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SUBSTITUTION AND REVENUE EFFECTS WITH ENDOGENOUS IMPATIENCE

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Building on a nonlinear additively separable representation for the generator of preferences, this article provides an extensive characterization of intertemporal consumption—leisure arbitrages with an endogenous measure of impatience. A joint characterization of the comparative dynamics of consumption and leisure is undertaken. An analytical approach first puts in evidence a classical Slutsky decomposition between revenue and substitution effects after various perturbations. Those conclusions are at a second stage illustrated by a three-dimensional geometric argument.

Keywords: Endogenous Rates of Time Preference, Substitution Effects, Revenue Effects, Consumption, Leisure, Intertemporal Arbitrages

1. INTRODUCTION

This article provides a comprehensive characterization of consumption and labor supply arbitrages when intertemporal preferences depart from additive separability over time. Does the relaxation of the invariant-rate-of-time-preference hypothesis, however, suffice to bring a more satisfactory understanding of intertemporal arbitrages? To provide a clearcut answer to this concern, we base our argument primarily on a particular formulation of dynamic preferences that is taken from the work of Epstein (1987) on continuous time-recursive representations. Though that method has not been considered recently, our twofold interest in it stems from its ease of use and the richness of its properties.

An exhaustive examination of the comparative dynamics of consumption and leisure is completed in a basic consumer problem. This proceeds through three distinct steps. First, the properties of the dynamic system in the neighborhood of the steady state are analyzed. Next, responses of consumption and labor supply to

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different sets of perturbations are successively considered. Specifically, the introduction of a lump-sum tax and modifications in the real wage rate and in the rate of interest are consecutively appraised. An understanding of those responses is achieved by performing a classical Slutsky distinction between substitution and revenue effects. Such a separation identifies the precise features of intertemporal arbitrages that govern, through the adaptation of the rate of time preference, current values of consumption and leisure. Finally, the dynamics of consumption and leisure are comprehended within a three-dimensional phase-diagram representation that permits an intuitive description of consumption and leisure adjustments.

The potential modifications of the standard consumption and leisure arbitrages brought about by a relaxation of separability axioms on preferences have not yet been analyzed in their full generality. Some work in that direction has been performed by Shy (1994) with an extended Uzawa (1968) representation for preferences due to Epstein et al. (1988) and within a complex open-economy environment. The current formulation for intertemporal preferences is much more tractable than the one they use. Indeed, the endogenous rates of time preference associated with both consumption and leisure arbitrages reduce to a simple expression that depends merely on future utility.

The main results are as follows: Three successive exercises in comparative dynamics of increasing complexity—namely the introduction of a permanent lump-sum tax on current income, an increase in the real wage rate, and a permanent rise in the rate of interest—are successively undertaken. The perturbation of the rate of interest noticeably implies long-run departures from intertemporal utility and the rate of time preference of the individual; it thus has permanent implications for long-run arbitrages. In the short run, intertemporal substitution and revenue effects will take place simultaneously, whereas only substitution effects will affect long-run arbitrages. Consumption and leisure will follow a parallel long-run rise. In contrast, the nature of instantaneous responses remains *a priori* undetermined, its sign resulting from the initial position of the individual on the credit market. Specifically, an initially heavily indebted agent will be detrimentally affected by a rise in the rate of interest and of the associated negative wealth effect. He will thus have to diminish both his consumption level and his leisure time. Conversely, a creditor will benefit from a positive wealth effect and raise his consumption level and leisure time. To sum up, as a direct outcome of an individual's initial position on the credit market, the short-run implications of a rise in the rate of interest can replicate or differ from the long-run ones.

The rest of this article is organized as follows. Section 2 introduces a generalized but simple formulation for the understanding of consumption and leisure arbitrages: It essentially adapts Epstein's (1987) separable generating function within a life-cycle savings setting with endogenous labor supply. Section 3 is interested in the comparative dynamic properties of this framework after the introduction of a lump-sum tax. Sections 4 and 5 complete a related exercise, respectively, after perturbations of the wage rate and the rate of interest,[1] the more technically demanding arguments are presented in the Appendix.

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2. GENERALIZED FORMULATION FOR INTERTEMPORAL ARBITRAGES

2.1. Model

Time is continuous. Consider an infinitely lived consumer. At date 0, he is endowed with an amount a_0 of the asset. He supplies at instant $t \geq 0$ an amount $\ell(t)$ of labor units, perceives a wage rate $w(t)$, and consumes an amount $c(t)$. He has access to a perfect financial market on which the rate of interest is $r(t)$. The intertemporal budget constraint of the individual is stated as

$$\dot{a}(t) = r(t)a(t) + w(t)\ell(t) - c(t),$$
$$\lim_{t\to+\infty} \exp\left[-\int_0^t r(s)\,ds\right]a(t) = 0, \tag{1}$$

the latter expression featuring a solvability condition.

Letting $\mathcal{C}$ and $\mathcal{L}$, respectively, denote the consumption and labor supply paths that associate at any date $t \in \mathbf{R}_+$ a consumption flow of $c(t)$ and a labor supply of $\ell(t)$. The restrictions of these paths to dates greater than a given t, or $\mathcal{C}|_{[t,+\infty[}$ and $\mathcal{L}|_{[t,+\infty[}$, are henceforth denoted by ${}_t\mathcal{C}$ and ${}_t\mathcal{L}$. At a given date, the recursive utility functional $U(\cdot,\cdot)$ is defined over these restricted paths and satisfies

$$\frac{d}{dt}U({}_t\mathcal{C}, {}_t\mathcal{L}) = -\mathcal{G}[c(t), \ell(t), U({}_t\mathcal{C}, {}_t\mathcal{L})], \tag{2a}$$
$$= -v[c(t), \ell(t)] + \mathcal{R}[U({}_t\mathcal{C}, {}_t\mathcal{L})] \tag{2b}$$

for $\mathcal{G}(\cdot,\cdot,\cdot)$, a function referred to as the *generator of preferences* in (2a). To facilitate the assessment of dynamical issues and along the lines of Epstein (1987), an additively separable representation of the type (2b) is henceforth retained for this generating function. Introducing a variable $\xi(t) = U({}_t\mathcal{C}, {}_t\mathcal{L})$ that features the current value of future utility at a given date, the definition of the intertemporal utility can be restated as

$$\dot{\xi}(t) = -v[c(t), \ell(t)] + \mathcal{R}[\xi(t)]. \tag{3}$$

The properties of the generator thus bring a *recursive utility* structure. The following assumptions are made on the components of the generating function:

Assumption 1. The function $v(\cdot)$ is strictly concave over $\mathbf{R}_+ \times \mathbf{R}_+$ with $v'_c > 0$, $v''_{cc} < 0$, $v'_\ell < 0$, $v''_{\ell\ell} < 0$, $v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2 > 0$ for all $(c, \ell) \in \mathbf{R}^*_+ \times \mathbf{R}^*_+$.

Assumption 2. Consumption and leisure are normal goods:

(i) $v'_c v''_{\ell\ell} - v'_\ell v''_{c\ell} \leq 0, \forall\,(c,\ell) \in \mathbf{R}^*_+ \times \mathbf{R}^*_+$;
(ii) $v'_\ell v''_{cc} - v''_{\ell c} v'_c \geq 0, \forall\,(c,\ell) \in \mathbf{R}^*_+ \times \mathbf{R}^*_+$.

Assumption 3. $v''_{c\ell} \geq 0\ \forall (c,\ell) \in \mathbf{R}^*_+ \times \mathbf{R}^*_+$.

Assumption 4. $\mathcal{R}(\cdot)$ satisfies $\mathcal{R}'(\xi) > 0$, $\mathcal{R}''(\xi) > 0$ for any $\xi \in \mathbf{R}^*_+$.

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Epstein (1987) established in a general way that equation (2) has a bounded solution for $U({}_o\mathcal{C}, {}_o\mathcal{L})$ as soon as the following restriction is imposed on the generator.

Assumption 5. $\mathcal{R}'(\xi) > \underline{\varrho} > 0$ for all $\xi \in \mathbf{R}_+$.

The *time perspective* condition pictured by Assumption 5 is to be understood, along the ideas of Koopmans et al. (1964), as the need for the individual to discount at a minimal and positive rate.

