THE JOURNAL OF SYMBOLIC LOGIC
Volume 84, Number 2, June 2019

DESTRUCTIBILITY OF THE TREE PROPERTY AT $\aleph_{\omega+1}$

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Abstract. We construct a model in which the tree property holds in $\aleph_{\omega+1}$ and it is destructible under $\mathrm{Col}(\omega, \omega_1)$. On the other hand we discuss some cases in which the tree property is indestructible under small or closed forcings.

§1. Introduction. A partial order $\langle T, \leq_T\rangle$ is called a *tree*, if it has a minimal element and for every $t \in T$, the set $\{s \in T \mid s \leq_T t\}$ is well ordered by $\leq_T$. The order type of the chain of elements that lie below t in the tree order is called the *level* of t and denoted by $\mathrm{Lev}_T(t)$. For a cardinal κ, T is called a κ-tree if $\sup_{t\in T}(\mathrm{Lev}_T(t) + 1) = \kappa$ and the cardinality of each level of T is strictly below κ.

By a theorem of Kőnig, every ω-tree has a cofinal branch (namely, a cofinal chain). On the other hand, a theorem of Aronszajn states that there is an ω_1-tree that has no cofinal branches. Such a tree is called *Aronszajn tree*. For any larger successor cardinal, $\kappa > \omega_1$, it is independent of ZFC whether there is a κ-tree with no cofinal branches. This question is related to other combinatorial topics and in order to get the consistency of the nonexistence of κ-Aronszajn trees, one must assume the consistency of some large cardinals. If every κ tree has a cofinal branch, we say that κ has the *tree property*.

By a theorem of Silver, if uncountable cardinal κ has the tree property then κ is weakly compact in L. On the other end, Mitchell proved that if κ is weakly compact and $\mu < \kappa$ is regular then there is a generic extension in which $\kappa = \mu^{++}$ and the tree property holds at κ, thus showing that the tree property at the double successor of a regular cardinal is equiconsistent with the existence of a weakly compact cardinal. Where κ is a successor of a singular cardinal, the situation is more complicated. In [4], Magidor and Shelah showed that it is consistent, relative to some large cardinals, that the tree property holds at $\aleph_{\omega+1}$. The large cardinal assumption was later reduced by Sinapova and Neeman to the existence of an ω-sequence of supercompact cardinals (see, e.g., [5] for the Prikry-free version). In both constructions, $\aleph_1$ plays a special role. It reflects, in some sense, the properties of $\aleph_{\omega+1}$.

In Section 3 we will show that it is consistent to have a model in which the tree property holds at $\aleph_{\omega+1}$, but after collapsing $\aleph_1$, it fails. This extends a work by Cummings, Foreman and the second author [2, Theorem 14]. In this article they

Received October 4, 2017.
2010 *Mathematics Subject Classification.* 03E35, 03E55.
Key words and phrases. tree property, successors of singulars, forcing.

0022-4812/19/8402-0008
DOI:10.1017/jsl.2019.4

show that it is possible that a weak square is added by a small forcing. Our arguments are very similar to the arguments there. In [6], Rinot shows that it is consistent that there is no special Aronszajn tree on $\aleph_{\omega_1+1}$ and a σ-closed $\aleph_2$-Knaster forcing of cardinality $\aleph_3$ introduces one. We note that we do not know how to apply a similar argument for this case.

In Section 4 we discuss three cases in which the tree property at a successor of a singular cardinal is somewhat indestructible. In 4.1 we will show that it is consistent that the tree property holds at $\aleph_{\omega^2+1}$ and it is indestructible under any forcing of cardinality $<\aleph_{\omega^2}$. In 4.2 we will show that the tree property at $\aleph_{\omega+1}$ can be made indestructible under small σ-closed forcings.

§2. Preliminaries.

The following notation, due to Magidor and Shelah [4], plays an important role in the investigation of the tree property at successors of singular cardinals. For more information about narrow systems and their connections to squares we refer to [3].

