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Magnetorotational stability in a self-consistent three dimensional axisymmetric magnetized warm plasma equilibrium with a gravitational field

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Magnetorotational stability is revisited for self-consistent three-dimensional magnetized hot plasma equilibria in a gravitational field. The eikonal analysis presented finds that magnetorotational stability analysis must be performed with some care to retain compressibility and density gradient effects, and departures from strict Keplerian motion. Indeed, retaining these effects highlights differences between the magnetorotational instability found in the absence of gravity (Velikhov, *Sov. Phys. JETP*, vol. 36, 1959, pp. 995–998) and that found the presence of gravity (Balbus & Hawley, *Astrophys. J.*, vol. 376, 1991, pp. 214–222). In the non-gravitational case, compressibility and density variation alter the stability condition, while these effects only enter for departures from strict Keplerian motion in a gravitational field. The conditions for instability are made more precise by employing recent magnetized equilibrium results (Catto *et al.*, *J. Plasma Phys.*, vol. 81, 2015, 515810603), rather than employing a hydrodynamic equilibrium. We focus on the stability of the $\beta > 1$ limit for which equilibria were found in the absence of a toroidal magnetic field, where $\beta =$ plasma/magnetic pressure.

Key words: astrophysical plasmas, plasma instabilities

1. Background

The groundbreaking effort of Velikhov (1959) and the pioneering work of Balbus & Hawley (1991) has led to a generally accepted explanation that the magnetorotational instability (MRI) is responsible for the existence of turbulence in accretion disks and provides a possible anomalous transport mechanism for particle accretion by compact astrophysical objects while angular momentum is transport outward. Cylindrical numerical simulations retaining the drive for magnetorotational instability indicate that the flow in the washer-shaped disk becomes turbulent with angular momentum redistribution (Hawley & Balbus 1991). Later simulations observed turbulent angular momentum distribution by continuing to omit the vertical component of gravity and

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neglecting pressure and density gradients (Hawley, Gammie & Balbus 1995). These early analytic results focused on the incompressible limit and the early simulation of Hawley & Balbus (1991) indicated insensitivity to compressibility. Even more recent simulations (Stone, Pringle & Begelman 1999; Stone & Pringle 2001) apply a vertical magnetic field to the polytropic (or adiabatic) hydrodynamic equilibrium of Papaloizou & Pringle (1984). Other treatments exist (Ogilvie 1998), but most treatments seem to assume strict Keplerian motion by which we mean the gravitational and centrifugal forces are in exact balance at the equatorial plane. When this assumption is removed, recent stratified shearing box simulations exhibit some sensitivity to compressibility (McNally & Pessah 2015). Here we assume a magnetized equilibrium and allow for general compressibility, more general rotational motion allowing an imbalance between the gravitational and centrifugal forces, and a fully self-consistent equilibrium with density variation. We find that all these ingredients are important in the overall stability threshold. However, when the motion is strictly Keplerian we show that both compressibility and density variation effects are absent, but departures from an axial magnetic field provide further destabilization. Interestingly, in the non-gravitational, axial magnetic field limit of Velikhov (1959) we find that density gradient effects are always present for a careful treatment of compressibility.

The MRI is thought to play a role in the formation of stars (Stone *et al.* 2000), the production of X-rays in neutron star and black hole systems (Blaes 2004) and the creation of gamma-ray bursts (Wheeler 2004). Local numerical investigations indicate that the magnetorotational instability allows mass to be accreted by the central compact object while angular momentum is transported radially outwards (Hawley *et al.* 1995; Hawley, Gammie & Balbus 1996; Stone *et al.* 1996) and that the instability might result in a magnetohydrodynamic (MHD) accretion disk dynamo generating a self-sustaining magnetic field. A possible implication is that a re-examination of stability for a fully self-consistent rotating and magnetized global gravitational equilibrium with compressibility and density gradient effects retained is a natural and needed step.

The lack of a fully self-consistent magnetized three-dimensional equilibrium in the presence of gravity is a shortcoming of previous work that can now be better addressed thanks to the self-similar equilibrium model recently formulated by Catto & Krasheninnikov (2015) and solved analytically and numerically by Catto, Pusztai & Krasheninnikov (2015). This global equilibrium model allows, but does not require, strict Keplerian motion in the equatorial plane. It fully retains the imbalance between the cylindrical centrifugal force and the central gravitational force that makes disk equilibria possible for some parameter ranges. Moreover, the condition for unstable modes to fit in the plasma disk results in the disk width entering the condition for instability. Catto *et al.* (2015) have shown numerically that the $\beta \ll 1$ equilibria found analytically by Catto & Krasheninnikov (2015) actually require a toroidal magnetic field as well. However, in the more interesting $\beta > 1$ limit, both numerical and analytic equilibrium solutions are found without a toroidal magnetic field. The work herein only considers magnetorotational stability in this zero toroidal magnetic field limit, since retaining it substantially complicates the analysis.

In the following sections we re-consider ideal magnetohyrodynamic magneto-rotational stability. We begin by briefly summarizing in § 2 the axisymmetric equilibrium equations for a general poloidal magnetic field satisfying the constraints imposed by the kinetic equation for a drifting Maxwellian (Hinton & Wong 1985; Catto, Bernstein & Tessarotto 1987; Helander 2014). In § 3 we then derive the linearized equations for an arbitrary axisymmetric perturbation in a hot, compressible, finite β, magnetized plasma rotating in a gravitational field. We then investigate the general magnetorotational stability condition for arbitrary β when the magnetic field is assumed axial. The results without gravity in § 4 are consistent with Velikhov (1959), and the eikonal results of §§ 5 and 6 with gravity reduce to Balbus & Hawley (1991) in the strict Keplerian limit. The Velikhov case without gravity is considered to illustrate our analysis technique in the simplest possible limit and to contrast its differences with the gravitational cases that follow. A self-consistent equilibrium without gravity (Catto & Krasheninnikov 2015) allows us to make a further connection to the seminal work of Velikhov (1959) in the unbounded plasma limit (most of Velikhov's analysis is for plasma bounded by two rotating cylindrical surfaces). Unlike Velikhov, we allow compressibility and radial density variation. To make further connections to the insightful astrophysical work of Balbus & Hawley (1991) we retain compressibility and density gradient effects when the rotation is not strictly Keplerian as both impact the stability threshold. Moreover, we also apply our stability results to the self-consistent magnetized gravitational equilibrium recently found by Catto *et al.* (2015) In this case we account for the thinness of the self-consistent plasma disk width when $\beta \gg 1$. In § 7 and the appendix we consider the stability of arbitrary β non-axial, magnetic field equilibria.

2. Equilibrium

We consider the stability of a hot, axisymmetric magnetized plasma rotating toroidally in a gravitational field. We assume the unperturbed axisymmetric magnetic field, $\boldsymbol{B}_0$, has no toroidal component so we can write

$$\boldsymbol{B}_0 = \nabla\psi_0 \times \nabla\zeta, \tag{2.1}$$

where ζ is the toroidal angle and ψ_0 is the unperturbed poloidal flux function. The unperturbed velocity $\boldsymbol{V}_0$ must be toroidal to satisfy the constraints ($\boldsymbol{B}_0 \cdot \nabla \boldsymbol{V}_0 \cdot \boldsymbol{B}_0 = 0$ and $\boldsymbol{B}_0 \cdot \nabla T_0 = 0$) imposed on a drifting Maxwellian solution in an axisymmetric system by the gyro-averaged kinetic equation in the presence of Coulomb collisions (Hinton & Wong 1985; Catto *et al.* 1987; Helander 2014):

$$\boldsymbol{V}_0 = \Omega R^2 \nabla\zeta, \tag{2.2}$$

where the toroidal rotation frequency Ω is related to the electrostatic potential Φ_0 by $\Omega = c\, \mathrm{d}\Phi_0/\mathrm{d}\psi_0$ with the electric field given by $\boldsymbol{E}_0 = -\nabla\Phi_0 = -\nabla\psi_0\, \mathrm{d}\Phi_0/\mathrm{d}\psi_0$, R the cylindrical radius from the axis of symmetry and c the speed of light. We have used $c\nabla\Phi_0 = \boldsymbol{V}_0 \times \boldsymbol{B}_0$ and $\boldsymbol{B}_o \cdot \nabla\Phi_0 = 0$ to find that the electrostatic potential Φ_0, and therefore Ω, must be a flux function to lowest order. The ideal gas relation between the total unperturbed pressure p_0 and the ion density n_0, $p_0 = n_0(T_{0i} + ZT_{0e}) = 2n_0T_0$, defines the effective temperature T_0, with T_{0i} and T_{0e} the unperturbed ion and electron temperatures.

We write the gravitational potential G as

$$G = -G_0 M_0/r, \tag{2.3}$$

with G_0 the gravitational constant, r the spherical radius and M_0 the mass of the astrophysical body that is assumed to be a massive localized source centred at $r = 0$, such that $R = r \sin\vartheta$ with ϑ the angle from the axis of symmetry. We define our spherical and cylindrical coordinates to satisfy $r\nabla\vartheta = R\nabla\zeta \times \nabla r$ and $\nabla z = R\nabla R \times \nabla\zeta$.

The hot magnetized plasma must adjust its flux surfaces to satisfy Ampere's law and total momentum balance under the influence of the attractive force of gravity and the outward centrifugal and plasma pressure forces:

$$c^{-1}\boldsymbol{J}_0\times\boldsymbol{B}_0=\nabla p_0+Mn_0(\nabla G+\boldsymbol{V}_0\boldsymbol{\cdot}\nabla\boldsymbol{V}_0)=Mn_0(\nabla G-\Omega^2R\nabla R)+\nabla(2n_0T_0), \quad (2.4)$$

where M is the mass of the plasma ions and the plasma is quasi-neutral ($Zn_0=n_{0e}=$ electron density, with Z the charge number of the ions). Importantly, the density must satisfy total parallel momentum balance

$$\boldsymbol{B}_0\boldsymbol{\cdot}\nabla[2G-\Omega^2R^2+(4T_0/M)\ell n(n_0/\eta_0)]=0, \quad (2.5)$$

with both T_0 and the unperturbed normalizing density η_0 (which we refer to as the psuedo-density to distinguish it from n_0) flux functions,

$$\boldsymbol{B}_0\boldsymbol{\cdot}\nabla T_0=0=\boldsymbol{B}_0\boldsymbol{\cdot}\nabla\eta_0, \quad (2.6)$$

and ℓn the natural logarithm. The density and pseudo-density are related by a poloidally varying Maxwell–Boltzmann exponential factor κ_0 that conveniently accounts for the mismatch between the centrifugal and gravitational potentials

$$n_0=n_0(\psi,\vartheta)=\eta_0(\psi)\mathrm{e}^{\kappa_0(\psi,\vartheta)}. \quad (2.7)$$

The parallel momentum constraint determines κ_0 within a flux function that can be set to zero by absorbing it into the pseudo-density η_0. As a result, (2.5) and (2.7) yield

$$(4T_0/M)\kappa_0=\Omega^2R^2-2G, \quad (2.8)$$

showing that the flux surfaces and poloidal variation of the density in the plasma pressure must adjust to maintain parallel force balance between the inward central gravitational force and the outward cylindrical rotation.

The preceding constraints differ from a true hydrodynamic equilibrium. For a drifting Maxwellian to satisfy the Boltzmann equation, constant T_0, Ω and η_0 are required in the Maxwell–Boltzmann density relation $n_0=\eta_0\exp[(M/2T_0)(\Omega^2R^2-2G)]$. If we allow $\Omega=\Omega(R)$, using $\nabla\times\nabla G=0$ in the momentum conservation, $n_0^{-1}\nabla p_0=M(\nabla G-\Omega^2R\nabla R)$, gives $\nabla n_0\times\nabla T_0=0$. However, we desire a solution with $n_0=n_0(R,z)$, so we need to keep T_0 constant. Keeping η_0 constant, then $n_0=\eta_0\exp[(M/T_0)(\int\mathrm{d}R\,\Omega^2R-G)]$. Then f or example, angular momentum can be preserved by assuming constant ΩR^2, to obtain $n_0=\eta_0\exp[-(M/2T_0)(\Omega^2R^2+2G)]$. Perhaps, more interestingly, assuming $\Omega^2R^3=MM_0G_0$ and defining $2g=MM_0G_0/T_0R_0$, with R_0 an arbitrary equatorial plane reference location, gives $n_0=\eta_0\exp\{-2g[(R_0/R)-(R_0/r)]\}$, with $r>R$ so that $n_0/\eta_0<1$. Notice that $n_0(r=R)=\eta_0$ is the density at the equatorial plane, and that for $g\gg 1$ a thin plasma disk occurs.

The $\nabla\psi_0$ and ∇r components of Ampere's law, $c\nabla\times\boldsymbol{B}_0=4\pi\boldsymbol{J}_0$, require $\boldsymbol{J}_0\boldsymbol{\cdot}\nabla\psi_0=0$ and $\boldsymbol{J}_0\boldsymbol{\cdot}\nabla r=0$ giving $\boldsymbol{J}_0\boldsymbol{\cdot}\boldsymbol{B}_0=0$. As a result, there is unperturbed current only in the toroidal direction

$$\boldsymbol{J}_0=J_0R\nabla\zeta=(cMn_0/B_0^2)\nabla\psi_0\boldsymbol{\cdot}[(2/Mn_0)\nabla(n_0T_0)+\nabla G-\Omega^2R\nabla R]\nabla\zeta. \quad (2.9)$$

The toroidal component of Ampere's law gives the Grad–Shafranov equation,

$$\nabla\boldsymbol{\cdot}(R^{-2}\nabla\psi_0)=-(4\pi Mn_0/R^2B_0^2)\nabla\psi_0\boldsymbol{\cdot}[(1/Mn_0)\nabla p_0+\nabla G-\Omega^2R\nabla R]=-4\pi J_0/cR, \quad (2.10)$$

that must be solved to find the equilibrium flux surfaces.

