

PAPER

Extensional constructive real analysis via locators

Auke B. Booij 

School of Computer Science, University of Birmingham, Birmingham, UK
Email: abb538@cs.bham.ac.uk

(Received 3 August 2018; revised 21 June 2020; accepted 19 July 2020; first published online 2 September 2020)

Abstract

Real numbers do not admit an extensional procedure for observing discrete information, such as the first digit of its decimal expansion, because every extensional, computable map from the reals to the integers is constant, as is well known. We overcome this by considering real numbers equipped with additional structure, which we call a locator. With this structure, it is possible, for instance, to construct a signed-digit representation or a Cauchy sequence, and conversely, these intensional representations give rise to a locator. Although the constructions are reminiscent of computable analysis, instead of working with a notion of computability, we simply work constructively to extract observable information, and instead of working with representations, we consider a certain locatedness structure on real numbers.

Keywords: constructive mathematics, constructive analysis, homotopy type theory, dependent type theory, exact real arithmetic

1. Introduction

It is well known how to compute with real numbers intensionally, with equality of real numbers specified by an imposed equivalence relation on representations (Bishop and Bridges, 1985; Cruz-Filipe, 2004; O'Connor, 2009), such as Cauchy sequences or streams of digits. It has to be checked explicitly that functions on the representations preserve such equivalence relations. Discrete observations, such as finite decimal approximations, can be made because representations are given, but a different representation of the same real number can result in a different observation, and hence discrete observations are necessarily non-extensional.

In univalent mathematics, equality of real numbers can be captured by identity types directly, rather than by an imposed equivalence relation, thus avoiding the use of setoids. Preservation of equality of real numbers is automatic, but the drawback is that we are prevented from making any discrete observations of arbitrary real numbers. This kind of problem is already identified by Hofmann (1995, Section 5.1.7.1) for an extensional type theory. Discrete observations of real numbers are made by breaking extensionality using a *choice operator*, which does not give rise to a *function*.

To avoid breaking extensionality, the central idea of this paper is to restrict our attention to real numbers that can be equipped with a simple structure called a *locator*. Such a locator is a strengthening of the locatedness property of Dedekind cuts. While the locatedness of a real number x says that for rational numbers $q < r$ we have the property $q < x$ or $x < r$, a locator produces a specific

selection of one of $q < x$ and $x < r$. In particular, the same real number can have different locators, and it is in this sense that locators are structure rather than property.

In a constructive setting such as ours, not all real numbers have locators, and we prove that the ones that do are the ones that have Cauchy representations in Section 3.9. However, working with locators rather than Cauchy representations gives a development which is closer to that of traditional real analysis. For example, we can prove that if x has a locator, then so does e^x , and this allows to compute e^x when working constructively, so that we say that the exponential function *lifts to locators*. As another example, if f is given a modulus of continuity and lifts to locators, then $\int_0^1 f(x) dx$ has a locator and we can compute the integral in this way.

Thus the difference between locatedness and locators is that one is property and the other is structure. Plain Martin-Löf type theory is not enough to capture this distinction because, for example, it allows to define the notion of locator as structure but not the notion of locatedness as property, and therefore it does not allow to define the type of Dedekind reals we have in mind, whose identity type should capture directly the intended notion of equality of real numbers. A good foundational system to account for such distinctions is univalent type theory (UTT), also known as homotopy type theory (HoTT) (The Univalent Foundations Program, 2013). For us, it is enough to work in the fragment consisting of Martin-Löf type theory with propositional truncation, propositional extensionality, and function extensionality (see Section 2). The need for univalence would arise only when considering types of sets with structure such as the type of metric spaces or the type of Banach spaces for the purposes of functional analysis.

We believe that our constructions can also be carried out in other constructive foundations such as Constructive Zermelo-Fraenkel the internal language of an elementary topos with a natural numbers object, or Heyting arithmetic of finite types. Our choice of UTT is to some extent a practical one, as it is a constructive system with sufficient extensionality, which admits, at least in theory, applications in proof assistants allowing for computation using the techniques in this paper.

In summary, the work has two aspects. One aspect is that instead of working with functions on intensional representations, we work with functions on real numbers that lift representations. The second aspect is the particular representation that seems suitable.

We describe the assumptions on the foundational system in Section 2.

The definition and basic theory of locators is given in Section 3. We construct locators for rationals in Section 3.3. We discuss preliminaries for observing data from locators in Sections 3.4 and 3.5, which is then used to compute rational bounds in Section 3.6. We compute locators for algebraic operations in Sections 3.7 and for limits in Section 3.8. We compute signed digit representations for reals with locators in Section 3.9. Given a real and a locator, we strengthen the properties for being a Dedekind cut into structure in Section 3.10.

We show some ways of using locators in constructive analysis in Section 4. We compute locators for integrals in Section 4.2. We discuss how locators can help computing roots of functions in Section 4.3.

2. Preliminaries

We work in type theory with universes \mathcal{U} and \mathcal{U}' with $\mathcal{U} : \mathcal{U}'$, identity types $x =_X y$ for $x, y : X$, a unit type $\mathbf{1}$, an empty type $\mathbf{0}$, a natural numbers type \mathbb{N} , dependent sum types Σ , dependent product types Π , and propositional truncation $\| \cdot \|$ (see Section 2.1). We assume function extensionality, which can be stated as the claim that all pointwise equal functions are equal. We assume propositional extensionality, namely, the claim that if P and Q are propositions in the sense of Section 2.1, and $P \Rightarrow Q$ and $Q \Rightarrow P$, then $P = Q$.

2.1 Propositions

Definition 2.1.1. A proposition is a type P all whose elements are equal, which is expressed type-theoretically as

$$\text{isHProp}(P) := \Pi(p, q : P) . (p =_P q).$$

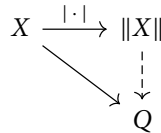
We have the type $\text{HProp} := \Sigma(P : \mathcal{U}) . \text{isHProp}(P)$ of all propositions, and we conflate elements of HProp with their underlying type, that is, their first projection.

We assume that every type has a propositional truncation.

Definition 2.1.2. The propositional truncation $\|X\|$ of a type X is a proposition together with a truncation map $|\cdot| : X \rightarrow \|X\|$ such that for any other proposition Q , given a map $g : X \rightarrow Q$, we obtain a map $h : \|X\| \rightarrow Q$.

Remark. The uniqueness of the obtained map $\|X\| \rightarrow Q$ follows from the fact that Q is a proposition, and function extensionality.

We may also think of propositional truncations categorically, in which case they have the universal property that given a map $X \rightarrow Q$ as in the diagram below, we obtain the vertical map, which automatically makes the diagram commute because Q is a proposition, and which is automatically equal to any other map that fits in the diagram.



Propositional truncations can be defined as higher-inductive types, or constructed via impredicative encodings assuming propositional resizing.

Even though the elimination rule in Definition 2.1.2 only constructs maps into propositions, we can *sometimes* get a map $\|X\| \rightarrow X$, as we discuss in Theorem 3.5.1.

Definition 2.1.3. Truncated logic is defined by the following, where $P, Q : \text{HProp}$ and $R : X \rightarrow \text{HProp}$ (The Univalent Foundations Program, 2013, Definition 3.7.1):

$$\begin{aligned} \top &:= \mathbf{1} \\ \perp &:= \mathbf{0} \\ P \wedge Q &:= P \times Q \\ P \Rightarrow Q &:= P \rightarrow Q \\ P \Leftrightarrow Q &:= P = Q \\ \neg P &:= P \rightarrow \mathbf{0} \\ P \vee Q &:= \|P + Q\| \\ \forall(x : X) . R(x) &:= \Pi(x : X) . R(x) \\ \exists(x : X) . R(x) &:= \|\Sigma(x : X) . R(x)\|. \end{aligned}$$

We use the following terminological conventions throughout the work.

Definition 2.1.4. We refer to types that are propositions as properties. We refer to types which may have several inhabitants as data or structures. We indicate the use of truncations with the verb “to exist”: so the claim “there exists an A satisfying B ” is to be interpreted as $\exists(a : A) . B(a)$, and “there exists an element of X ” is to be interpreted as $\|X\|$. Most other verbs, including “to have,” “to find,” “to

construct,” “to obtain,” “to get,” “to give,” “to equip,” “to yield,” and “to compute,” indicate the absence of truncations.

Example 2.1.5. One attempt to define when $f : X \rightarrow Y$ is a surjection is

$$\prod(y : Y) . \Sigma(x : X) . fx = y.$$

In fact, this is rather called split surjective, as from that structure, we obtain a map $Y \rightarrow X$ which is inverse to f : so we have defined when a function is a *section*. Rather defining surjectivity as

$$\forall(y : Y) . \exists(x : X) . fx = y,$$

by virtue of using the *property* $\exists(x : X) . fx = y$, does not yield an inverse map.

In words, we say that f is a surjection if for every $y : Y$ there *exists* a pre-image. The terminology that every $y : Y$ has a pre-image means a choice of pre-images, which formalizes sections.

Example 2.1.6. Given a function $f : A \rightarrow B$, the *image* of f is the collection of elements $b : B$ that are reached by f , that is, for which there is an element $a : A$ such that $fa =_B b$. The propositions-as-types interpretation would formalize this as

$$\Sigma(b : B) . \Sigma(a : A) . fa =_B b.$$

However, because the type $\Sigma(b : B) . fa =_B b$ is contractible (The Univalent Foundations Program, 2013, Lemma 3.11.8), in fact this type is equivalent to the type A itself, in the sense that there is a map with a left pointwise inverse and a right pointwise inverse, and so it does not adequately represent the image of f .

Using truncations, we instead formalize the image of f as the collection of elements of B for which there *exists* a pre-image along f , that is, in UTT the image of f is formalized as

$$\Sigma(b : B) . \exists(a : A) . fa =_B b,$$

noting that the inner Σ is truncated whereas the outer is not: we want to distinguish elements in the image of f , but we do not want to distinguish those elements based on a choice of pre-image in A .

Example 2.1.7. We may compute the integral of a uniformly continuous function f as

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + k \cdot \frac{b-a}{n}\right).$$

The construction of the limit value, for example, as in Lemma 2.2.4, uses the modulus of uniform continuity of f as in Definition 4.1.4. However, since the integral is independent of the choice of modulus, by unique choice, for example, as in Theorem 5.4 of Kraus et al. (2017), the *existence* (defined constructively as in Definition 2.1.3) of a modulus of uniform continuity suffices to compute the integral. We discuss this further in Sections 3.8 and 4.2.

2.2 Dedekind reals

Although the technique of equipping numbers with locators can be applied to any archimedean ordered field, for clarity and brevity we will work with the Dedekind reals \mathbb{R}_D as defined in The Univalent Foundations Program (2013). A more general description is given in Booiij (2020b).

Definition 2.2.1. A predicate B on a type $X : \mathcal{U}$ is a map $B : X \rightarrow \text{HProp}$. For $x : X$, we write $(x \in B) := B(x)$.

A Dedekind real is defined by a pair (L, U) of predicates on \mathbb{Q} with some properties. To phrase these properties succinctly, we use the following notation for $x = (L, U)$:

$$(q < x) := (q \in L) \quad \text{and} \\ (x < r) := (r \in U).$$

This is justified by the fact that $q \in L$ holds iff $i(q) < x$, with $i : \mathbb{Q} \hookrightarrow \mathbb{R}_{\mathbb{D}}$ the canonical inclusion of the rationals into the Dedekind reals.