As shown by Geoffard (1990), there exists a unique positive bounded solution to (3) when the discounted sum of utility further satisfies

$$\lim_{t\to\infty} \exp\left\{ \int_{s=0}^{t} -\mathcal{R}'[\xi(s)]\, ds \right\} \times \xi(t) = 0. \quad \mathbf{(4)}$$

Central features of the intertemporal utility functional $U({}_o\mathcal{C}, {}_o\mathcal{L})$ are the *rates of time preference* associated with consumption and leisure arbitrages and are defined, respectively, by Epstein and Hynes (1983), at date $T \geq 0$, by

$$\rho^c[c(T), \ell(T), \xi(T)] = -\frac{d}{dT} \ln[U'_{\mathcal{C}T}(\mathcal{C}, \mathcal{L})]|_{\dot{c}(T)=\dot{\ell}(T)=0} = \mathcal{R}'[\xi(t)], \quad \mathbf{(5a)}$$

$$\rho^\ell[c(T), \ell(T), \xi(T)] = -\frac{d}{dT} \ln[U'_{\mathcal{L}T}(\mathcal{C}, \mathcal{L})]|_{\dot{c}(T)=\dot{\ell}(T)=0} = \mathcal{R}'[\xi(t)], \quad \mathbf{(5b)}$$

with $U'_{\mathcal{C}T}$ and $U'_{\mathcal{L}T}$ the respective Volterra derivatives of consumption and labor supply.

It is then noted that Assumption 4 corresponds to a rate of time preference that is an increasing function of future utility, a property that is referred to as an *increasing impatience* assumption and that plays a central role in the stability properties of accumulation models with recursive preferences.[2]

2.2. Consumer Maximization Program

The program of the consumer is stated as the maximization of his utility at date 0 under his intertemporal budget constraint:

$$\begin{aligned}
&\operatorname*{Max}_{\{c,\ell\}} \xi(0),\\
&\text{s.t. } \dot{\xi} = -v(c,\ell) + \mathcal{R}(\xi),\\
&\quad \dot{a} = ra + w\ell - c,\\
&\quad \lim_{t\to\infty} \exp\left\{ \int_{s=0}^{t} -\mathcal{R}'[\xi(s)]\, ds \right\} \times \xi(t) = 0,\\
&\quad \lim_{t\to\infty} \exp\left[-\int_{s=0}^{t} r(s)\, ds \right] a(t) = 0,
\end{aligned}$$

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$$a(0), \text{ given}$$

$$c \geq 0, \ell \geq 0.$$

This problem being of the Mayer type, the associated Hamiltonian is stated as

$$\mathcal{H} = \beta[-v(c, \ell) + \mathcal{R}(\xi)] + \lambda(ra + w\ell - c). \tag{6}$$

Letting $\chi = -\lambda/\beta$, the first-order conditions are

$$v'_c(c, \ell) = \chi, \tag{7a}$$

$$v'_\ell(c, \ell) = -\chi w, \tag{7b}$$

$$\dot{\chi} = -\chi[r - \mathcal{R}'(\xi)], \tag{7c}$$

$$\dot{\xi} = -v(c, \ell) + \mathcal{R}(\xi), \tag{7d}$$

$$\dot{a} = ra + w\ell - c. \tag{7e}$$

In the preceding equations, χ should be understood as the price of instantaneous consumption while χw is the price of leisure. For a given value for χ, c and ℓ will emerge as the solutions of a static arbitrage described by (7a) and (7b). Under Assumption 1, solving this subsystem leads to defining c and ℓ as functions of χ and w; that is, $c = \varphi(\chi, w)$ and $\ell = \psi(\chi, w)$. These functions in turn allow us to define an "indirect immediate utility" function $u(\chi, w) := v[\varphi(\chi, w), \psi(\chi, w)]$ in terms of these two variables. Finally, it is convenient to introduce an extra function $\mathcal{E}(\cdot, \cdot)$ of these two variables χ and w that describes the opposite of the current savings of the agent: $\mathcal{E}(\chi, w) = \varphi(\chi, w) - w\psi(\chi, w)$. Under Assumptions 1, 2, and 3, these functions display clear-cut properties whose derivations have been gathered in Section A.1 of the Appendix: $\varphi'_\chi \leq 0$, $\varphi'_w \geq 0$, $\psi'_\chi \geq 0$, $\psi'_w \geq 0$, $u'_\chi \leq 0$, $u'_w \leq 0$, $\mathcal{E}'_\chi \leq 0$, $\mathcal{E}'_w \leq 0$.

An autonomous form of the dynamical system is then available as

$$\dot{\xi} = -u(\chi, w) + \mathcal{R}(\xi), \tag{8a}$$

$$\dot{\chi} = -\chi[r - \mathcal{R}'(\xi)], \tag{8b}$$

$$\dot{a} = ra - \mathcal{E}(\chi, w), \tag{8c}$$

where the sole consideration of the subsystem (8a), (8b) itself provides an autonomous block in the variables χ and ξ. The whole system also possesses a unique backward variable, a, and two forward variables, χ and ξ. Assuming then the existence of a unique stationary point $(\bar{\xi}, \bar{\chi}, \bar{a})$ for the system (8), the latter will satisfy

$$-u(\bar{\chi}, w) + \mathcal{R}(\bar{\xi}) = 0, \tag{9a}$$

$$\mathcal{R}'(\bar{\xi}) = r, \tag{9b}$$

$$r\bar{a} - \mathcal{E}(\bar{\chi}, w) = 0. \tag{9c}$$

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Under the aforementioned list of assumptions, the obtention of a saddle-point property is immediate and checked in Section A.2 of the Appendix. Since this is clearly apparent from (8) and (9), it may be worth stressing that a steady state directly results from the adjustment of the rate of time preference and thus emerges as an explicit outcome of its endogenous determination.

Actually, within the current infinite-horizon environment, both χ and ξ are located on the stable manifold, denoted as $\mathcal{W}^s(\bar{\chi}, \bar{\xi})$, of the saddle-point configuration. The initial point $(\xi(0), \chi(0))$, however, is still to be characterized. Noticing that the dynamics of the subsystem (9a), (9b) *a priori* allow for an infinity of equilibrium trajectories,

$$\chi(t) = \mathcal{X}(\chi(0), t),$$

$$\xi(t) = \Xi(\xi(0), t),$$

the coordinates of the initial point $\chi(0)$ and $\xi(0)$ are determined by two extra equations: the intertemporal budget constraint, written as

$$\int_0^\infty \mathcal{E}[\mathcal{X}(\chi(0), t), w] \exp\left[-\int_0^t r(s)\, ds\right] dt = a(0); \qquad \textbf{(10)}$$

and the condition $(\chi(0), \xi(0)) \in \mathcal{W}^s(\bar{\chi}, \bar{\xi})$. The values of $c(t)$ and $\ell(t)$ then are derived in a static way from the one of $\chi(t)$: $c(t) = \varphi[\chi(t), w(t)]$ and $\ell(t) = \psi[\chi(t), w(t)]$.

3. COMPARATIVE DYNAMICS AFTER THE INTRODUCTION OF A LUMP-SUM TAX

When lump-sum taxation is introduced, the accumulation law of the asset (8c) becomes

$$\dot{a} = ra - \mathcal{E}(\chi, w) - \tau(t), \qquad \textbf{(8c}'\textbf{)}$$

the remaining components of the dynamical system, or (8a), (8b), being unaffected.

Comparative dynamics analytical methods are first called through an approach due to Aoki (1980) and d'Autume (1991) that is based upon differential calculus techniques. Geometrical methods that build on three-dimensional phase portraits in the spirit of Obstfeld (1990) then enlighten the content of these formal conclusions.

3.1. Adjustment Dynamics after a Fiscal Perturbation

It is assumed that at $t = 0$, an infinitesimal tax is introduced. The ensuing perturbation is subject to the following law of motion:

$$\delta\tau(t) = \delta\tau(0)\exp(-\lambda t), \quad \text{with } \delta\tau(0) > 0 \text{ and } \lambda \geq 0. \qquad \textbf{(11)}$$

With this formulation, $\lambda = 0$ corresponds to a permanent perturbation, whereas $\lambda > 0$ is associated with a transitory perturbation that disappears at a rate λ.

