now $q > 6$ we iterate the same argument. Indeed, because Ω is bounded, we have in particular that $j \in L^6(\Omega)$ so that the previous reasoning applies yielding:

$$\|u\|_{2,6} \leqslant C\|j\|_{L^6(\Omega)} \leqslant C'\|j\|_{L^q(\Omega)}.$$

We may then apply the continuous embedding $W^{2,6}(\Omega) \subset W^{1,\infty}(\Omega)$. Hence, we obtain now again that $j - \rho u \in L^q(\Omega)$ and we conclude by application of the regularity theorem for Stokes equations as previously. □

Keeping in mind that we want to compare the N-solution $E_\Omega[u^N]$ with $\bar{u}$ on Ω, we do not expect to be able to use a regular theory for the Stokes (or Stokes–Brinkman) equations as above. Indeed, the u^N are solutions to the Stokes equations on $\mathcal{F}^N$ only. Even if we were extending the pressure π^N to $E_\Omega[\pi^N]$ by fixing a constant on the $B(x_i^N, r_i^N/N)$ (say 0 for instance), we expect that $\Delta E_\Omega[u^N] - \nabla E_\Omega[\pi^N]$ contains single layer distributions on the interfaces fluid/holes. Fortunately, these single layer distributions are regular enough to compute L^p-estimates as depicted below. These L^p-estimates are adapted from weak-regularity statements for stationary Stokes equations that have been obtained in the study of fluid–structure interaction problems [12, Appendix 1].

Given $p \in]1, 6[$, we introduce the following norm of $v \in H^1_0(\Omega)$:

$$[v]_{p,\Omega} := \sup \left\{ \left| \int_\Omega \nabla v : \nabla w \right|, w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega), \operatorname{div} w = 0, \|w\|_{2,p'} = 1 \right\}.$$

We have then:

Lemma 2.3. *Let $p \in]1, 6[$. There exists a nonnegative $C = C(\Omega, p)$ such that:*

$$\|v\|_p \leqslant C[v]_{p,\Omega},$$

for all divergence-free $v \in H^1_0(\Omega)$.

Similarly, we define the following norm based on the weak formulation for the Stokes–Brinkman equations. Given $\rho \in L^\infty(\Omega)$ such that $\rho \geqslant 0$, we set:

$$[v]_{p,\Omega,\rho} := \sup \left\{ \left| \int_\Omega \nabla v : \nabla w + \int_\Omega \rho v \cdot w \right|, \right.$$
$$\left. w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega), \operatorname{div} w = 0, \|w\|_{2,p'} = 1 \right\}.$$

Then, there holds:

Lemma 2.4. *Let $p \in]1, 6[$ and $\rho \in L^\infty(\Omega)$ such that $\rho \geqslant 0$. There exists a nonnegative $C = C(\Omega, p, \|\rho\|_\infty)$ that satisfies:*

$$\|v\|_p \leqslant C[v]_{p,\Omega,\rho},$$

for all divergence-free $v \in H^1_0(\Omega)$.

As Lemma 2.3 can be obtained by setting $\rho = 0$ in Lemma 2.4, we prove only the second one.

Proof. The idea is to use the following equality:

$$\|v\|_p = \sup \left\{ \left| \int_\Omega v \cdot \phi \right|, \phi \in L^{p'}(\Omega)\ \|\phi\|_{L^{p'}(\Omega)} = 1 \right\}.$$

Let $p \leqslant 6$ and $\phi \in L^{p'}(\Omega)$, $\|\phi\|_{p'} = 1$. Because $p' \geqslant 6/5$, we introduce the unique solution (u_ϕ, π_ϕ) to the problem

$$\begin{cases} -\Delta u_\phi + \nabla \pi_\phi + \rho u_\phi = \phi, \\ \operatorname{div} u_\phi = 0, \end{cases} \quad \text{on } \Omega, \tag{12}$$

completed with the boundary condition $u_\phi = 0$ on $\partial\Omega$. According to Lemma 2.2, this solution satisfies $u_\phi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$, $p_\phi \in W^{1,p'}(\Omega)$ and

$$\|u_\phi\|_{2,p'} \leqslant C\|\phi\|_{p'} \leqslant C.$$

Moreover, we have that $W^{2,p'}(\Omega) \subset W^{1,2}(\Omega)$. This yields that, using an integration by parts:

$$\int_\Omega v \cdot \phi = \int_\Omega v \cdot (-\Delta u_\phi + \nabla \pi_\phi + \rho u_\phi) = \int_\Omega \nabla u_\phi : \nabla v + \int_\Omega \rho u_\phi \cdot v.$$

This entails:

$$\left| \int_\Omega v \cdot \phi \right| \leqslant [v]_{p,\Omega,\rho} \|u_\phi\|_{2,p'} \leqslant C[v]_{p,\Omega,\rho}.$$

We obtain:

$$\left| \int_\Omega v \cdot \phi \right| \leqslant C[v]_{p,\Omega,\rho}, \quad \forall \phi \in L^{p'}(\Omega), \|\phi\|_{p'} = 1.$$

□

2.2. The Stokes problem in an exterior domain

In this part, we focus on the case $\Omega = \mathbb{R}^3 \setminus \overline{B(0,r)}$ where $r > 0$. Given $V \in \mathbb{R}^3$, we consider the Stokes problem on Ω:

$$\begin{cases} -\Delta u + \nabla \pi = 0, \\ \operatorname{div} u = 0, \end{cases} \quad \text{on } \Omega, \tag{13}$$

completed with boundary conditions:

$$u(x) = V, \quad \text{on } \partial B(0,r), \quad \lim_{|x| \to \infty} |u(x)| = 0. \tag{14}$$

We investigate here the convergence of Stokes solutions on annuli to the Stokes solution on the exterior domain. Precisely, let $R > r$ and $\Omega_R = B(0,R) \setminus \overline{B(0,r)} = A(0,r,R)$. We denote by (u_R, π_R) the solution to:

$$\begin{cases} -\Delta u_R + \nabla \pi_R = 0, \\ \operatorname{div} u_R = 0, \end{cases} \quad \text{on } \Omega_R, \tag{15}$$

completed with boundary conditions:

$$u(x) = V, \quad \text{on } \partial B(0,r), \quad u(x) = 0, \quad \text{on } \partial B(0,R). \tag{16}$$

We emphasize that we only consider constant boundary conditions. In this particular case existence theory for (13)–(14) is well known since explicit formulas for the solutions are part of the folklore (see [10] and more recently [5]). Explicit solutions for (15)–(16) are also available following the same construction scheme as in the unbounded case. We refer here to [5, § 6.2] for more details. On the basis of these formulas, the convergence of (u_R, π_R) to (u, π) is studied in [5]. For later purpose, we complement here this study with two supplementary properties of this convergence.

First, we denote

$$F_R^r = \int_{\partial B(0,r)} (\nabla u_R - \pi_R \mathbb{I}) n \, d\sigma, \quad F^r = \int_{\partial B(0,r)} (\nabla u - \pi \mathbb{I}) n \, d\sigma.$$

We use the symbol $\mathbb{I}$ here for the identity matrix in $\mathbb{R}^3$. These quantities are related to the force exerted by the flow (u_R, π_R) (respectively (u, π)) on the hole $B(0, R)$

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(see Appendix A for more details). We recall that Stokes law states that $F^r = 6\pi r V$. The following lemma shows that the sequence F^r_R converges to F^r. Moreover, explicit formulas for u_R and u allow to compute the rate of this convergence:

Lemma 2.5. *There holds:*

$$|F^r_R - F^r| \lesssim r^2 \frac{|V|}{R}.$$

Proof. We show the inequality for $r = 1$. The result extends to any $r > 0$ by a standard scaling argument that we recall afterward.