Definition 2.2.2. A pair $x = (L, U)$ of predicates on the rationals is a Dedekind cut or Dedekind real if it satisfies the four Dedekind properties:

1. bounded: $\exists(q : \mathbb{Q}) . q < x$ and $\exists(r : \mathbb{Q}) . x < r$;
2. rounded: for all $q, r : \mathbb{Q}$,

$$q < x \Leftrightarrow \exists(q' : \mathbb{Q}) . (q < q') \wedge (q' < x), \quad \text{and} \\ x < r \Leftrightarrow \exists(r' : \mathbb{Q}) . (r' < r) \wedge (x < r');$$

3. transitive: $(q < x) \wedge (x < r) \Rightarrow (q < r)$ for all $q, r : \mathbb{Q}$;
4. located: $(q < r) \Rightarrow (q < x) \vee (x < r)$ for all $q, r : \mathbb{Q}$.

The collection $\mathbb{R}_{\mathbb{D}} : \mathcal{U}'$ of pairs of predicates (L, U) together with proofs of the four properties, collected in a Σ -type, is called the Dedekind reals.

Remark. The Univalent Foundations Program (2013) has disjointness

$$\forall(q : \mathbb{Q}) . \neg(x < q \wedge q < x)$$

instead of the transitivity property, which is equivalent to it in the presence of the other conditions, and it is this disjointedness condition that we use most often in proofs.

Proof. Assuming transitivity, if $x < q \wedge q < x$, then transitivity yields $q < q$, which contradicts irreflexivity of $<$ on the rationals, which shows disjointness.

Conversely, if $q < x$ and $x < r$, apply trichotomy of the rationals on q and r : in case that $q < r$ we are done, and in the other two cases we obtain $x < q$, contradicting disjointness. \square

Definition 2.2.3. For Dedekind reals x and y , we define the strict ordering relation by

$$x < y := \exists(q : \mathbb{Q}) . x < q < y,$$

where $x < q < y$ means $(x < q) \wedge (q < y)$, and their apartness by

$$x \# y := (x < y) \vee (y < x).$$

As is typical in constructive analysis, we have $x \# y \Rightarrow \neg(x = y)$, but not the converse.

The following proof that $\mathbb{R}_{\mathbb{D}}$ is Cauchy complete is based on The Univalent Foundations Program (2013, Theorem 11.2.12).

Lemma 2.2.4. The Dedekind reals are Cauchy complete. More explicitly, given a modulus of Cauchy convergence M for a sequence x of Dedekind reals, that is, a map $M : \mathbb{Q}_+ \rightarrow \mathbb{N}$ such that

$$\forall(\varepsilon : \mathbb{Q}_+) . \forall(m, n : \mathbb{N}) . m, n \geq M(\varepsilon) \Rightarrow |x_m - x_n| < \varepsilon,$$

we can compute $l : \mathbb{R}_{\mathbb{D}}$ as the Dedekind cut defined by:

$$(q < l) := \exists(\varepsilon, \theta : \mathbb{Q}_+) . (q + \varepsilon + \theta < x_{M(\varepsilon)}), \\ (l < r) := \exists(\varepsilon, \theta : \mathbb{Q}_+) . (x_{M(\varepsilon)} < r - \varepsilon - \theta),$$

and l is the limit of x in the usual sense:

$$\forall(\varepsilon : \mathbb{Q}_+) . \exists(N : \mathbb{N}) . \forall(n : \mathbb{N}) . n \geq N \Rightarrow |x_n - l| < \varepsilon.$$

Proof. Inhabitedness and roundedness of l are straightforward. For transitivity, suppose $q < l < r$, then we wish to show $q < r$. There exist $\varepsilon, \theta, \varepsilon', \theta' : \mathbb{Q}_+$ with $q + \varepsilon + \theta < x_{M(\varepsilon)}$ and $x_{M(\varepsilon')} < r - \varepsilon' - \theta'$. Now $|x_{M(\varepsilon)} - x_{M(\varepsilon')}| \leq \max(\varepsilon, \varepsilon')$, so either $q + \theta < x_{M(\varepsilon')}$ or $x_{M(\varepsilon)} < r - \theta$, and in either case $q < r$.

For locatedness, suppose $q < r$. Set $\varepsilon := \frac{r-q}{5}$, so that $q + 2\varepsilon < r - 2\varepsilon$. By locatedness of x_ε , we have $(q + 2\varepsilon < x_\varepsilon) \vee (x_\varepsilon < r - 2\varepsilon)$, hence $(q < l) \vee (l < r)$.

In order to show convergence, let $\varepsilon : \mathbb{Q}_+$, set $N := M(\varepsilon)$, and let $n \geq N$. We need to show $|x_n - l| \leq \varepsilon$, or equivalently, $-\varepsilon \leq x_n - l \leq \varepsilon$. For $x_n - l \leq \varepsilon$, suppose that $\varepsilon < x_n - l$, or equivalently, $l < x_n - \varepsilon$. There exist $\varepsilon', \theta' : \mathbb{Q}_+$ with $x_{M(\varepsilon')} < x_n - \varepsilon - \varepsilon' - \theta'$, or equivalently, $\varepsilon + \varepsilon' + \theta' < x_n - x_{M(\varepsilon')}$, which contradicts M being a modulus of Cauchy convergence. We can similarly show $-\varepsilon \leq x_n - l$. □

We denote limits of sequences by $\lim_{n \rightarrow \infty} x_n$.

Example 2.2.5 (Exponential function). We can define the exponential function $\exp : \mathbb{R}_D \rightarrow \mathbb{R}_D$ as $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. We obtain the *existence* of a modulus of Cauchy convergence by boundedness (as in Definition 2.2.2) of x .

3. Locators

The basic idea is that we equip real numbers with the structure of a *locator*, defined in Section 3.1. The purpose of the work is to show *how* to extract discrete information from an existing theory of real analysis in UTT.

The following example, which will be fully proved in Theorem 4.3.5, illustrates how we are going to use locators. Suppose f is a pointwise continuous function, and $a < b$ are real numbers with locators. Further suppose that f is locally non-constant, that $f(x)$ has a locator whenever x has a locator, and that $f(a) \leq 0 \leq f(b)$. Then we can find a root of f , which comes equipped with a locator. For the moment, we provide a proof sketch, to motivate the techniques that we are going to develop in this section. We define sequences $a, b : \mathbb{N} \rightarrow \mathbb{R}_D$ with $a_n < a_{n+1} < b_{n+1} < b_n$, with $f(a_n) \leq 0 \leq f(b_n)$, with $b_n - a_n \leq (b - a) \left(\frac{2}{3}\right)^n$, and such that all a_n and b_n have locators. Set $a_0 = a, b_0 = b$. Suppose a_n and b_n are defined. We will explain in the complete proof of Theorem 4.3.5 how to find q_n with $\frac{2a_n + b_n}{3} < q_n < \frac{a_n + 2b_n}{3}$ and $f(q_n) \neq 0$. The important point for the moment is that this is possible precisely because we have locators.

- If $f(q_n) > 0$, then set $a_{n+1} := a_n$ and $b_{n+1} := q_n$.
- If $f(q_n) < 0$, then set $a_{n+1} := q_n$ and $b_{n+1} := b_n$.

The sequences converge to a number x . For any positive rational ε , we have $|f(x)| \leq \varepsilon$, hence $f(x) = 0$. This completes our sketch.

We need to explain why the sequences a and b come equipped with locators and why their limit x has a locator. In fact, all q_n are rationals and hence have locators, as discussed in Section 3.3. The number q_n is constructed using the central techniques for observing data from locators, see Sections 3.4 and 3.5. These techniques can then also be used in Section 3.6 to compute rational bounds. Locators for $\frac{2a_n + b_n}{3}$ and $\frac{a_n + 2b_n}{3}$ can be constructed as locators for algebraic operations, as in Section 3.7. Locators for limits are discussed in Section 3.8.

We compute signed-digit representations for reals with locators in Section 3.9. Given a real and a locator, we strengthen the properties for being a Dedekind cut into structure in Section 3.10.

3.1 Definition

Recall that there is a canonical embedding of the rationals into \mathbb{R}_D . Throughout the remainder of this paper, we identify $q : \mathbb{Q}$ with its embedding $i(q) : \mathbb{R}_D$.

Recall from Definition 2.2.2 that a pair of predicates on the rationals $x = (L, U)$ is *located* if $\forall (q, r : \mathbb{Q}) . (q < r) \Rightarrow (q < x) \vee (x < r)$. Indeed, this property holds for an arbitrary $x : \mathbb{R}_D$ by cotransitivity of $<$.

Definition 3.1.1. A locator for $x : \mathbb{R}_D$ is a function $\ell : \Pi(q, r : \mathbb{Q}) . q < r \rightarrow (q < x) + (x < r)$. We denote by *locator* (x) the type of locators on x . That is, we replace the logical disjunction in locatedness by a disjoint sum, so that we get structure rather than property, allowing us to compute.

A locator for x can be thought of as falling in the Dedekind tradition of considering the rationals to the left and right of x , in contrast with Cauchy-style representations such as sequences of nested intervals. Whereas existing Dedekind-style developments directly define a fixed notion of real number (Bridges and Vita, 2006), locators are a structure that can be defined for an arbitrary type of reals.

A locator can be seen as an analogue to a Turing machine representing a computable real number, in the sense that it will provide us with enough data to be able to type-theoretically compute, for instance, signed-digit expansions. However, a locator does not express that a given real is a computable real: in the presence of excluded middle, there exists a locator for every $x : \mathbb{R}_D$, despite not every real being computable. To make this precise, we first formalize the principle of excluded middle type-theoretically.

Definition 3.1.2. A decidable proposition is a proposition P such that $P + \neg P$. We have the collection

$$\text{DHProp} := \Sigma(P : \text{HProp}) . P + \neg P$$

of decidable propositions. We identify elements of DHProp with their underlying proposition, and hence with their underlying types.

Remark. If P and Q are decidable, then so is $P \wedge Q$, and we use this fact in later developments.

Definition 3.1.3. The principle of excluded middle (PEM) is the claim that every proposition is decidable, that is,

$$\text{PEM} := \Pi(P : \text{HProp}) . P + \neg P.$$

Lemma 3.1.4. Assuming PEM, for every $x : \mathbb{R}_D$, we can construct a locator for x .

Proof. For given rationals $q < r$, use PEM to decide $q < x$. If $q < x$ holds, we can simply return the proof given by our application of PEM. If $\neg(q < x)$ holds, then we get $x \leq q < r$ so that we can return a proof of $x < r$. □

Remark. Note that we use the word “proof” also to refer to type-theoretic constructions of types that are not propositions. This section contains many such proofs that do not prove propositions in the sense of Definition 2.1.1.

In Section 4, we will define when a function $f : \mathbb{R}_D \rightarrow \mathbb{R}_D$ lifts to locators, which can be seen as an analogue to a computable function on the reals. There, the contrast with the theory of computable analysis becomes more pronounced, as the notion of lifting to locators is neither stronger nor weaker than continuity.

The structure of a locator has been used previously by The Univalent Foundations Program in a proof that assuming either countable choice or excluded middle, the Cauchy reals, and the Dedekind reals coincide (The Univalent Foundations Program, 2013, Section 11.4).

The reader may wonder why we only choose to modify one of the Dedekind properties to become structure. We show in Theorem 3.10.4 that given only a locator, we can obtain the remaining structures, corresponding to boundedness, roundedness, and transitivity, automatically.