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The differentiation of the dynamical system (8a), (8b), (8c′) in the neighborhood of the steady state $(\bar{\xi}, \bar{\chi}, \bar{a})$ leads to

$$\delta\dot{\xi} = -u'_{\chi}(\bar{\chi}, w)\delta\chi + \mathcal{R}'(\bar{\xi})\delta\xi, \quad \textbf{(12a)}$$

$$\delta\dot{\chi} = \bar{\chi}\mathcal{R}''(\bar{\xi})\delta\xi, \quad \textbf{(12b)}$$

$$\delta\dot{a} = r\delta a - \mathcal{E}'_{\chi}(\bar{\chi}, w)\delta\chi - \delta\tau. \quad \textbf{(12c)}$$

Denoting as μ_1 and μ_2 the eigenvalues associated with the subsystem (12a), (12b) and letting, as a convention, $\mu_1 > 0$ and $\mu_2 < 0$,[3]

$$\mu_{1,2} = (1/2)\left\{\mathcal{R}'(\bar{\xi}) \pm \sqrt{[\mathcal{R}'(\bar{\xi})]^2 - 4u'_{\chi}(\bar{\chi}, w)\bar{\chi}\mathcal{R}''(\bar{\xi})}\right\}. \quad \textbf{(13)}$$

As detailed in Section A.3 of the Appendix, solving (12) produces the following resulting variations of the price, indirect utility, and value of the asset:

$$\delta\chi = -\frac{(r-\mu_2)}{\mathcal{E}'_{\chi}(\bar{\chi}, w)(\lambda + r)} \cdot \delta\tau(0)\exp(\mu_2 t), \quad \textbf{(14a)}$$

$$\delta\xi = -\frac{\mu_2(r-\mu_2)}{\bar{\chi}\mathcal{R}''(\bar{\xi})\mathcal{E}'_{\chi}(\bar{\chi}, w)(\lambda + r)} \cdot \delta\tau(0)\exp(\mu_2 t), \quad \textbf{(14b)}$$

$$\delta a = \frac{[\exp(-\lambda t) - \exp(\mu_2 t)]}{(\lambda + r)} \cdot \delta\tau(0). \quad \textbf{(14c)}$$

The transitional dynamics of consumption and leisure thus finally are derived as

$$\delta c = \varphi'_{\chi}(\bar{\chi}, w)\delta\chi, \quad \text{with } \varphi'_{\chi}(\bar{\chi}, w) \le 0, \quad \textbf{(15a)}$$

$$\delta\ell = \psi'_{\chi}(\bar{\chi}, w)\delta\chi, \quad \text{with } \psi'_{\chi}(\bar{\chi}, w) \ge 0. \quad \textbf{(15b)}$$

3.2. Instantaneous and Long-Run Responses

3.2.1. Instantaneous response to a fiscal perturbation. Computing the initial values of the jumps of the variables [henceforth $\delta\chi(0)$, $\delta\xi(0)$, $\delta c(0)$, and $\delta\ell(0)$] and recalling that $\delta a(0) = 0$, it is clear from (14a), (14b) together with Section A.1 of the Appendix that $\delta\chi(0) > 0$ and $\delta\xi(0) < 0$. From (15), then, $\delta c(0) \le 0$ and $\delta\ell(0) \ge 0$ are derived. A fiscal perturbation hence initially entails a drop in consumption together with a rise in labor supply.

3.2.2. The long run. The long-run values of χ and ξ are left unaffected by the fiscal perturbation; that is, $\delta\chi(\infty) = \delta\xi(\infty) = 0$. The long-run values of c and ℓ are similarly left unchanged after the introduction of a fiscal perturbation. Finally, the long-run variation of the value of the asset held by the consumer satisfies $\delta a(\infty) = 0$ for $\lambda > 0$ and $\delta a(\infty) = \delta\tau(0)/r > 0$ for $\lambda = 0$.

The behavior of the individual may thus be understood as follows: In the short run, the introduction of a lump-sum tax implies a negative intertemporal wealth

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effect. The individual will decrease his consumption level and his leisure time (he raises his labor supply). In the long run, both the consumption level and the labor supply recover their initial values. In fact, in the long run, the introduction of a lump-sum tax implies a modification of neither the price of consumption χ nor of the level of utility ξ. The long-run rate of time preference thus remains unmodified.

Specifically, the effort completed by the agent after the perturbation has enabled him to accumulate a positive auxiliary wealth that exactly fits the level that was required for recovering his initial permanent income in the long run. Hence long-run consumption and leisure can recover their original value.

3.3. A Geometric Argument

The dynamics of the adjustment path are pictured by the means of a three-dimensional phase diagram for the variables ξ, a, and χ. It is convenient to first consider the analysis of the projection of the adjustment path over the plane (ξ, χ). In Figure 1, the locus $\dot{\xi} = 0$ is defined from the equation $u(\chi, w) = \mathcal{R}(\xi)$ and entails a decreasing relation in the space (χ, ξ). The locus $\dot{\chi} = 0$ is similarly defined from $\mathcal{R}'(\xi) = r$ and expresses ξ as a function of r.

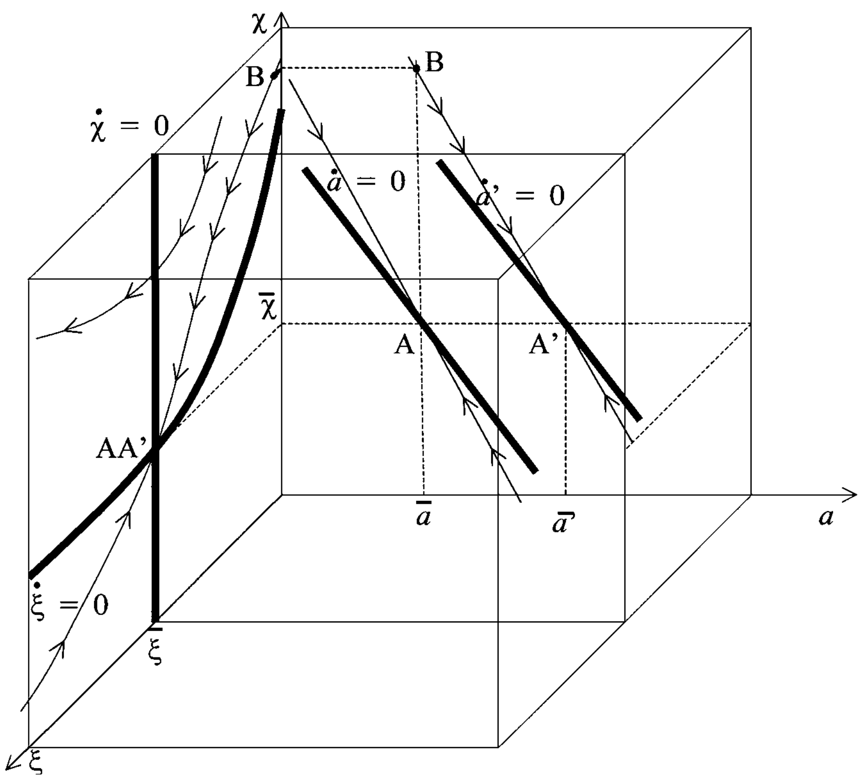

FIGURE 1. Effect of a lump-sum tax τ, where ABA$' =$ a permanent tax income.

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The system initially jumps to B, but then follows a saddle path and finally converges toward $A' = A$.

In the same manner, a parallel study based upon the projection of the adjustment process on the plane (a, χ) illustrates the implications of the introduction of a permanent lump-sum tax on the definition of the locus of the stationary values of the asset; the projection of $\dot{a} = 0$, of equation $ra = \mathcal{E}(\chi, w) + \tau$, is translated on the right-hand side of the three-dimensional phase portrait, the new long run being described by A'. The dynamics of the adjustment process summarize to an initial jump onto B and then a progress on the saddle path toward the new stationary state. It is worthwhile to emphasize that, in such a representation, only the price variable χ may be subject to jumps, the value of the asset being for itself determined by its past values.

From the previous computations, c initially assumes a downward jump, whereas ℓ is first subject to an upward movement; afterward, both converge to their constant long-run values. Finally, a progressively rises toward a greater new long-run value. Those conclusions may be understood as follows: The introduction of a lump-sum tax is associated with a short-run negative wealth effect, which will disappear in the long run. If both the long-run consumption and the labor supply levels of the individual remain unaffected by this lump-sum tax, then his response to this fiscal perturbation will be a joint short-run decrease in consumption and increase in labor supply.

4. COMPARATIVE DYNAMICS AFTER AN INCREASE IN THE WAGE RATE

The experience builds from a variation of a parameter that will modify the long-run levels of both consumption and labor supply.