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3. Linearized equations

Linear stability is normally examined by perturbing about a presumed equilibrium to find the conditions for instability. However, a self-consistent equilibrium normally imposes additional constraints that can determine whether these unstable conditions are accessible. Consequently, the actual stability of a system can only be determined for a fully self-consistent equilibrium. Said another way, we want to avoid instability being an artefact of perturbing around a state that has not fully relaxed to a self-consistent magnetized equilibrium. Consequently, we perturb about an equilibrium satisfying (2.5) and (2.10)

We investigate the stability of our axisymmetric equilibrium to axisymmetric ideal MHD perturbations with adiabatic index $\gamma = 5/3$. We denote linearized quantities by a subscript '1' except for B_ζ, the toroidal component of the perturbed magnetic field $\boldsymbol{B}_1$, and the displacement vector $\boldsymbol{\xi}$, which is related to the perturbed velocity $\boldsymbol{V}_1$ by

$$\partial \boldsymbol{\xi}/\partial t = \boldsymbol{V}_1. \tag{3.1}$$

The total (unperturbed plus perturbed) conservation of number and energy equations,

$$\partial n/\partial t + \nabla \boldsymbol{\cdot} (n\boldsymbol{V}) = 0 \quad \text{and} \quad \partial p/\partial t + \boldsymbol{V} \boldsymbol{\cdot} \nabla p + \gamma p \nabla \boldsymbol{\cdot} \boldsymbol{V} = 0, \tag{3.2a,b}$$

are then conveniently used to obtain the perturbed density and pressure, n_1 and p_1, as

$$n_1 + \nabla \boldsymbol{\cdot} (n_0 \boldsymbol{\xi}) = 0 \quad \text{and} \quad p_1 + \boldsymbol{\xi} \boldsymbol{\cdot} \nabla p_0 + \gamma p_0 \nabla \boldsymbol{\cdot} \boldsymbol{\xi} = 0, \tag{3.3a,b}$$

since $\nabla \boldsymbol{\cdot} \boldsymbol{V}_0 = 0 = \nabla \boldsymbol{\cdot} (n_1 \boldsymbol{V}_0) = \boldsymbol{V}_0 \boldsymbol{\cdot} \nabla p_1$. The unperturbed particle and energy conservation equations are satisfied by the unperturbed flow so there is no need to assume an unperturbed adiabatic or polytropic relation between the unperturbed density and pressure. If n_1 and p_1 are combined to eliminate $\nabla \boldsymbol{\cdot} \boldsymbol{\xi}$ the result is the entropy constraint,

$$n_0(p_1 + \boldsymbol{\xi} \boldsymbol{\cdot} \nabla p_0) = \gamma p_0 (n_1 + \boldsymbol{\xi} \boldsymbol{\cdot} \nabla n_0), \tag{3.4}$$

which differs in form from Balbus & Hawley (1991) who are missing the perturbed pressure term in their equations (2.2*h*) and (2.3*a*). They also employ incompressibility by using their equation (2.3*c*).

We desire to find a set of three equations in which only the three components of the displacement enter, as in ideal MHD without an unperturbed flow. To begin, we consider Faraday's law for ideal MHD, $\partial \boldsymbol{B}/\partial t = -c\nabla \times \boldsymbol{E} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B})$. Linearization gives the equation for the perturbed magnetic field $\boldsymbol{B}_1$ to be

$$\partial \boldsymbol{B}_1/\partial t = \nabla \times [(\partial \boldsymbol{\xi}/\partial t) \times \boldsymbol{B}_0 + \varOmega R^2 \nabla \zeta \times \boldsymbol{B}_1]. \tag{3.5}$$

The $\nabla \zeta$ component of (3.5) gives the equation for $B_\zeta \equiv R\boldsymbol{B}_1 \boldsymbol{\cdot} \nabla \zeta$ to be

$$\partial B_\zeta/\partial t = R\boldsymbol{B}_0 \boldsymbol{\cdot} \nabla (R^{-1} \partial \xi_\zeta/\partial t) + R\boldsymbol{B}_1 \boldsymbol{\cdot} \nabla \varOmega, \tag{3.6}$$

where $\xi_\zeta \equiv R\boldsymbol{\xi} \boldsymbol{\cdot} \nabla \zeta$, and we have used $\nabla \boldsymbol{\cdot} \boldsymbol{B}_0 = 0 = \nabla \boldsymbol{\cdot} \boldsymbol{B}_1 = 0 = \nabla \boldsymbol{\cdot} \boldsymbol{V}_0 = 0 = \boldsymbol{V}_0 \boldsymbol{\cdot} \nabla (\boldsymbol{B}_1 \boldsymbol{\cdot} \nabla \zeta)$. For the poloidal components of Faraday's law we use (3.5) to form the equation for $\boldsymbol{B}_1 \boldsymbol{\cdot} \nabla Q$, where Q is an arbitrary axisymmetric scalar function so that $\nabla \zeta \boldsymbol{\cdot} \nabla Q = 0$. Taking advantage of axisymmetry Faraday's law gives the simple result

$$\boldsymbol{B}_1 \boldsymbol{\cdot} \nabla Q = \boldsymbol{B}_0 \boldsymbol{\cdot} \nabla (\boldsymbol{\xi} \boldsymbol{\cdot} \nabla Q) - \nabla \boldsymbol{\cdot} (\boldsymbol{\xi} \boldsymbol{B}_0 \boldsymbol{\cdot} \nabla Q). \tag{3.7}$$

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Using $Q=\psi_0$, the preceding gives $\boldsymbol{B}_1\cdot\nabla\psi_0=\boldsymbol{B}_0\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0)$. Axisymmetry and $\nabla\cdot\boldsymbol{B}_1=0$ mean we can write

$$\boldsymbol{B}_1=RB_\zeta\nabla\zeta+\nabla\psi_1\times\nabla\zeta, \tag{3.8}$$

with ψ_1 the perturbed flux function defined so that $\boldsymbol{B}_1\cdot\nabla\psi_0+\boldsymbol{B}_0\cdot\nabla\psi_1=0$. Consequently, $\psi_1=-\boldsymbol{\xi}\cdot\nabla\psi_0$ and we may write

$$\boldsymbol{B}_1=RB_\zeta\nabla\zeta+\nabla\zeta\times\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0). \tag{3.9}$$

Then we can form $R^2\nabla\zeta\times\boldsymbol{B}_1=-\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0)$ and

$$\partial B_\zeta/\partial t=R\boldsymbol{B}_0\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\Omega+R^{-1}\partial\xi_\zeta/\partial t). \tag{3.10}$$

Finally, we turn our attention to the linearization of the momentum conservation equation,

$$cMn(\partial\boldsymbol{V}/\partial t+\boldsymbol{V}\cdot\nabla\boldsymbol{V}+\nabla G)+c\nabla p=\boldsymbol{J}\times\boldsymbol{B}. \tag{3.11}$$

To carry out the linearization we use

$$\begin{aligned}(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0&=\nabla\psi_0\nabla\cdot(\boldsymbol{B}_1\times\nabla\zeta)-\nabla\zeta\nabla\cdot(\boldsymbol{B}_1\times\nabla\psi_0)\\&=\nabla\psi_0\nabla\cdot[R^{-2}\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0)]+\nabla\zeta\boldsymbol{B}_0\cdot\nabla(RB_\zeta),\end{aligned} \tag{3.12}$$

$$\boldsymbol{J}_0\times\boldsymbol{B}_1=J_0R\nabla\zeta\times[\nabla\zeta\times\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0)]=-R^{-1}J_0\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0), \tag{3.13}$$

$$R\nabla\nabla\zeta=-\nabla\zeta\nabla R-\nabla R\nabla\zeta, \tag{3.14}$$

and

$$\boldsymbol{V}_0\cdot\nabla\boldsymbol{V}_0=-\Omega^2R\nabla R, \tag{3.15}$$

as well as axisymmetry. We obtain

$$\begin{aligned}&\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}+\Omega R^2\nabla\zeta\cdot\nabla\frac{\partial\boldsymbol{\xi}}{\partial t}+\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot[\nabla(\Omega R^2)\nabla\zeta-\Omega R(\nabla\zeta\nabla R+\nabla R\nabla\zeta)]\\&\quad-\frac{n_1}{n_0}\left(\Omega^2R\nabla R-\frac{M_0G_0}{r^2}\nabla r\right)=-\frac{\nabla p_1}{Mn_0}-\frac{J_0\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0)}{cMn_0R}\\&\quad+\frac{\nabla\psi_0\nabla\cdot(R^{-2}\boldsymbol{\xi}\cdot\nabla\psi_0)+\nabla\zeta\boldsymbol{B}_0\cdot\nabla(RB_\zeta)}{4\pi Mn_0}.\end{aligned} \tag{3.16}$$

Recalling (3.3) and (3.10) we see that a time derivative of (3.16) will result in an equation in which only the three components of the displacement enter.

The preceding observation leads us to taking a time derivative of the $\nabla\zeta$ component of (3.16) and inserting (3.10) yields a form in which only the displacement enters

$$R\partial^3\xi_\zeta/\partial t^3+\nabla(\Omega R^2)\cdot\partial^2\boldsymbol{\xi}/\partial t^2=(4\pi Mn_0)^{-1}\boldsymbol{B}_0\cdot\nabla[R^2\boldsymbol{B}_0\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\Omega+R^{-1}\partial\xi_\zeta/\partial t)]. \tag{3.17}$$

Using parallel and radial pressure balance allows us to rewrite equation (3.16) and then dot it by $\boldsymbol{B}_0$ to obtain a second component, namely

$$\left[\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}-2\frac{\partial\xi_\zeta}{\partial t}\Omega\nabla R-\frac{n_1}{n_0}\frac{\nabla p_0}{Mn_0}+\frac{\nabla p_1}{Mn_0}\right]\cdot\boldsymbol{B}_0=\frac{\nabla\cdot(R^{-2}\nabla\psi_0)}{4\pi Mn_0}\boldsymbol{B}_0\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0), \tag{3.18}$$

where using that the unperturbed current density is azimuthal we find $\nabla \boldsymbol{B}_0 \boldsymbol{\cdot} \nabla \zeta = \nabla \zeta \boldsymbol{\cdot} \nabla \boldsymbol{B}_0$, giving

$$\nabla \zeta \boldsymbol{\cdot} \nabla \boldsymbol{U} \boldsymbol{\cdot} \boldsymbol{B}_0 = -\nabla \zeta \boldsymbol{\cdot} \nabla \boldsymbol{B}_0 \boldsymbol{\cdot} \boldsymbol{U} = -\boldsymbol{U} \boldsymbol{\cdot} \nabla \boldsymbol{B}_0 \boldsymbol{\cdot} \nabla \zeta = \boldsymbol{U} \boldsymbol{\cdot} \nabla \nabla \zeta \boldsymbol{\cdot} \boldsymbol{B}_0 = -R^{-1} \boldsymbol{U} \boldsymbol{\cdot} \nabla \zeta \boldsymbol{B}_0 \boldsymbol{\cdot} \nabla R, \tag{3.19}$$

with $\boldsymbol{U}$ an arbitrary vector.

For the final component we use $0 = \nabla \zeta \times (\nabla \times \nabla Q) = \nabla \nabla Q \boldsymbol{\cdot} \nabla \zeta - \nabla \zeta \boldsymbol{\cdot} \nabla \nabla Q$ to form the ∇Q component of the momentum equation by noting that $\nabla \zeta \boldsymbol{\cdot} \nabla \nabla Q = R^{-1} \nabla \zeta \, \nabla R \boldsymbol{\cdot} \nabla Q$. The result is

$$\left[\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - 2\frac{\partial \xi_\zeta}{\partial t} \Omega \nabla R - \frac{n_1}{n_0}\left(\Omega^2 R \nabla R - \frac{M_0 G_0}{r^2} \nabla r\right) + \frac{\nabla p_1}{M n_0}\right] \boldsymbol{\cdot} \nabla Q \\ = -\frac{J_0 \nabla Q \boldsymbol{\cdot} \nabla(\boldsymbol{\xi} \boldsymbol{\cdot} \nabla \psi_0)}{c M n_0 R} + \frac{\nabla \psi_0 \boldsymbol{\cdot} \nabla Q}{4\pi M n_0} \nabla \boldsymbol{\cdot} [R^{-2} \nabla(\boldsymbol{\xi} \boldsymbol{\cdot} \nabla \psi_0)]. \tag{3.20}$$

Equations (3.17), (3.18) and (3.20) are the key equations we will make use of to obtain our stability condition. It has only been necessary to take the extra time derivative when forming (3.17).

Only the displacement enters in equations (3.18) and (3.20), since n_1 and p_1 are known in terms of $\boldsymbol{\xi}$. Equations (3.16)–(3.20), with (3.3) inserted, are the key equations for the three displacement components from the momentum equation. They are given in vector forms to allow us to work in any axisymmetric coordinate system we desire.