3.2 Terminology for locators

A locator ℓ for a real x can be evaluated by picking $q, r : \mathbb{Q}$ and $v : q < r$. The value $\ell(q, r, v)$ has type $(q < x) + (x < r)$, and so $\ell(q, r, v)$ can be either in the left summand or the right summand. We say that “we locate $q < x$ ” when the locator gives a value in the left summand, and similarly we say “we locate $x < r$ ” when the locator gives a value in the right summand.

We often do case analysis on $\ell(q, r, v) : (q < x) + (x < r)$ by constructing a value $c : C(q <_x r)$ for some type family $C : (q < x) + (x < r) \rightarrow \mathcal{U}$. To construct c , we use the elimination principle of $+$, for which we need to specify two values corresponding to the disjuncts $q < x$ and $x < r$, so the two values have corresponding types $\Pi(\xi : q < x) . C(\text{inl}(\xi))$ and $\Pi(\zeta : x < r) . C(\text{inr}(\zeta))$. These two values correspond to the two possible answers of the locator, and we will often indicate this by using the above terminology: the expression “we locate $q < x$ ” corresponds to constructing a value of the former type, and the expression “we locate $x < r$ ” corresponds to constructing a value of the latter type.

For example, for every real x with a locator ℓ , we can output a Boolean depending on whether ℓ locates $0 < x$ or $x < 1$. Namely, if we locate $0 < x$, we output true, and if we locate $x < 1$, we output false. We use this construction in the proof of Lemma 3.10.1.

3.3 Locators for rationals

Lemma 3.3.1. *Suppose $x : \mathbb{R}_{\mathbb{D}}$ is a rational, or more precisely, that $\exists(s : \mathbb{Q}) . (x = i(s))$, with $i : \mathbb{Q} \hookrightarrow \mathbb{R}_{\mathbb{D}}$ the canonical embedding of the rationals into the Dedekind reals, then x has a locator.*

We give two constructions to emphasize that locators are not unique. We use trichotomy of the rationals, namely, for all $a, b : \mathbb{Q}$,

$$(a < b) + (a = b) + (a > b).$$

First proof. Let $q < r$ be arbitrary, then we want to give $(q < s) + (s < r)$. By trichotomy of the rationals applied to q and s , we have

$$(q < s) + (q = s) + (q > s).$$

In the first case $q < s$, we can locate $q < s$. In the second case $q = s$, we have $s = q < r$, so we locate $s < r$. In the third case, we have $s < q < r$, so we locate $s < r$. □

Second proof. Let $q < r$ be arbitrary, then we want to give $(q < s) + (s < r)$. By trichotomy of the rationals applied to s and r , we have

$$(s < r) + (s = r) + (s > r).$$

In the first case $s < r$, we can locate $s < r$. In the second case $s = r$, we have $q < r = s$, so we locate $q < s$. In the third case, we have $q < r = s$, so we locate $q < s$. □

In the case that $q < s < r$, the first construction locates $s < r$, whereas the second construction locates $q < s$. In particular, given a pair $q < r$ of rationals, the first proof locates $q < 0$ if q is indeed negative, and $0 < r$ otherwise. The second proof locates $0 < r$ if r is indeed positive, and $q < 0$ otherwise. Note that these locators disagree when $q < 0 < r$, illustrating that locators are not unique.

3.4 The logic of locators

Our aim is to combine properties of real numbers with the structure of a locator to make discrete observations.

If one represents reals by Cauchy sequences, one obtains lower bounds immediately from the fact that any element in the sequence approximates the real up to a known error. As a working

example, we show, perhaps surprisingly, that we can get a lower bound for an real x , that is an element of $\Sigma(q : \mathbb{Q}) . q < x$, from the locator alone.

Recall that Dedekind reals are bounded from below, so that $\exists(q : \mathbb{Q}) . q < x$. We will define a proposition P which gives us a bound, in the sense that we can use the elimination rule for propositional truncations to get a map

$$(\exists(q : \mathbb{Q}) . q < x) \rightarrow P,$$

and then we can extract a bound using a simple projection map

$$P \rightarrow (\Sigma(q : \mathbb{Q}) . q < x).$$

More concretely, we define a type of rationals which are bounds for x and which are *minimal* in a certain sense. The minimality is *not* intended to find tight bounds but is intended to make this collection of rationals into a proposition: in other words, minimality ensures that the answer is unique, so that we may apply the elimination rule for propositional truncations.

Our technique has two central elements: reasoning about the structure of locators using propositions and the construction of a unique answer using bounded search (Section 3.5).

Given a locator $\ell : \text{locator}(x)$, $q, r : \mathbb{Q}$ and $v : q < r$, we have the notation

$$q <_x^\ell r := \ell(q, r, v) : (q < x) + (x < r),$$

leaving the proof of $q < r$ implicit. We further often drop the choice of locator, writing $q <_x r$ for $q <_x^\ell r$.

Lemma 3.4.1. *For types A and B , we have*

$$A + B \simeq \Sigma(P : \text{DHPProp}) . (P \rightarrow A) \times (\neg P \rightarrow B).$$

Proof. For a given element $x : A + B$, the proposition P is defined to hold when x is given by an element of A , and false otherwise, so that the two conditions on P hold. Vice versa, for a given proposition P , we simply decide P to obtain the respective element of $A + B$. It has to be checked that these two constructions result in an equivalence. □

Lemma 3.4.2. *The type locator (x) of Definition 3.1.1 is equivalent to the type*

$$\begin{aligned} &\Sigma(\text{locatesRight} : \Pi(q, r : \mathbb{Q}) . q < r \rightarrow \text{DHPProp}) . \\ &(\Pi(q, r : \mathbb{Q}) . \Pi(v : q < r) . \text{locatesRight}(q, r, v) \rightarrow q < x) \\ &\times (\Pi(q, r : \mathbb{Q}) . \Pi(v : q < r) . \neg \text{locatesRight}(q, r, v) \rightarrow x < r). \end{aligned}$$

Proof. The previous lemma yields the equivalence

$$\begin{aligned} \text{locator}(x) &\simeq \Pi(q, r : \mathbb{Q}) . q < r \rightarrow \\ &\Sigma(P : \text{DHPProp}) . (P \rightarrow q < x) \times (\neg P \rightarrow x < r), \end{aligned}$$

and then we can apply Theorems 2.15.5 and 2.15.7 in The Univalent Foundations Program (2013) to distribute the Π -types over Σ and \times . □

Remark. We emphasize that, confusingly, $\text{locatesRight}(q, r, v)$ is defined type-theoretically as $\text{isLeft}(q <_x^\ell r)$.

Definition 3.4.3. *For a real x with a locator ℓ and rationals $q < r$, we write*

$$\text{locatesRight}(q <_x^\ell r) \quad \text{or} \quad \text{locatesRight}(q <_x r)$$

for the decidable proposition $\text{locatesRight}(q, r, v)$ obtained from Lemma 3.4.2. We write

$$\text{locatesLeft}(q <_x^\ell r) \quad \text{or} \quad \text{locatesLeft}(q <_x r)$$

to be the negation of $\text{locatesRight}(q <_x r)$: so it is the proposition which is true if we locate $x < r$.

Remark. In general, if we have $q' < q < r$, then $\text{locatesRight}(q <_x r)$ does *not* imply $\text{locatesRight}(q' <_x r)$.

Lemma 3.4.4. For any real x with a locator ℓ and rationals $q < r$,

$$\neg(q < x) \Rightarrow \text{locatesLeft}(q <_x^\ell r), \quad \text{and}$$

$$\neg(x < r) \Rightarrow \text{locatesRight}(q <_x^\ell r).$$

Proof. From the defining properties of locatesRight in Lemma 3.4.2, we know

$$\text{locatesRight}(q <_x^\ell r) \Rightarrow (q < x), \quad \text{and}$$

$$\neg \text{locatesRight}(q <_x^\ell r) \Rightarrow (x < r).$$

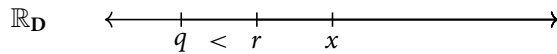
The contrapositives of these are, respectively,

$$\neg(q < x) \Rightarrow \neg \text{locatesRight}(q <_x^\ell r) \quad \text{and}$$

$$\neg(x < r) \Rightarrow \neg \neg \text{locatesRight}(q <_x^\ell r).$$

Using the fact that $\neg \neg A \Rightarrow A$ when A is decidable, this is the required result. □

Example 3.4.5. Let x be a real equipped with a locator. We can type-theoretically express that the locator must give certain answers. For example, if we have $q < r < x$, shown visually as



we must locate $q < x$, because $\neg(x < r)$. In other words, we obtain truth of the proposition $\text{locatesRight}(q <_x r)$: the property $\neg(x < r)$ yielded a property of the structure $q <_x r$.

Continuing our working example of computing a lower bound, for any $q : \mathbb{Q}$, we have the claim

$$P(q) := \text{locatesRight}(q - 1 <_x q)$$

that we locate $q - 1 < x$. This claim is a decidable proposition. And from the existence $\exists(q : \mathbb{Q}). q < x$ of a lower bound for x , we can deduce that $\exists(q : \mathbb{Q}). P(q)$, because if $q < x$ then $\neg(x < q)$ and hence the above lemma applies. If we manage to *find* a $q : \mathbb{Q}$ for which $P(q)$ holds, then we have certainly found a lower bound of x , namely $q - 1$.

3.5 Bounded search

Even though the elimination rule for propositional truncation in Definition 2.1.2 only constructs maps into propositions, we can use elements of propositional truncations to obtain witnesses of non-truncated types – in other words, we can sometimes obtain structure from property.

Theorem 3.5.1 (Escardó 2013; Escardó and Xu 2015; The Univalent Foundations Program 2013, Exercise 3.19). *Let $P : \mathbb{N} \rightarrow \text{DHProp}$. If $\exists(n : \mathbb{N}). P(n)$, then we can construct an element of $\Sigma(n : \mathbb{N}). P(n)$.*

Remark. In general, we do not have $\|X\| \rightarrow X$ for all types X , as this would imply excluded middle (Kraus et al., 2017). But for some types X , we do have $\|X\| \rightarrow X$, namely when X has a constant endomap (Kraus et al., 2017).

Even without univalence, Theorem 3.5.1 also works for any type equivalent to \mathbb{N} .

Corollary 3.5.2. *Let A be a type and $e : \mathbb{N} \simeq A$ be an equivalence, that is, a function $\mathbb{N} \rightarrow A$ with a left inverse and right inverse. Let $P : A \rightarrow \text{DHPProp}$. If $\exists(a : A) . P(a)$, then we can construct an element of $\Sigma(a : A) . P(n)$.*

Proof. Use Theorem 3.5.1 with $P'(n) := P(e(n))$. In order to show $\exists(n : \mathbb{N}) . P'(n)$, it suffices to show $(\Sigma(a : A) . P(a)) \rightarrow (\Sigma(n : \mathbb{N}) . P'(n))$, so let $a : A$ and $p : P(a)$. Then since $a = e(e^{-1}(a))$, we get $P(e(e^{-1}(a)))$ by transport.

Hence from Theorem 3.5.1, we obtain some $(n, p') : \Sigma(n : \mathbb{N}) . P'(e(n))$, so we can output $(e(n), q)$. □

3.6 Computing bounds

We are now ready to finish our running example of computing a lower bound for x .

Lemma 3.6.1. *Given a real $x : \mathbb{R}_{\mathbb{D}}$ equipped with a locator, we get bounds for x , that is, we can find $q, r : \mathbb{Q}$ with $q < x < r$.*

Proof. We pick any enumeration of \mathbb{Q} , that is, an equivalence $\mathbb{N} \simeq \mathbb{Q}$. Set

$$P(q) := \text{locatesRight } (q - 1 <_x q).$$

From Section 3.4, we know that $\exists(q : \mathbb{Q}) . P(q)$, and so we can apply Corollary 3.5.2. We obtain $\Sigma(q : \mathbb{Q}) . P(q)$, and in particular $\Sigma(q : \mathbb{Q}) . q - 1 <_x q$.