Definition 2.1. Let λ be a regular cardinal. A *system* is a triplet $\mathcal{S} = \langle I, \kappa, \mathcal{R}\rangle$ such that:

(1) $I \subseteq \lambda$ unbounded. $\kappa < \lambda$.
(2) $\mathcal{R}$ is a collection of partial order relations on $I \times \kappa$.
(3) Each $R \in \mathcal{R}$ is a tree like partial order. R respects the lexicographic order on $I \times \kappa$. Namely, $\langle \alpha, \zeta\rangle R \langle \beta, \xi\rangle$ implies $\alpha \leq \beta$ and if $\alpha = \beta$ then $\zeta = \xi$. Moreover, if $\langle \beta, \xi\rangle, \langle \gamma, \rho\rangle R \langle \alpha, \zeta\rangle$ and $\beta \leq \gamma$ then $\langle \beta, \xi\rangle R \langle \gamma, \rho\rangle$.
(4) For every $\alpha < \beta$ in I there are $\zeta, \xi < \kappa$ and $R \in \mathcal{R}$ such that $\langle \alpha, \zeta\rangle R \langle \beta, \xi\rangle$.

A *branch* through $\mathcal{S}$ is a set of elements on $I \times \kappa$ which is a chain relative to some $R \in \mathcal{R}$. We say that a branch b meets the α-th level of $\mathcal{S}$ if $b \cap \{\alpha\} \times \kappa \neq \emptyset$. A branch is *cofinal* if it meets cofinally many levels.

A system $\mathcal{S}$ is *narrow* if $\max(\kappa^+, |\mathcal{R}|^+) < \lambda$.

Definition 2.2. Let λ be a regular cardinal. We say that the *narrow system property* holds at λ if every narrow system of height λ has a cofinal branch.

Unlike the tree property, the narrow system property is indestructible by any small forcing. Let $\mathbb{P}$ be a forcing notion with $|\mathbb{P}|^+ < \lambda$ and let $\dot{\mathcal{S}}$ be a name for a narrow system. Let $\dot{\mathcal{R}}$ be the collection of names of relations in $\mathcal{S}$ and let I be the set of all ordinals that can be levels of the $\mathbb{P}$. Let us define the narrow system $\hat{\mathcal{S}}$ in the natural way: the relations of $\hat{\mathcal{S}}$ are indexed by $\mathbb{P} \times \dot{\mathcal{R}}$, and let $\langle \alpha, \beta\rangle (p, R) \langle \gamma, \delta\rangle$ iff $p \Vdash \langle \alpha, \beta\rangle R \langle \gamma, \delta\rangle$ for $R \in \dot{\mathcal{R}}$. A branch in the system $\hat{\mathcal{S}}$ corresponds to a condition $p \in \mathbb{P}$ and a set of element in $\mathcal{S}$ which are forced to be a branch in the generic extension by p.

§3. Destructible tree property.

Theorem 3.1. *Let $\kappa = \kappa_0 < \kappa_1 < \cdots$ be an ω-sequence of supercompact cardinals. Then there is a forcing extension in which the tree property holds at $\aleph_{\omega+1}$ and the forcing* $\mathrm{Col}(\omega, \omega_1)$ *adds a special $\aleph_{\omega+1}$-Aronszajn tree.*

We will prove something slightly stronger. We will define a forcing poset that forces that in the generic extension there is a partial weak square on $\aleph_{\omega+1}$ whose domain contains all ordinals with cofinality above ω_1, while the tree property holds at $\aleph_{\omega+1}$. If we further extend the universe and collapse ω_1 to be countable, then we can complete all the missing places in this square sequence by just adding ω sequences. By a theorem of Shelah and Ben-David [1, Theorem 3], without violating the continuum hypothesis at $\aleph_\omega$, we cannot hope to have this kind of partial square with only one club at each ordinal, while having the tree property.

Let $\mu = \sup \kappa_n$ and let $\lambda = \mu^+$.

We begin with some definitions:

Definition 3.2. A partial square on a set $S \subseteq \lambda$ with width $< \eta$ is a sequence $\mathcal{C} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ such that:

(1) For every $\alpha < \lambda$, $\mathcal{C}_\alpha$ is a set of cardinality $< \eta$. If $\alpha \in S$ then $\mathcal{C}_\alpha \neq \emptyset$.
(2) Every $D \in \mathcal{C}_\alpha$ is a closed and unbounded subset of α and $\operatorname{otp} D < \alpha$.
(3) If $\beta \in \operatorname{acc} D$, $D \in \mathcal{C}_\alpha$ then $D \cap \beta \in \mathcal{C}_\beta$.