4. Cylindrically symmetric magnetic field in the absence of gravity ($G = 0$)

To illustrate the subtleties of retaining compressibility and unperturbed radial variation in a self-consistent equilibrium it is informative to first consider the simplest limit of rotation in an axial magnetic field without gravity ($G = 0$) or any unperturbed axial variation. In this limit the equilibrium magnetic field $\boldsymbol{B}_0$ is axial and of the form

$$\boldsymbol{B}_0 = B_0(R) \nabla z, \tag{4.1}$$

with $\psi_0 = \psi_0(R)$ and $R B_0 = \mathrm{d}\psi_0/\mathrm{d}R$. In this cylindrical geometry limit $\boldsymbol{J}_0 \times \boldsymbol{B}_0 = J_0 B_0 \nabla R$ so equilibrium pressure balance has only the ∇R component

$$c^{-1} J_0 B_0 = \partial p_0/\partial R - M n_0 R \Omega^2 = -(8\pi)^{-1} \partial B_0^2/\partial R, \tag{4.2}$$

and T_0, p_0 and n_0 depend only on R since the equilibrium has no axial variation.

Defining the Alfvén speed by

$$v_A^2 = B_0^2/4\pi M n_0, \tag{4.3}$$

the equation for (3.17) becomes

$$\partial^3 \xi_\zeta/\partial t^3 + [R^{-1}\, \mathrm{d}(\Omega R^2)/\mathrm{d}R]\, \partial^2 \xi_R/\partial t^2 = v_A^2 (\partial^2/\partial z^2)[\xi_R R(\mathrm{d}\Omega/\mathrm{d}R) + (\partial \xi_\zeta/\partial t)], \tag{4.4}$$

where we now write the displacement as $\boldsymbol{\xi} = \xi_R \nabla R + \xi_z \nabla z + \xi_\zeta R \nabla \zeta$. We may use $\Omega = \Omega(R)$ because $\Omega = \Omega(\psi_0)$.

To get the remaining components we use the parallel or $Q = z$ form of (3.20) to find the ξ_z equation and then insert p_1 and use unperturbed pressure balance to find

$$\frac{\partial^2 \xi_z}{\partial t^2} = \left[R \Omega^2 \frac{\partial \xi_R}{\partial z} + c_S^2 \frac{\partial}{\partial z}(\nabla \boldsymbol{\cdot} \boldsymbol{\xi})\right], \tag{4.5}$$

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where

$$c_S^2 = \gamma p_0/Mn_0, \tag{4.6}$$

$$\beta = 8\pi p_0/B_0^2, \tag{4.7}$$

and

$$\gamma\beta = 2c_S^2/v_A^2. \tag{4.8}$$

The last equation for ξ_R follows by using $Q=R$ in (3.20) and then substituting in n_1 and p_1 from (3.3) to obtain

$$\begin{aligned}&\frac{\partial^2\xi_R}{\partial t^2} - 2\Omega\frac{\partial\xi_\zeta}{\partial t} + R\Omega^2\frac{\partial\xi_z}{\partial z} - R\xi_R\frac{\partial\Omega^2}{\partial R}\\ &\quad = \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] + v_A^2\frac{\partial^2\xi_R}{\partial z^2} + \frac{\gamma}{n_0}\frac{\partial}{\partial R}\left(\frac{p_0}{M}\nabla\cdot\boldsymbol{\xi}\right),\end{aligned} \tag{4.9}$$

where we eliminate $\partial p_0/\partial R$ by using unperturbed pressure balance.

The preceding two equations (4.5) and (4.9), along with (4.4) for ξ_ζ are the general set of equations for a cylindrically symmetric magnetic field configuration with no axial variation in the absence of gravity. The final forms of the equations for ξ_R and ξ_z allow us to be sure that we are carefully treating all terms in n_1 and p_1.

To simplify the description we Fourier transform in z and seek solutions harmonic in time by assuming

$$\boldsymbol{\xi} \propto \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}k_z z}. \tag{4.10}$$

We obtain the three equations for the three components of the displacement

$$\mathrm{i}\omega(\omega^2 - k_z^2v_A^2)\xi_\zeta = \xi_R[\omega^2R^{-1}\,\mathrm{d}(\Omega R^2)/\mathrm{d}R - k_z^2v_A^2R\,\mathrm{d}\Omega/\mathrm{d}R], \tag{4.11}$$

$$(\omega^2 - k_z^2c_S^2)\xi_z = -\mathrm{i}k_z\left[R\Omega^2\xi_R + \frac{c_S^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right], \tag{4.12}$$

and

$$\begin{aligned}&\left(\omega^2 - k_z^2v_A^2 + R\frac{\partial\Omega^2}{\partial R}\right)\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]\\ &\quad = 2\mathrm{i}\omega\Omega\xi_\zeta + \mathrm{i}k_z\left[R\Omega^2\xi_z - \frac{1}{n_0}\frac{\partial}{\partial R}(n_0c_S^2\xi_z)\right].\end{aligned} \tag{4.13}$$

First, eliminating ξ_z by substituting (4.12) into (4.13) gives

$$\begin{aligned}&\left(\omega^2 - k_z^2v_A^2 + R\frac{\partial\Omega^2}{\partial R}\right)\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]\\ &\quad = 2\mathrm{i}\omega\Omega\xi_\zeta + \frac{k_z^2}{n_0c_S^2}\left(R\Omega^2 - c_S^2\frac{\partial}{\partial R}\right)\left\{\frac{n_0c_S^2}{(\omega^2 - k_z^2c_S^2)}\left[R\Omega^2\xi_R + \frac{c_S^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]\right\}.\end{aligned} \tag{4.14}$$

Then eliminating ξ_ζ from the preceding by using (4.11) gives the full radial differential equation for ξ_R to be

$$\begin{aligned}&\left[\omega^2 - k_z^2v_A^2 - \frac{R}{n_0}\frac{\partial(n_0\Omega^2)}{\partial R} - \frac{4\omega^2\Omega^2}{(\omega^2 - k_z^2v_A^2)}\right]\xi_R + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]\\ &\quad = \frac{k_z^2R^2\Omega^4\xi_R}{(\omega^2 - k_z^2c_S^2)} - \frac{\omega^2R\xi_R}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0\Omega^2}{(\omega^2 - k_z^2c_S^2)}\right] - \frac{\omega^2}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0c_S^2R^{-1}\partial(R\xi_R)/\partial R}{(\omega^2 - k_z^2c_S^2)}\right],\end{aligned} \tag{4.15}$$

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where we have combined the $n_0 c_S^2 \partial(R\xi_R)/\partial R$ terms to exactly cancel some first derivatives of $R\xi_R$ terms. Equation (4.15) is the generalization of the radial differential form of Velikhov (1959) to retain compressibility. Its solution for a self-consistent equilibrium with the appropriate boundary conditions will give the exact eigenfrequencies, eigenfunctions and stability threshold condition. To retain Alfvén and sound waves and their coupling it allows $k_z^2 c_S^2 \sim \Omega^2 \sim k_z^2 v_A^2 \gtrsim \omega^2$, $k_z^2 R^2 \sim 1$, and $\partial/\partial R \sim 1/R$. These orderings will be further verified shortly when we consider the integral constraint associated with the self-adjoint form (4.15).

To perform a Wentzel–Kramers–Brillouin (WKB) analysis we must assume

$$R\xi_R^{-1}\partial\xi_R/\partial R \gg R n_0^{-1}\partial n_0/\partial R \sim R p_0^{-1}\partial p_0/\partial R \sim R\Omega^{-1}\partial\Omega/\partial R \sim 1, \tag{4.16}$$

so that $k_R^2 R^2 \gg 1$, where $\xi_R \propto \mathrm{e}^{\mathrm{i}\int \mathrm{d}R\, k_R}$. To allow $k_z^2 R^2 \gtrsim 1$ and correctly treat marginal stability, we order

$$k_z^2 c_S^2 \sim R^2\Omega^4/c_S^2 \sim \Omega^2 \sim k_R^2 v_A^2 \gtrsim k_z^2 v_A^2 \gtrsim \omega^2 \tag{4.17}$$

and thereby assume $\beta \gg 1$. The geometric optics dispersion relation allowing $k_R^2 \gtrsim k_z^2$ is then

$$k^2 v_A^2 + \frac{4\omega^2\Omega^2}{(\omega^2 - k_z^2 v_A^2)} - \frac{k_z^2 R^2 \Omega^4 + \omega^2 k_R^2 c_S^2}{k_z^2 c_S^2} + \frac{R}{n_0}\frac{\partial}{\partial R}(n_0\Omega^2) = 0, \tag{4.18}$$

where $k^2 = k_z^2 + k_R^2$. All terms in (4.18) are allowed to be the same order. Notice that at marginality ($\omega^2 \rightarrow 0$) equation (4.18) recovers a radial density gradient term as well as the usual rotation frequency gradient

$$\frac{k^2 v_A^2}{\Omega^2} < \frac{R^2\Omega^2}{c_S^2} - \frac{R}{n_0\Omega^2}\frac{\partial(n_0\Omega^2)}{\partial R}. \tag{4.19}$$

Marginal stability is most easily determined graphically from (4.18) by examining the behaviour near $\omega^2 = 0$. The instability window for this new result is larger than if we had assumed strict incompressibility ($c_S^2 \rightarrow \infty$). Also, (4.19) does not assume constant n_0 as in Velikhov (1959) or $\Omega^2 \propto R^{-3}$.

In the hydrodynamic limit ($v_A^2 \equiv 0$), WKB still requires $k_R R \gg 1$, but in (4.15) we can allow $c_S^2/R^2 \sim k_z^2 c_S^2 \sim \Omega^2 \sim R^2\Omega^4/c_S^2 \gtrsim \omega^2$ to find $\omega^2 k^2 k_z^{-2} = R^{-3} n_0^{-1}\partial(n_0\Omega^2 R^4)/\partial R - R^2\Omega^4/c_S^2$ when $\omega^2 \ll k_z^2 c_S^2$. For a hydrodynamic equilibrium at constant Ω, T_0 and η_0 stability is assured. In fact, for $\Omega = \Omega(R)$ and constant T_0 and η_0, an inviscid Couette flow instability requires $-R\Omega^{-2}\partial\Omega^2/\partial R > 4 + (\gamma - 1)M\Omega^2 R^2/\gamma T_0$, since in the fluid limit $c_S^2 \rightarrow \gamma T_0/M$ and $\gamma \rightarrow 5/3$.

The preceding results are consistent with the following integral constraint obtained by multiplying (4.15) by $n_0 R\xi_R^*$:

$$\begin{aligned} &\int_0^\infty \mathrm{d}R\, n_0|\xi_R^2| \left\{ \omega^2 - k_z^2 v_A^2 - \frac{R}{n_0}\frac{\partial(n_0\Omega^2)}{\partial R} - \frac{4\omega^2\Omega^2}{(\omega^2 - k_z^2 v_A^2)} \right. \\ &\quad \left. - \frac{k_z^2 R^2 \Omega^4}{(\omega^2 - k_z^2 c_S^2)} + \frac{k_z^2 R}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0\Omega^2 c_S^2}{(\omega^2 - k_z^2 c_S^2)}\right] \right\} \\ &\quad = \int_0^\infty \mathrm{d}R \frac{n_0}{R}\left[v_A^2 + \frac{\omega^2 c_S^2}{(\omega^2 - k_z^2 c_S^2)}\right]\left|\frac{\partial}{\partial R}(R\xi_R)\right|^2, \end{aligned} \tag{4.20}$$

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where ξ_R^* is the complex conjugate of ξ_R and we assume the total derivative terms vanish at $R=0$ and infinity. At marginality this gives the threshold constraint

$$\int_0^\infty \mathrm{d}R\, n_0 \left\{ R\left|\xi_R^2\right| \left[k_z^2 v_A^2 + \frac{R}{n_0}\frac{\partial (n_0 \Omega^2)}{\partial R} - \frac{R^2\Omega^4}{c_S^2}\right] - \frac{v_A^2}{R}\left|\frac{\partial}{\partial R}(R\xi_R)\right|^2 \right\} = 0. \tag{4.21}$$

These integral constraints are consistent with the WKB dispersion relation, stability threshold condition, and our orderings, and remain valid even when geometric optics fails for $\beta \lesssim 1$ so that the full differential equation (4.15) must be solved. Indeed, (4.21) indicates there are no other unstable modes besides the MRI. At marginality (4.15) becomes

$$\left[\frac{R^2\Omega^4}{c_S^2} - \frac{R}{n_0}\frac{\partial(n_0\Omega^2)}{\partial R} - k_z^2 v_A^2\right]\xi_R + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] = 0, \tag{4.22}$$

which is consistent with our WKB treatment. We can see from (4.20) to (4.22) that we need to order $k_z^2 c_S^2 \sim R^2\Omega^4/c_S^2 \sim \Omega^2 \sim k_R^2 v_A^2 \gtrsim k_z^2 v_A^2 \gtrsim \omega^2$ for $\beta \gg 1$. These orderings based on the cylindrical limit provide much of the basis for the orderings in limits with gravity.

The Velikhov (1959) condition for marginality ($\omega \to 0$) is obtained from

$$\mathrm{i}\omega\xi_\zeta = \xi_R R\, \mathrm{d}\Omega/\mathrm{d}R, \tag{4.23}$$

$$k_z c_S^2 \xi_z = \mathrm{i}\left[R\Omega^2\xi_R + \frac{c_S^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right], \tag{4.24}$$

and

$$R\frac{\partial\Omega^2}{\partial R}\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] = 2\mathrm{i}\omega\Omega\xi_\zeta + \mathrm{i}k_z\left[R\Omega^2\xi_z - \frac{1}{n_0}\frac{\partial}{\partial R}(n_0 c_S^2\xi_z)\right]. \tag{4.25}$$

Inserting (4.23) into (4.25) leaves (4.24) and the radial momentum balance equation

$$\frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] = \mathrm{i}k_z\left[R\Omega^2\xi_z - \frac{1}{n_0}\frac{\partial}{\partial R}(n_0 c_S^2\xi_z)\right]. \tag{4.26}$$

Then eliminating ξ_z by inserting (4.24) into (4.26) yields (4.19). To recover the incompressible limit we see that a careful treatment of the $c_S^2 \to \infty$ terms is required.