4.1. Adjustment Dynamics

At the initial date $t = 0$, an infinitesimal perturbation on the wage rate is introduced. Its evolution is governed by the following law of motion:

$$\delta w(t) = \delta w(0) \exp(-\lambda t), \tag{16}$$

with $\delta w(0) > 0$ and $\lambda \geq 0$. The eventual transitional dynamics of consumption and leisure are derived as

$$\delta c = \varphi'_{\chi}(\bar{\chi}, w)\delta\chi + \varphi'_{w}(\bar{\chi}, w)\delta w, \tag{17a}$$

$$\delta \ell = \psi'_{\chi}(\bar{\chi}, w)\delta\chi + \psi'_{w}(\bar{\chi}, w)\delta w. \tag{17b}$$

Henceforth letting CSE and ISE, respectively, denote the contemporaneous and intertemporal substitution effects and similarly letting IRE designate the intertemporal revenue effect, it is shown in Sections A.4 and A.5 of the Appendix that the

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perturbated expressions of c and ℓ may be put under a form for which a decomposition between substitution and revenue effects is recovered:

$$\delta c = \underbrace{\varphi'_w \cdot \delta w(0)\exp(-\lambda t)}_{\text{CSE}} - \underbrace{\varphi'_\chi \frac{u'_w \mathcal{R}''\chi}{(\lambda+\mu_1)(\lambda+\mu_2)}\delta w(0)\left[\exp(-\lambda t) + \frac{\lambda}{\mu_2}\exp(\mu_2 t)\right]}_{\text{ISE}}$$
$$+ \underbrace{\left[\frac{(\mu_2 - r)}{\lambda + r}\frac{\mathcal{E}'_w}{\mathcal{E}'_\chi} - \frac{(\mu_2 - r)u'_w\mathcal{R}''\chi}{(\lambda+r)(\lambda+\mu_1)(\lambda+\mu_2)} + \frac{u'_w\mathcal{R}''\chi\lambda}{(\lambda+\mu_1)(\lambda+\mu_2)\mu_2}\right]\varphi'_\chi\delta w(0)\exp(\mu_2 t)}_{\text{IRE}}, \tag{18a}$$

$$\delta \ell = \underbrace{\psi'_w \cdot \delta w(0)\exp(-\lambda t)}_{\text{CSE}} - \underbrace{\psi'_\chi \frac{u'_w \mathcal{R}''\chi}{(\lambda+\mu_1)(\lambda+\mu_2)}\delta w(0)\left[\exp(-\lambda t) + \frac{\lambda}{\mu_2}\exp(\mu_2 t)\right]}_{\text{ISE}}$$
$$+ \underbrace{\left[\frac{(\mu_2 - r)}{\lambda + r}\frac{\mathcal{E}'_w}{\mathcal{E}'_\chi} - \frac{(\mu_2 - r)u'_w\mathcal{R}''\chi}{(\lambda+r)(\lambda+\mu_1)(\lambda+\mu_2)} + \frac{u'_w\mathcal{R}''\chi\lambda}{(\lambda+\mu_1)(\lambda+\mu_2)\mu_2}\right]\psi'_\chi\delta w(0)\exp(\mu_2 t)}_{\text{IRE}}. \tag{18b}$$

In the above formulas, the direct effect of δw on c and ℓ for a given χ is referred to *contemporaneous substitution effects*. The remaining component is then identified as the *intertemporal* substitution component.

Remark 1. Comparison of the above conclusions with the ones that would have been obtained within the standard model with a fixed rate of time preference $\rho > 0$ is particularly instructive. Formulation (8) subsumes the latter as a special case for which $\mathcal{R}'(\xi) = \rho$ and $\mathcal{R}''(\xi) = 0$. Nonetheless, to make a sensible comparison, the special case of the *permanent income hypothesis* $r = \rho$ is to be considered; indeed, it is only in this limit case that the consumption path of the life-cycle model gives rise to a steady state. According to note 3, this also implies $\mu_2 = 0$ and the dynamical features of the adjustment path become noticeably poorer. In particular, when a permanent shock is considered and $\lambda = 0$, the ajustment is no longer taking place: *The dynamics of c and ℓ are the ones of the perturbations*. ■

4.2. Instantaneous and Long-Run Responses

Specializing the argument to a permanent shock and thus letting $\lambda = 0$,

$$\delta c = \underbrace{-\varphi'_\chi \frac{u'_w}{u'_\chi}\delta w(0)}_{\text{ISE}_{(\geq 0)}} + \underbrace{\varphi'_w \delta w(0)}_{\text{CSE}_{(\geq 0)}} \underbrace{-\varphi'_\chi \frac{\mu_2 - r}{r}\frac{\psi}{\mathcal{E}'_\chi}\exp(\mu_2 t)\delta w(0)}_{\text{IRE}_{(\geq 0)}}, \tag{19a}$$

$$\delta \ell = \underbrace{-\psi'_\chi \frac{u'_w}{u'_\chi}\delta w(0)}_{\text{ISE}_{(\leq 0)}} + \underbrace{\psi'_w \delta w(0)}_{\text{CSE}_{(\geq 0)}} \underbrace{-\psi'_\chi \frac{\mu_2 - r}{r}\frac{\psi}{\mathcal{E}'_\chi}\exp(\mu_2 t)\delta w(0)}_{\text{IRE}_{(\leq 0)}}. \tag{19b}$$

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Remark 2. The intertemporal revenue effect no longer appears in the long run. This results from the adjustment of the rate of time preference $\mathcal{R}'(\xi)$, which is intimately related to the dynamics of the variable ξ. At the opposite, for a model with a fixed rate of time preference, $\mu_2 = 0$, so that the intertemporal revenue effect now takes place in a *permanent* way. ■

4.2.1. Instantaneous responses. Computing the corresponding instantaneous response in consumption in (19a), it appears that any of the effects is positive, hence $\delta c(0) > 0$ and the sign of the jump is unambiguous.

Completing the same approach for the corresponding instantaneous response in labor supply in (19b), it appears that, whereas the intertemporal substitution effect and the intertemporal revenue effect favor a decrease in labor supply, the contemporaneous substitution effect indicates an upward jump. The actual sign of $\delta\ell(0)$ hence remains undetermined.

Finally, the instantaneous variation of future utility is similarly computed to be $\delta\xi(0) = -[\psi(\mu_2 - r)\mu_2/r\mathcal{E}'_\chi\mathcal{R}''\chi]\delta w_0 > 0$ (see Sections A.1 and A.4 in the Appendix).

4.2.2. Long-run responses. Immediate computations on (19a) and (19b) indicate that $\delta c(\infty) > 0$ and $\delta\ell(\infty) > 0$. Similarly, from Section A.4 of the Appendix, $\delta\xi(\infty) = 0$, whereas $\delta a(\infty) = [\delta w(0)/r](-\psi) < 0$.

One may further verify that $\delta c(0) > \delta c(\infty)$, whereas $\delta\ell(0) < \delta\ell(\infty)$.

The increase in the wage rate w leads to a positive contemporaneous substitution effect between c and ℓ as a result of the static arbitrage between the two for a given χ. This is a direct outcome of the separable representation for the generating function. As for the positiveness of the substitution effect, it derives from the normality of consumption and leisure imposed by Assumption 2.

An increase of w also leads to a revenue effect in the short run: The ensuing rise in the wealth level of the individual will encourage him to consume more and to take more leisure time. Finally, the last effect involved is an intertemporal substitution effect. For a permanent shock, the subsequent decrease in the current price of the asset χ will modify the consumption–leisure arbitrage through an increase in c and a decrease in ℓ. Otherwise stated, the individual will substitute higher future values for consumption and leisure for current ones. In this perspective, it is noticed that the long-run value of the asset will assume a long-run decrease along $\delta a(\infty) = -\ell\delta w(0)/r$. This adjustment corresponds to the parallel increase in consumption and leisure; that is, $\delta c = -w\delta\ell$.