We have that:

$$F^1_R - F^1 = \int_{\partial B(0,1)} (\nabla(u_R - u))n \, d\sigma + \int_{\partial B(0,1)} (\pi - \pi_R)n \, d\sigma.$$

Adopting the notations introduced in [5] we set $r = |x|$, $\omega = \frac{x}{|x|}$ and $P_\omega V = (\omega \cdot V)\omega$. We have then, for arbitrary $x \in A(0, 1, R)$ that:

$$\begin{aligned} u_R(x) &= -\left[4A(R)r^2 + 2B(R) + \frac{C(R)}{r} - \frac{D(R)}{r^3}\right](\mathbb{I} - P_\omega)V \\ &\quad - 2\left[A(R)r^2 + B(R) + \frac{C(R)}{r} + \frac{D(R)}{r^3}\right]P_\omega V \end{aligned}$$

where:

$$\begin{aligned} A(R) &= -\frac{3}{8R^3} + O\left(\frac{1}{R^4}\right), \quad B(R) = \frac{9}{8R} + O\left(\frac{1}{R^2}\right), \\ C(R) &= -\frac{3}{4} + O\left(\frac{1}{R}\right), \quad D(R) = \frac{1}{4} + O\left(\frac{1}{R}\right). \end{aligned}$$

The formula for u is obtained by replacing $A(R), B(R), C(R), D(R)$ by their limits when $R \to \infty$ in the formula defining u.

In the same spirit as on [5, p. 965], we have that, for arbitrary $x \in A(0, 1, R)$:

$$\begin{aligned} u_R(x) - u(x) &= \left[\frac{3}{2R^3}r^2 - r^2 O\left(\frac{1}{R^4}\right) - \frac{9}{4R} + O\left(\frac{1}{R}\right) + O\left(\frac{1}{R}\right)\frac{1}{r} + \frac{1}{r^3}O\left(\frac{1}{R}\right)\right](\mathbb{I} - P_\omega)V \\ &\quad + \left[\frac{3}{4R^3}r^2 + r^2 O\left(\frac{1}{R^4}\right) - \frac{9}{4R} + O\left(\frac{1}{R}\right) + O\left(\frac{1}{R}\right)\frac{1}{r} + \frac{1}{r^3}O\left(\frac{1}{R}\right)\right]P_\omega V. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\partial B(0,1)} \nabla(u_R - u)n \, d\sigma &= \left(\frac{3}{2R^3} + O\left(\frac{1}{R^4}\right) + O\left(\frac{1}{R}\right)\right)\left(4\pi^2 V - \int_{\partial B(0,1)} V \cdot xx \, d\sigma\right) \\ &\quad + \left(\frac{3}{2R^3} + O\left(\frac{1}{R^4}\right) + O\left(\frac{1}{R}\right)\right)\int_{\partial B(0,1)} V \cdot xx \, d\sigma, \end{aligned}$$

and consequently:

$$\left|\int_{\partial B(0,1)} \nabla(u_R - u)n \, d\sigma\right| \lesssim \frac{|V|}{R}.$$

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By using [5, § 6.2] we get a similar formula for the pressures:

$$\pi_R(x) - \pi(x) = \left(-20A(R)|x| + \frac{5A(R)+3B(R)}{|x|^2} \right) \frac{x \cdot V}{|x|}, \quad \forall x \in A(0,1,R).$$

This entails that:

$$\begin{aligned}\left| \int_{\partial B(0,1)} (\pi - \pi_R) n \, d\sigma \right| &\lesssim (25|A(R)| + 3|B(R)|) \left| \int_{\partial B(0,1)} V \cdot x x \, d\sigma \right| \\ &\lesssim \frac{|V|}{R}.\end{aligned}$$

Finally, we get that:

$$|F_R^1 - F^1| \lesssim \frac{|V|}{R}.$$

We obtain the inequality for arbitrary r by remarking that, denoting $(\tilde{u}, \tilde{\pi})$ the solution to the Stokes problem on $\mathbb{R}^3 \setminus B(0,r)$ (respectively $(\tilde{u}_R, \tilde{\pi}_R)$ the solution to the Stokes problem on $A(0,r,R)$), we have:

$$\begin{aligned}(\tilde{u}(x), \tilde{\pi}(x)) &= \left(u\left(\frac{x}{r}\right), \frac{1}{r}\pi\left(\frac{x}{r}\right) \right), && \text{for all } x \in \mathbb{R}^3 \setminus \overline{B(0,r)}, \\ (\tilde{u}_R(x), \tilde{\pi}_R(x)) &= \left(u_{R/r}\left(\frac{x}{r}\right), \frac{1}{r}\pi_{R/r}\left(\frac{x}{r}\right) \right), && \text{for all } x \in A(0,r,R).\end{aligned}$$

Introducing this scaling in the formulas for F_R^r, we get that:

$$F_R^r - F^r = r(F_{R/r}^1 - F^1).$$

This entails finally that:

$$\begin{aligned}|F_R^r - F^r| &= r|F_{R/r}^1 - F^1| \\ &\lesssim r^2 \frac{|V|}{R}.\end{aligned}$$

□

We conclude this section by an error estimate for the velocity gradient:

Lemma 2.6. *There holds:*

$$\int_{A(0,R/2,R)} |\nabla u_R|^2 \lesssim \frac{r^2}{R}|V|^2.$$

Proof. We obtain the result for $r = 1$ by plugging the explicit formulas for u^R and the coefficients $A(R), B(R), C(R), D(R)$ in the previous proof and generalize it to arbitrary $r > 0$ by a scaling argument. The details are left to the reader. □

3. Proofs of Theorems 1.1 and 1.2

We proceed in this section with the proofs of our main theorems. In this section, we fix Ω, R_0, C_0, and $p \in]1, 3/2[$, $q \in (3, \infty)$ as in the assumptions of our theorems. When using the symbol $\lesssim$, we allow the implicit constant to depend on these values R_0, C_0, p, q, Ω.

Let $N \geqslant N_0 := (4R_0/C_0)^{3/2}$. we recall that $E_\Omega[u^N]$ is the solution to the Stokes problem (1)–(2) on the perforated domain $\mathcal{F}^N$ with boundary data $V_1, \ldots, V_N$. With similar arguments to [9, § 3] we have:

Proposition 3.1. *There exists a constant K depending only on R_0 and C_0 for which:*

$$\|\nabla E_\Omega[u^N]\|_2 \leqslant K E_0, \quad \forall N \geqslant N_0.$$

We also introduce $\bar{u}$ the solution to the Stokes–Brinkman problem (3)–(4) associated with the data j, ρ that may be computed from the particle distribution function to which the sequence of empirical measures describing the N-configurations converges.

The main idea is common to both proofs: we apply duality arguments reported in Lemma 2.3 or in Lemma 2.4 in order to estimate the L^p-norm of the vector field $v^N := E_\Omega[u^N] - \bar{u}$. Hence, the core of the proof is the computation of

$$\left| \int_\Omega \nabla v^N : \nabla w \right|,$$

for an arbitrary divergence-free vector field $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$.

In the two next parts, we prepare these computations by fixing a divergence-free vector field $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. We compute equivalent formulas for

$$\int_\Omega \nabla v^N : \nabla w,$$

and provide some bounds that are relevant for both proofs.