4.1. *Self-consistent equilibrium for an axial magnetic field without gravity*

For the exact self-similar equilibrium solution of Catto & Krasheninnikov (2015)

$$\psi_0 = CH(\mu)/r^\alpha, \tag{4.27}$$

where $\mu = \cos\vartheta$, C is a constant and α is an eigenvalue determined by solving the nonlinear Grad–Shafranov equation for H. The density, temperature and rotation are of

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the form $n_0 \propto R^{-2\alpha-3}$, $T_0 \propto R^{-1}$ and $\Omega \propto R^{-3/2}$, and the solution of the Grad–Shafranov equation requires α to satisfy

$$\alpha + 2 = -\gamma \Omega^2 R^2/(2c_S^2 + \gamma v_A^2). \tag{4.28}$$

For $\alpha + 2 < 0$ the axial magnetic field of this equilibrium model increases away from the axis of symmetry for instability. Using the preceding gives

$$-\frac{R}{n_0\Omega^2}\frac{\partial(n_0\Omega^2)}{\partial R} = 2(\alpha+3) = 2 - \frac{2\gamma\Omega^2R^2}{2c_S^2+\gamma v_A^2}. \tag{4.29}$$

Then for $\beta \gg 1$, inserting (4.29) our condition for instability for this self-consistent equilibrium becomes

$$1 > \frac{\Omega^2R^2}{3c_S^2} + \frac{k^2v_A^2}{2\Omega^2} = \frac{2\Omega^2R^2}{3\gamma\beta v_A^2} + \frac{k^2R^2v_A^2}{2\Omega^2R^2}, \tag{4.30}$$

where we must keep $k_R^2R^2 \gg 1$ to satisfy our geometric optics treatment in the radial direction. Therefore, compressibility stabilizes the MRI for all wavelengths when $\gamma\beta > 2\Omega^2R^2/3v_A^2 \gg 1$. Also, density variation is stabilizing since it results in the standard Balbus & Hawley (1991) result $k^2v_A^2/3\Omega^2$ being replaced by $k^2v_A^2/2\Omega^2$. It is interesting to notice that the compressibility effects hidden in the self-consistent variation of $n\Omega^2$ from (4.29) give rise to an implicit stabilizing compressibility effect larger than the explicit destabilizing one in (4.19).

Our incompressible condition for instability $2\Omega^2 > k^2v_A^2$ is consistent with the findings of Velikhov (1959) who worked in the $\beta \gg 1$ limit by assuming $\nabla \cdot \boldsymbol{\xi} = 0$ so that $c_S^2 \to \infty$. In the absence of an azimuthal unperturbed magnetic field he considered the full radial eigenvalue problem. However, he only mentions in passing 'the flow near a cylinder rotating in an unbounded medium' as his work focused on plasma being driven unstable between two concentric rotating cylinders. In the unbounded limit Velikhov's quasi-classical calculation for a radial mode number of 10 at the end of his section 2 estimated magnetic fields $B_0 < (4\pi M n_0)^{1/2}\Omega R/18$ as being unstable – a condition in rough agreement with our result of $2\Omega^2 > k_R^2v_A^2$ for $k_RR = 10$ when $3c_S^2 \gg \Omega^2R^2$.

5. Axial magnetic field with gravity and strict Keplerian rotation

In this strict Keplerian limit in which the equilibrium magnetic field $\boldsymbol{B}_0$ is axial and any unperturbed axial variation caused by the imbalance of the centrifugal and gravitational forces ignored, the calculation of the previous section can be repeated by keeping gravity in the radial and axial pressure balance conditions. We order $c_S^2 \sim \Omega^2R^2$ so that we can consider the thin disk limit $c_S^2 \ll \Omega^2R^2$, as well as the extended, non-disk global equilibrium that exist more generally (Catto *et al.* 2015).

We first obtain the equations for general Keplerian rotation by letting $\Omega^2 \to \Omega^2 - M_0G_0r^{-3}$ in (4.12) and (4.13) except in the $2\mathrm{i}\omega\Omega\xi_\zeta$ term in (4.13). No changes are needed in (4.11). The equations for the components of the displacement then become

$$\mathrm{i}\omega(\omega^2 - k_z^2v_A^2)\xi_\zeta = [(\omega^2 - k_z^2v_A^2)R\,\mathrm{d}\Omega/\mathrm{d}R + 2\omega^2\Omega]\xi_R, \tag{5.1}$$

$$(\omega^2 - k_z^2c_S^2)\xi_z = -\frac{\mathrm{i}k_zc_S^2}{R}\frac{\partial}{\partial R}(R\xi_R) - \mathrm{i}k_zR(\Omega^2 - M_0G_0r^{-3})\xi_R, \tag{5.2}$$

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and

$$(\omega^2 - k_z^2 v_A^2)\,\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]$$
$$= 2\mathrm{i}\omega\Omega\xi_\zeta - \frac{\mathrm{i}k_z}{n_0}\frac{\partial}{\partial R}(n_0 c_S^2 \xi_z) + \mathrm{i}k_z R(\Omega^2 - M_0 G_0 r^{-3})\xi_z - \xi_R R\frac{\partial}{\partial R}(\Omega^2 - M_0 G_0 r^{-3}). \tag{5.3}$$

Next we assume strict Keplerian rotation by using $\Omega^2 = M_0 G_0 r^{-3}$ in (5.2) and (5.3). Again, eliminating ξ_z first gives

$$(\omega^2 - k_z^2 v_A^2)\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]$$
$$= 2\mathrm{i}\omega\Omega\xi_\zeta - \frac{k_z^2}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^4}{(\omega^2 - k_z^2 c_S^2)R}\frac{\partial}{\partial R}(R\xi_R)\right]. \tag{5.4}$$

Then, eliminating ξ_ζ gives the radial eigenvalue equation

$$\left[\omega^2 - k_z^2 v_A^2 - \frac{4\omega^2\Omega^2}{(\omega^2 - k_z^2 v_A^2)} - R\frac{\partial\Omega^2}{\partial R}\right]\xi_R$$
$$+ \frac{\omega^2}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{(\omega^2 - k_z^2 c_S^2)R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] = 0. \tag{5.5}$$

Solving by geometric optics ($k_R^2 R^2 \gg 1$) the dispersion relation that results is

$$k^2 v_A^2 + R\frac{\partial\Omega^2}{\partial R} + \frac{4\omega^2\Omega^2}{(\omega^2 - k_z^2 v_A^2)} - \frac{\omega^2 k_R^2}{k_z^2} = 0, \tag{5.6}$$

where $k^2 = k_z^2 + k_R^2$. We allow $k_z^2 c_S^2 \sim R^2\Omega^4/c_S^2 \sim \Omega^2 \sim k_R^2 v_A^2 \gtrsim k_z^2 v_A^2 \gtrsim \omega^2$ and $k_z^2 R^2 \gtrsim 1$, and note that $\beta \gg 1$. Graphically, (5.6) gives magnetorotational (or equivalently, magnetogravitational) instability to be the standard result of Balbus & Hawley (1991),

$$\frac{k^2 v_A^2}{\Omega^2} < -\frac{R}{\Omega^2}\frac{\partial\Omega^2}{\partial R}, \tag{5.7}$$

where for Keplerian rotation $-R\Omega^{-2}\partial\Omega^2/\partial R = 3$. The preceding is in agreement with Balbus & Hawley (1991) provided their small $\partial p_0/\partial R$ and $\partial p_0/\partial z$ terms are ignored. In Hawley *et al.* (1995) the entropy equation is corrected to be consistent with energy conservation (recall our entropy equation after (3.2)). Condition (5.7) gives the MRI as $B_0 \to 0$. Stability may be possible for strong magnetic fields since $k_R^2 R^2 \lesssim 1$ gives $3\Omega^2 R^2/v_A^2 \lesssim 1$, implying $\beta \lesssim 2c_S^2/3\gamma\Omega^2 R^2 \sim 1$ for our $c_S^2 \sim \Omega^2 R^2$ ordering. Interestingly, in the hydrodynamic limit ($v_A^2 \equiv 0$) if $\omega^2 \ll k_z^2 c_S^2$, (5.4) reduces to $\omega^2 k^2 k_z^{-2} = R^{-3}\partial(\Omega^2 R^4)/\partial R$.

This strict Keplerian case is the one that the insightful physical interpretation by Balbus (2006) of masses attached by springs is most appropriate. The bare bones strict Keplerian WKB treatment reduces (5.1)–(5.3) to

$$\mathrm{i}\omega(\omega^2 - k_z^2 v_A^2)\xi_\zeta = [(\omega^2 - k_z^2 v_A^2)R\,\mathrm{d}\Omega/\mathrm{d}R + 2\omega^2\Omega]\xi_R, \tag{5.8}$$

$$(\omega^2 - k_z^2 c_S^2)\xi_z = -\frac{\mathrm{i}k_z c_S^2}{R}\frac{\partial}{\partial R}(R\xi_R), \tag{5.9}$$

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and

$$\frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial (R\xi_R)}{\partial R}\right]+\frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]=2\mathrm{i}\omega\Omega\xi_\zeta-\frac{\mathrm{i}k_z}{n_0}\frac{\partial}{\partial R}(n_0 c_S^2\xi_z), \tag{5.10}$$

from which we obtain

$$\frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]=\left[R\frac{\partial\Omega^2}{\partial R}+\frac{4\omega^2\Omega^2}{(\omega^2-k_z^2v_A^2)}\right]\xi_R-\frac{\omega^2}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{(\omega^2-k_z^2c_S^2)R}\frac{\partial}{\partial R}(R\xi_R)\right], \tag{5.11}$$

where we can now use $k_z^2c_S^2\gg\omega^2$ to further simplify the last term to obtain (5.6). If we simply want to get the MRI threshold ($\omega\to 0$) we may assume incompressibility in (5.9) to remove the c_S^2 terms in (5.10) to obtain

$$\mathrm{i}\omega\xi_\zeta=\xi_R R\,\mathrm{d}\Omega/\mathrm{d}R \tag{5.12}$$

and

$$\frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]=2\mathrm{i}\omega\Omega\xi_\zeta. \tag{5.13}$$

The only non-Alfvénic term is from the inertial term $2\mathrm{i}\omega\Omega\xi_\zeta$ in (5.13) since (5.12) is from the right side of (3.17). The simple toroidal momentum conservation equation of (5.12) is recovered from the vanishing of the toroidal component of the perturbed magnetic field for a frozen-in flow, (3.6), or equivalently, the vanishing of the perturbed radial current density, $\nabla\cdot(\boldsymbol{B}_1\times\nabla\psi_0)=0$. Equation (5.13) is radial momentum balance between the inertial term and $\boldsymbol{J}\times\boldsymbol{B}$ force to lowest order:

$$Mn_0(\boldsymbol{V}_1\cdot\nabla\boldsymbol{V}_0+\boldsymbol{V}_0\cdot\nabla\boldsymbol{V}_1)\cdot\nabla\psi_0\simeq[c^{-1}\boldsymbol{J}_0\times\boldsymbol{B}_1+(4\pi)^{-1}(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0]\cdot\nabla\psi_0. \tag{5.14}$$

Equations (5.12) and (5.13) are the ones for which the Balbus (2006) physical interpretation is valid. For a marginally unstable mode $\omega\to\mathrm{i}\,\mathrm{Im}\,\omega$ with $\mathrm{Im}\,\omega>0$ we see from (5.12) that $\xi_\zeta\to 3\Omega\xi_R/2\,\mathrm{Im}\,\omega$, where Im denotes the imaginary part. Using $\partial^2\xi_R/\partial R^2\to -k_R^2\xi_R$ in (5.13) gives $\xi_R\to 2\Omega\xi_\zeta\mathrm{Im}\,\omega/k_R^2v_A^2$, where the signs of two displacements ξ_R and ξ_ζ are seen to be the same. Therefore, two incremental axisymmetric toroidal mass rings stacked one above the other with oppositely directed initial radial displacements rotate toroidally in opposite directions and continue to separate radially due to the MRI. Noticing $\xi_\zeta\mathrm{Im}\,\omega\to -R\nabla\zeta\cdot\boldsymbol{V}_1$, the upper mass ring with an initial radial perturbation $\xi_R>0$ slows ($\xi_\zeta>0$) when moved outward, thereby decreasing its angular rotation frequency ($\Omega\propto R^{-3/2}$) and increasing its angular momentum ($\Omega R^2\propto R^{1/2}$). Similarly, the initial $\xi_R<0$ radial perturbation of the lower mass ring increases its angular rotation frequency ($\xi_\zeta<0$) as it moves inward, causing its angular momentum to drop. In the unstable limit, the tension of the frozen-in magnetic field lines resists any decrease (or increase) in angular rotation frequency of the upper (lower) ring by a positive (negative) torque that increases its angular momentum, thereby moving the ring to still larger (smaller) R and further lowering (raising) its angular frequency. As a result, the upper mass ring continues to move to a larger radius, thereby lowering its angular frequency, while the lower mass ring moves to smaller radii in an attempt to gain angular frequency. The presence of a magnetic field importantly changes the azimuthal component (3.17) of the conservation of angular momentum equation. In the absence of a magnetic field only the left side of (3.17) enters and acts to try to keep the angular momentum ΩR^2 constant. If a magnetic field is present then field line bending dominates at marginality and tries to keep the angular frequency Ω fixed.