To sum up, *consumption will be subject to an initial overshooting mechanism* that results from a positive intertemporal revenue effect. *Labor supply will, at the opposite, have to assume an initial undershooting mechanism* as a result of a negative revenue effect. Such an effect explains how *the initial adjustment in labor supply can take place in a direction that is in opposition to its long-run move.*

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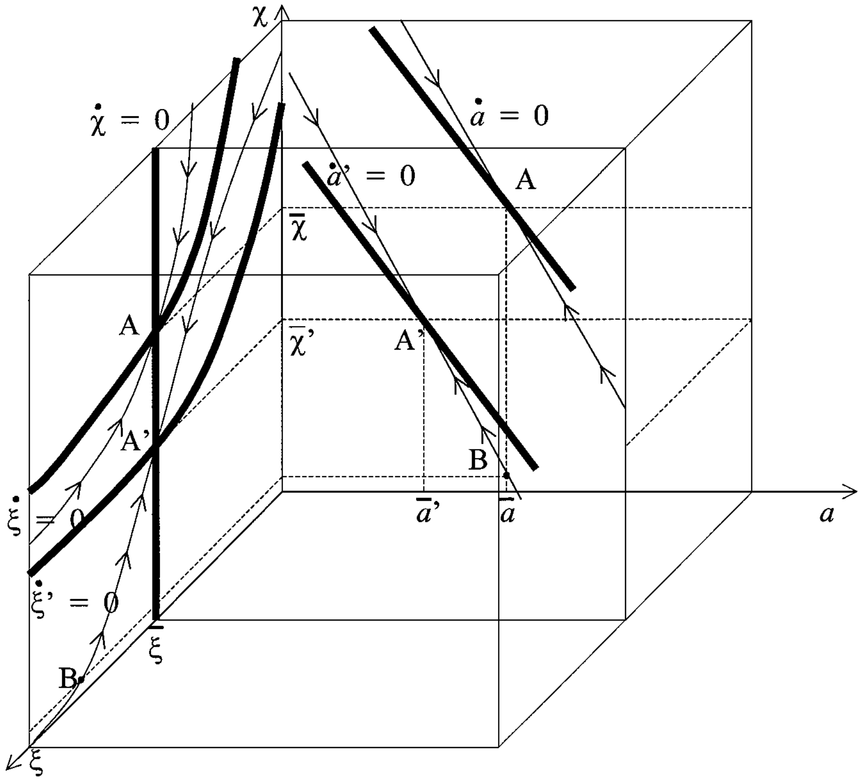

FIGURE 2. Effect of a permanent increase of w: ABA′.

4.3. A Geometric Argument

Representing the dynamics of the adjustment path in a three-dimensional diagram, its examination over the plane (ξ, χ) gives a simple picture of the phenomenon. As represented in Figure 2, the locus of the stationary values of the price, or $\dot{\chi} = 0$, remains unchanged, whereas the locus of the stationary values of future utility, or $\dot{\xi} = 0$, follows a downward translation.

More precisely, at the time of the increase of the wage rate w, the system follows a downward jump from A to B and then converges on the saddle path toward the new long-run value A'.

An analysis of the adjustment process over the plane (a, χ) allows us to detail this mechanism: The locus $\dot{a} = 0$ is translated to the left, so that the long-run value of the financial asset satisfies $\bar{a}' < \bar{a}$ since $\partial \bar{a}/\partial w < 0$. Similarly, the asset a cannot jump, but the price χ jumps downward. The system moves from A to B and then converges toward the new long-run value A' along the stable manifold of the saddle.

From the previous computations, c initially jumps upward and then converges to a new, greater, long-run steady state. To sum up, consumption will be subject to an overshooting phenomenon after a positive perturbation on the wage rate.

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At the opposite, ℓ initially will be subject to a downward jump and then will converge to a long-run state that is located above its initial position. The direction of the instantaneous response hence contrasts with its long-run counterpart: The revenue effect is predominant in the short run but it is the substitution effect that prevails in the long run.

5. COMPARATIVE DYNAMICS AFTER AN INCREASE IN THE INTEREST RATE

The new experience is now an increase in the interest rate. It is the one that implies a long-run departure of future utility with respect to its initial level.

5.1. Adjustment Dynamics after a Perturbation on the Interest Rate

At $t = 0$, an infinitesimal perturbation on the interest rate is introduced. It follows that

$$\delta r(t) = \delta r(0)\exp(-\lambda t), \tag{20}$$

with $\delta r(0) > 0$ and $\lambda \geq 0$.

It is shown in Section A.6 of the Appendix that the resulting variations of c and ℓ can be put in a form that reveals the decomposition between substitution and revenue effects:

$$\begin{aligned}\delta c = {} & \underbrace{\frac{\varphi'_\chi \chi \delta r(0)}{(\lambda+\mu_1)(\lambda+\mu_2)}\left[(\lambda+\mathcal{R}')\exp(-\lambda t) - \frac{\mathcal{R}'' u'_\chi \chi}{\mu_2}\exp(\mu_2 t)\right]}_{\text{ISE}} \\ & + \underbrace{\varphi'_\chi \cdot \delta r(0)\left[\left(\frac{\mathcal{E}'_\chi \chi(\lambda+\mathcal{R}')}{(\lambda+\mu_1)(\lambda+\mu_2)} - a\right)\frac{(\mu_2 - r)}{(\lambda+r)\mathcal{E}'_\chi}\exp(\mu_2 t)\right.}_{\text{IRE}} \\ & \left. + \underbrace{\frac{\chi\mathcal{R}'' u'_\chi \chi}{\mu_2(\lambda+\mu_1)(\lambda+\mu_2)}\exp(\mu_2 t)}_{\text{IRE}}\right],\end{aligned} \tag{21a}$$

$$\begin{aligned}\delta \ell = {} & \underbrace{\frac{\psi'_\chi \chi \delta r(0)}{(\lambda+\mu_1)(\lambda+\mu_2)}\left[(\lambda+\mathcal{R}')\exp(-\lambda t) - \frac{\mathcal{R}'' u'_\chi \chi}{\mu_2}\exp(\mu_2 t)\right]}_{\text{ISE}} \\ & + \underbrace{\psi'_\chi \cdot \delta r(0)\left[\left(\frac{\mathcal{E}'_\chi \chi(\lambda+\mathcal{R}')}{(\lambda+\mu_1)(\lambda+\mu_2)} - a\right)\frac{(\mu_2 - r)}{(\lambda+r)\mathcal{E}'_\chi}\exp(\mu_2 t)\right.}_{\text{IRE}} \\ & \left. + \underbrace{\frac{\chi\mathcal{R}'' u'_\chi \chi}{\mu_2(\lambda+\mu_1)(\lambda+\mu_2)}\exp(\mu_2 t)}_{\text{IRE}}\right].\end{aligned} \tag{21b}$$

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Notice that, as occurs after an increase in the wage rate, the intertemporal revenue effect will disappear in the long run. This is still the direct outcome of the adjusment of the rate of time preference $\mathcal{R}'(\xi)$. In the case of a permanent shock, this ajustment will still exist in the long run. The role of the term $\mathcal{R}''(\xi)$ should also be emphasized in the previous decomposition because it is associated with the endogenous determination of the rate of time preference.

Remark 3. A direct comparison with the standard framework associated with an exogenous rate of time preference ρ is no longer available. Starting from a steady-state situation where the relation $r = \rho$ prevails, an increase in r for a given value of ρ would indeed translate into a nonstationary long-run consumption path. ■

5.2. Instantaneous and Long-Run Responses with a Permanent Shock

Computing the initial values of the variables δc and $\delta \ell$ for $t = 0$ and $\lambda = 0$, we get

$$\delta c(0) = \underbrace{\varphi'_\chi \frac{\mathcal{R}'}{u'_\chi \mathcal{R}''}\delta r(0)}_{\text{ISE}_{(>0)}} \underbrace{-\varphi'_\chi(\chi/\mu_2)\delta r(0)}_{\text{ISE}_{(<0)}} + \underbrace{\varphi'_\chi \delta r(0)\left[\frac{\chi}{\mu_2} + \left(\frac{\mathcal{R}'}{\chi \mathcal{R}''} - a\right)\frac{\mu_2 - r}{r\mathcal{E}'_\chi}\right]}_{\text{IRE}_{(\lessgtr 0)}} \lessgtr 0,$$

$$\delta \ell(0) = \underbrace{\psi'_\chi \frac{\mathcal{R}'}{u'_\chi \mathcal{R}''}\delta r(0)}_{\text{ISE}_{(<0)}} \underbrace{-\psi'_\chi(\chi/\mu_2)\delta r(0)}_{\text{ISE}_{(>0)}} + \underbrace{\varphi'_\chi \delta r(0)\left[\frac{\chi}{\mu_2} + \left(\frac{\mathcal{R}'}{\chi \mathcal{R}''} - a\right)\frac{\mu_2 - r}{r\mathcal{E}'_\chi}\right]}_{\text{IRE}_{(\gtrless 0)}} \gtrless 0,$$

having made use of $ra = \mathcal{E}$ and $(u'_\chi/\mathcal{E}'_\chi) = \chi$. Similarly,

$$\delta\xi(0) = \delta r(0)\left[1 + \left(\frac{\mathcal{R}'}{\mathcal{R}''} - \chi a\right)\frac{(\mu_2 - r)\mu_2}{r\chi u'_\chi}\right].$$

Computing the long-run values of δc, $\delta \ell$, $\delta \xi$, and δa for $\lambda = 0$ and $t \to +\infty$, we get

$$\delta a(\infty) = \frac{\delta r(0)}{r}\left(\frac{\mathcal{R}'}{\mathcal{R}''}\frac{1}{\chi} - \frac{\mathcal{E}}{r}\right) \lessgtr 0, \qquad \delta\xi(\infty) = \frac{\delta r(0)}{\mathcal{R}''} > 0,$$

$$\delta c(\infty) = \underbrace{\varphi'_\chi \frac{\mathcal{R}'}{\mathcal{R}'' u'_\chi}\delta r(0)}_{\text{ISE}} > 0, \qquad \delta \ell(\infty) = \underbrace{\psi'_\chi \frac{\mathcal{R}'}{\mathcal{R}'' u'_\chi}\delta r(0)}_{\text{IRE}} < 0.$$

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In the long run, the sign of $\delta a(\infty)/\delta r(0)$ is given by the one of $(\mathcal{R}'/\mathcal{R}'')(\mathcal{E}'_\chi/u'_\chi) - a$, an expression that also appears in the formulas for $\delta\chi(0)$ and $\delta\xi(0)$.