We remind the classical embedding that we use repeatedly below: since $p \in [1, \frac{3}{2}[$, there holds:

$$W^{2,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega}).$$

3.1. Extraction of first order terms

Let $N \geqslant N_0$ and $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ be divergence-free. We have:

$$\int_\Omega \nabla v^N : \nabla w = \int_\Omega \nabla E_\Omega[u^N] : \nabla w - \int_\Omega \nabla \bar{u} : \nabla w,$$

where

$$\int_\Omega \nabla \bar{u} : \nabla w = \int_\Omega (j(x) - \rho(x)\bar{u}(x)) \cdot w(x)\, dx,$$
$$\int_\Omega \nabla E_\Omega[u^N] : \nabla w = \int_{\mathcal{F}^N} \nabla u^N : \nabla w.$$

In what follows we use the shortcuts:

$$A_j^N := A(x_j^N, C_0/4N^{1/3}, C_0/2N^{1/3}), \quad \bar{u}_j^N := \fint_{A_j^N} u^N, \tag{17}$$

and

$$\Omega_j^N = B\left(x_j^N, \frac{C_0}{2N^{1/3}}\right) \setminus \overline{B_j^N} \quad \forall j, N.$$

Because of the definition (H1) of C_0, the sets Ω_j^N are disjoint and cover a subset of Ω. These sets are also annuli but they play a special role to our proof hence the different

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name. Because $N \geqslant N_0$, the sets Ω_j^N are not empty so that their boundaries are made of two concentric spheres. The internal sphere is ∂B_j^N while we denote below $\partial_e \Omega_j^N$ the external sphere.

We first decompose the scalar product $\int_{\mathcal{F}^N} \nabla u^N : \nabla w$ into N integrals on the disjoint annuli Ω_j^N. To this end, given $j \in \{1, \dots, N\}$, we define $(\widehat{w}_j^N, \widehat{\pi}_j^N)$ the unique solution to the Stokes problem

$$\begin{cases} -\Delta \widehat{w}_j^N + \nabla \widehat{\pi}_j^N = 0, \\ \qquad\quad \operatorname{div} \widehat{w}_j^N = 0, \end{cases} \quad \text{on } \Omega_j^N, \tag{18}$$

completed with boundary conditions:

$$\begin{cases} \widehat{w}_j^N(x) = w(x), & \text{on } \partial B_j^N, \\ \widehat{w}_j^N(x) = 0, & \text{on } \partial_e \Omega_j^N. \end{cases} \tag{19}$$

We still denote $\widehat{w}_j^N$ the trivial extension of $\widehat{w}_j^N$ to $\mathcal{F}^N$ and we set

$$w^N := \sum_{j=1}^N \widehat{w}_j^N.$$

We remark then that w^N satisfies:

$$\begin{cases} w^N \in H^1(\mathcal{F}^N), & \\ \operatorname{div} w^N = 0, & \text{on } \mathcal{F}^N, \\ w^N = w, & \text{on } \partial \mathcal{F}^N. \end{cases}$$

We have then:

$$\int_{\mathcal{F}^N} \nabla u^N : \nabla w = \int_{\mathcal{F}^N} \nabla u^N : \nabla (w - w^N) + \int_{\mathcal{F}^N} \nabla u^N : \nabla w^N.$$

Because u^N is the solution to the Stokes problem on $\mathcal{F}^N$ and $w - w^N \in D_0(\mathcal{F}^N)$, the first term on the right-hand side vanishes:

$$\int_{\mathcal{F}^N} \nabla u^N : \nabla w = \sum_{j=1}^N \int_{\mathcal{F}^N} \nabla u^N : \nabla \widehat{w}_j^N := \sum_{j=1}^N I_j^N.$$

Let denote now by (w_j^N, π_j^N) the unique solution to:

$$\begin{cases} -\Delta w_j^N + \nabla \pi_j^N = 0, \\ \qquad\quad \operatorname{div} w_j^N = 0, \end{cases} \quad \text{on } \Omega_j^N, \tag{20}$$

completed with boundary conditions:

$$\begin{cases} w_j^N(x) = = w(x_j^N), & \text{on } \partial B_j^N, \\ \quad w_j^N(x) = 0, & \text{on } \partial_e \Omega_j^N. \end{cases} \tag{21}$$

For arbitrary $j = 1, \dots, N$, we have

$$I_j^N = \int_{\Omega_j^N} \nabla u^N : \nabla (\widehat{w}_j^N - w_j^N) + \int_{\Omega_j^N} \nabla u^N : \nabla w_j^N,$$

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and we set

$$R1_j^N := \int_{\Omega_j^N} \nabla u^N : \nabla(\widehat{w}_j^N - w_j^N).$$

Because w_j^N is a solution to (20) and $u \in H^1(\Omega_j^N)$ is divergence-free we have also that:

$$I_j^N = \int_{\partial B_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N \, d\sigma + \int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N \, d\sigma + R1_j^N.$$

In the first integral, we note that $u^N = V_j^N$ on ∂B_j^N. We then introduce:

$$F_j^N = \int_{\partial B_j^N} (\nabla w_j^N - p\mathbb{I}) \cdot n \, d\sigma$$

to rewrite the first term:

$$\int_{\partial B_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N \, d\sigma = F_j^N \cdot V_j^N.$$

As for the second term, we have (recall (17) for the definition of $\bar{u}_j^N$):

$$\int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N \, d\sigma = \int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot \bar{u}_j^N \, d\sigma + R2_j^N,$$

where

$$R2_j^N = \int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot (u^N - \bar{u}_j^N) \, d\sigma.$$

At this point, we remark that the Stokes system is the divergence form of the conservation of the normal stresses. This yields that:

$$\int_{\partial B_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n \, d\sigma + \int_{\partial_e \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n \, d\sigma = 0.$$

Consequently, we obtain that:

$$\int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N \, d\sigma = R2_j^N - F_j^N \cdot \bar{u}_j^N.$$

Eventually, plugging the identities above in $\int_{\mathcal{F}^N} \nabla u^N : \nabla w$ yields that:

$$\begin{aligned} \int_\Omega \nabla v^N : \nabla w = & \sum_{j=1}^N F_j^N \cdot V_j^N - \int_\Omega j(x) \cdot w(x) \, dx \\ & - \left[\sum_{j=1}^N F_j^N \cdot \bar{u}_j^N - \int_\Omega \rho(x) \bar{u}(x) \cdot w(x) \, dx \right] + R1^N + R2^N, \end{aligned} \tag{22}$$

where:

$$R1^N := \sum_{j=1}^N R1_j^N, \quad R2^N := \sum_{j=1}^N R2_j^N.$$

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3.2. Estimates applied in both proofs

We state and prove here several propositions that are useful in the proof of both theorems.