5.1. *Self-consistent strict Keplerian equilibrium for an axial magnetic field*

In the strict Keplerian limit, Catto *et al.* (2015) obtain $\beta > 1$ analytic and numerical results with the equatorial plane plasma density increasing linearly with R for a thin equatorial plane disk when $M\Omega^2R^2/4T = M_0G_0M/4TR \gg 1$. In this case the disk width Δ is given by $\Delta/R = 2T^{1/2}/M^{1/2}\Omega R \ll 1$. Perturbation wavelengths in the z direction are limited to the disk width and therefore must satisfy $k_z\Delta > 2\pi$, while our eikonal treatment requires $k_R^2R^2 \gg 1$. Assuming $k_R^2 > k_z^2$ in the condition for instability requires $3\gamma\beta\Omega^2R^2/2c_S^2 > k_R^2R^2 > k_z^2R^2 > 4\pi^2R^2/\Delta^2 = 2\pi^2\gamma\Omega^2R^2/c_S^2$ or

$$\beta > 4\pi^2/3. \tag{5.15}$$

Therefore, the high β ordering is appropriate with large β needed for instability as might be expected. Moreover, thin disk equilibria exist for the self-similar equilibria of Catto *et al.* (2015) in the strict Keplerian limit when $H \simeq (1-\mu^2)^{-\alpha/3}$ inside the thin disk, transitions to $H \simeq (1-\mu^2)^{-\alpha/2}$ outside the disk and $\alpha \approx -2$. Global extended or non-disk strict Keplerian self-similar solutions are found when $M\Omega^2R^2/4T = M_0G_0M/4TR \lesssim 1$.

The preceding treatment might be viewed as somewhat unsatisfactory since we seem to have assumed $\Omega^2 = M_0G_0r^{-3}$ for all z. To counter the imbalance between the centrifugal and gravitational forces the density must depend on z and satisfy parallel pressure balance. Consequently, in the next section we justify the preceding limit by extending the analysis of (5.1)–(5.3) to a fully self-consistent configuration allowing z variation without assuming strict Keplerian rotation.

6. Axial magnetic field with gravity for general rotation

When the equilibrium magnetic field is axial as in (4.1) and unperturbed axial variation of the plasma density retained, the calculation of the previous sections can be repeated for general Keplerian motion by using a geometric optics approach in the z direction that seeks solutions of the form $\boldsymbol{\xi} \propto e^{-i\omega t + ik_z z}$. The equations obtained are identical to (5.1)–(5.3) which retain gravity in the radial and axial pressure balance conditions. Parallel momentum balance accounts for the difference between Ω^2R^2 and M_0G_0/r and results in the unperturbed pressure gradient term in (3.18). The axial eikonal treatment then requires $n_1\boldsymbol{B}_0 \cdot \nabla p_0 \ll n_0\boldsymbol{B}_0 \cdot \nabla p_1$ or $k_z\Delta \gg 1$, with Δ the disk thickness.

For the general Keplerian case self-similar solutions are found to exist when $\Omega^2R^2 \geqslant M_0G_0/R$ in the equatorial plane, but not for $\Omega^2R^2 < M_0G_0/R$ as gravity is so strong it cannot be balanced by the centrifugal force. In this general Keplerian limit thin disk equilibria require $M_0G_0M/4TR \gg 1$, but when $\Omega^2R^2 > M_0G_0/R$ the Maxwell–Boltzmann behaviour of the density as given by (2.7) only seems to allow thin disks with $\beta \gg 1$ in the strict Keplerian limit $\Omega^2R^2 = M_0G_0/R$ (Catto *et al.* 2015). We continue to order $c_S^2 \sim \Omega^2R^2$ to be able to treat the thin disks of the strict Keplerian limit as well as the global solutions of more general Keplerian motion.

Eliminating ξ_z by inserting (5.2) into (5.3) gives

$$\begin{aligned}\left[\omega^2 - k_z^2v_A^2 + R\frac{\partial}{\partial R}\left(\Omega^2 - \frac{M_0G_0}{r^3}\right)\right]\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] \\ + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] = 2i\omega\Omega\xi_\zeta + \frac{1}{n_0}\left[\frac{R}{c_S^2}\left(\Omega^2 - \frac{M_0G_0}{r^3}\right) - \frac{\partial}{\partial R}\right] \\ \times\left\{\frac{n_0k_z^2c_S^2}{(\omega^2 - k_z^2c_S^2)}\left[R\left(\Omega^2 - \frac{M_0G_0}{r^3}\right)\xi_R + \frac{c_S^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]\right\}.\end{aligned} \tag{6.1}$$

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Then using (5.1) to eliminate ξ_ζ from (6.1) gives the radial differential equation

$$\left[\omega^2 - k_z^2 v_A^2 - R\frac{\partial}{\partial R}\left(\frac{M_0 G_0}{r^3}\right)\right]\xi_R + \frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]$$
$$= \frac{4\omega^2\Omega^2\xi_R}{(\omega^2 - k_z^2 v_A^2)} + \frac{1}{n_0}\left[\frac{R}{c_S^2}\left(\Omega^2 - \frac{M_0 G_0}{r^3}\right) - \frac{\partial}{\partial R}\right]$$
$$\times \left\{\frac{n_0 k_z^2 c_S^2}{(\omega^2 - k_z^2 c_S^2)}\left[R\left(\Omega^2 - \frac{M_0 G_0}{r^3}\right)\xi_R + \frac{c_S^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]\right\}. \tag{6.2}$$

Equation (6.2) is the radial differential equation for general rotation in an axial magnetic field for the MRI instability when an axial eikonal treatment is employed.

We seek a WKB, geometric optics or eikonal radial solution by using $\xi_R \propto \mathrm{e}^{\mathrm{i}\int k_R\,\mathrm{d}R}$ and $\partial\xi_R/\partial R = \mathrm{i}k_R\xi_R$ with

$$k_R R \sim R\xi_R^{-1}\partial\xi_R/\partial R \gg R n_0^{-1}\partial n_0/\partial R \sim R p_0^{-1}\partial p_0/\partial R \sim R\Omega^{-1}\partial\Omega/\partial R \sim 1, \tag{6.3}$$

by allowing

$$k_z^2 c_S^2 \sim R^2\Omega^4/c_S^2 \sim \Omega^2 \sim M_0 G_0 r^{-3} \sim k_R^2 v_A^2 \gtrsim k_z^2 v_A^2 \gtrsim \omega^2, \tag{6.4}$$

$k_z^2 R^2 \gtrsim 1$ and $\beta \gg 1$. From (6.2) we obtain the dispersion relation

$$k^2 v_A^2 + R\frac{\partial}{\partial R}\left(\frac{M_0 G_0}{r^3}\right) + \frac{4\omega^2\Omega^2}{(\omega^2 - k_z^2 v_A^2)} - \frac{\omega^2 k_R^2}{k_z^2} - \frac{R^2(\Omega^2 - M_0 G_0 r^{-3})^2}{c_S^2}$$
$$+ \frac{R}{n_0}\frac{\partial}{\partial R}[n_0(\Omega^2 - M_0 G_0 r^{-3})] = 0, \tag{6.5}$$

where $k^2 = k_z^2 + k_R^2$. Graphically, the behaviour near $\omega^2 = 0$ gives gravitational magnetorotational instability when

$$k^2 v_A^2 + R\frac{\partial\Omega^2}{\partial R} < \frac{R^2(M_0 G_0 r^{-3} - \Omega^2)^2}{c_S^2} + (M_0 G_0 r^{-3} - \Omega^2)\frac{R}{n_0}\frac{\partial n_0}{\partial R}. \tag{6.6}$$

Clearly, the preceding reduces to the Velikhov (1959) result (4.18) in the absence of gravity and the Balbus & Hawley (1991) result (5.7) when $\Omega^2 = M_0 G_0 r^{-3}$. More importantly, it is clear that unless the motion is strictly Keplerian, compressibility and density gradient effects will impact the stability threshold.

For a hydrodynamic equilibrium ($v_A^2 = 0$), equation (6.2) yields the dispersion relation

$$\omega^2 k^2 k_z^{-2} = R^{-3}\partial(\Omega^2 R^4)/\partial R - (\Omega^2 - M_0 G_0 r^{-3}) n_0^{-1}\partial n_0/\partial R - R^2(\Omega^2 - M_0 G_0 r^{-3})^2/c_S^2. \tag{6.7}$$

For strict Keplerian motion we see that instability occurs whenever angular momentum decreases with radius.

We can interpret (6.6) by first considering radial force balance for two adjacent, axisymmetric, incompressible plasma rings ($j = 1, 2$) near the equatorial plane with radius R_j, plasma mass density $M n_j = \rho_j$, rotation frequency Ω_j and pressure p_j:

$$\partial p_j/\partial R = \rho_j(\Omega_j^2 R_j - M_0 G_0/R_j^2). \tag{6.8}$$

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If the first ring is displaced outward to the location of the second ring ($R_1 \rightarrow R_2 > R_1$ and $\rho_1 \rightarrow \rho_1$), the frozen-in magnetic field lines moves with it. Therefore, the ring continues to rotate at the original angular frequency Ω_1 of these frozen in field lines. Consequently, the resulting torques due to field line bending act on the first ring to keep its angular frequency fixed while its fluid element is moved ($\Omega_1 \rightarrow \Omega_1$), as pointed out by Velikhov (1959) and Balbus & Hawley (1991). After being displaced the first ring is acted on by the pressure difference across its new location $\partial p_2/\partial R$. However, force balance is no longer satisfied as a restoring force ΔF arises due to its movement:

$$\Delta F + \partial p_2/\partial R = \rho_1(\Omega_1^2 R_2 - M_0 G_0/R_2^2). \tag{6.9}$$

Subtracting force balance for the initial second ring from the preceding expression gives

$$\begin{aligned} \Delta F &= \rho_1(\Omega_1^2 R_2 - M_0 G_0/R_1^2) - \rho_2(\Omega_2^2 R_2 - M_0 G_0/R_2^2) \\ &= -\left[\rho R \frac{\partial \Omega^2}{\partial R} + \frac{\partial \rho}{\partial R}(\Omega^2 R - M_0 G_0/R^2)\right](R_2 - R_1). \end{aligned} \tag{6.10}$$

If $\Delta F < 0$, then the new plasma pressure difference across the displaced ring returns it to its original location. If instead, $\Delta F > 0$ then the plasma pressure across the ring is too small to prevent the displaced ring from continuing to move outward. Therefore, in agreement with (6.6) the incompressible instability condition is

$$nR\frac{\partial \Omega^2}{\partial R} + \frac{\partial n}{\partial R}(\Omega^2 R - M_0 G_0/R^2) < 0. \tag{6.11}$$

To get the marginal stability condition in general rotation case we need only combine the following $\omega \rightarrow 0$ versions of (5.1)–(5.3):

$$\mathrm{i}\omega\xi_\zeta = \xi_R R\, \mathrm{d}\Omega/\mathrm{d}R, \tag{6.12}$$

$$\mathrm{i}k_z\xi_z = -\frac{1}{R}\frac{\partial}{\partial R}(R\xi_R) - (\Omega^2 - M_0 G_0 r^{-3})\frac{R\xi_R}{c_S^2}, \tag{6.13}$$

and

$$\begin{aligned} &\frac{1}{n_0}\frac{\partial}{\partial R}\left[\frac{n_0 c_S^2}{R}\frac{\partial(R\xi_R)}{\partial R}\right] + \frac{v_A^2}{B_0^2}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right] = 2\mathrm{i}\omega\Omega\xi_\zeta - \frac{\mathrm{i}k_z}{n_0}\frac{\partial}{\partial R}(n_0 c_S^2 \xi_z) \\ &\quad + \mathrm{i}k_z R(\Omega^2 - M_0 G_0 r^{-3})\xi_z - \xi_R R\frac{\partial}{\partial R}(\Omega^2 - M_0 G_0 r^{-3}). \end{aligned} \tag{6.14}$$

Of course these equations are more complicated than (5.12) and (5.13) because compressibility effects must be retained when $\Omega^2 \neq M_0 G_0 r^{-3}$.