Upper bounds are constructed by a symmetric argument, using

$$P(r) := \text{locatesLeft } (r <_x r + 1). \quad \square$$

We emphasize that even though we cannot decide $q < x$ in general, we *can* decide what the locator tells us, and this is what is exploited in our development. Given a real x with a locator, the above construction of a lower bound searches for a rational q for which we locate $q - 1 <_x q$. We emphasize once more that the rational thus found is minimal in the sense that it appears first in the chosen enumeration of \mathbb{Q} , and *not* a tight bound.

Remark. The proof of Theorem 3.5.1 works by an exhaustive, but bounded, search. So our construction for Lemma 3.6.1 similarly exhaustively searches for an appropriate rational q . The efficiency of the algorithm thus obtained can be improved:

- (1) We do not need to test every rational number: it suffices to test, for example, bounds of the form $\pm 2^{k+1}$ for $k : \mathbb{N}$, as there always exists a bound of that form. Formally, such a construction is set up by enumerating a subset of the integers instead of enumerating all rationals, and showing the existence of a bound of the chosen form, followed by application of Corollary 3.5.2.
- (2) More practically, Lemma 3.6.1 shows that we may as well additionally equip bounds to reals that already have locators. Then, any later constructions that use rational bounds can simply use these equipped rational bounds. This is essentially the approach of interval arithmetic with open nondegenerate intervals. We can also see this equipping of bounds as a form of memoization, which we can apply more generally.

Lemma 3.6.2. *For a real x equipped with a locator and any positive rational ε , we can find $u, v : \mathbb{Q}$ with $u < x < v$ and $v - u < \varepsilon$.*

Proof. The construction of bounds in Lemma 3.6.1 yields $q, r : \mathbb{Q}$ with $q < x < r$. We can compute $n : \mathbb{N}$ such that $r < q + \frac{n\varepsilon}{3}$. Consider the equidistant subdivision

$$q - \frac{\varepsilon}{3}, q, q + \frac{\varepsilon}{3}, q + \frac{2\varepsilon}{3}, \dots, q + \frac{n\varepsilon}{3}, q + \frac{(n+1)\varepsilon}{3}.$$

By Lemma 3.4.4, necessarily $\text{locatesRight}(q - \frac{\epsilon}{3} <_x q)$ because $q < x$. Similarly, we have $\text{locatesLeft}(q + \frac{n\epsilon}{3} <_x q + \frac{(n+1)\epsilon}{3})$ because $x < q + \frac{n\epsilon}{3}$.

For some i , which we can find by a finite search using a one-dimensional version of Sperner's lemma, we have

$$\text{locatesRight}\left(q + \frac{i\epsilon}{3} <_x q + \frac{(i+1)\epsilon}{3}\right) \wedge \text{locatesLeft}\left(q + \frac{(i+1)\epsilon}{3} <_x q + \frac{(i+2)\epsilon}{3}\right).$$

For this i , we can output $u = q + \frac{i\epsilon}{3}$ and $v = q + \frac{(i+2)\epsilon}{3}$. □

Remark. The above result allows us to compute arbitrarily precise bounds for a real number x with a locator. But, as in Remark 3.6, the above theorem shows that we may as well equip an appropriate algorithm for computing arbitrarily precise lower and upper bounds to real numbers. This may be a better idea when efficiency of the computation matters.

3.7 Locators for algebraic operations

If x and y are reals that we can compute with in an appropriate sense, then we expect to be able to do so with $-x, x + y, x \cdot y, x^{-1}$ (assuming $x \neq 0$), $\min(x, y)$ and $\max(x, y)$ as well. In our case, that means that if x and y come equipped with locators, then so should the previously listed values.

If one works with intensional real numbers, such as when they are given as Cauchy sequences, then the algebraic operations are specified directly on the representations. This means that the computational data are automatically present. Since in our case the algebraic operations are specified extensionally, they do not give any discrete data, and so the construction of locators has to be done explicitly in order to compute.

The algebraic operations can be defined for Dedekind cuts as in The Univalent Foundations Program (2013, Section 11.2.1). Recall from Section 2.2 that for a Dedekind cut $x = (L, U)$, we write $q < x$ for the claim that $q : \mathbb{Q}$ is in the left cut L . In fact, now that we have identified $q : \mathbb{Q}$ with its canonical embedding $i(q) : \mathbb{R}_{\mathbb{D}}$ in the reals, we can simply understand $q < x$ as $i(q) <_{\mathbb{R}_{\mathbb{D}}} x$, which coincides with the notation for Dedekind cuts. In summary, we have the following relations for $x, y, z, w : \mathbb{R}_{\mathbb{D}}$ with $w < 0 < z$ and $q, r : \mathbb{Q}$:

$$\begin{aligned} q < -x &\Leftrightarrow x < -q \\ -x < r &\Leftrightarrow -r < x \\ q < x + y &\Leftrightarrow \exists(s : \mathbb{Q}). s < x \wedge (q - s) < y \\ x + y < r &\Leftrightarrow \exists(t : \mathbb{Q}). x < t \wedge y < (r - t) \\ q < xy &\Leftrightarrow \exists(a, b, c, d : \mathbb{Q}). q < \min(ac, ad, bc, bd) \\ &\quad \wedge a < x < b \wedge c < y < d \\ xy < r &\Leftrightarrow \exists(a, b, c, d : \mathbb{Q}). \max(ac, ad, bc, bd) < r \\ &\quad \wedge a < x < b \wedge c < y < d \\ q < z^{-1} &\Leftrightarrow qz < 1 \\ z^{-1} < r &\Leftrightarrow 1 < rz \\ q < w^{-1} &\Leftrightarrow 1 < qw \\ w^{-1} < r &\Leftrightarrow rw < 1 \\ q < \min(x, y) &\Leftrightarrow q < x \wedge q < y \\ \min(x, y) < r &\Leftrightarrow x < r \vee y < r \\ q < \max(x, y) &\Leftrightarrow q < x \vee q < y \\ \max(x, y) < r &\Leftrightarrow x < r \wedge y < r \end{aligned}$$

The Dedekind reals satisfy the Archimedean property, which can be succinctly stated as the claim that for all $x, y : \mathbb{R}_{\mathbb{D}}$,

$$x < y \Rightarrow \exists(q : \mathbb{Q}) . x < q < y.$$

We will use the following variation of the Archimedean property. We write \mathbb{Q}_+ for the positive rationals.

Lemma 3.7.1. *For real numbers $x < y$, there exist $q : \mathbb{Q}$ and $\varepsilon : \mathbb{Q}_+$ with $x < q - \varepsilon < q + \varepsilon < y$.*

Proof. By a first application of the Archimedean property, we know $\exists(s : \mathbb{Q}) . x < s < y$. Since we are showing a proposition, we may assume to have such an $s : \mathbb{Q}$. Now for $s < y$, by the Archimedean property, we know $\exists(t : \mathbb{Q}) . s < t < y$, and again we may assume to have such a t . Now set $q := \frac{s+t}{2}$ and $\varepsilon := \frac{t-s}{2}$. □

In particular, the above variation can be used to strengthen the \exists of the Archimedean property into Σ when the reals involved come equipped with locators. Its corollary, Corollary 3.7.3, is used to compute locators for multiplicative inverses.

Lemma 3.7.2. *For reals x and y equipped with locators, we have the Archimedean structure*

$$x < y \rightarrow \Sigma(q : \mathbb{Q}) . x < q < y.$$

Proof. Let x and y be reals equipped with locators. By Lemma 3.7.1, there exist $q : \mathbb{Q}$ and $\varepsilon : \mathbb{Q}_+$ with $x < q - \varepsilon < q + \varepsilon < y$. The following proposition is decidable for any (q', ε') , and we have $\exists((q, \varepsilon) : \mathbb{Q} \times \mathbb{Q}_+) . P(q, \varepsilon)$:

$$P(q', \varepsilon') := \text{locatesLeft}(q' - \varepsilon' <_x q') \wedge \text{locatesRight}(q' <_y q' + \varepsilon').$$

Using Corollary 3.5.2, we can find (q', ε') with $P(q', \varepsilon')$ and hence $x < q' < y$. □

Corollary 3.7.3. *For reals x and y equipped with locators, and $s : \mathbb{Q}$ a rational, if $x < y$ then we have a choice of $x < s$ or $s < y$, that is,*

$$\Pi(s : \mathbb{Q}) . x < y \rightarrow (x < s) + (s < y).$$

Proof. By Lemma 3.7.2, we can find $q : \mathbb{Q}$ with $x < q < y$. Apply trichotomy of the rationals: if $q < s$ or $q = s$, then we locate $x < s$, and if $s < q$, then we locate $s < y$. □

Remark. Instead of the rational $s : \mathbb{Q}$, we can have any real z equipped with a locator in the above corollary, so that we obtain a form of strong cotransitivity of the strict ordering relation on the real numbers, but we will not be using this.

Having developed such a strong cotransitivity, we could characterize the algebraic operations on the Dedekind reals using the Archimedean structure of Lemma 3.7.2 rather than using the Archimedean property. This would then yield a structural characterization of the algebraic operations for $x, y : \mathbb{R}_{\mathbb{D}}$ equipped with locators, along the lines of:

$$\begin{aligned} q < x + y &\Leftrightarrow \Sigma(s, t : \mathbb{Q}) . (q = s + t) \wedge s < x \wedge t < y \\ q < x \cdot y &\Leftrightarrow \Sigma(a, b, c, d : \mathbb{Q}) . q < \min(a \cdot c, a \cdot d, b \cdot c, b \cdot d) \\ &\quad \wedge a < x < b \wedge c < y < d \\ q < \max(x, y) &\Leftrightarrow q < x + q < y \\ &\quad \vdots \end{aligned}$$

Theorem 3.7.4. *If reals $x, y : \mathbb{R}_{\mathbb{D}}$ are equipped with locators, then we can also equip $-x, x + y, x \cdot y, x^{-1}$ (assuming $x \neq 0$), $\min(x, y)$ and $\max(x, y)$ with locators.*

Remark. As we define absolute values by $|x| = \max(x, -x)$, as is common in constructive analysis, if x has a locator, then so does $|x|$, and we use this fact in the proof of the above theorem.

Proof of Theorem 3.7.4. Throughout this proof, we assume x and y to be reals equipped with locators, and $q < r$ to be rationals.

We construct a locator for $-x$. We can give $(q < -x) + (-x < r)$ by considering $-r <_x -q$.

We construct a locator for $x + y$. We need to show $(q < x + y) + (x + y < r)$. Note that $q < x + y$ iff there exists $s : \mathbb{Q}$ with $q - s < x$ and $s < y$. Similarly, $x + y < r$ iff there exists $t : \mathbb{Q}$ with $x < r - t$ and $y < t$.

Set $\varepsilon := (r - q)/2$, such that $q + \varepsilon = r - \varepsilon$. By Lemma 3.6.2, we can find $u, v : \mathbb{Q}$ such that $u < x < v$ and $v - u < \varepsilon$, so in particular $x < u + \varepsilon$. Set $s := q - u$, so that $q - s < x$. Now consider $s <_y s + \varepsilon$. If we locate $s < y$, we locate $q < x + y$. If we locate $y < s + \varepsilon$, we have $x < q - s + \varepsilon = r - s - \varepsilon$, that is, we can set $t := s + \varepsilon$ to locate $x + y < r$.