As a result of the appearance of a in the above responses, an increase in the interest rate r will have different revenue effects according to whether the initial asset corresponds to a borrower or a lender position. Two distinct terms are to be considered in the analysis of the intertemporal revenue effect. An increase in r will first imply a long-run decrease in the shadow price of the asset. More interestingly, it will also translate as a modification of the valuation of future outcomes through the component $\mathcal{R}'(\xi)/\mathcal{R}''(\xi)$.

The instantaneous and long-run implications of the increase in the interest rate can be understood in the following way. First, and as a result of Assumption 2 retained on $v(\cdot)$, consumption and labor supply move in opposite directions. In contrast to the earlier analysis of a modification in the wage rate, an increase in the interest rate will not modify the instantaneous consumption–leisure arbitrage, and only intertemporal effects will occur. Further, the intertemporal revenue effect, which can be of any sign according to whether the individual is a creditor or a debtor, will only take place in the short run. When the agent is a creditor, both his consumption level and his leisure time may increase in the short run. The converse is true when the agent is indebted on the credit market: The revenue effect may then imply a parallel decrease in consumption and leisure.

In the long run, the sole persistent effect is the substitution one and it will have unequivocal implications: Consumption and leisure jointly increase. The increase in the interest rate has been associated with an upward adjustment of the rate of time preference. Hence, the agent substitutes higher future levels of consumption and amounts of leisure time for their current values.

5.3. A Geometric Argument

The long-run steady state is modified with respect to its reference definition, that is system (9). Further, and as a result of the increasing property of $\mathcal{R}'(\cdot)$—this is ensured by the retention of Assumption 4—this long-run steady state will be characterized by a greater long-run value for future utility $\bar{\xi}'$, for $\bar{\xi}' > \bar{\xi}$.

Two particular cases allow for a characterization of the signs of the expressions at $t = 0$:

Case 1 $a \le 0$.

One then necessarily gets $\delta a(\infty)/\delta r(0) \ge 0$ and it can be verified that $\delta\chi(0) \ge 0$ is similarly satisfied. Remembering then that $\delta c(0) = \varphi'_\chi \delta\chi(0)$ and $\delta\ell(0) = \psi'_\chi \delta\chi(0)$, one finally obtains $\delta c(0) \le 0$ and $\delta\ell(0) \ge 0$.

Case 2 $\delta a(\infty)/\delta r(0) \le 0$.

It is clear that this requires $a > 0$. One then gets $\delta\chi(0) \le 0$ and $\delta c(0) \ge 0$, $\delta\ell(0) \le 0$. Asymptotically, $\delta\chi(\infty) \le 0$, whence $\delta c(\infty) \ge 0$ and $\delta\ell(\infty) \le 0$.

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The eventual response of c and ℓ to a perturbation on r is hence conditional on the initial value of the asset. This follows the intuition that an increase in the interest rate will disadvantage an indebted individual but favor a creditor.

In the long run, an increase in the interest rate increases the consumption flow and decreases the number of worked hours, whatever the sign of the holding of the asset. At the opposite, in the short run, an indebted agent will have to make an instantaneous effort, that is, to consume less but to work more.

The dynamics on the transition path to the stationary state are illustrated as follows in the case $a \leq 0$.

A projection of the dynamics of the adjustment path in the plane (ξ, χ) indicates, as illustrated through Figure 3, that the locus of stationary values of future utility, or $\dot{\xi} = 0$, remains unchanged. At the opposite, the locus of the stationary values of the price, or $\dot{\chi} = 0$, follows a translation in the direction of the front of the picture.

As illustrated in Figure 3, when the interest rate r follows a variation, the system jumps from A to B. It then converges on the stable manifold toward its long-run position A'.

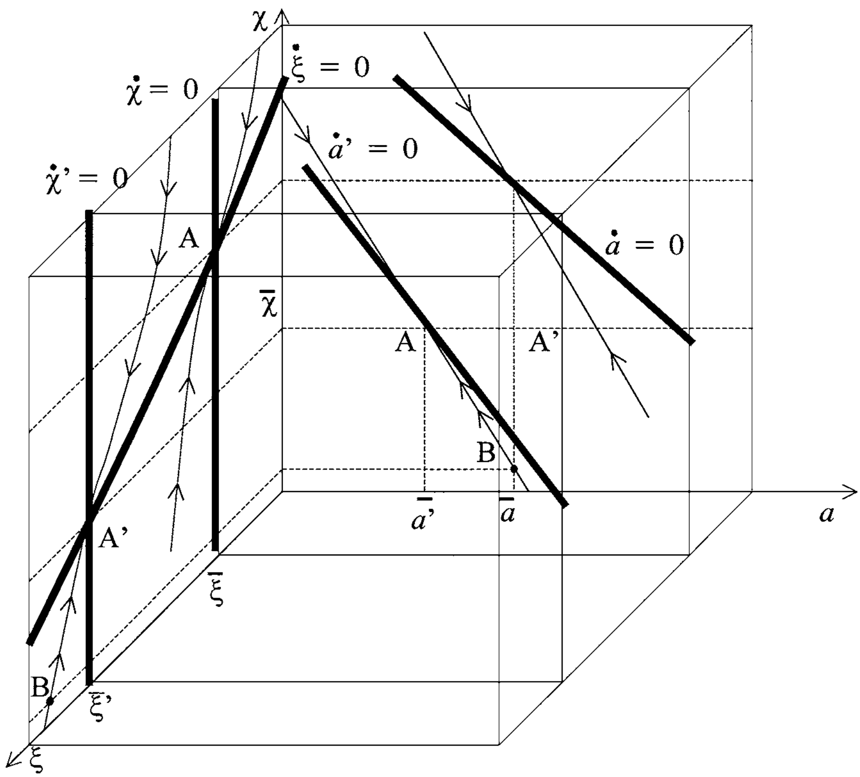

Figure 3. Effect of a permanent increase of r under the assumption $d\bar{a}/dr < 0$: ABA′.

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The projection of the adjustment process over the plane (a, χ) indicates that the locus of the stationary values of the asset $\dot{a} = 0$ assumes a clockwise right-hand rotation with $\bar{a}' > \bar{a}$—this is precisely the configuration that had been retained for the three-dimensional phase portrait. As before, the value of the financial asset a is not allowed to jump. Finally, the price χ jumps upward and the system initially changes from A to B. It afterward converges on the stable manifold to its new long-run value A'.

NOTES

1. Various remarks at different stages of the exposition should ease the comparison with the standard case.
2. *Vide* Epstein (1987).
3. All the comparative dynamics properties of the consumption and leisure paths in the subsequent analysis are ruled by μ_2. The standard model with an exogenous rate of time preference $\rho > 0$, or $\mathcal{R}'(\xi) = \rho$ and $\mathcal{R}''(\xi) = 0$, reduces to $\mu_2 = 0$. As expected, one would then recover an infinite persistence for any given exogenous perturbation.