Proposition 3.2. *There holds:*

$$\left|\sum_{k=1}^{N} F_k^N \cdot v_k^N - \int_{\Omega} j(x)\cdot w(x)\,dx\right| \lesssim \left(\frac{E_0}{N^{2/3}} + \|j - j^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*}\right)\|w\|_{2,p'}.$$

Proof. We define (W_j^N, Π_j^N) by:

$$(w_j^N(x), \pi_j^N(x)) = (W_j^N(N(x - x_j^N)), N\Pi_j^N(N(x - x_j^N))), \quad \forall x \in \Omega_j^N. \tag{23}$$

We note that, substituting in the integral yields:

$$\int_{\partial B_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I})\cdot n\,d\sigma = \frac{1}{N}\int_{\partial B(0,r_j^N)} (\nabla W_j^N - \Pi_j^N \mathbb{I})\cdot n\,d\sigma,$$

and:

$$\begin{aligned} F_j^N &= \frac{1}{N}\int_{\partial B(0,r_j^N)} (\nabla W_j^N - \Pi_j^N \mathbb{I})\cdot n\,d\sigma \\ &= \frac{1}{N}\left(\int_{\partial B(0,r_j^N)} (\nabla W_j^N - \Pi_j^N \mathbb{I})\cdot n\,d\sigma - 6\pi r_j^N w(x_j^N)\right) + \frac{6\pi}{N} r_j^N w(x_j^N). \end{aligned}$$

We remark then that (W_j^N, Π_j^N) is the solution to:

$$\begin{cases} -\Delta W_j^N + \nabla \Pi_j^N = 0, \\ \operatorname{div} W_j^N = 0, \end{cases} \quad \text{on } B(0, C_0/N^{2/3}) \setminus \overline{B(0, r_j^N)}, \tag{24}$$

completed with boundary conditions:

$$W_j^N(x) = w(x_j^N), \quad \text{on } \partial B(0, r_j^N), \quad W_j^N(x) = 0, \quad \text{on } \partial B(0, C_0/N^{2/3}), \tag{25}$$

so that Lemma 2.5 applies. Assumptions (H2) and (H3) then entail that:

$$\begin{aligned} \left|\sum_{k=1}^{N} F_k^N \cdot v_k^N - \int_{\Omega} j\cdot w\right| &= \left|\frac{1}{N}\left(\sum_{j=1}^{N}\int_{\partial B(0,r_j^N)} (\nabla W_j^N - P_j^N \mathbb{I})\cdot n\,d\sigma - 6\pi r_j^N w(x_j^N)\right)\cdot v_k^N \right. \\ &\quad \left. + \frac{6\pi}{N}\sum_{k=1}^{N} r_k^N [w(x_k^N)]\cdot v_k^N - \int_{\Omega} j\cdot w\right| \\ &\lesssim \frac{1}{N}\sum_{k=1}^{N}\frac{|r_k^N|^2}{N^{2/3}}|w(x_k^N)||v_k^N| + |\langle j^N - j, w\rangle| \\ &\lesssim \frac{E_0}{N^{2/3}}\|w\|_\infty + \|j^N - j\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*}\|w\|_{\mathcal{C}^{0,1}(\bar{\Omega})}. \end{aligned}$$

We conclude the proof by applying the embedding $W^{2,p'}(\Omega) \subset C^{0,1}(\bar{\Omega})$. □

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Remark 3.1. A more general estimate can be proved when $p \in]3/2, 3[$. Indeed, in this case we have the Sobolev embedding $W^{2,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,\min(1,\alpha_p)}(\bar{\Omega})$ with $\alpha_p := 2 - \frac{3}{p'} = -1 + \frac{3}{p} \in (0, 1)$. Hence, in the last list of inequality, we may bound:

$$|\langle j^N - j, w\rangle| \leqslant \|j^N - j\|_{(\mathcal{C}^{0,\alpha_p}(\bar{\Omega}))^*} \|w\|_{2,p'}.$$

We complete the joint part of our main proofs by showing that both $R1^N$ and $R2^N$ vanish when $N \to \infty$. First, we have the following proposition:

Proposition 3.3. *There holds*

$$|R1^N| \lesssim \frac{E_0 \|w\|_{2,p'}}{N}.$$

Proof. We remind that:

$$R1^N = \sum_{j=1}^N \int_\Omega \nabla E_\Omega(u^N) : \nabla(\widehat{w}_j^N - w_j^N).$$

We set $\tilde{w}_j^N$ the difference $\widehat{w}_j^N - w_j^N$, hence:

$$\begin{aligned} |R1^N| &\leqslant \|\nabla E_\Omega(u^N)\|_{L^2(\Omega)} \left(\sum_{j=1}^N \|\nabla \tilde{w}_j^N\|^2_{L^2(\Omega_j^N)} \right)^{1/2} \\ &\lesssim E_0 \left(\sum_{j=1}^N \|\nabla \tilde{w}_j^N\|^2_{L^2(\Omega_j^N)} \right)^{1/2}, \end{aligned} \tag{26}$$

because of the bound on $E_\Omega[u^N]$ that we obtained in Proposition 3.1.

At this point, we remark that, for $j \in \{1, \ldots, N\}$ the $\tilde{w}_j^N$ can be associated with a pressure $\tilde{\pi}_j^N$ (namely $\widehat{\pi}_j^N - \pi_j^N$) to get the unique solution to the Stokes problem:

$$\begin{cases} -\Delta \tilde{w}_j^N + \nabla \tilde{\pi}_j^N = 0, \\ \operatorname{div} \tilde{w}_j^N = 0, \end{cases} \quad \text{on } \Omega_j^N, \tag{27}$$

completed with boundary conditions:

$$\begin{cases} \tilde{w}_j^N(x) = w(x) - w(x_j), & \text{on } \partial B_j^N, \\ \tilde{w}_j^N = 0 & \text{on } \partial_e \Omega_j^N. \end{cases} \tag{28}$$

The aim is to bound the $H_0^1(\Omega_j^N)$-norm of $\tilde{w}_j^N$ by constructing a lifting of boundary conditions (28) and using the variational characterization of $\tilde{w}_j^N$ solution to (27)–(28).

Let χ be a truncation function equal to 1 on $B(0, R_0)$ and vanishing outside $B(0, 2R_0)$. We set $\chi^N := \chi(N(x - x_j^N))$ and we denote $v = v_1 + v_2$ where:

$$\begin{aligned} v_1(x) &= \chi^N(x)(w(x) - w(x_j^N)), \quad \forall x \in \Omega_j^N, \\ v_2 &= \mathfrak{B}_{x_j, R_0/N, 2R_0/N}[-\operatorname{div}(v_1)], \end{aligned}$$

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with $\mathfrak{B}$ the Bogovskii operator (see [6, § III.3]). Because $\mathrm{div}(v_1) = \nabla \chi^N \cdot (w - w(x_j^N))$ has mean 0 on Ω_j^N, the vector field v_2 is well defined. We may then apply [9, Appendix A, Lemma 15] to get that:

$$\begin{aligned}\int_{\Omega_j^N} |\nabla v|^2 &\lesssim \int_{A(x_j^N, R_0/N, 2R_0/N)} |\nabla w(x)|^2 \\ &\quad + N^2 \int_{A(x_j^N, R_0/N, 2R_0/N)} |w(x) - w(x_j^N)|^2 \sup_{x \in B(0,2R_0)} |\nabla \chi(x)|^2 \\ &\lesssim \frac{1}{N^3} \|w\|^2_{W^{2,p'}(\Omega)}.\end{aligned}$$

We applied here again the embedding $W^{2,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega})$ for $p' > 3$. Finally, we have

$$|R1^N| \lesssim \frac{E_0}{N} \|w\|_{W^{2,p'}(\Omega)}.$$

This ends the proof of our estimate. □

Remark 3.2. As in Remark 3.1, a more general result can be obtained for all $p \in]3/2, 3[$. In this case, we have that $W^{2,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha_p}(\bar{\Omega})$, which provides a more general bound for the error term $R1^N$ of the form $\frac{1}{N^{\alpha_p}}$.

In order to compute the second error term, we need the following lemma. We recall that the annuli A_j^N are defined in (17). We keep the convention that $\partial_e A_j^N$ stands for the external sphere bounding A_j^N.