We construct a locator for $\min(x, y)$. We consider both $q <_x r$ and $q <_y r$. If we locate $x < r$ or $y < r$, we can locate $\min(x, y) < r$. Otherwise, we have located both $q < x$ and $q < y$, so we can locate $q < \min(x, y)$.

The locator for $\max(x, y)$ is symmetric to the case of $\min(x, y)$.

We construct a locator for xy . We need to show $(q < xy) + (xy < r)$. Note that $q < xy$ means:

$$\exists(a, b, c, d : \mathbb{Q}) . (a < x < b) \wedge (c < y < d) \wedge (q < \min\{ac, ad, bc, bd\}).$$

Similarly, $xy < r$ means

$$\exists(a, b, c, d : \mathbb{Q}) . (a < x < b) \wedge (c < y < d) \wedge (\max\{ac, ad, bc, bd\} < r).$$

Using Lemma 3.6.2, we can find $z, w : \mathbb{Q}$ with $|x| + 1 < z$ and $|y| + 1 < w$, since we have already constructed locators for $\max, +, -$ and all rationals.

Set $\varepsilon := r - q$, $\delta := \min\{1, \frac{\varepsilon}{2z}\}$ and $\eta := \min\{1, \frac{\varepsilon}{2w}\}$. Find $a < x < b$ and $c < y < d$ such that $b - a < \eta$ and $d - c < \delta$. Note that $|a| < |x| + \eta \leq |x| + 1 < z$ and similarly $|b| < z, |c| < w$ and $|d| < w$. Then, the distance between any two elements of $\{ac, ad, bc, bd\}$ is less than ε . For instance, $|ac - bd| < \varepsilon$ because $|ac - bd| \leq |ac - ad| + |ad - bd|$, and $|ac - ad| = |a||c - d| < |a|\delta < \frac{\varepsilon}{2}$ and similarly $|ad - bd| < \frac{\varepsilon}{2}$. Hence, $\max\{ac, ad, bc, bd\} - \min\{ac, ad, bc, bd\} < \varepsilon$. Thus, by dichotomy of the rationals, one of $q < \min\{ac, ad, bc, bd\}$ and $\max\{ac, ad, bc, bd\} < r$ must be true, yielding a corresponding choice of $(q < xy) + (xy < r)$.

We construct a locator for x^{-1} . Consider the case that $x > 0$. Given $q < r$, we need $(q < x^{-1}) + (x^{-1} < r)$, or equivalently $(qx < 1) + (1 < rx)$. By the previous case, qx and rx have locators, so we can apply Corollary 3.7.3. The case $x < 0$ is similar. □

This proof works whether we use a definition of algebraic operations as in The Univalent Foundations Program (2013), or whether we work with the archimedean field axioms, because from the archimedean field axioms we deduce the same properties as the definitions.

Remark. Locators for reciprocals can also be constructed by more elementary methods, as follows. For $x > 0$, we use dichotomy of the rationals for 0 and q . If $q \leq 0$, we may locate $q < x$, and otherwise, we have $0 < 1/r < 1/q$, so that by considering $1/r <_x 1/q$ we may either locate $x < r$ or $q < x$. There is a similar construction for $x < 0$.

By using the techniques of Sections 3.4 and 3.5, we have computed locators for algebraic operations applied to reals equipped with locators.

3.8 Locators for limits

In a spirit similar to the previous section, if we have a Cauchy sequence of reals, each of which equipped with a locator, then we can compute a locator for the limit of the sequence.

Lemma 3.8.1. *Suppose $x : \mathbb{N} \rightarrow \mathbb{R}_{\mathbb{D}}$ has modulus of Cauchy convergence M , and suppose that every value in the sequence $x : \mathbb{N} \rightarrow \mathbb{R}_{\mathbb{D}}$ comes equipped with a locator, that is, suppose we have an element of $\Pi(n : \mathbb{N}) . \text{locator}(x_n)$. Then, we have a locator for $\lim_{n \rightarrow \infty} x_n$.*

Proof. Let $q < r$ be arbitrary rationals. We need $(q < \lim_{n \rightarrow \infty} x_n) + (\lim_{n \rightarrow \infty} x_n < r)$. Set $\varepsilon := \frac{r-q}{3}$ so that $q + \varepsilon < r - \varepsilon$. Since M is a modulus of Cauchy convergence, we have $|x_{M(\varepsilon/2)} - \lim_{n \rightarrow \infty} x_n| < \varepsilon$, that is,

$$x_{M(\varepsilon/2)} - \varepsilon < \lim_{n \rightarrow \infty} x_n < x_{M(\varepsilon/2)} + \varepsilon.$$

We consider the locator equipped to $x_{M(\varepsilon/2)}$ and do case analysis on $q + \varepsilon < x_{M(\varepsilon/2)} < r - \varepsilon$. If we locate $q + \varepsilon < x_{M(\varepsilon/2)}$, then we can locate $q < \lim_{n \rightarrow \infty} x_n$. If we locate $x_{M(\varepsilon/2)} < r - \varepsilon$, then we can locate $\lim_{n \rightarrow \infty} x_n < r$. □

Remark. We emphasize that Lemma 3.8.1 requires the sequence to be *equipped* with a modulus of Cauchy convergence, whereas existence suffices for the computation of the limit $\lim_{n \rightarrow \infty} x_n$ itself, namely the element of \mathbb{R}_D .

Example 3.8.2 (Locators for exponentials). Given a locator for x , we can use Lemma 3.6.1 to obtain a modulus of Cauchy convergence of $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Hence, $\exp(x)$ has a locator.

Example 3.8.3. Many constants such as π and e have locators, which can be found by examining their construction as limits of sequences.

We can now construct locators for limits of sequences whose elements have locators, and so using Lemma 3.3.1, in particular, limits for sequences of rationals. As we will make precise in Theorem 3.9.7, this covers all the cases.

3.9 Calculating digits

Example 3.9.1. We would like to print digits for numbers equipped with locators, such as π . Such a digit expansion gives rise to rational bounds of the number in question: if a digit expansion of π starts with $3.1\dots$, then we have the bounds $3.0 < \pi < 3.3$.

We now wish to generate the entire sequence of digits of a real number x equipped with a locator. As in computable analysis and other settings where one works intensionally, with reals given as Cauchy sequences or streams of digits, we wish to extract digit representations from a real equipped with a locator.

In fact, various authors including Brouwer (1921) and Turing (1937) encountered problems with computing decimal expansions of real numbers in their work. As is common in constructive analysis, we instead consider *signed-digit* representations. Wiedmer shows how to calculate directly on the signed-digit representations in terms of computability theory (Wiedmer, 1977).

Definition 3.9.2. A *signed-digit representation* for $x : \mathbb{R}_D$ is given by $k : \mathbb{Z}$ and a sequence a of signed digits $a_i \in \{ \bar{9}, \bar{8}, \dots, \bar{1}, 0, 1, \dots, 9 \}$, with $\bar{a} := -a$, such that

$$x = k + \sum_{i=0}^{\infty} a_i \cdot 10^{-i-1}.$$

Example 3.9.3. The number π may be given by a signed-decimal expansion as $3.1415\dots$, or as $4.\bar{8}\bar{6}15\dots$, or as $3.2\bar{5}\bar{8}\bar{5}\dots$

Lemma 3.9.4. For any x equipped with a locator, we can find $k : \mathbb{Z}$ such that $x \in (k - 1, k + 1)$.

Proof. Use Lemma 3.6.2 with $\varepsilon = 1$ to obtain rationals $u < v$ with $u < x < v$ and $v < 1 + u$. Set $k = \lfloor u \rfloor + 1$. Then,

$$k - 1 = \lfloor u \rfloor \leq u < x < v < u + 1 < k + 1. \quad \square$$

Theorem 3.9.5. For a real number x , locators and signed-digit representations are interdefinable.

Proof. If a real number has a signed-digit representation, then it is the limit of a sequence of rational numbers, and so by Lemma 3.8.1 it has a locator.

Conversely, assume a real x has a locator. By Lemma 3.9.4, we get $k : \mathbb{Z}$ with $x \in (k - 1, k + 1)$. Consider the equidistant subdivision

$$k - 1 < k - \frac{9}{10} < \dots < k - \frac{1}{10} < k < k + \frac{1}{10} < \dots < k + 1.$$

By applying the locator several times, we can find a signed-digit a_0 such that

$$k + \frac{a_0 - 1}{10} < x < k + \frac{a_0 + 1}{10}.$$

We find subsequent digits in a similar way. □

Note that since \mathbb{R}_D is Cauchy complete, there is a canonical inclusion $\mathbb{R}_C \rightarrow \mathbb{R}_D$ from the Cauchy reals into \mathbb{R}_D .

Definition 3.9.6. We write $\text{isCauchyReal}(x)$ for the claim that a given real $x : \mathbb{R}_D$ is in the image of the canonical inclusion of the Cauchy reals into \mathbb{R}_D . Equivalently, $\text{isCauchyReal}(x)$ holds when there is a rational Cauchy sequence with limit x .

We emphasize that $\|\text{locator}(x)\|$ is *not* equivalent to the locatedness property of Definition 2.2.2.

Theorem 3.9.7. The following are equivalent for $x : \mathbb{R}_D$:

1. $\|\text{locator}(x)\|$, that is, there exists a locator for x .
2. There exists a signed-digit representation of x .
3. There exists a Cauchy sequence of rationals that x is the limit of.
4. $\text{isCauchyReal}(x)$.

Proof. Items 1 and 2 are equivalent by Theorem 3.9.5. Item 2 implies item 3 since a signed-digit representation gives rise to a sequence with a modulus of Cauchy convergence. Item 3 implies item 1 because a sequence of rational numbers with modulus of Cauchy convergence has a locator by Lemma 3.8.1. Equivalence of items 3 and 4 is a standard result. □

Remark. The notion of locator can be truncated into a proposition in three ways:

$$\|\Pi(q, r : \mathbb{Q}). q < r \rightarrow (q < x) + (x < r)\| \tag{1}$$

$$\Pi(q, r : \mathbb{Q}). \|q < r \rightarrow (q < x) + (x < r)\| \tag{2}$$

$$\Pi(q, r : \mathbb{Q}). q < r \rightarrow \|(q < x) + (x < r)\|. \tag{3}$$

Now (1) is $\|\text{locator}(x)\|$, and (3) is the locatedness property of Definition 2.2.2, which holds for all $x : \mathbb{R}_D$ as mentioned in Section 3.1. In summary, we have

$$(1) \implies (2) \iff (3),$$

where the implications to the right can be shown using the induction rule for propositional truncations, and the implication to the left follows from the fact that $q < r$ is a decidable proposition for $q, r : \mathbb{Q}$.

In other words, we cannot expect to be able to equip every real with a locator, as this would certainly imply that the Cauchy reals and the Dedekind reals coincide, which is not true in general (Lubarsky, 2007).

Corollary 3.9.8. *The following are equivalent:*

1. *For every Dedekind real, there exists a signed-digit representation of it.*
2. *The Cauchy reals and the Dedekind reals coincide.*

The types \mathbb{R}_C and \mathbb{R}_D do not coincide in general, but they do assuming excluded middle or countable choice. We are not aware of a classical principle that is equivalent with the coincidence of \mathbb{R}_C and \mathbb{R}_D .