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APPENDIX

A.1. DETERMINANTS OF SAVINGS BEHAVIOR

(i) Under Assumptions 1, 2, and 3, consumption and leisure are such that

$$\varphi'_\chi = \frac{v''_{\ell\ell} - (v'_\ell/v'_c)v''_{c\ell}}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2} \leq 0;$$

$$\psi'_\chi = \frac{v''_{cc}(v'_\ell/v'_c) - v''_{\ell c}}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2} \geq 0;$$

$$\varphi'_w = \frac{v''_{c\ell}}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2} \geq 0;$$

$$\psi'_w = \frac{-v''_{cc}\chi}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2} \geq 0.$$

(ii) Building on (i), the properties of the indirect utility function follow as a simple corollary:

$$u'_\chi = \frac{\chi\left(v''_{\ell\ell} + 2wv''_{c\ell} + w^2v''_{cc}\right)}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2};$$

$$u'_w = \frac{\chi^2[v''_{c\ell} - (v'_\ell/v'_c)v''_{cc}]}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2}.$$

Hence, and under Assumptions 1 and 2, $u'_\chi \leq 0$ and $u'_w \leq 0$ prevail.

(iii) Finally, considering the *opposite* of the savings of the agents $\mathcal{E}(\chi, w) = c - w\ell = \varphi(\chi, w) - w\psi(\chi, w)$ and completing the same kind of verifications,

$$\mathcal{E}'_\chi = \frac{w^2v''_{cc} + 2wv''_{c\ell} + v''_{\ell\ell}}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2} \leq 0;$$

$$\mathcal{E}'_w = -\psi + \frac{\chi[v''_{c\ell} - (v'_\ell/v'_c)v''_{cc}]}{v''_{cc}v''_{\ell\ell} - (v''_{c\ell})^2} \leq 0.$$

The argument is complete. ■

A.2. SADDLE-POINT PROPERTY

The Jacobian matrix considered in a neighborhood of $(\bar{\xi}, \bar{\chi}, \bar{a})$ is given by

$$J = \begin{pmatrix} \mathcal{R}' & -u'_\chi & 0 \\ \chi\mathcal{R}'' & 0 & 0 \\ 0 & -\mathcal{E}'_\chi & r \end{pmatrix}.$$

The characteristic polynomial is expressed as

$$\mathcal{P}(\lambda) = (r - \lambda)\left(\lambda^2 - \lambda\mathcal{R}' + u'_\chi\bar{\chi}\mathcal{R}''\right).$$

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Immediately, r appears to be a positive root. The polynomial $\lambda^2 - \mathcal{R}'\lambda + u'_\chi \bar{\chi}\mathcal{R}''$ has two real roots, of positive and negative signs since their sum is given by $\mathcal{R}' > 0$ whereas their product is given by $u'_\chi \bar{\chi}\mathcal{R}'' \leq 0$. Thus, there are two positive and one negative eigenvalue, or a saddle-point structure. ■

A.3. COMPARATIVE DYNAMICS AFTER A FISCAL SHOCK

Under the convention $\mu_1 > 0$, $\mu_2 < 0$, the law of motion of the perturbed patterns of the price and of prospective utility are stated as

$$\delta\chi = \delta\chi(0)\exp(\mu_2 t),$$
$$\delta\xi = \delta\xi(0)\exp(\mu_2 t).$$

The corresponding law of motion of the asset follows as

$$\delta\dot{a} = r\delta a - \delta\chi(0)\exp(\mu_2 t)\mathcal{E}'_\chi - \exp(-\lambda t)\delta\tau(0).$$

This is a first-order differential equation that can be solved through standard methods. The general form of the solution is given by

$$\delta a = \exp(rt) \times f(t).$$

Reincorporating into the equation and simplifying, we obtain

$$\dot{f} = -\delta\chi_0 \exp[(\mu_2 - r)t]\mathcal{E}'_\chi - \exp[-(\lambda + r)t]\delta\tau(0).$$

Integrating produces

$$f(t) = f_0 + \frac{\delta\chi(0)}{r - \mu_2}\mathcal{E}'_\chi \exp[(\mu_2 - r)t] + \frac{\delta r(0)}{\lambda + r}\exp[-(\lambda + r)t]$$

for f_0 a constant. Replacing in the law of motion of the asset gives

$$\delta a = f_0 \exp(rt) + \frac{\delta\chi_0}{r - \mu_2}\mathcal{E}'_\chi \exp(\mu_2 t) + \frac{\delta\tau(0)}{\lambda + r}\exp(-\lambda t).$$

Invoking the solvability condition of the asset in (1), $f_0 = 0$. Considering the initial condition, one gets $\delta a(0) = 0$. Whence, finally,

$$\delta\chi(0) = -\frac{\delta\tau(0)(r - \mu_2)}{\mathcal{E}'_\chi(\lambda + r)} > 0.$$

The expression of $\delta\xi$ is then available:

$$\delta\xi(0) = -\frac{\mu_2(r - \mu_2)\delta\tau(0)}{\bar{\chi}\mathcal{R}''(\bar{\chi})\mathcal{E}'_\chi(\lambda + r)} < 0.$$

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A.4. COMPARATIVE DYNAMICS AFTER A PERTURBATION OF THE WAGE RATE

Differentiating (8) in the neighborhood of the steady state $(\bar{\xi}, \bar{\chi}, \bar{a})$, one obtains

$$\begin{aligned}
\delta\dot{\xi} &= -u'_\chi(\bar{\chi}, w)\delta\chi - u'_w(\bar{\chi}, w)\delta w + \mathcal{R}'(\bar{\xi})\delta\xi,\\
\delta\dot{\chi} &= \bar{\chi}\mathcal{R}''(\bar{\xi})\delta\xi,\\
\delta\dot{a} &= r\delta a - \mathcal{E}'_\chi(\bar{\chi}, w)\delta\chi - \mathcal{E}'_w(\bar{\chi}, w)\delta w.
\end{aligned}$$

The eigenvalues associated with the subsystem defined from the two first equations are still given by the expressions of μ_1 and μ_2 in (13). It can be solved under the form

$$\begin{aligned}
\delta\xi &= g_0 \exp(\mu_2 t) + \frac{f_0\mu_1}{\mu_1 - \mu_2}\exp(\mu_1 t) + u'_w \frac{\lambda}{\lambda + \mu_1}\frac{1}{\lambda + \mu_2}\exp(-\lambda t)\delta w(0),\\
\delta\chi &= g_0 \frac{\mathcal{R}''\chi}{\mu_2}\exp(\mu_2 t) + \chi\mathcal{R}'' f_0 \exp(\mu_1 t) - \frac{u'_w\mathcal{R}''\chi}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\delta w(0),
\end{aligned}$$

for f_0 and g_0 the given constants. Resting on the relation $\mu_1 > \mathcal{R}' > 0 > \mu_2$, from the transversality condition $\lim_{t\to\infty}\xi(t)\exp\{\int_{s=0}^{t} -\mathcal{R}'[\xi(s)]\,ds\} = 0$, it is then derived that $f_0 = 0$ is to hold. Whence,

$$\begin{aligned}
\delta\xi &= g_0 \exp(\mu_2 t) + u'_w \frac{\lambda}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\delta w(0),\\
\delta\chi &= g_0 \frac{\mathcal{R}''\chi}{\mu_2}\exp(\mu_2 t) - \frac{u'_w\mathcal{R}''\chi}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\delta w(0),
\end{aligned}$$

Finally, replacing the preceding expression of $\delta\chi$ in $\delta\dot{a}$ and solving the differential equation, one obtains

$$\begin{aligned}
\delta a(t) = {} & h_0 \exp(rt) + \left[\mathcal{E}'_w - \frac{\mathcal{E}'_\chi u'_w \mathcal{R}''\chi}{(\lambda + \mu_1)(\lambda + \mu_2)}\right]\frac{1}{\lambda + r}\exp(-\lambda t)\delta w(0)\\
& - \frac{g_0\mathcal{E}'_\chi\mathcal{R}''\chi}{\mu_2(\mu_2 - r)}\exp(\mu_2 t),
\end{aligned}$$

for h_0 a given constant. From the solvability condition, $h_0 = 0$ is derived. The initial condition $\delta a(0) = 0$ determines g_0. The eventual expressions of δa, $\delta\chi$ and $\delta\xi$ are derived as:

$$\begin{aligned}
\delta a = {} & \left[\mathcal{E}'_w(\bar{\chi}, w) - \frac{\mathcal{E}'_\chi(\bar{\chi}, w)u'_w(\bar{\chi}, w)\mathcal{R}''(\bar{\xi})\bar{\chi}}{(\lambda + \mu_1)(\lambda + \mu_2)}\right][\exp(-\lambda t) - \exp(\mu_2 t)]\frac{\delta w(0)}{\lambda + r},\\
\delta\chi = {} & \delta w(0)\left\{\frac{(\mu_2 - r)}{\mathcal{E}'_\chi(\bar{\chi}, w)(\lambda + r)}\left[\mathcal{E}'_w(\bar{\chi}, w) - \frac{\mathcal{E}'_\chi(\bar{\chi}, w)u'_w(\bar{\chi}, w)\mathcal{R}''(\bar{\xi})\bar{\chi}}{(\lambda + \mu_1)(\lambda + \mu_2)}\right]\exp(\mu_2 t)\right.\\
& \left. - \frac{u'_w(\bar{\chi}, w)\mathcal{R}''(\bar{\xi})\bar{\chi}}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\right\}\\
\delta\xi = {} & \delta w(0)\left\{\frac{(\mu_2 - r)\mu_2}{\mathcal{E}'_\chi(\bar{\chi}, w)\mathcal{R}''(\bar{\xi})(\lambda + r)\bar{\chi}}\left[\mathcal{E}'_w(\bar{\chi}, w) - \frac{\mathcal{E}'_\chi(\bar{\chi}, w)u'_w(\bar{\chi}, w)\mathcal{R}''(\bar{\xi})\bar{\chi}}{(\lambda + \mu_1)(\lambda + \mu_2)}\right]\right.\\
& \left. \times \exp(\mu_2 t) + \frac{u'_w(\bar{\chi}, w)\lambda}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\right\}.
\end{aligned}$$

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A.5. CLASSICAL DECOMPOSITION: SUBSTITUTION AND REVENUE EFFECTS WITH NONSEPARABLE INTERTEMPORAL PREFERENCES

This section will consider the duals of the problems which have been considered in Sections 1 and 2. The focus is now turned to the minimisation of the total expense subject to a constraint on the value of intertemporal utility.

A.5.1. Statement of the Dual Problem

The *actualization variable* q is introduced as

$$q(t) = \exp\left[-\int_{s=0}^{t} r(s)\, ds\right].$$

The dual problem consists of minimizing the expense subject to a minimum level of intertemporal utility $\bar{\xi}$:

$$\begin{aligned}
&\min \int_{t=0}^{\infty} [q(c - w\ell)]\, dt \\
&\text{s.t. } \xi(0) \geq \bar{\xi}, \\
&\quad \dot{\xi} = -\mathcal{G}(c, \ell, \xi), \\
&\quad c \geq 0, \ell \geq 0, \\
&\quad \lim_{t\to\infty} \xi(t) \exp\left\{\int_{s=0}^{t} -\mathcal{R}'[\xi(s)]\, ds\right\} = 0.
\end{aligned}$$

The first-order conditions for this optimization problem remain:

$$\begin{aligned}
\upsilon_c'(c, \ell) &= \chi, \\
\upsilon_\ell'(c, \ell) &= -\chi w, \\
\dot{\chi} &= \chi[\mathcal{R}'(\xi) - r].
\end{aligned}$$

A.5.2. Characterization of Substitution Effects After a Perturbation of the Wage Rate

The perturbed system is given by

$$\begin{aligned}
\delta\dot{\xi} &= -u_\chi' \delta\chi - u_w' \delta w + \mathcal{R}' \delta\xi, \\
\delta\dot{\chi} &= \chi \mathcal{R}'' \delta\xi, \\
\delta w &= \delta w(0) \exp(-\lambda t), \\
\delta\xi(0) &= 0.
\end{aligned}$$

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Following the same approach as in Section A.4, one derives

$$\delta\xi = \frac{u'_w \lambda}{(\lambda+\mu_1)(\lambda+\mu_2)} \cdot \delta w(0)[\exp(-\lambda t) - \exp(\mu_2 t)],$$

$$\delta\chi = -\frac{u'_w \mathcal{R}'' \chi}{(\lambda+\mu_1)(\lambda+\mu_2)} \cdot \delta w(0)\left[\exp(-\lambda t) + \frac{\lambda}{\mu_2}\exp(\mu_2 t)\right].$$

The eventual expressions of δc and $\delta \ell$ that correspond to the substitution effect satisfy

$$\delta c = \varphi'_\chi \delta\chi + \varphi'_w \delta w,$$

$$\delta \ell = \psi'_\chi \delta\chi + \psi'_w \delta w.$$

A.6. COMPARATIVE DYNAMICS AFTER A PERTURBATION OF THE INTEREST RATE

Differentiating (8) in the neighborhood of the steady state $(\bar{\xi}, \bar{\chi}, \bar{a})$, one obtains

$$\delta\dot{\xi} = -u'_\chi(\bar{\chi}, w)\delta\chi + \mathcal{R}'(\bar{\xi})\delta\xi,$$

$$\delta\dot{\chi} = -\bar{\chi}\delta r + \bar{\chi}\mathcal{R}''(\bar{\xi})\delta\xi,$$

$$\delta\dot{a} = a \cdot \delta r + r \cdot \delta a - \mathcal{E}'_\chi(\bar{\chi}, w)\delta\chi.$$

The eigenvalues associated with the subsystem defined from the two first equations remain the ones of system (13). This subsystem can be solved independently as follows:

$$\delta\xi = \frac{f_0 \mu_1}{\mu_1 - \mu_2}\exp(\mu_1 t) + g_0 \exp(\mu_2 t) - \frac{\alpha\chi\mu_1}{(\lambda+\mu_1)(\lambda+\mu_2)}\exp(-\lambda t)\delta r(0),$$

$$\delta\chi = f_0 \chi \mathcal{R}'' \exp(\mu_1 t) + \frac{\chi \mathcal{R}'' g_0}{\mu_2}\exp(\mu_2 t) + \exp(-\lambda t)\frac{\delta r(0)\chi(\lambda + \mathcal{R}')}{(\lambda+\mu_1)(\lambda+\mu_2)},$$

with f_0 and g_0 two given constants. From the transversality condition, $\lim_{t\to\infty}\xi(t)\exp\{\int_{s=0}^{t} -\mathcal{R}'[\xi(s)]ds\} = 0$, one derives $f_0 = 0$. Finally, replacing the expression of $\delta\chi$ in $\delta\dot{a}$ and solving the differential equation, one derives that

$$\delta a(t) = h_0 \exp(rt) + \frac{\delta r(0)\exp(-\lambda t)}{\lambda + r}\left[-a + \frac{\mathcal{E}'_\chi \chi(\lambda + \mathcal{R}')}{(\lambda+\mu_1)(\lambda+\mu_2)}\right]$$

$$-\frac{\mathcal{E}'_\chi \chi \mathcal{R}''}{\mu_2(\mu_2 - r)} g_0 \exp(\mu_2 t),$$

with h_0 a constant. The holding of $h_0 = 0$ then follows from the solvability condition on consumption.

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The initial condition $\delta a(0) = 0$ determines g_0. One finally gets

$$\delta a = \left[\frac{\mathcal{E}'_\chi(\bar{\chi}, w)\bar{\chi}(\lambda + \mathcal{R}'(\bar{\xi}))}{(\lambda + \mu_1)(\lambda + \mu_2)} - a\right][\exp(-\lambda t) - \exp(\mu_2 t)]\frac{\delta r(0)}{\lambda + r},$$

$$\delta\chi = \delta r(0)\left\{\left[\frac{\mathcal{E}'_\chi(\bar{\chi}, w)\bar{\chi}[\lambda + \mathcal{R}'(\bar{\xi})]}{(\lambda + \mu_1)(\lambda + \mu_2)} - a\right]\frac{(\mu_2 - r)}{(\lambda + r)\mathcal{E}'_\chi(\bar{\chi}, w)}\exp(\mu_2 t)\right.$$

$$\left. + \frac{\bar{\chi}[\lambda + \mathcal{R}'(\bar{\chi})]}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\right\}$$

$$\delta\xi = \delta r(0)\left\{\left[\frac{\mathcal{E}'_\chi(\bar{\chi}, w)\bar{\chi}[\lambda + \mathcal{R}'(\bar{\xi})]}{(\lambda + \mu_1)(\lambda + \mu_2)} - a\right]\frac{(\mu_2 - r)\mu_2}{(\lambda + r)\mathcal{E}'_\chi(\bar{\chi}, w)\bar{\chi}\mathcal{R}''(\bar{\xi})}\exp(\mu_2 t)\right.$$

$$\left. + \frac{\bar{\chi} \cdot u'_\chi(\bar{\chi}, w)}{(\lambda + \mu_1)(\lambda + \mu_2)}\exp(-\lambda t)\right\}.$$

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