Lemma 3.4. *For $j = 1, \ldots, N$, let $v_j^N \in H^1(A_j^N)$ satisfy:*

- $\mathrm{div}\, v_j^N = 0$ *on* A_j^N*;*
- *the flux of v_j^N through the exterior boundary of A_j^N vanishes:*

$$\int_{\partial_e A_j^N} v_j^N \cdot n \, d\sigma = 0;$$

- *the mean of v_j^N on A_j^N vanishes.*

Then, there holds:

$$\left| \sum_{j=1}^N \int_{\partial_e \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot v_j^N \, d\sigma \right| \lesssim \left(\sum_{j=1}^N \|\nabla v_j^N\|^2_{L^2(A_j^N)} \right)^{1/2} \left(\sum_{j=1}^N \|\nabla w_j^N\|^2_{L^2(A_j^N)} \right)^{1/2}.$$

Proof. We begin by introducing a suitable lifting of $v_j^N|_{\partial_e A_j^N}$.

Namely, we introduce a truncation function χ such that χ vanishes on $B(0, C_0/4)$ and is equal to 1 outside $B(0, C_0/3)$. For $j \in \{1, \ldots, N\}$, we denote $\chi_j^N = \chi(N^{1/3}(x - x_j^N))$ and we set:

$$\tilde{v}_j = \tilde{v}_{j,1} + \tilde{v}_{j,2},$$

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where:

$$\tilde{v}_{j,1} = \chi_j^N v_j^N, \quad \tilde{v}_{j,2} = \mathfrak{B}_{x_j^N, C_0/4N^{1/3}, C_0/2N^{1/3}}[-\operatorname{div} \tilde{v}_{j,1}].$$

As, by assumption, we have that v_j^N has flux zero on $\partial_e A_j^N$ we obtain that $\operatorname{div} \tilde{v}_{j,1}$ has mean zero on A_j^N and $\tilde{v}_{j,2}$ is well defined. For convenience, we also set:

$$w^N = \sum_{j=1}^N \mathbf{1}_{\Omega_j^N} w_j^N, \quad D\tilde{v} = \sum_{j=1}^N \mathbf{1}_{\Omega_j^N} \nabla \tilde{v}_j.$$

At this point, we note that:

- on $\partial B_j^N \subset B(x_j^N, C_0/4N^{1/3})$ we have $\chi_j^N = 0$ so that $\tilde{v}_{j,1} = 0$. As $\tilde{v}_{j,2} = 0$ by construction, we get $\tilde{v}_j = 0$;
- on $\partial_e A_j^N = \partial_e \Omega_j^N$, we have $\chi_j^N = 1$ so that $\tilde{v}_{j,1} = v_j^N$. As, by construction, $\tilde{v}_{j,2} = 0$, we get $v_j^N = \tilde{v}_j$.

These remarks entail that:

$$\begin{aligned}\sum_{j=1}^N \int_{\partial_e A_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot v_j^N \, d\sigma &= \sum_{j=1}^N \int_{\partial \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot \tilde{v}_j \, d\sigma \\ &= \sum_{j=1}^N \int_{\Omega_j^N} \nabla w_j^N : \nabla \tilde{v}_j, \\ &= \int_\Omega \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} \nabla w^N : D\tilde{v}.\end{aligned}$$

Consequently, we have:

$$\left| \sum_{j=1}^N \int_{\partial_e \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot v_j^N \, d\sigma \right| \leqslant \left(\int_\Omega \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} |\nabla w^N|^2 \right)^{1/2} \left(\int_\Omega |D\tilde{v}|^2 \right)^{1/2}.$$

Due to the fact that the supports of the Ω_j^N are disjoint and cover the support of $D\tilde{v}$, we have:

$$\int_\Omega \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} |\nabla w^N|^2 = \sum_{j=1}^N \int_{\Omega_j^N} \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} |\nabla w_j^N|^2$$

where $\operatorname{Supp}(D\tilde{v}) \cap \Omega_j^N = A_j^N$ so that

$$\int_\Omega \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} |\nabla w^N|^2 = \sum_{j=1}^N \int_{A_j^N} |\nabla w_j^N|^2.$$

With a similar decomposition, we obtain also:

$$\begin{aligned}\int_\Omega |D\tilde{v}|^2 &= \sum_{j=1}^N \int_{A_j^N} |\nabla \tilde{v}_j|^2, \\ &\leqslant 2 \sum_{j=1}^N \int_{A_j^N} |\nabla \tilde{v}_{j,1}|^2 + |\nabla \tilde{v}_{j,2}|^2.\end{aligned}$$

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As in the proof of the previous proposition, we compute the terms $\nabla\tilde{v}_{j,1}$, $\nabla\tilde{v}_{j,2}$ and use estimates on Bogovskii operator (see [9, Appendix A, Lemma 15]) to get that there exists a positive constant K_χ such that:

$$\int_{A_j^N} |\nabla\tilde{v}_{j,1}|^2 + |\nabla\tilde{v}_{j,2}|^2 \leqslant K_\chi\left(\int_{A_j^N} N^{2/3}|v_j^N|^2 + \int_{A_j^N} |\nabla v_j^N|^2\right).$$

Finally, we apply the Poincaré–Wirtinger inequality in the case of annuli (see [9, Appendix A, Lemma 13]): there exists a constant $C > 0$ independent of N for which:

$$\int_{A_j^N} |v_j^N|^2 \leqslant C\left(\frac{C_0}{2N^{1/3}}\right)^2 \int_{A_j^N} |\nabla v_j^N|^2.$$

Finally we get that

$$\int_\Omega |D\tilde{v}|^2 \lesssim \sum_{j=1}^N \int_{A_j^N} |\nabla v_j^N|^2. \qquad \square$$

We may now state the result on the control of the second error term R_2^N:

Proposition 3.5. *There holds*

$$|R2^N| \lesssim \frac{E_0\|w\|_{2,p'}}{N^{1/3}}.$$

Proof. The main idea to compute $R2^N$ is to apply the previous lemma to

$$v_j^N = \mathbf{1}_{A_j^n}\left[u^N - \fint_{A(x_j^N, C_0/4N^{1/4}, C_0/2N^{1/3})} u^N\right].$$

This entails that

$$|R2^N| \leqslant K\|\nabla u^N\|_{L^2(\mathcal{F}^N)}\left(\sum_{i=1}^N \|\nabla w_j^N\|^2_{L^2(A_j^N)}\right)^{1/2}. \tag{29}$$

At this point, we recall the definition of W_j^N (see (23)) and use the change of variable $y = N(x - x_j^N)$:

$$\begin{aligned}\|\nabla w_j^N\|^2_{L^2(A_j^N)} &= \int_{A_j^N} N^2|\nabla W_j^N(N(x-x_j^N))|^2\,dx \\ &= \frac{1}{N}\int_{A(0,C_0N^{2/3}/4,C_0N^{2/3}/2))} |\nabla W_j^N(y)|^2\,dy \\ &= \frac{1}{N}\|\nabla W_j^N\|^2_{L^2(A(0,C_0N^{2/3}/4,C_0N^{2/3}/2))}.\end{aligned}$$

We may then apply Lemma 2.6 to get that:

$$\|\nabla W_j^N\|^2_{L^2(A(0,C_0N^{2/3}/4,C_0N^{2/3}/2))} \lesssim |r_j^N|^2\frac{|w(x_j^N)|^2}{C_0N^{2/3}}.$$

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Plugging these identities into (29), applying the fact that $E_\Omega(u^N)$ is bounded for the $D_0(\Omega)$-norm and assumption (H2), we obtain:

$$|R2^N| \lesssim E_0\left(\frac{1}{N}\sum_{j=1}^N |r_j^N|^2 \frac{|w(x_j^N)|^2}{C_0 N^{2/3}}\right)^{1/2} \lesssim \frac{E_0}{N^{1/3}}\|w\|_\infty. \qquad \square$$