3.10 Dedekind cuts structure

Let $x = (L, U)$ be a pair of predicates on the rationals, that is, $L, U : \mathcal{P}\mathbb{Q}$. In Definition 2.2.2, we specified the necessary *properties* for x to be a Dedekind cut. More explicitly, we have $\text{isDedekindCut} : \mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q} \rightarrow \text{HProp}$ defined by:

$$\begin{aligned} \text{isDedekindCut}(x) := & \text{boundedLower}(x) \wedge \text{boundedUpper}(x) \\ & \wedge \text{closedLower}(x) \wedge \text{closedUpper}(x) \\ & \wedge \text{openLower}(x) \wedge \text{openUpper}(x) \\ & \wedge \text{transitive}(x) \wedge \text{located}(x), \end{aligned}$$

where

$$\begin{aligned} \text{boundedLower}(x) &:= \exists(q : \mathbb{Q}). q < x, \\ \text{boundedUpper}(x) &:= \exists(r : \mathbb{Q}). x < r, \\ \text{closedLower}(x) &:= \forall(q, q' : \mathbb{Q}). (q < q') \wedge (q' < x) \Rightarrow q < x, \\ \text{closedUpper}(x) &:= \forall(r, r' : \mathbb{Q}). (r' < r) \wedge (x < r') \Rightarrow x < r, \\ \text{openLower}(x) &:= \forall(q : \mathbb{Q}). q < x \Rightarrow \exists(q' : \mathbb{Q}). (q < q') \wedge (q' < x), \\ \text{openUpper}(x) &:= \forall(r : \mathbb{Q}). x < r \Rightarrow \exists(r' : \mathbb{Q}). (r' < r) \wedge (x < r'), \\ \text{transitive}(x) &:= \forall(q, r : \mathbb{Q}). (q < x) \wedge (x < r) \Rightarrow (q < r), \\ \text{located}(x) &:= \forall(q, r : \mathbb{Q}). (q < r) \Rightarrow (q < x) \vee (x < r). \end{aligned}$$

We may also consider when x has these data as *structure*, that is, when it is equipped with the structure $\text{isDedekindCut}^\S : \mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q} \rightarrow \mathcal{U}$ defined by:

$$\begin{aligned} \text{isDedekindCut}^\S(x) := & \text{boundedLower}^\S(x) \times \text{boundedUpper}^\S(x) \\ & \times \text{closedLower}^\S(x) \times \text{closedUpper}^\S(x) \\ & \times \text{openLower}^\S(x) \times \text{openUpper}^\S(x) \\ & \times \text{transitive}^\S(x) \times \text{located}^\S(x), \end{aligned}$$

where

$$\begin{aligned} \text{boundedLower}^\S(x) &:= \Sigma(q : \mathbb{Q}). q < x, \\ \text{boundedUpper}^\S(x) &:= \Sigma(r : \mathbb{Q}). x < r, \\ \text{closedLower}^\S(x) &:= \Pi(q, q' : \mathbb{Q}). (q < q') \times (q' < x) \rightarrow q < x, \\ \text{closedUpper}^\S(x) &:= \Pi(r, r' : \mathbb{Q}). (r' < r) \times (x < r') \rightarrow x < r, \end{aligned}$$

$$\begin{aligned} \text{openLower}^{\S}(x) &:= \Pi(q : \mathbb{Q}) . q < x \rightarrow \Sigma(q' : \mathbb{Q}) . (q < q') \times (q' < x), \\ \text{openUpper}^{\S}(x) &:= \Pi(r : \mathbb{Q}) . x < r \rightarrow \Sigma(r' : \mathbb{Q}) . (r' < r) \times (x < r'), \\ \text{transitive}^{\S}(x) &:= \Pi(q, r : \mathbb{Q}) . (q < x) \times (x < r) \rightarrow (q < r), \\ \text{located}^{\S}(x) &:= \Pi(q, r : \mathbb{Q}) . (q < r) \rightarrow (q < x) + (x < r) = \text{locator}(x). \end{aligned}$$

In this section, we investigate when $x = (L, U)$ has the property $\text{isDedekindCut}(x)$, and when it has the data $\text{isDedekindCut}^{\S}(x)$. First, note that we cannot expect all Dedekind cuts to come equipped with that data.

Lemma 3.10.1. *Suppose given a choice $\Pi(x : \mathbb{R}_{\mathbf{D}}) . \text{locator}(x)$ of locator for each $x : \mathbb{R}_{\mathbf{D}}$. Then, we can define a strongly non-constant function $f : \mathbb{R}_{\mathbf{D}} \rightarrow \mathbf{2}$ in the sense that there exist reals $x, y : \mathbb{R}_{\mathbf{D}}$ with $f(x) \neq f(y)$.*

Proof. Given a locator for $x : \mathbb{R}_{\mathbf{D}}$, we can output true or false depending on whether the locator return the left or the right summand for $0 < 1$, as follows.

$$f(x) = \begin{cases} \text{true} & \text{if } \text{locatesRight}(0 <_x 1) \\ \text{false} & \text{if } \text{locatesLeft}(0 <_x 1). \end{cases}$$

The map thus constructed must give a different answer for the real numbers 0 and 1. □

Since any strongly non-constant map from the reals to the Booleans gives rise to a discontinuous map on the reals, we have violated the continuity principle that every map on the reals is continuous. Following Ishihara (2001), we can derive weak limited principle of omniscience (WLPO) from it.

Definition 3.10.2. *The weak limited principle of omniscience is the following consequence of PEM: for every decidable predicate $P : \mathbb{N} \rightarrow \text{DHProp}$ on the naturals, we can decide $\neg \exists (n : \mathbb{N}) . P(n)$:*

$$\text{WLPO} := \Pi(P : \mathbb{N} \rightarrow \text{DHProp}) . \neg(\exists(n : \mathbb{N}) . P(n)) + \neg\neg(\exists(n : \mathbb{N}) . P(n)).$$

Note that this is a proposition in the sense of Definition 2.1.1 because if Q is one then $Q + \neg Q$ is one.

Lemma 3.10.3. *If there exists a strongly non-constant function $\mathbb{R}_{\mathbf{D}} \rightarrow \mathbf{2}$, then WLPO holds.*

Proof. Since WLPO is a proposition, we may assume to have $f : \mathbb{R}_{\mathbf{D}} \rightarrow \mathbf{2}$ and $x, y : \mathbb{R}_{\mathbf{D}}$ with $f(x) \neq f(y)$. Let $P : \mathbb{N} \rightarrow \text{DHProp}$ be a decidable predicate.

We start by setting up a decision procedure. We define two sequences $a, b : \mathbb{N} \rightarrow \mathbb{R}_{\mathbf{D}}$ with $f(a_i) = \text{ff}$ and $f(b_i) = \text{tt}$ for each i , and so that a and b converge to the same real l .

Without loss of generality, assume $f(x) = \text{ff}$ and $f(y) = \text{tt}$, and set

$$\begin{aligned} a_0 &:= x & b_0 &:= y \\ a_{n+1} &:= \begin{cases} \frac{a_n + b_n}{2} & \text{if } f\left(\frac{a_n + b_n}{2}\right) = \text{ff} \\ a_n & \text{otherwise} \end{cases} & b_{n+1} &:= \begin{cases} \frac{a_n + b_n}{2} & \text{if } f\left(\frac{a_n + b_n}{2}\right) = \text{tt} \\ b_n & \text{otherwise.} \end{cases} \end{aligned}$$

In words, with a_n and b_n defined, we decide the next point by considering f evaluated at the midpoint $\frac{a_n + b_n}{2}$, and correspondingly updating one of the points. The sequences converge to the same point l . Without loss of generality, we have $f(a_n) = f(l) = \text{ff}$ and $f(b_n) = \text{tt}$ for all $n : \mathbb{N}$.

We may now decide $\neg \exists (n : \mathbb{N}) . P(n)$. We first define a sequence $c : \mathbb{N} \rightarrow \mathbb{R}_{\mathbf{D}}$ as follows. For a given $n : \mathbb{N}$, we decide if there is any $i < n$ for which $P(i)$ holds, and if so, we set $c_n = b_i$ for the least such i . Otherwise, we set $c_n = l$.

The sequence c converges, giving a limit $m : \mathbb{R}_{\mathbf{D}}$. Consider $f(m)$.

If $f(m) = \text{ff}$, then $\neg \exists (n : \mathbb{N}) . P(n)$, since if there did exist n with $P(n)$, then $m = b_i$ for some $i \leq n$, so that $f(m) = f(b_i) = \text{tt}$.

If $f(m) = \text{tt}$, then $\neg \exists (n : \mathbb{N}) . P(n)$, since if $\forall (n : \mathbb{N}) . \neg P(n)$, then $m = l$ and so $f(m) = \text{ff}$. \square

The following key theorem explains the relationships between being a Dedekind cut, having the Dedekind data $\text{isDedekindCut}^\S(x)$, and equipping a real with a locator.

Theorem 3.10.4. *For a pair $x = (L, U)$ of predicates on the rationals, we have the following:*

1. $\text{isDedekindCut}^\S(x) \rightarrow \text{isDedekindCut}(x)$,
2. $\|\text{isDedekindCut}^\S(x)\| \Rightarrow \text{isDedekindCut}(x)$,
3. $\text{isDedekindCut}(x) \times \text{locator}(x) \rightarrow \text{isDedekindCut}^\S(x)$,
4. $\text{isDedekindCut}(x) \times \|\text{locator}(x)\| \Rightarrow \text{isCauchyReal}(x)$, and
5. $\|\text{isDedekindCut}^\S(x)\| \Rightarrow \text{isCauchyReal}(x)$.

The third item tells us that for a given Dedekind real x , in order to obtain the structures that make up $\text{isDedekindCut}^\S(x)$, we only require $\text{locator}(x)$.

Proof. We show the first item by considering all property/structure pairs above.

$\text{boundedLower}^\S(x) \rightarrow \text{boundedLower}(x)$ follows by applying the truncation map $|\cdot|$ of Definition 2.1.2, and similarly for boundedUpper .

$\text{closedLower}^\S(x) \rightarrow \text{closedLower}(x)$ is trivial since, following Definition 2.1.3, their definitions work out to the same thing: we do not need to make any changes to make closedLower^\S structural.

$\text{openLower}^\S(x) \rightarrow \text{openLower}(x)$ by a pointwise truncation: let $q : \mathbb{Q}$ be arbitrary and assume $q < x$, then we get $\Sigma(q' : \mathbb{Q}) . (q < q') \times (q' < x)$, and hence $\exists(q' : \mathbb{Q}) . (q < q') \wedge (q' < x)$.

Again following Definition 2.1.3, $\text{transitive}(x)$ and $\text{transitive}^\S(x)$ are defined equally.

$\text{locator}(x) \rightarrow \text{located}(x)$ again by a pointwise truncation.

The second item follows using the elimination rule for propositional truncations since $\text{isDedekindCut}(x)$ is a proposition.

For the third item, it remains to construct bounds, and to construct $\text{openLower}^\S(x)$ and $\text{openUpper}^\S(x)$. The former is Lemma 3.6.1. The latter follows from the Archimedean structure of Lemma 3.7.2 and the fact that we have locators for rationals, as in Lemma 3.3.1.

The fourth item follows from Theorem 3.9.7.

The fifth item follows by combining the second and the fourth. \square

Theorem 3.10.5. *For an arbitrary pair $x = (L, U)$ of predicates on the rationals, it is not provable that $\text{isDedekindCut}(x)$ implies $\|\text{isDedekindCut}^\S(x)\|$.*

Proof. By Theorem 3.10.4, $\|\text{isDedekindCut}^\S(x)\|$ implies that x is a Cauchy real. However, in general, the Cauchy reals and the Dedekind reals do not coincide (Lubarsky, 2007). \square

4. Some Constructive Analysis with Locators

We show some ways of using locators in an existing theory of constructive analysis. We re-emphasize that although the technique of equipping numbers with locators can be applied to any archimedean ordered field, for clarity and brevity we will work with the Dedekind reals $\mathbb{R}_\mathbb{D}$, with more general description given in Boij (2020b).