6.1. *Self-consistent equilibrium in an axial magnetic field with gravity at $\beta \gg 1$*

We consider the limit of the self-consistent equilibrium of Catto *et al.* (2015) with B_0 axial, $\Omega^2 \propto R^{-3}$, $T_0 \propto R^{-1}$ and axial pressure balance in the vicinity of the equatorial plane requiring

$$p_0 = 2n_0 T_0 = \bar{p}_0 \mathrm{e}^{-(z/\Delta)^2}, \tag{6.15}$$

with $\bar{p}_0$ a constant and $\Delta/R = 2(TR/MM_0G_0)^{1/2}$. In this limit the motion cannot be strictly Keplerian since $\Omega^2 > M_0 G_0 R^{-3}$ is required to find a solution. Also, disk

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formation is unlikely for the nearly axial equilibrium solutions found as β is required to be extremely large: $\beta \gg [\Omega^2/(\Omega^2 - M_0 G_0 R^{-3})]^2 \mathrm{e}^{-3(R/\Delta)^2}$. Taking $\Delta/R \sim 1$, we can use $n_0 \propto R$ to find $R\partial n_0/\partial R \simeq R$. In addition, we have $R\partial \Omega^2/\partial R = -3\Omega^2$ so that using (6.6) and $k_R^2 R^2 \gg 1$ for $\beta \gg 1$ gives instability when

$$\frac{2c_S^2}{\gamma\beta\,\Omega^2 R^2} \ll \frac{k^2 v_A^2}{\Omega^2} < 2 + \frac{M_0 G_0}{\Omega^2 r^3} + \frac{R^2(\Omega^2 - M_0 G_0 r^{-3})^2}{\Omega^2 c_S^2}, \tag{6.16}$$

where $\beta \gg 2c_S^2/\gamma\,\Omega^2 R^2$. Consequently, the departure from strict Keplerian rotation increases the size of the instability window when

$$\frac{R^2(\Omega^2 - M_0 G_0 r^{-3})^2}{\Omega^2 c_S^2} > 1 - \frac{M_0 G_0}{\Omega^2 r^3} > 0, \tag{6.17}$$

and decreases it otherwise. The smallest non-disk window of instability occurs in the Velikhov limit (4.24) when $\Omega^2 R^2/c_S^2 \ll 1$ and $M_0 G_0/\Omega^2 r^3 \ll 1$. Stability may only be possible for strong magnetic fields when WKB fails ($k^2 R^2 \lesssim 1$) and

$$\frac{\Omega^2 R^2}{v_A^2}\left[2 + \frac{M_0 G_0}{\Omega^2 r^3} + \frac{R^2(\Omega^2 - M_0 G_0 r^{-3})^2}{\Omega^2 c_S^2}\right] < k_R^2 R^2 \lesssim 1, \tag{6.18}$$

implying $\beta \lesssim 2c_S^2/3\gamma\,\Omega^2 R^2 \sim 1$ for our $c_S^2 \sim \Omega^2 R^2$ and $M_0 G_0/\,\Omega^2 r^3 \sim 1$ ordering.

Often the magnetic field is not cylindrically symmetric so the self-similar form of (4.14), $\psi_0 = CH(\cos\vartheta)/r^\alpha$, allows solutions away from $\alpha \approx -2$ (see figures 1, 3 and 4 of Catto *et al.* (2015), for example, where $\alpha \approx -3$, -4 and -6 respectively). For smooth, slowly varying departures from cylindrical symmetry for which the radial magnetic field component is sufficiently weak, (6.16) can be generalized slightly by using $n_0 \propto R^{-2\alpha-3}$. In such cases

$$\frac{2c_S^2}{\gamma\beta\Omega^2 R^2} \ll \frac{k^2 v_A^2}{\Omega^2} < 2 + \frac{M_0 G_0}{\Omega^2 r^3} + (2\alpha + 4)\frac{(\Omega^2 - M_0 G_0 r^{-3})}{\Omega^2} + \frac{R^2(\Omega^2 - M_0 G_0 r^{-3})^2}{\Omega^2 c_S^2}, \tag{6.19}$$

with $\alpha \leqslant -2$ an eigenvalue determined by solving the self-similar form of the Grad–Shafranov equation, and where $\alpha = -2$ corresponds to the cylindrically symmetric limit. When $\alpha < -2$, stronger density variation further narrows the instability window; however, once $|\alpha|$ becomes too large the radial magnetic field may no longer be negligible.

It is important to realize that equilibrium solutions can be found for $\beta \gg 1$ and $\Omega^2 > M_0 G_0 R^{-3}$ with very strong departures from cylindrical symmetry (see figures 2 and 5 of Catto *et al.* (2015) for examples). The general, but rather complicated, magnetorotational stability treatment in the next section is intended to deal these and other non-cylindrically symmetric equilibrium solutions for which the radial magnetic field cannot be ignored.

7. Stability in a general poloidal magnetic field for $\beta > 1$

In a strong gravitational field at high β, the magnetic field is comparatively weak. The rotating frozen-in plasma is more easily distorted, with the field lines bulging outward substantially and the possibility of quite substantial departures from cylindrical symmetry.

To perform the stability analysis for an arbitrary axisymmetric poloidal magnetic field it is convenient to use the azimuthal, parallel and $Q=\psi_0$ components of the $\partial^2\boldsymbol{\xi}/\partial t^2$ equation:

$$R\partial^3\xi_\zeta/\partial t^3+\nabla(\Omega R^2)\cdot\partial^2\boldsymbol{\xi}/\partial t^2=v_A^2B_0^{-2}\boldsymbol{B}_0\cdot\nabla[R^2\boldsymbol{B}_0\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\Omega+R^{-1}\partial\xi_\zeta/\partial t)], \tag{7.1}$$

$$\left[\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}-2\frac{\partial\xi_\zeta}{\partial t}\Omega\nabla R-\frac{n_1}{n_0}\frac{\nabla p_0}{Mn_0}+\frac{\nabla p_1}{Mn_0}\right]\cdot\boldsymbol{B}_0=\frac{v_A^2}{B_0^3}\left[\nabla\cdot(R^{-2}\nabla\psi_0)\right]\boldsymbol{B}_0\cdot\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0), \tag{7.2}$$

and

$$\left[\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}-2\frac{\partial\xi_\zeta}{\partial t}\Omega\nabla R-\frac{n_1}{n_0}\left(\Omega^2R\nabla R-\frac{M_0G_0}{r^2}\nabla r\right)+\frac{\nabla p_1}{Mn_0}\right]\cdot\nabla\psi_0 \\ =v_A^2R^2\left[\nabla\cdot(R^{-2}\nabla\psi_0)\right]\frac{\partial}{\partial\psi_0}(\boldsymbol{\xi}\cdot\nabla\psi_0)+v_A^2R^2\nabla\cdot[R^{-2}\nabla(\boldsymbol{\xi}\cdot\nabla\psi_0)], \tag{7.3}$$

where to remove J_0 in the last equation we used the Grad–Shafranov equation

$$4\pi J_0/cR=-\nabla\cdot(R^{-2}\nabla\psi_0)=R^{-2}v_A^{-2}\nabla\psi_0\cdot[(1/Mn_0)\nabla p_0+M_0G_0r^{-2}\nabla r-\Omega^2R\nabla R]. \tag{7.4}$$

To complete the description we only need n_1 and p_1 as given in § 3.

For the general analysis we write

$$\nabla=\nabla\psi_0\partial/\partial\psi_0+B_0^{-2}\boldsymbol{B}_0\boldsymbol{B}_0\cdot\nabla+\nabla\zeta\,\partial/\partial\zeta, \tag{7.5}$$

$$\begin{aligned}\boldsymbol{\xi}&=(RB_0)^{-2}(\boldsymbol{\xi}\cdot\nabla\psi_0)\nabla\psi_0+B_0^{-2}\boldsymbol{\xi}\cdot\boldsymbol{B}_0\boldsymbol{B}_0+\xi_\zeta R\nabla\zeta\\&=(RB_0)^{-2}\xi_\psi\nabla\psi_0+B_0^{-1}\xi_\parallel\boldsymbol{B}_0+\xi_\zeta R\nabla\zeta,\end{aligned} \tag{7.6}$$

and

$$\nabla\cdot\boldsymbol{\xi}=\boldsymbol{B}_0\cdot\nabla(B_0^{-1}\xi_\parallel)+\nabla\cdot(R^{-2}B_0^{-2}\xi_\psi\nabla\psi_0)=\boldsymbol{B}_0\cdot\nabla(B_0^{-1}\xi_\parallel)+\nabla\cdot(B_0^{-2}\xi_\psi\nabla\zeta\times\boldsymbol{B}_0). \tag{7.7}$$

Using the preceding, the equations for ξ_ζ and $\xi_\parallel$ become

$$R\partial^3\xi_\zeta/\partial t^3+R^2(\mathrm{d}\Omega/\mathrm{d}\psi_0)\partial^2\xi_\psi/\partial t^2+2\Omega\left(B_{0z}B_0^{-2}\partial^2\xi_\psi/\partial t^2+RB_{0R}B_0^{-1}\partial^2\xi_\parallel/\partial t^2\right) \\ =v_A^2B_0^{-2}\boldsymbol{B}_0\cdot\nabla\left\{R^2\boldsymbol{B}_0\cdot\nabla\left[\xi_\psi(\mathrm{d}\Omega/\mathrm{d}\psi_0)+R^{-1}\partial\xi_\zeta/\partial t\right]\right\} \tag{7.8}$$

and, after using the Grad–Shafranov equation and inserting p_1 and n_1,

$$\frac{\partial^2\xi_\parallel}{\partial t^2}-2\frac{\partial\xi_\zeta}{\partial t}\Omega B_{0R}B_0^{-1}-\frac{\gamma\boldsymbol{B}_0\cdot\nabla(p_0\nabla\cdot\boldsymbol{\xi})}{Mn_0B_0}-\frac{\xi_\psi\boldsymbol{B}_0\cdot\nabla(\partial p_0/\partial\psi_0)}{Mn_0B_0} \\ -\frac{\xi_\parallel\boldsymbol{B}_0\cdot\nabla(n_0\boldsymbol{B}_0\cdot\nabla p_0)}{Mn_0^2B_0^2}=-\frac{\boldsymbol{B}_0\cdot\nabla p_0}{Mn_0^2B_0}\nabla\cdot\left(\frac{n_0\xi_\psi}{R^2B_0^2}\nabla\psi_0\right) \\ -R^{-2}B_0^{-3}\nabla\psi_0\cdot[M_0G_0r^{-2}\nabla r-\Omega^2R\nabla R]\boldsymbol{B}_0\cdot\nabla\xi_\psi, \tag{7.9}$$

where

$$B_{0R}=\boldsymbol{B}_0\cdot\nabla R \tag{7.10}$$

and

$$RB_{0z} = \nabla R \cdot \nabla \psi_0. \tag{7.11}$$

Inserting p_1 and n_1 and using

$$\begin{aligned}\frac{R^2B_0^2}{Mn_0}\frac{\partial}{\partial\psi_0}\left(\xi_\psi\frac{\partial p_0}{\partial\psi_0}\right) &= -R^2v_A^2\frac{\partial}{\partial\psi_0}\left[\xi_\psi\nabla\cdot(R^{-2}\nabla\psi_0)\right] \\ &\quad -\frac{R^2B_0^2}{n_0}\frac{\partial}{\partial\psi_0}\left[\frac{\xi_\psi n_0}{R^2B_0^2}\left(\frac{M_0G_0}{r^2}\nabla r - \Omega^2R\nabla R\right)\cdot\nabla\psi_0\right],\end{aligned} \tag{7.12}$$

we obtain a convenient form for the third equation:

$$\begin{aligned}&\frac{\partial^2\xi_\psi}{\partial t^2} - 2\frac{\partial\xi_\zeta}{\partial t}\Omega RB_{0z} - \frac{R^2B_0^2}{Mn_0}\frac{\partial}{\partial\psi_0}\left(\gamma p_0\nabla\cdot\boldsymbol{\xi} + \xi_\parallel B_0^{-1}\boldsymbol{B}_0\cdot\nabla p_0\right) \\ &\quad -(n_0^{-1}\boldsymbol{\xi}\cdot\nabla n_0 + \nabla\cdot\boldsymbol{\xi})\left(\frac{M_0G_0}{r^2}\nabla r - \Omega^2R\nabla R\right)\cdot\nabla\psi_0 \\ &\quad + n_0^{-1}\nabla\psi_0\cdot\nabla\left[\frac{\xi_\psi n_0}{R^2B_0^2}\left(\frac{M_0G_0}{r^2}\nabla r - \Omega^2R\nabla R\right)\cdot\nabla\psi_0\right] \\ &= v_A^2R^2\left[\boldsymbol{B}_0\cdot\nabla\left(\frac{\boldsymbol{B}_0\cdot\nabla\xi_\psi}{R^2B_0^2}\right) - \xi_\psi\frac{\partial}{\partial\psi_0}\nabla\cdot(R^{-2}\nabla\psi_0) + \nabla\cdot\left(\frac{\partial\xi_\psi}{\partial\psi_0}R^{-2}\nabla\psi_0\right)\right],\end{aligned} \tag{7.13}$$

where in the axial magnetic field limit

$$\nabla\cdot\left(\frac{\partial\xi_\psi}{\partial\psi_0}R^{-2}\nabla\psi_0\right) - \xi_\psi\frac{\partial}{\partial\psi_0}\nabla\cdot(R^{-2}\nabla\psi_0) \to \frac{1}{B_0^2R}\frac{\partial}{\partial R}\left[\frac{B_0^2}{R}\frac{\partial}{\partial R}(R\xi_R)\right]. \tag{7.14}$$

Using

$$\xi_\psi n_0^{-1}\partial n_0/\partial\psi_0 + \nabla\cdot(B_0^{-2}R^{-2}\xi_\psi\nabla\psi_0) = n_0^{-1}\nabla\cdot(B_0^{-2}R^{-2}n_0\xi_\psi\nabla\psi_0) \tag{7.15}$$

to simplify further gives

$$\begin{aligned}&\frac{\partial^2\xi_\psi}{\partial t^2} - \frac{R^2B_0^2}{Mn_0}\frac{\partial}{\partial\psi_0}\left(\gamma p_0\nabla\cdot\boldsymbol{\xi} + \xi_\parallel B_0^{-1}\boldsymbol{B}_0\cdot\nabla p_0\right) \\ &\quad + \xi_\psi B_0^{-1}\nabla\psi_0\cdot\nabla\left[\frac{(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0}{R^2B_0}\right] - 2\frac{\partial\xi_\zeta}{\partial t}\Omega RB_{0z} \\ &\quad - \left[\xi_\psi B_0^{-1}\nabla\cdot(B_0^{-1}R^{-2}\nabla\psi_0) + n_0^{-1}\boldsymbol{B}_0\cdot\nabla(B_0^{-1}n_0\xi_\parallel)\right]\left(\frac{M_0G_0}{r^2}\nabla r - \Omega^2R\nabla R\right)\cdot\nabla\psi_0 \\ &= v_A^2R^2\left[\boldsymbol{B}_0\cdot\nabla\left(\frac{\boldsymbol{B}_0\cdot\nabla\xi_\psi}{R^2B_0^2}\right) - \xi_\psi\frac{\partial}{\partial\psi_0}\nabla\cdot(R^{-2}\nabla\psi_0) + \nabla\cdot\left(\frac{\partial\xi_\psi}{\partial\psi_0}R^{-2}\nabla\psi_0\right)\right],\end{aligned} \tag{7.16}$$

where in the axial magnetic field limit $\nabla\cdot(B_0^{-1}R^{-2}\nabla\psi_0) \to 0$.