3.3. Proof of Theorem 1.1

We now turn to the proof of the theorem including a smallness assumption on the size of the holes. For this proof, we first complement the computations in the previous section by estimating the term on the second line of (22):

Proposition 3.6. *Under the further assumption that $j \in L^q(\Omega)$ for some $q > 3$, there exists $K_{p,\Omega}$ depending only on p and Ω such that:*

$$\left|\sum_{k=1}^N F_k^N \cdot \bar{u}_k^N - \int_\Omega \rho(x) w(x) \cdot \bar{u}(x)\right| - K_{p,\Omega}\frac{R_0}{C_0^3}\|w\|_{2,p'}\|u^N - \bar{u}\|_p$$
$$\lesssim \left[\frac{E_0}{N^{2/3}} + \left(\|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}}\right)\|\bar{u}\|_{2,3}\right]\|w\|_{2,p'}.$$

Proof. We may write:

$$\sum_{k=1}^N F_k^N \cdot \bar{u}_k^N = \sum_{k=1}^N \left(F_k^N - \frac{6\pi}{N} r_k^N w(x_k^N)\right) \cdot \bar{u}_k^N + \frac{6\pi}{N} r_k^N w(x_k^N) \cdot \bar{u}_k^N.$$

We remind that given $k \in \{1, \dots, N\}$:

$$|A_k^N| = |B(x_k^N, C_0/2N^{1/3})| - |B(x_k^N, C_0/4N^{1/3})| = \frac{4}{3}\pi\left(\frac{C_0^3}{8N} - \frac{C_0^3}{64N}\right) = \frac{7C_0^3\pi}{48N}.$$

According to the same computations as in the proof of Proposition 3.2:

$$\left|\sum_{k=1}^N \left(F_k^N - \frac{6\pi}{N} r_k^N w(x_k^N)\right) \cdot \bar{u}_k^N\right| \lesssim \frac{1}{N^{2/3}} \sum_{k=1}^N \frac{1}{N}|\bar{u}_k^N| \|w\|_\infty$$
$$\lesssim \frac{1}{N^{2/3}}\|w\|_\infty \frac{1}{N}\sum_{k=1}^N \frac{1}{|A_k^N|}\int_{A_k^N} |u^N|$$
$$\lesssim \frac{E_0}{N^{2/3}}\|w\|_\infty.$$

In order to compute the remaining term we introduce the linear mapping:

$$\Pi_N : \begin{cases} \mathcal{C}_c^\infty(\bar{\Omega}) & \longrightarrow \quad \mathbb{R} \\ \quad \phi & \longmapsto \quad \langle \Pi_N, \phi\rangle := \dfrac{6\pi}{N}\displaystyle\sum_{k=1}^N r_k^N w(x_k^N) \cdot ⨍_{A_k^N} \phi. \end{cases}$$

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We also set:

$$\langle \Pi, \phi \rangle := \int_\Omega \rho(x) w(x) \cdot \phi(x)\, dx,$$

to rewrite the term:

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} 6\pi r_k^N w(x_k^N)\cdot \bar{u}_k^N - \int_\Omega \rho(x)w(x)\cdot \bar{u}(x)\, dx &= \langle \Pi_N, u^N\rangle - \langle \Pi, \bar{u}\rangle \\ &= \langle \Pi_N, u^N - \bar{u}\rangle + \langle \Pi_N - \Pi, \bar{u}\rangle.\end{aligned}$$

By straightforward computations, we show that $(\Pi_N)_N$ is a bounded family of linear mappings on $L^p(\Omega)$. Indeed, recalling the definition of R_0 in (H2) and the above computation of $|A_k^N|$, we obtain:

$$\begin{aligned}|\langle \Pi_N, \phi\rangle| &\leqslant \|w\|_\infty \frac{6\pi}{N|A_1^N|} \max_k r_k^N \int_{\bigsqcup_k A_k^N} |\phi| \\ &\leqslant K_{p,\Omega} \frac{R_0}{C_0^3} \|w\|_\infty \|\phi\|_p,\end{aligned}$$

with $K_{p,\Omega}$ depending only on Ω and p. Hence, applying the embedding $W^{2,p'}(\Omega) \subset L^\infty(\Omega)$ (with a constant depending only on p, Ω) we obtain with a possibly different constant $K_{p,\Omega}$, keeping the same dependencies:

$$|\langle \Pi_N, u^N - \bar{u}\rangle| \leqslant K_{p,\Omega} \frac{R_0}{C_0^3} \|w\|_{2,p'} \|u^N - \bar{u}\|_p.$$

To compute the last term we use the regularity of $\bar{u}$ solution to the Brinkman problem. Indeed, if $j \in L^q$ for some $q > 3$ then Theorem 2.2 shows that $\bar{u} \in W^{2,q}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega})$, thus, there holds:

$$\begin{aligned}|\langle \Pi_N - \Pi, \bar{u}\rangle| &= \left| \frac{6\pi}{N} \sum_{k=1}^N r_k^N w(x_k^N)\cdot \bar{u}(x_k^N) - \int_\Omega \rho(x) w(x)\cdot \bar{u}(x)\, dx \right. \\ &\qquad \left. + \frac{6\pi}{N}\sum_{k=1}^N r_k^N w(x_k^N)\cdot \fint_{A_k^N} (\bar{u} - \bar{u}(x_k^N)) \right| \\ &\lesssim |\langle \rho^N - \rho, w\cdot \bar{u}\rangle| \\ &\qquad + \frac{6\pi}{N}\sum_{k=1}^N r_k^N |w(x_k^N)| \frac{C_0}{2N^{1/3}} \|\bar{u}\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \\ &\lesssim \|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} \|w\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \|\bar{u}\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \\ &\qquad + \frac{1}{N^{1/3}} \|w\|_\infty \|\bar{u}\|_{\mathcal{C}^{0,1}(\bar{\Omega})}.\end{aligned}$$

□

Remark 3.3. When $j \in L^q(\Omega)$ with $q \in]3/2, 3[$, a similar estimate holds involving the distance between ρ and ρ^N in the dual of $\mathcal{C}^{0,\alpha_q}(\bar{\Omega})$. This restriction is due to the embedding $\bar{u} \in W^{2,2}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha_q}(\bar{\Omega})$. The case $q \in]3/2, 3[$ involves also a remainder term that converges to zero like $\frac{1}{N^{\alpha_q/3}}$.

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To complete the proof of Theorem 1.1, we remind that we introduced an exponent $p \in]1,3/2[$, and an arbitrary divergence-free test function $w \in W_0^{1,p'}(\Omega) \cap W^{2,p'}(\Omega)$; inspired by Lemma 2.3, we computed (22) which we recall here:

$$\int_\Omega \nabla(E_\Omega[u^N] - \bar{u}) : \nabla w = \left(\sum_{j=1}^N F_j^N \cdot v_j^N - \int_\Omega j(x) \cdot w(x)\, dx \right) + \left(\int_\Omega \rho(x)\bar{u}(x) \cdot w(x)\, dx - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N \right) + R1^N + R2^N. \quad (30)$$

At this point, we apply now Propositions 3.2, 3.3, 3.5 and 3.6. This entails that there exists a constant K depending only on p, Ω, R_0, C_0 for which

$$\left| \int_\Omega \nabla(E_\Omega[u^N] - \bar{u}) : \nabla w \right| \leqslant \|w\|_{2,p'} \times \left(K \left[\|j^N - j\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{E_0 + \|\bar{u}\|_{2,q}}{N^{1/3}} + \|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\Omega))^*} \right] + K_{p,\Omega} \frac{R_0}{C_0^3} \|u^N - \bar{u}\|_p \right).$$

Consequently, applying Lemma 2.3 and regularity theory for Stokes–Brinkman problem, we obtain a constant $K_{p,\Omega}$ which may differ from the previous ones, but still depending only on p and Ω, such that:

$$\left(1 - K_{p,\Omega}\frac{R_0}{C_0^3}\right)\|u^N - \bar{u}\|_p \lesssim \|j^N - j\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\Omega))^*} + \frac{E_0 + \|j\|_q}{N^{1/3}}.$$

This yields the expected result assuming that R_0/C_0^3 is sufficiently small.