The central notion is that of functions on the reals that *lift to locators*, discussed in Section 4.1, which is neither weaker nor stronger than continuity. We compute locators for integrals in Section 4.2. We discuss how locators can help computing roots of functions in Section 4.3.

4.1 Preliminaries

What are the functions on the reals that allow us to compute? When such a function $f : \mathbb{R}_\mathbb{D} \rightarrow \mathbb{R}_\mathbb{D}$ is applied to an input real number $x : \mathbb{R}_\mathbb{D}$ that we can compute with, then we should be able

to compute with the output $f(x)$. This can be formalized in terms of locators in the following straightforward way, which we use as an abstract notion of computation.

Definition 4.1.1. A function $f : \mathbb{R}_{\mathbb{D}} \rightarrow \mathbb{R}_{\mathbb{D}}$ lifts to locators if it comes equipped with a method for constructing a locator for $f(x)$ from a locator for x . This means that f lifts to locators if it is equipped with an element of the type

$$\Pi(x : \mathbb{R}_{\mathbb{D}}) . \text{locator}(x) \rightarrow \text{locator}(f(x)).$$

Another way to say this is that f lifts to locators iff we can find the top edge in the diagram

$$\begin{array}{ccc} \mathbb{R}_{\mathbb{D}}^{\Sigma} & \longrightarrow & \mathbb{R}_{\mathbb{D}}^{\Sigma} \\ \downarrow \text{pr}_1 & \circ & \downarrow \text{pr}_1 \\ \mathbb{R}_{\mathbb{D}} & \xrightarrow{f} & \mathbb{R}_{\mathbb{D}} \end{array}$$

where $\mathbb{R}_{\mathbb{D}}^{\Sigma} := \Sigma(x : \mathbb{R}_{\mathbb{D}}) . \text{locator}(x)$ is the type of real numbers equipped with locators. “Lifting to locators” itself is structure.

Remark. If the reals are defined intensionally, for example, as the collection of all Cauchy sequences without quotienting, then every function on them is defined completely by its behavior on those intensional reals. However, in our case, given only the lifting structure $\mathbb{R}_{\mathbb{D}}^{\Sigma} \rightarrow \mathbb{R}_{\mathbb{D}}^{\Sigma}$, we cannot recover the function $f : \mathbb{R}_{\mathbb{D}} \rightarrow \mathbb{R}_{\mathbb{D}}$, because we do not have a locator for every $x : \mathbb{R}_{\mathbb{D}}$.

In other words, well-behaved maps are specified by two pieces of data, namely a function $f : \mathbb{R}_{\mathbb{D}} \rightarrow \mathbb{R}_{\mathbb{D}}$ representing the extensional value of the function, and a map $\mathbb{R}_{\mathbb{D}}^{\Sigma} \rightarrow \mathbb{R}_{\mathbb{D}}^{\Sigma}$ that tells us how to compute.

Example 4.1.2. The exponential function $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ of Examples 2.2.5 and 3.8.2 lifts to locators, for example, using our construction of locators for limits as in Lemma 3.8.1.

In order to start developing analysis, we define some notions of continuity.

Definition 4.1.3. A function $f : \mathbb{R}_{\mathbb{D}} \rightarrow \mathbb{R}_{\mathbb{D}}$ is continuous at $x : \mathbb{R}_{\mathbb{D}}$ if

$$\forall(\varepsilon : \mathbb{Q}_+) . \exists(\delta : \mathbb{Q}_+) . \forall(y : \mathbb{R}_{\mathbb{D}}) . |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

f is pointwise continuous if it is continuous at all $x : \mathbb{R}_{\mathbb{D}}$.

Definition 4.1.4. A modulus of uniform continuity for f on $[a, b]$, with $a, b : \mathbb{R}_{\mathbb{D}}$, is a map $\omega : \mathbb{Q}_+ \rightarrow \mathbb{Q}_+$ with:

$$\forall(x, y \in [a, b]) . |x - y| < \omega(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Example 4.1.5 (Continuity of \exp). For any a, b , there exists a modulus of uniform continuity for \exp on the range $[a, b]$. If a and b have locators, then we can find a modulus of uniform continuity for \exp on that interval.

From a constructive viewpoint in which computation and continuity align, it would be desirable if some form of continuity of $f : \mathbb{R}_{\mathbb{D}} \rightarrow \mathbb{R}_{\mathbb{D}}$ would imply that it lifts to locators. Alas, this is not the case, not even for constant functions.

Lemma 4.1.6. If it holds that all constant functions lift to locators, then every $x : \mathbb{R}_{\mathbb{D}}$ comes equipped with a locator.

Using Lemmas 3.10.1 and 3.10.3, this then yields the constructive taboo WLPO.

Proof. For $x : \mathbb{R}_D$, let $f : \mathbb{R}_D \rightarrow \mathbb{R}_D$ be the constant map at x , and note that f is continuous, so that by assumption it lifts to locators. Since the rational number 0 has a locator, $f(0) = x$ has a locator. \square

The converse direction that lifting to locators would imply continuity also fails dramatically.

Lemma 4.1.7. *Assuming PEM, we can define a discontinuous map $f : \mathbb{R}_D \rightarrow \mathbb{R}_D$ that lifts to locators.*

Proof. We can use PEM to define a discontinuous function, which automatically lifts to locators by applying Lemma 3.1.4. \square

It may be the case that the structure of lifting to locators can be used to strengthen certain *properties* of continuity into *structures*. For example, does every function that lifts to locators and is pointwise continuous come equipped with the structure

$$\Pi(x : \mathbb{R}_D) . \Pi(\varepsilon : \mathbb{Q}_+) . \Sigma(\delta : \mathbb{Q}_+) . \forall(y : \mathbb{R}_D) . |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

of structural pointwise continuity at every $x : \mathbb{R}_D$? We leave this as an open question.

For the above reasons, the theorems in this section and the next assume continuity *and* a structure of lifting to locators: the former to make the constructive analysis work, and the latter to compute.

4.2 Integrals

We can compute definite integrals of uniformly continuous functions in the following way.

Theorem 4.2.1. *Suppose $f : \mathbb{R}_D \rightarrow \mathbb{R}_D$ has a modulus of uniform continuity on $[a, b]$, and a and b are real numbers with locators. Suppose that f lifts to locators. Then $\int_a^b f(x) dx$ has a locator.*

Proof. For uniformly continuous functions, the integral $\int_a^b f(x) dx$ can be computed as the limit

$$\lim_{n \rightarrow \infty} \frac{b - a}{n} \sum_{k=0}^{n-1} f\left(a + k \cdot \frac{b - a}{n}\right).$$

Now every value

$$\frac{b - a}{n} \sum_{k=0}^{n-1} f\left(a + k \cdot \frac{b - a}{n}\right),$$

in the sequence comes equipped with a locator using Lemmas 3.3.1 and 3.7.4, and using the fact that a and b have locators and f lifts to locators. From the modulus of uniform continuity of f , and the computation of a rational B with $b - a \leq B$ using Lemmas 3.7.4 and 3.6.1, we can compute a modulus of Cauchy convergence of the sequence. Hence, the limit has a locator using Lemma 3.8.1. \square

Combining this with the calculation of signed-digit representations of reals with locators in Theorem 3.9.5, the above means we can generate the digit sequence of certain integrals. Through the construction of close bounds in Lemma 3.6.2, we can in principle verify the value of integrals up to arbitrary precision.

Remark. Integrals, as elements of \mathbb{R}_D , can be defined given only the *existence* of a modulus of uniform continuity. To get a locator, we use the modulus of uniform continuity to find a modulus of Cauchy convergence.

Example 4.2.2. The integral $\int_0^8 \sin(x + \exp(x)) dx$ has a locator (where \sin is defined, and shown to lift to locators, in a way similar to \exp). This integral is often incorrectly approximated by computer algebra systems. Mahboubi et al. (2016, Section 6.1) have formally verified approximations of this integral, and in principle, our work gives an alternative method to do so. However, our constructions are not efficient enough to do so in practice, and we give some possible remedies in the conclusions in Section 5.

4.3 Intermediate value theorems

We may often compute locators of real numbers simply by analysing the proof of existing theorems in constructive analysis. The following construction of the root of a function is an example of us being able to construct locators simply by following the proof in the literature.

Theorem 4.3.1. *Suppose f is pointwise continuous on the interval $[a, b]$ and $f(a) < 0 < f(b)$ with $a, b : \mathbb{R}_D$. Then for every $\varepsilon : \mathbb{Q}_+$ we can find $x : \mathbb{R}_D$ with $|f(x)| < \varepsilon$. If f lifts to locators, and a and b are equipped with locators, then x is equipped with a locator.*

Proof. The first claim is shown as in Frank (2017) by defining sequences $c, d, z, w : \mathbb{N} \rightarrow \mathbb{R}_D$:

$$\begin{aligned} z_0 &= a & c_n &= (z_n + w_n)/2 & z_{n+1} &= c_n - d_n(b - a)/2^{n+1} \\ w_0 &= b & d_n &= \max\left(0, \min\left(\frac{1}{2} + \frac{f(c_n)}{\varepsilon}, 1\right)\right) & w_{n+1} &= w_n - d_n(b - a)/2^{n+1} \end{aligned}$$

with x defined as the limit of $c : \mathbb{N} \rightarrow \mathbb{R}_D$, which converges since $z, w : \mathbb{N} \rightarrow \mathbb{R}_D$ are monotone sequences with $z_n \leq c_n \leq w_n$ and $z_n - w_n = (b - a)/2^n$. Because f lifts to locators, and a and b have a locator, all c_n have locators. For a modulus of Cauchy convergence, Lemma 3.7.4 gives a locator for $b - a$ so that we can use Lemma 3.6.1 to compute a rational B with $|z_n - w_n| \leq B/2^n$. So by Lemma 3.8.1, x has a locator. □

We will now work toward an intermediate value theorem in which the locators help us with the computation of the root itself, avoiding any choice principles. We stated this intermediate value theorem and its proof informally in the introduction to Section 3.

Definition 4.3.2. *A function $f : \mathbb{R}_D \rightarrow \mathbb{R}_D$ is locally nonconstant if for all $x < y$ and $t : \mathbb{R}_D$, there exists $z : \mathbb{R}_D$ with $x < z < y$ and $f(z) \# t$, recalling that $(f(z) \# t) = (f(z) > t) \vee (f(z) < t)$.*

Example 4.3.3. Every strictly monotone function is locally nonconstant, but not every locally nonconstant function is strictly monotone.

Lemma 4.3.4. *Suppose f is a pointwise continuous function, and x, y , and t are real numbers with locators with $x < y$. Further suppose that f is locally nonconstant, and lifts to locators. Then, we can find $r : \mathbb{Q}$ with $x < r < y$ and $f(r) \# t$.*

Proof. Since f is locally nonconstant, there exist $z : \mathbb{R}_D$ and $\varepsilon : \mathbb{Q}_+$ with $|f(z) - t| > \varepsilon$. Since f is continuous at z , there exists $q : \mathbb{Q}$ with $|f(q) - t| > \varepsilon/2$. Since \mathbb{Q}_+ and \mathbb{Q} are denumerable, we can find $r : \mathbb{Q}$ such that there exists $\eta : \mathbb{Q}_+$ with $|f(r) - t| > \eta$. In particular, r satisfies $|f(r) - t| > 0$, that is, $f(r) \# t$. □

The above result can be thought of as saying that if f is a pointwise continuous function that lifts to locators, then the *property* of local nonconstancy implies a certain *structure* of local nonconstancy: for given reals with locators $x < y$ and t , we do not just get the existence of a real z , but we can explicitly choose a point z where f is apart from t .