The three equations (7.8), (7.9) and (7.16) are the most general forms of our equations for the three components of the displacement ξ_ζ, $\xi_\parallel$ and ξ_ψ. They are the full ideal MHD representation for the magneto-rotational instability for axisymmetric

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perturbations to a self-consistent axisymmetric magnetic field having both radial and axial components at high β.

Our three displacement equations are now in the desired form to begin to employ the eikonal approximation by assuming

$$\boldsymbol{\xi} \propto \mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}S}, \tag{7.17}$$

with S independent of ζ,

$$\boldsymbol{k} = \nabla S = k_\psi \nabla\psi_0 + k_\| B_0^{-1}\boldsymbol{B}_0, \tag{7.18}$$

$$k_\| = B_0^{-1}\boldsymbol{B}_0\cdot\nabla S, \tag{7.19}$$

$$k_\perp = k_\psi R B_0, \tag{7.20}$$

$k_\psi = \partial S/\partial\psi_0$, and the wave vector assumed to be slowly varying. As in earlier sections we only perform the eikonal treatment in the parallel direction to start. We need to be careful with ψ_0 derivatives in c_S^2 terms as before, but we also want to retain departures from cylindrical symmetry for the magnetic field.

For a general poloidal magnetic field we assume $k_\perp^2 R^2 \gg 1$ and

$$c_S^2/R^2 \sim k_\|^2 c_S^2 \sim \Omega^2 \sim k_\perp^2 v_A^2 \gg k_\|^2 v_A^2 \gtrsim \omega^2, \tag{7.21}$$

$k_\|^2 R^2 \gtrsim 1$ and $\beta \gg 1$ and use $k^2 = k_\|^2 + k_\perp^2$. We assume small wavelengths parallel unperturbed magnetic field,

$$k_\| \sim \xi_\psi^{-1} B_0^{-1}\boldsymbol{B}_0\cdot\nabla\xi_\psi \gg B_0^{-1}\boldsymbol{B}_0\cdot\nabla p_0 \sim n_0^{-1}B_0^{-2}\boldsymbol{B}_0\cdot\nabla B_0 \sim R^{-1}B_0^{-1}\boldsymbol{B}_0\cdot\nabla R, \tag{7.22}$$

so that

$$\boldsymbol{B}_0\cdot\nabla(\gamma p_0 \nabla\cdot\boldsymbol{\xi}) \gg n_1 n_0^{-1}\boldsymbol{B}_0\cdot\nabla p_0 \gg \xi_\| B_0^{-1} n_0^{-1}\boldsymbol{B}_0\cdot\nabla(n_0\boldsymbol{B}_0\cdot\nabla p_0) \sim \xi_\psi \boldsymbol{B}_0\cdot\nabla(\partial p_0/\partial\psi_0), \tag{7.23}$$

and recall $\boldsymbol{B}_0\cdot\nabla(\mathrm{d}\Omega/\mathrm{d}\psi_0) = 0$. Then the ξ_ζ equation gives

$$\mathrm{i}\omega(\omega^2 - k_\|^2 v_A^2)\xi_\zeta = \left[(\omega^2 - k_\|^2 v_A^2)(R\,\mathrm{d}\Omega/\mathrm{d}\psi_0) + 2\omega^2\Omega R^{-1}B_0^{-2}B_{0z}\right]\xi_\psi + 2\omega^2\Omega B_0^{-1}B_{0R}\xi_\|. \tag{7.24}$$

The ξ_ζ equation contains $\xi_\|$ dependence which accounts for a new non-axial field effect.

Using the eikonal approximation on the equation for $\xi_\|$ and neglecting the small terms already noted we obtain

$$\begin{aligned}(\omega^2 - k_\|^2 c_s^2)\xi_\| &= 2\mathrm{i}\omega\Omega B_0^{-1}B_{0R}\xi_\zeta \\ &\quad + \mathrm{i}k_\|[\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0\cdot(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R) - c_S^2\nabla\cdot(\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0)],\end{aligned} \tag{7.25}$$

where once again all components of the displacement enter. In the $\xi_\|$ equation we have been careful to retain seemingly small terms. We will eventually find that the $\partial\xi_\psi/\partial\psi_0$ terms will cancel exactly and the remaining large $\partial^2\xi_\psi/\partial\psi_0^2$ term will enter with a small coefficient at marginality ($\omega^2 \to 0$) as in earlier sections. For this reason we must retain $\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0\cdot(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)$, as well as $c_S^2\nabla\cdot(\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0)$.

For the third equation we use

$$\partial\xi_\psi/\partial\psi_0 \gg \xi_\psi B_0^{-1}\partial B_0/\partial\psi_0 \sim \xi_\psi p_0^{-1}\partial p_0/\partial\psi_0 \sim \xi_\psi R^{-1}\partial R/\partial\psi_0, \tag{7.26}$$

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for v_A^2 terms, but we again keep seemingly small terms to carefully deal with the later cancelation in c_S^2 terms:

$$
\begin{aligned}
&(\omega^2 - k_\parallel^2 v_A^2)\xi_\psi + R^2 B_0^2 v_A^2 \frac{\partial^2 \xi_\psi}{\partial \psi_0^2} + \frac{c_S^2 R^2 B_0^2}{p_0}\frac{\partial}{\partial \psi_0}\left[p_0 \nabla\boldsymbol{\cdot}\left(\frac{\xi_\psi \nabla\psi_0}{B_0^2 R^2}\right)\right] \\
&\quad + \frac{\xi_\psi[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]^2}{R^2 B_0^2}\nabla\boldsymbol{\cdot}\left\{\frac{\nabla\psi_0}{[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]}\right\} \\
&= 2\mathrm{i}\omega\Omega R B_{0z}\xi_\zeta - \mathrm{i}k_\parallel\left[\xi_\parallel\left(\frac{M_0 G_0}{r^2}\nabla r - \Omega^2 R\nabla R\right)\boldsymbol{\cdot}\nabla\psi_0 + \frac{c_S^2 R^2 B_0^2}{p_0}\frac{\partial}{\partial\psi_0}(p_0\xi_\parallel)\right],
\end{aligned}
\tag{7.27}
$$

where to combine terms we use

$$
\begin{aligned}
R^2 B_0^2 (\partial/\partial\psi_0)(R^{-2}B_0^{-1}Q) &= \nabla\psi_0\boldsymbol{\cdot}\nabla(R^{-2}B_0^{-1}Q) \\
&= Q\nabla\boldsymbol{\cdot}(R^{-2}B_0^{-1}\nabla\psi_0) - Q^2 R^{-2}B_0^{-1}\nabla\boldsymbol{\cdot}(Q^{-1}\nabla\psi_0).
\end{aligned}
\tag{7.28}
$$

We retain all the $\xi_\parallel$ terms because of the cancelation that will occur as already noted.

To highlight a new effect due to the non-axial magnetic fields we consider marginality ($\omega\to 0$) for strict Keplerian rotation ($M_0 G_0 r^{-2}\nabla r\boldsymbol{\cdot}\nabla\psi_0 = \Omega^2 R\nabla R\boldsymbol{\cdot}\nabla\psi_0$), to reduce these equations to

$$
\mathrm{i}\omega\xi_\zeta = \xi_\psi R\,\mathrm{d}\Omega/\mathrm{d}\psi_0, \tag{7.29}
$$

$$
k_\parallel^2 c_s^2 \xi_\parallel = -2\mathrm{i}\omega\Omega B_0^{-1}B_{0R}\xi_\zeta + \mathrm{i}k_\parallel c_S^2\nabla\boldsymbol{\cdot}(\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0), \tag{7.30}
$$

and

$$
R^2 B_0^2 v_A^2 \frac{\partial^2\xi_\psi}{\partial\psi_0^2} + \frac{c_S^2 R^2 B_0^2}{p_0}\frac{\partial}{\partial\psi_0}\left[p_0\nabla\boldsymbol{\cdot}\left(\frac{\xi_\psi\nabla\psi_0}{B_0^2 R^2}\right)\right] = 2\mathrm{i}\omega\Omega R B_{0z}\xi_\zeta - \frac{\mathrm{i}k_\parallel c_S^2 R^2 B_0^2}{p_0}\frac{\partial}{\partial\psi_0}(p_0\xi_\parallel). \tag{7.31}
$$

Combining to eliminate ξ_ζ and $\xi_\parallel$ and using the radial eikonal form instability occurs if

$$
k^2 v_A^2 < -R^2[B_{0z} - k_\perp k_\parallel^{-1}B_{0R}]\,\mathrm{d}\Omega^2/\mathrm{d}\psi_0. \tag{7.32}
$$

The new term that appears in the preceding is always destabilizing when $k_\parallel^{-1}k_\perp B_{0R}B_{0z}^{-1} < 0$ and accounts for non-axial magnetic field effects. Moreover, instability occurs for $\mathrm{d}\Omega^2/\mathrm{d}\psi_0 > 0$ when $k_\parallel^{-1}k_\perp B_{0R}B_{0z}^{-1} > 1$, a new limit not previously noted.

The eikonal treatment for the general axisymmetric poloidal magnetic field case is quite complex so the details are relegated to the appendix A. The general condition for instability is found from $\omega^2 = 0$ to be

$$
\begin{aligned}
k^2 v_A^2 \;<\; &\frac{[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]^2}{R^2 B_0^2 c_S^2} - \left(B_{0z} - \frac{k_\perp}{k_\parallel}B_{0R}\right)R^2\frac{\mathrm{d}\Omega^2}{\mathrm{d}\psi_0} \\
&+ (M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0\frac{\partial n_0}{\partial\psi_0},
\end{aligned}
\tag{7.33}
$$

where $\nabla\psi_0\boldsymbol{\cdot}\nabla n_0 = R^2 B_0^2\partial n_0/\partial\psi_0$. For our ordering $(k_\perp/k_\parallel)^2 \gg 1$, making the new non-axial magnetic field term potentially important when $|k_\perp B_{0R}| \gtrsim |k_\parallel|B_{0z}$. The preceding is our most general expression for gravitational magnetorotational instability. Once again strict Keplerian motion removes compressibility and density gradient effects.

8. Discussion and conclusions

We have re-examined magnetorotational stability with and without gravity by retaining compressibility and density variation, and allowing general axisymmetric poloidal magnetic fields in fully self-consistent equilibria. Within the limitations of a geometric optics or eikonal analysis, we have obtained general expressions for the stability threshold, which is found to have a different form in the absence of gravity than in the presence of strict Keplerian motion, as can be seen by comparing (4.19) and (5.7). Further subtle and important modifications arise when gravity is present if the rotation frequency is general as can be seen by the appearance of compressibility and density gradient terms in (6.6). For the $\beta \gg 1$ equilibria of Catto *et al.* (2015) with rotation stronger than Keplerian these compressibility and density gradient effects become important as seen by (6.16). Interestingly, the incompressible limit without gravity, but with density variation is unstable for the model equilibrium of Catto & Krasheninnikov (2015) for a geometric optics approach when $2\Omega^2 > k^2 v_A^2$. However, in the strict Keplerian limit of Balbus & Hawley (1991), compressibility and density variation do not matter and $3\Omega^2 > k^2 v_A^2$ is required for instability.

In the absence of gravity our WKB results are consistent with the unbounded medium case of Velikhov (1959) in the incompressible limit he considered. Our general compressibility results modify his results and are illustrated by applying them to the self-consistent equilibrium of Catto & Krasheninnikov (2015). This analysis highlights the need to treat the radial WKB analysis carefully to fully retain compressibility and radial density variation as well as the radial variation of the rotation frequency. Compressibility is found to completely stabilize the Velikhov (1959) limit when $3c_S^2/\Omega^2 R^2 < 1$. However, when $3c_S^2/\Omega^2 R^2 > 1$ instability occurs when

$$1 - \frac{\Omega^2 R^2}{3c_S^2} > \frac{k^2 v_A^2}{2\Omega^2} = \frac{k^2 c_S^2}{\gamma \beta \Omega^2}, \tag{8.1}$$

thereby requiring $\beta \gg 1$. The self-similar equilibrium for this case without gravity has a magnetic field strength that increases with radius as in most of the cases with gravity.

In the presence of gravity our results are in agreement with Balbus & Hawley (1991) and Hawley *et al.* (1995) for strict Keplerian motion, but there are subtle and significant differences whenever there are departures from strict Keplerian motion or non-axial magnetic field effects. Departure from strict Keplerian motion gives rise to compressibility and density gradient effects in our cylindrically symmetric and general axisymmetric magnetic field treatments for $\beta > 1$. These extensions allow us to obtain a general threshold for gravitational magnetorotational instability in thermal magnetized plasmas. The results depend explicitly on gravity, compressibility, and density variation as well as rotation frequency. In the presence of a sufficiently strong radial magnetic field at $\beta > 1$, the radial component of the magnetic field provides further destabilization based on (7.33), and the density gradient term depends on axial as well as radial gradients.