3.4. Proof of Theorem 1.2

We proceed with the proof of our second main result. We do not consider in this case any particular restriction on the ratio R_0/C_0^3. We want to apply now Lemma 2.4. So, we remind that for a fixed divergence-free test function $w \in W_0^{1,p'}(\Omega) \cap W^{2,p'}(\Omega)$, by using again formula (22), we get:

$$\int_\Omega \nabla[E_\Omega[u^N] - \bar{u}] : \nabla w + \int_\Omega \rho[E_\Omega[u^N] - \bar{u}] \cdot w = \left(\sum_{j=1}^N F_j^N \cdot v_j^N - \int_\Omega j \cdot w \right) + \left(\int_\Omega \rho E_\Omega[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N \right) + R1^N + R2^N. \quad (31)$$

In order to treat the new term

$$\int_\Omega \rho E_\Omega[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N,$$

we apply the following proposition:

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Proposition 3.7. *There holds:*

$$\left| \int_\Omega \rho E_\Omega[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N \right| \lesssim \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right)^{1/3} E_0 \|w\|_{W^{2,p'}(\Omega)}.$$

Proof. Using the same notations Π_N and Π as in the previous proof, we write:

$$\int_\Omega \rho E_\Omega[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N = \sum_{k=1}^N \left(\frac{6\pi}{N} r_k^N w(x_k^N) - F_k^N \right) \cdot \bar{u}_k^N + \langle \Pi - \Pi_N, E_\Omega[u^N] \rangle,$$

where the first quantity on the right-hand side is treated as in Proposition 3.2:

$$\left| \sum_{k=1}^N \left(F_k^N - \frac{6\pi}{N} r_k^N w(x_k^N) \right) \cdot \bar{u}_k^N \right| \lesssim \frac{E_0}{N^{2/3}} \|w\|_\infty.$$

To compute the second term $|\langle \Pi_N - \Pi, u^N \rangle|$, we remark that for arbitrary smooth test function ϕ there holds:

$$\begin{aligned} \langle \Pi_N - \Pi, \phi \rangle &= \frac{6\pi}{N} \sum_{j=1}^N r_j^N w(x_j^N) \cdot ⨏_{A_j^N} \phi(x)\, dx - \langle \rho, \phi \cdot w \rangle \\ &= \langle \rho^N - \rho, \phi \cdot w \rangle + \frac{6\pi}{N} \sum_{j=1}^N r_j^N w(x_j^N) ⨏_{A_j^N} (\phi(x) - \phi(x_j^N))\, dx, \end{aligned}$$

and consequently,

$$\begin{aligned} |\langle \Pi_N - \Pi, \phi \rangle| &\lesssim |\langle \rho^N - \rho, \phi \cdot w \rangle| + \frac{1}{N^{1/3}} \|w\|_\infty \|\phi\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \\ &\lesssim \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right) \|\phi\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \|w\|_{\mathcal{C}^{0,1}(\bar{\Omega})}. \end{aligned}$$

On the other hand for all $\phi \in L^2(\Omega)$, we have that:

$$\begin{aligned} |\langle \Pi_N - \Pi, \phi \rangle| &= \left| \frac{6\pi}{N} \int_\Omega \sum_{j=1}^N \frac{1_{A_j^N}(x)}{|A_j^N|} w(x_j^N) \cdot \phi(x)\, dx - \int_\Omega \rho \phi \cdot w \right| \\ &\lesssim \|w\|_\infty \sum_{j=1}^N \int_{A_j^N} |\phi| + \|w\|_\infty \|\rho\|_2 \|\phi\|_2 \\ &\lesssim \|w\|_\infty \|\phi\|_2. \end{aligned}$$

We now propose to interpolate the results above as we want to apply the previous inequalities with $\phi = E_\Omega[u^N] \in H_0^1(\Omega)$. So, let χ be a mollifier having support in $B(0,1)$. We construct then the approximation of unity

$$\chi_\delta(\cdot) = \frac{1}{\delta^3} \chi\left(\frac{\cdot}{\delta}\right), \quad \forall \delta > 0.$$

Thanks to the previous computations, we have that:

$$\begin{aligned}|\langle \Pi_N - \Pi, E_\Omega[u^N]\rangle| &\leqslant |\langle \Pi_N - \Pi, E_\Omega[u^N] * \chi_\delta\rangle| + |\langle \Pi_N - \Pi, E_\Omega[u^N] - E_\Omega[u^N] * \chi_\delta\rangle| \\ &\lesssim \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}}\right)\|E_\Omega[u^N] * \chi_\delta\|_{\mathcal{C}^{0,1}(\bar{\Omega})}\|w\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \\ &\quad + \|w\|_\infty \|E_\Omega[u^N] - E_\Omega[u^N] * \chi_\delta\|_2.\end{aligned}$$

At this point we remark that $E_\Omega[u^N] * \chi_\delta \in H^3(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega})$ with continuous embedding. Furthermore, straightforward computations show that:

$$\begin{aligned}\|E_\Omega[u^N] - E_\Omega[u^N] * \chi_\delta\|_{L^2(\mathbb{R}^3)} &\lesssim \delta \|u\|_{H^1_0(\Omega)}, \\ \|E_\Omega[u^N] * \chi_\delta\|_{H^3(\mathbb{R}^3)} &\lesssim \frac{1}{\delta^2}\|u\|_{H^1_0(\Omega)}.\end{aligned}$$

Plugging these estimates in the previous inequality yields that:

$$\begin{aligned}&|\langle \Pi_N - \Pi, E_\Omega[u^N]\rangle| \\ &\quad \leqslant \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}}\right)\frac{1}{\delta^2}\|\nabla E_\Omega[u^N]\|_2\|w\|_{\mathcal{C}^{0,1}(\bar{\Omega})} + \delta\|w\|_{L^\infty}\|\nabla E_\Omega[u^N]\|_{L^2(\Omega)}.\end{aligned}$$

We may then set $\delta = (\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}})^{1/3}$ and again apply that $W^{2,p'}(\Omega) \subset \mathcal{C}^{0,1}(\bar{\Omega})$ with the uniform control on $\|\nabla E_\Omega[u^N]\|_2$ to get that

$$|\langle \Pi_N - \Pi, E_\Omega[u^N]\rangle| \lesssim \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}}\right)^{1/3} E_0 \|w\|_{W^{2,p'}(\Omega)}. \qquad \square$$

Similarly as in the proof of the previous theorem, we complete the proof of Theorem 1.2 by applying Propositions 3.2, 3.3, 3.5 and 3.7 to control the right-hand side of (31) and referring to Lemma 2.4 to conclude.