Exact intermediate value theorems based on local nonconstancy usually assume dependent choice, see, for example, Bridges and Richman (1987, Chapter 3, Theorem 2.5) or Troelstra and

van Dalen (1988, Chapter 6, Theorem 1.5). The following result holds in the absence of such choice principles. It can perhaps be compared to developments in which the real numbers are represented directly as Cauchy sequences (Hendtlass, 2012; Schuster and Schwichtenberg, 2004; Schwichtenberg, 2017) or with Taylor (2010). Note, however, that

1. we assume local nonconstancy rather than monotonicity and
2. we use the *property* of local nonconstancy to compute roots, rather than assuming this as structure.

Theorem 4.3.5. *Suppose f is a pointwise continuous function, and $a < b$ are real numbers with locators. Further suppose that f is locally nonconstant, and lifts to locators, with $f(a) \leq 0 \leq f(b)$. Then, we can find a root of f , which comes equipped with a locator.*

Proof. We define sequences $a, b : \mathbb{N} \rightarrow \mathbb{R}_D$ with $a_n < a_{n+1} < b_{n+1} < b_n$, with $f(a_n) \leq 0 \leq f(b_n)$, with $b_n - a_n \leq (b - a) \left(\frac{2}{3}\right)^n$, and such that all a_n and b_n have locators. Set $a_0 = a, b_0 = b$. Suppose a_n and b_n are defined, and use Lemma 4.3.4 to find q_n with $\frac{2a_n + b_n}{3} < q_n < \frac{a_n + 2b_n}{3}$ and $f(q_n) \neq 0$.

- If $f(q_n) > 0$, then set $a_{n+1} := a_n$ and $b_{n+1} := q_n$.
- If $f(q_n) < 0$, then set $a_{n+1} := q_n$ and $b_{n+1} := b_n$.

For a modulus of Cauchy convergence, we can compute a locator for $b - a$ and from this we can compute a rational B with $|b_n - a_n| \leq B \left(\frac{2}{3}\right)^n$. The sequences converge to a number x . For any ε , we have $|f(x)| \leq \varepsilon$, hence $f(x) = 0$. □

Remark. Since we only appealed to Lemma 4.3.4 with $t = 0$, that is, since we were only interested in points where f is apart from 0, Theorem 4.3.5 may be strengthened by only requiring that f is locally nonzero.

Theorem 4.3.5 is an improvement on existing exact intermediate value theorems (Schuster and Schwichtenberg, 2004; Taylor, 2010) since it assumes the *property* of local nonconstancy to compute roots.

Example 4.3.6. The function \exp is strictly increasing, and hence locally nonconstant. So if $y > 0$ has a locator, then $\exp(x) = y$ has a solution x with a locator.

5. Closing Remarks

We have paid attention to the difference between property and structure while defining the real numbers and other foundations of constructive analysis. We have introduced the term *locator* to mean the structure that is the focus of this paper and have introduced a basic theory of locators. The fact that the results about locators have equivalents in terms of intensional representations of reals suggests that we are not doing anything new. This is desirable: we merely introduced a particular representation that seems suitable for computation. The presence of the locators is not to make the constructive analysis work; rather, it is to make the computation work. In this sense, we have made the computation work without a conceptual burden of intensional representations.

The constructions and results remind of computable analysis. But our development is orthogonal to computability: even reals that are not computable in some semantics can have locators, for example, in the presence of choice axioms, in which case all reals have locators.

Locators allow to observe information of real numbers, such as signed-digit expansions. We have shown the interdefinability of locators with Cauchy sequences, and in this way, we characterized the Cauchy reals as those Dedekind reals for which a locator exists.

The new notion of *lifting to locators* grew out of a naive desire to have locators for the output of a function whenever we have a locator for the input. We have left the following open question: given that $f : \mathbb{R}_D \rightarrow \mathbb{R}_D$ lifts to locators, do we obtain a certain *structure* of continuity from a *property* of continuity?

We have not spent much time finding an alternative notion of “functions that compute” with a closer relationship to continuity, and this could be the topic of further research. Such a notion could perhaps allow for more satisfying formulations of the theorems in Sections 4.2 and 4.3.

Our work allows to obtain signed-digit representations of integrals. These results are based on backwards error propagation, essentially due to our notion of lifting to locators. The advantage of this is that we are guaranteed to be able to find results. However, forward error propagation, as in Mahboubi et al. (2016), may be more efficient. It may be possible to combine the naturalness of locators with forward error propagation by equipping the real numbers involved with bounds as in the remark below Lemma 3.6.1. Having shown that we can compute arbitrarily precise approximations to reals with locators in Lemma 3.6.2, we may as well equip real numbers with an efficient method for doing so. Thus, in future work, some of the techniques of previous work on verified computation with exact reals may be developed in our setting as well.

Another possible future direction is to find a more general notion of locator that applies to more general spaces, such as the complex plane, function spaces, or metric spaces. This could then be a framework for observing information of differential equations, which are also discussed in a more general description of locators (Booij, 2020b).

The work lends itself to being formalized in proof assistants such as Agda or Coq. In this way, we can automatically obtain algorithms from proofs. Part of the work has indeed been formalized in Coq (Booij, 2020a). Results in the work above correspond with the formalized proofs in theories/Analysis/Locator.v as follows: Lemma 3.1.4 as `all_reals_locators`, Lemma 3.3.1 as `locator_left` and `locator_right`, Lemma 3.4.2 as `equiv_locator_locator'`, Lemma 3.4.4 as `nltqx_locates_left` and `nltxr_locates_right`, Lemma 3.6.1 as `lower_bound` and `upper_bound`, Lemma 3.6.2 as `tight_bound`, Lemma 3.7.2 as `archimedean_structure`, and the majority of Theorem 3.7.4, as well as Lemma 3.8.1, as the terms starting with `locator_`. This development has been merged into the HoTT library (Bauer et al., 2017). But we may worry that the proofs we provided are not sufficiently efficient for useful calculations, and we intend to address this important issue in future work.

Acknowledgments

We would like to thank the reviewers for their invaluable feedback. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 731143.

References

- Bauer, A., Gross, J., Lumsdaine, P. L., Shulman, M., Sozeau, M. and Spitters, B. (2017). The HoTT library: a formalization of homotopy type theory in Coq. In: *Proceedings of the 6th ACM SIGPLAN Conference on Certified Programs and Proofs, CPP 2017, Paris, France, January 16–17, 2017*, 164–172. doi:10.1145/3018610.3018615.
- Bishop, E. and Bridges, D. (1985). *Constructive Analysis*, Berlin, Heidelberg, Springer-Verlag. doi:10.1007/978-3-642-61667-9.
- Booij, A. B. (2020a). Coq development of locators. <https://github.com/abooij/HoTT/tree/locators> (accessed February 2020).
- Booij, A. B. (2020b). *Analysis in Univalent Type Theory*. PhD thesis, University of Birmingham.
- Bridges, D. and Richman, F. (1987). *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes, vol. 97, Cambridge, Cambridge University Press.
- Bridges, D. and Vita, L. S. (2006). *Techniques of Constructive Analysis*, New York, Springer-Verlag. doi:10.1007/978-0-387-38147-3.

- Brouwer, L. E. J. (1921). Besitzt jede reelle zahl eine Dezimalbruchentwicklung? *Mathematische Annalen* **83**, 201–210. <http://eudml.org/doc/158869>.
- Cruz-Filipe, L. (2004). *Constructive Real Analysis: A Type-Theoretical Formalization and Applications*. PhD thesis, University of Nijmegen.
- Escardó, M. H. (2013). Message to the Univalent Foundations mailing list. <https://groups.google.com/d/msg/univalent-foundations/SA0dzenV1G4/d5iIGdKKNxMJ>.
- Escardó, M. H. and Xu, C. (2015). The inconsistency of a Brouwerian continuity principle with the Curry-Howard interpretation. In: *13th International Conference on Typed Lambda Calculi and Applications, TLCA 2015, July 1–3, 2015, Warsaw, Poland*, 153–164. doi:10.4230/LIPIcs.TLCA.2015.153.
- Frank, M. (2017). Interpolating between choices for the approximate intermediate value theorem. ArXiv e-prints, January 2017. arXiv:1701.02227.
- Hendtllass, M. (2012). The intermediate value theorem in constructive mathematics without choice. *Annals of Pure and Applied Logic* **163**(8), 1050–1056. doi:10.1016/j.apal.2011.12.026.
- Hofmann, M. (1995). *Extensional Concepts in Intensional Type theory*. PhD thesis, University of Edinburgh.
- Ishihara, H. (2001). Sequentially continuity in constructive mathematics. In: Calude, C. S., Dinneen, M. J. and Sburlan, S. (eds.) *Combinatorics, Computability and Logic*, London, Springer, 5–12. doi:10.1007/978-1-4471-0717-0_2.
- Kraus, N., Escardó, M. H., Coquand, T. and Altenkirch, T. (2017). Notions of anonymous existence in Martin-Löf type theory. *Logical Methods in Computer Science* **13**(1). doi:10.23638/LMCS-13(1:15)2017.
- Lubarsky, R. S. (2007). On the Cauchy completeness of the constructive Cauchy reals. *Electronic Notes in Theoretical Computer Science* **167**, 225–254. doi:10.1016/j.entcs.2006.09.012.
- Mahboubi, A., Melquiond, G. and Sibut-Pinote, T. (2016). Formally verified approximations of definite integrals. In: *Interactive Theorem Proving—7th International Conference, ITP 2016, Nancy, France, August 22–25, 2016, Proceedings*, 274–289. doi:10.1007/978-3-319-43144-4_17.
- O'Connor, R. (2009). *Incompleteness & Completeness: Formalizing Logic and Analysis in Type Theory*. PhD thesis, Radboud Universiteit Nijmegen.
- Schuster, P. and Schwichtenberg, H. (2004). Constructive solutions of continuous equations. In: Link, G. (eds.) *de Gruyter Series in Logic and Its Applications*, Walter de Gruyter. doi:10.1515/9783110199680.227.
- Schwichtenberg, H. (2017). Constructive analysis with witnesses. <http://www.math.lmu.de/~schwicht/seminars/semws16/constr16.pdf>.
- Taylor, P. (2010). A lambda calculus for real analysis. *Journal of Logic and Analysis* **2**. <http://logicandanalysis.org/index.php/jla/article/view/63/25>.
- The Univalent Foundations Program. (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study. <https://homotopytypetheory.org/book>.
- Troelstra, A. S. and van Dalen, D. (1988). *Constructivism in Mathematics, An Introduction*, Studies in Logic and the Foundations of Mathematics, vol. 1, North-Holland.
- Turing, A. M. (1936). On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society. Second Series* **42**, 230–265. See correction Turing (1937).
- Turing, A. M. (1937). On computable numbers, with an application to the Entscheidungsproblem. A correction. *Proceedings of the London Mathematical Society. Second Series* **43**, 544–546. See Turing (1936).
- Wiedmer, E. (1977). *Exaktes Rechnen mit reellen Zahlen und anderen unendlichen Objekten*. PhD thesis, ETH Zurich.