In the strict Keplerian limit these compressibility and density gradient effects vanish. For an axial magnetic field the disk width Δ is given by $\Delta/R = 2T^{1/2}/M^{1/2}\Omega R \ll 1$, limiting perturbation wavelengths in the z direction. Then, for $k_z\Delta > 2\pi$ and $k_R^2 R^2 \gg 1$, the instability requires large β,

$$\beta > 4\pi^2/3. \tag{8.2}$$

This limit is the case for which Balbus' (2006) intuitive physical picture of masses attached by springs is most appropriate.

Whether instability is responsible for shedding excess momentum in an accretion disk is unclear, of course, based on a linear eikonal analysis of axisymmetric perturbations. The nonlinear evolution will have to be determined by sophisticated nonlinear global simulations, presumably requiring full kinetic simulations with dissipation. Clearly much work remains if speculation is to be converted to substance To do so requires using fully self-consistent global magnetized equilibria when performing global simulations allowing non-axisymmetric fluctuations

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Appendix A. General poloidal field case

Equations (7.24) to (7.27) can be combined to obtain the general eikonal stability condition. We first eliminate $\xi_\parallel$ from the ξ_ζ equation to obtain

$$\begin{aligned}\mathrm{i}\omega\left[(\omega^2-k_\parallel^2v_A^2)-\frac{4\omega^2\varOmega^2B_{0R}^2}{B_0^2(\omega^2-k_\parallel^2c_S^2)}\right]\xi_\zeta &= [(\omega^2-k_\parallel^2v_A^2)(R\,\mathrm{d}\varOmega/\mathrm{d}\psi_0)+2\omega^2\varOmega R^{-1}B_0^{-2}B_{0z}]\xi_\psi \\ &+\frac{2\mathrm{i}k_\parallel\omega^2\varOmega B_{0R}}{B_0(\omega^2-k_\parallel^2c_S^2)}[\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0\boldsymbol{\cdot}(M_0G_0r^{-2}\nabla r-\varOmega^2R\nabla R)-c_S^2\nabla\boldsymbol{\cdot}(\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0)].\end{aligned} \tag{A 1}$$

Eliminating $\xi_\parallel$ from the ξ_ψ equation we find

$$\begin{aligned}&(\omega^2-k_\parallel^2v_A^2)\xi_\psi+R^2B_0^2v_A^2\frac{\partial^2\xi_\psi}{\partial\psi_0^2}+\frac{c_S^2R^2B_0^2}{p_0}\frac{\partial}{\partial\psi_0}\left[p_0\nabla\boldsymbol{\cdot}\left(\frac{\xi_\psi}{B_0^2R^2}\nabla\psi_0\right)\right]-2\mathrm{i}\omega\varOmega RB_{0z}\xi_\zeta \\ &\quad+\xi_\psi\frac{[(M_0G_0r^{-2}\nabla r-\varOmega^2R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]^2}{R^2B_0^2}\nabla\boldsymbol{\cdot}\left\{\frac{\nabla\psi_0}{[(M_0G_0r^{-2}\nabla r-\varOmega^2R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]}\right\} \\ &=+k_\parallel R^2B_0^2p_0^{-1}\left[R^{-2}B_0^{-2}(M_0G_0r^{-2}\nabla r-\varOmega^2R\nabla R)\boldsymbol{\cdot}\nabla\psi_0+c_S^2\frac{\partial}{\partial\psi_0}\right]\left\{\frac{2\omega\varOmega p_0B_{0R}\xi_\zeta}{B_0(\omega^2-k_\parallel^2c_S^2)}\right. \\ &\quad\left.+\frac{k_\parallel p_0}{(\omega^2-k_\parallel^2c_S^2)}[\xi_\psi R^{-2}B_0^{-2}(M_0G_0r^{-2}\nabla r-\varOmega^2R\nabla R)\boldsymbol{\cdot}\nabla\psi_0-c_S^2\nabla\boldsymbol{\cdot}(\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0)]\right\}.\end{aligned} \tag{A 2}$$

Next, we may use

$$c_S^2R^2B_0^2\partial\xi_\zeta/\partial\psi_0\gg\xi_\zeta(M_0G_0r^{-2}\nabla r-\varOmega^2R\nabla R)\boldsymbol{\cdot}\nabla\psi_0 \tag{A 3}$$

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for the ξ_ζ terms in this last equation, since cancelation of $\partial\xi_\zeta/\partial\psi_0$ terms is not an issue. Doing so we obtain

$$
\begin{aligned}
&(\omega^2 - k_\parallel^2 v_A^2)\xi_\psi + R^2B_0^2 v_A^2 \frac{\partial^2\xi_\psi}{\partial\psi_0^2} + \frac{c_S^2 R^2 B_0^2}{p_0}\frac{\partial}{\partial\psi_0}\left[p_0\nabla\cdot\left(\frac{\xi_\psi}{B_0^2R^2}\nabla\psi_0\right)\right] - 2\mathrm{i}\omega\Omega RB_{0z}\xi_\zeta \\
&\quad + \xi_\psi\frac{[(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0]^2}{R^2B_0^2}\nabla\cdot\left\{\frac{\nabla\psi_0}{[(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0]}\right\} \\
&= \frac{2k_\parallel c_S^2R^2B_0\omega\Omega B_{0R}}{(\omega^2 - k_\parallel^2c_S^2)}\frac{\partial\xi_\zeta}{\partial\psi_0} \\
&\quad + k_\parallel R^2B_0^2p_0^{-1}\left[R^{-2}B_0^{-2}(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0 + c_S^2\frac{\partial}{\partial\psi_0}\right] \\
&\quad \times\left\{\frac{k_\parallel p_0}{(\omega^2 - k_\parallel^2c_S^2)}[\xi_\psi R^{-2}B_0^{-2}(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0 - c_S^2\nabla\cdot(\xi_\psi R^{-2}B_0^{-2}\nabla\psi_0)]\right\}.
\end{aligned}
\tag{A 4}
$$

Combining $\partial^2\xi_\psi/\partial\psi_0^2$ terms and cancelling the $\partial\xi_\psi/\partial\psi_0$ terms in the c_S^2 terms leaves

$$
\begin{aligned}
&(\omega^2 - k_\parallel^2 v_A^2)\xi_\psi + R^2B_0^2 v_A^2 \frac{\partial^2\xi_\psi}{\partial\psi_0^2} + \frac{\omega^2 c_S^2 R^2 B_0^2}{p_0}\frac{\partial}{\partial\psi_0}\left[\frac{p_0\nabla\cdot(\xi_\psi B_0^{-2}R^{-2}\nabla\psi_0)}{(\omega^2 - k_\parallel^2c_S^2)}\right] \\
&\quad + \xi_\psi\frac{[(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0]^2}{R^2B_0^2}\nabla\cdot\left\{\frac{\nabla\psi_0}{[(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0]}\right\} \\
&= -\xi_\psi\frac{p_0k_\parallel^2c_S^2}{R^2B_0^2}\left[\frac{(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0}{(\omega^2 - k_\parallel^2c_S^2)}\right]^2 \\
&\quad \times\nabla\cdot\left[\frac{(\omega^2 - k_\parallel^2c_S^2)\nabla\psi_0}{p_0(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0}\right] \\
&\quad + \frac{k_\parallel^2[(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0]^2\xi_\psi}{R^2B_0^2(\omega^2 - k_\parallel^2c_S^2)} + 2\mathrm{i}\omega\Omega RB_{0z}\xi_\zeta \\
&\quad + \frac{2k_\parallel c_S^2R^2B_0\omega\Omega B_{0R}}{(\omega^2 - k_\parallel^2c_S^2)}\frac{\partial\xi_\zeta}{\partial\psi_0},
\end{aligned}
\tag{A 5}
$$

where we use

$$
\begin{aligned}
&\frac{R^2B_0^2}{p_0}\frac{\partial}{\partial\psi_0}\left(\frac{p_0Q}{R^2B_0^2}\right) - Q\nabla\cdot\left(\frac{\nabla\psi_0}{R^2B_0^2}\right) = \frac{1}{p_0B_0}\nabla\psi_0\cdot\nabla\left(\frac{p_0Q}{R^2B_0}\right) \\
&\quad - \frac{Q}{B_0}\nabla\cdot\left(\frac{\nabla\psi_0}{R^2B_0}\right) = -\frac{p_0Q^2}{R^2B_0^2}\nabla\cdot\left(\frac{\nabla\psi_0}{p_0Q}\right).
\end{aligned}
\tag{A 6}
$$

Notice that within the eikonal approximation it is tempting to assume

$$
\frac{\omega^2c_S^2R^2B_0^2}{p_0}\frac{\partial}{\partial\psi_0}\left[\frac{p_0\nabla\cdot(\xi_\psi B_0^{-2}R^{-2}\nabla\psi_0)}{(\omega^2 - k_\parallel^2c_S^2)}\right] \gg \frac{k_\parallel^2[(M_0G_0r^{-2}\nabla r - \Omega^2R\nabla R)\cdot\nabla\psi_0]^2\xi_\psi}{R^2B_0^2(\omega^2 - k_\parallel^2c_S^2)},
\tag{A 7}
$$

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however, at marginality ($\omega^2 \to 0$) this large term becomes unimportant so we have been extra careful to retain the next largest terms.

Inserting ξ_ζ in its lowest order form

$$\mathrm{i}\omega\left[(\omega^2 - k_\parallel^2 v_A^2) - \frac{4\omega^2\Omega^2 B_{0R}^2}{B_0^2(\omega^2 - k_\parallel^2 c_S^2)}\right]\xi_\zeta = \left[(\omega^2 - k_\parallel^2 v_A^2)R\frac{\mathrm{d}\Omega}{\mathrm{d}\psi_0} + \frac{2\omega^2\Omega B_{0z}}{RB_0^2}\right]\xi_\psi - \frac{2\mathrm{i}k_\parallel c_S^2\omega^2\Omega B_{0R}}{B_0(\omega^2 - k_\parallel^2 c_S^2)}\frac{\partial\xi_\psi}{\partial\psi_0}, \tag{A 8}$$

into the preceding equation for ξ_ψ we can finally use the remaining eikonal approximation $\partial\xi_\psi/\partial\psi_0 = \mathrm{i}k_\psi\xi_\psi$ to obtain the full stability relation

$$\begin{aligned}\omega^2 - k^2 v_A^2 - \frac{\omega^2 k_\perp^2 c_S^2}{(\omega^2 - k_\parallel^2 c_S^2)} &= \frac{k_\parallel^2[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]^2}{R^2 B_0^2(\omega^2 - k_\parallel^2 c_S^2)} \\ &- \frac{[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]^2}{R^2 B_0^2}\nabla\boldsymbol{\cdot}\left\{\frac{\nabla\psi_0}{[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]}\right\} \\ &- \frac{p_0 k_\parallel^2 c_S^2}{R^2 B_0^2}\left[\frac{(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0}{(\omega^2 - k_\parallel^2 c_S^2)}\right]^2\nabla\boldsymbol{\cdot}\left[\frac{(\omega^2 - k_\parallel^2 c_S^2)\nabla\psi_0}{p_0(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0}\right] \\ &+ \left[B_{0z} + \frac{k_\parallel k_\perp c_S^2 B_{0R}}{(\omega^2 - k_\parallel^2 c_S^2)}\right]\frac{\left\{(\omega^2 - k_\parallel^2 v_A^2)(R^2\,\mathrm{d}\Omega^2/\mathrm{d}\psi_0) + 4\omega^2\Omega^2 B_0^{-2}\left[B_{0z} + \dfrac{k_\parallel k_\perp c_S^2 B_{0R}}{(\omega^2 - k_\parallel^2 c_S^2)}\right]\right\}}{\{(\omega^2 - k_\parallel^2 v_A^2) - [4\omega^2\Omega^2 B_{0R}^2/(\omega^2 - k_\parallel^2 c_S^2)B_0^2]\}},\end{aligned} \tag{A 9}$$

where we have used $k_\perp = k_\psi R B_0$ and $k^2 = k_\parallel^2 + k_\perp^2 \simeq k_\perp^2$. Discarding small terms leaves

$$\begin{aligned}k^2 v_A^2 - \frac{\omega^2 k_\perp^2}{k_\parallel^2} &= \frac{[(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0]^2}{R^2 B_0^2 c_S^2} \\ &+ \frac{\nabla\psi_0\boldsymbol{\cdot}\nabla n_0}{n_0 R^2 B_0^2}(M_0 G_0 r^{-2}\nabla r - \Omega^2 R\nabla R)\boldsymbol{\cdot}\nabla\psi_0 \\ &+ \left[B_{0z} - \frac{k_\perp B_{0R}}{k_\parallel}\right]\frac{\left\{(\omega^2 - k_\parallel^2 v_A^2)(R^2\,\mathrm{d}\Omega^2/\mathrm{d}\psi_0) + 4\omega^2\Omega^2 B_0^{-2}\left[B_{0z} - \dfrac{k_\perp B_{0R}}{k_\parallel}\right]\right\}}{[(\omega^2 - k_\parallel^2 v_A^2) + (4\omega^2\Omega^2 B_{0R}^2/k_\parallel^2 c_S^2 B_0^2)]}.\end{aligned} \tag{A 10}$$

At marginality (A 10) reduces to (7.33).

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