4. Final remarks

In the main theorems of this paper, we introduce dilution assumptions reminiscent of [5] and we measure the distance $E_\Omega[u^N] - \bar{u}$ in L^p spaces with respect to the distances between (ρ^N, j^N) and (ρ, j) in the bounded-Lipschitz norms. In this final remark, we discuss the extension of these estimates to other distances and to more general dilution regimes.

Extension to Zolotarev-like distances

Let first consider Zolotarev-like distances of the data (see [11, § 2.5]):

$$\|\rho^N - \rho\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \|j^N - j\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*}, \quad \alpha \in (0,1).$$

In the computations of the previous section, we may replace the $(C^{0,1}(\Omega))^*$ distance by such distances. Thanks to the Remarks 3.1–3.3, we obtain then:

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Theorem 4.1. *Let* $\alpha \in (0,1)$ *and* $(p,q) \in (1, 3/(1+\alpha)) \times (3/(2-\alpha), \infty)$. *Assume that* $j \in L^q(\Omega)$ *and* R_0/C_0^3 *is sufficiently small, there exists a constant* $K > 0$ *depending only on* R_0, C_0, p, q, Ω *for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leqslant K\left[\|j^N - j\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \|\rho^N - \rho\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \frac{E_0}{N^{\min(1/3,\alpha)}} + \frac{\|j\|_{L^q(\Omega)}}{N^{\alpha/3}}\right],$$

for $N \geqslant (4R_0/C_0)^{3/2}$.

Theorem 4.2. *Let* $\alpha \in (0,1)$ *and* $p \in (1, 3/(1+\alpha))$. *There exists a constant* $K > 0$ *depending only on* $R_0, C_0, p, \|\rho\|_{L^\infty(\Omega)}, \Omega$ *for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leqslant K\left[\|j - j^N\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}}\right)^{1/3} E_0\right],$$

for $N \geqslant (4R_0/C_0)^{3/2}$.

Extension of the result to the dilution regime of [9]

One key point of the method we develop in this paper is that it relies on computing the size of error terms:

$$w \mapsto \int_\Omega (\nabla E_\Omega[u^N] - \nabla \bar{u}) : \nabla w + \rho(E_\Omega[u^N] - \bar{u}) \cdot w$$

seen as linear forms on Sobolev spaces with sufficiently high regularity. With the help of Lemma 2.4 we may then transform such computations into distance estimates between $E_\Omega[u^N]$ and $\bar{u}$ in some L^p spaces. This very method is not restricted by the dilution regime under consideration herein. In particular, we may extend our result to the dilution regime introduced in [9]. In this reference, the new assumptions on the particles distribution are the following. We denote by $d^N_{\min}$ the minimal distance between particles:

$$d^N_{\min} := \min\left\{\operatorname{dist}(x_i, \partial\Omega), \min_j \operatorname{dist}(x_i, x_j)\right\} > \frac{4R_0}{N}.$$

We assume that there exists a sequence $(\lambda^N)_{N\in\mathbb{N}}$ such that the following quantity:

$$M^N := \sup_{x\in\Omega}\left\{\#\left\{i \in \{1,\ldots,N\}, x_i \in \overline{B(x,\lambda^N)}\right\}\right\}$$

satisfies:

$$\lim_{N\to\infty} \frac{d^N_{\min}}{|\lambda^N|^3} = +\infty, \quad \left(\frac{N^{2/3}|\lambda^N|^5}{d^N_{\min}}\right)_{N\in\mathbb{N}} \text{ is bounded.} \tag{32}$$

$$\left(\frac{M^N}{N|\lambda^N|^3}\right)_{N\in\mathbb{N}} \text{ is bounded.} \tag{33}$$

Moreover, we keep the different radii's hypothesis on the balls and the associated assumption (H2). We also keep assumption (H3) by introducing a uniform bound $E_0 < \infty$ such that:

$$\left(\frac{1}{N}\sum_{i=1}^N |V_i^N|^2\right)^{1/2} \leqslant E_0.$$

Note that assumption (33) implies that there exists a positive constant M^∞ such that:

$$\frac{M^N}{|\lambda^N|^3 N} \leqslant M^\infty.$$

The approach of [9] is to define a specific covering of Ω with cubes of width λ^N which allows to focus on the study of each cell by using the fact that the maximal number of holes in each cell is well known thanks to (33). In particular, the parameter δ which appears in the following estimates is introduced to handle the centers that are too close to the boundaries of the covering. So, adapting the computation of error terms to the decomposition proposed in [9] one obtains the following result:

Theorem 4.3. *Given $p \in]1, 3/2[$ and $\delta \geqslant 4$ arbitrary large, there exist two positive constants $C = C(\Omega, p, \|\rho\|_\infty, R_0, E_0, M^\infty)$ and $C_\delta = C(\delta)$ such that:*

$$\begin{aligned}\|E_\Omega[u^N] - \bar{u}\|_p \leqslant C\Bigg(& \|j^N - j\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \delta^{1/3}(\|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + |\lambda^N|)^{1/3} + \sqrt{\frac{|\lambda^N|^3}{d^N_{\min}}} \\ & + \frac{1}{\sqrt{\delta}} + C_\delta \sqrt{\delta}|\lambda^N| + \frac{N^{1/3}|\lambda^N|^{5/2}}{\delta^{1/3}\sqrt{d^N_{\min}}} + \delta^{1/6} N^{1/3}|\lambda^N|^2\Bigg). \qquad (34)\end{aligned}$$

Appendix A. Fluid/solid interaction

In this part, we assume that $\Omega = \mathbb{R}^3 \setminus \overline{B(0,r)}$ where $r > 0$. Let $V \in \mathbb{R}^3$ be fixed in what follows. We consider the unique pair (u, π) solution to the Stokes problem:

$$\begin{cases} -\Delta u + \nabla \pi = 0, \\ \quad \operatorname{div} u = 0, \end{cases} \quad \text{on } \Omega, \qquad (35)$$

completed with boundary conditions:

$$u(x) = V, \quad \text{on } \partial B(0, r), \qquad \lim_{|x|\to\infty} u(x) = 0. \qquad (36)$$

We denote by F the reaction force applied by the obstacle $B(0, r)$ on the fluid, it is defined as:

$$F = \int_{\partial\Omega} (\nabla u + \nabla u^\top - \pi \mathbb{I}) \cdot n \, d\sigma. \qquad (37)$$

The following lemma provides us an equivalent definition of F:

Lemma A.1. *Let $R_0 \geqslant r$, there holds:*

$$F = \int_{\partial B(0,R_0)} [\nabla u - \pi \mathbb{I}] \cdot n \, d\sigma.$$

Proof. The aim is to prove that for arbitrary $W \in \mathbb{R}^3$:

$$F \cdot W = \int_{\partial B(0,R_0)} [(\nabla u - \pi \mathbb{I}) \cdot n] \cdot W \, d\sigma.$$

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Fix a vector field $w \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that $\operatorname{div} w = 0$, $w = W$ on $B(0, R_0)$, extend u by the value V on $B(0, r)$ and still denote by u the extension for simplicity. After integration by parts we obtain:

$$
\begin{aligned}
F \cdot W &= \int_{\mathbb{R}^3 \setminus B(0,r)} \nabla u : \nabla w + \nabla u : \nabla w^\top \\
&= \int_{\mathbb{R}^3} \nabla u : \nabla w \\
&= \int_{\mathbb{R}^3 \setminus B(0,R_0)} \nabla u : \nabla w \\
&= \int_{\partial B(0,R_0)} [(\nabla u - \pi \mathbb{I}) \cdot n] \cdot W \, d\sigma.
\end{aligned}
$$

As $\operatorname{div} u = 0$ and $w = W$ on $B(0, r)$. □

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