When $\lambda = \mu^+$, we may assume that $\operatorname{otp} D \leq \mu$ for every $D \in \mathcal{C}_\alpha$.

Since successor ordinals are never accumulation points of a club, the values of the square sequence at successor points are irrelevant. We will assume that $\mathcal{C}_{\alpha+1} = \{\alpha\}$ for every α, for consistency.

We want to force a partial square for the set $S^\lambda_{\geq\kappa}$ with width $< \mu$.

Definition 3.3. Let $\mathbb{S}$ be the following forcing notion. A condition $s \in \mathbb{S}$ is a sequence $s = \langle c_i \mid i \leq \gamma \rangle$ for some ordinal $\gamma < \mu^+$ such that all three requirements for the partial square sequence hold for every $\alpha \leq \gamma$. Namely,

(1) $\forall \alpha \leq \gamma$, c_α is a set of less than μ sets. If $\operatorname{cf} \alpha \geq \kappa$, then $c_\alpha \neq \emptyset$.
(2) For every $D \in c_\alpha$, $\operatorname{otp} D \leq \mu$ and D is a closed and unbounded subset of α.
(3) If $\beta \in \operatorname{acc} D$, $D \in c_\alpha$ then $D \cap \beta \in c_\beta$.

We order $\mathbb{S}$ by end extension.

We will think of the conditions $s \in \mathbb{S}$ as functions, so for $s = \langle c_i \mid i \leq \gamma \rangle$ we will write $\operatorname{dom} s = \gamma + 1$ and $s(i) = c_i$ for $i \in \operatorname{dom} s$.

Lemma 3.4. *$\mathbb{S}$ is κ-directed closed.*

Given a partial square $\mathcal{C}$, we will define a threading forcing, $\mathbb{T}_\eta$. This forcing will add a club at λ with order type η such that all its initial segments are from $\mathcal{C}$.

Definition 3.5. Let $\mathbb{T}_\eta = \{D \mid \exists \alpha,\ D \in \mathcal{C}_\alpha,\ 1 < \operatorname{otp} D < \eta\}$, ordered by end extension.

The following lemma is standard:

Lemma 3.6. *Let $\mathbb{S}, \mathbb{T}_\eta$ be as above. Then*:

(1) *$\mathbb{S}$ is λ-distributive.*
(2) *Let $\mathcal{C}$ be the generic partial square added by $\mathbb{S}$, and let η be a regular cardinal. $\mathbb{S} * \mathbb{T}_\eta$ is equivalent to an η-directed closed forcing. Moreover, for every $\rho < \mu$, $\mathbb{S} * \mathbb{T}_\eta^\rho$ (where we use full support power in $V^{\mathbb{S}}$) contains an η-directed closed dense subset.*

Proof. Let us show that $\mathbb{S}$ is λ-distributive. We will show that it is η-strategically closed for every regular $\eta < \lambda$. We will do this by showing the second part of the lemma—that $\mathbb{S} * \mathbb{T}_\eta$ contains a η-closed dense set.

https://doi.org/10.1017/jsl.2019.4 Published online by Cambridge University Press

Let us observe first that the set of conditions $\langle s, \check{t}\rangle \in \mathbb{S} * \mathbb{T}_\eta$, $\operatorname{dom}(s) = \gamma + 1$, $t \in s(\gamma)$ is dense. For every condition $\langle s, \dot{t}\rangle$,

$$s \Vdash \text{“}\dot{t} \text{ is a member of some set in the square sequence”},$$

and therefore $\dot{t}$ is forced to be a member of the ground model.

Thus, there is an extension of s, s', which decides the value of $\dot{t}$ to be equal to an element in V, that we will denote by t. The closed set t might have no extension in $s'(\max \operatorname{dom} s')$ but we can extend s' to s'' where $\operatorname{dom} s'' = \operatorname{dom} s' + \omega + 1$, and t has an extension in the top element of s''. Let call this extension t'. Thus we have a condition $\langle s'', t'\rangle \leq \langle s, t\rangle$ and $\langle s'', t'\rangle$ has the desired form.

The set

$$D = \{\langle s, \check{t}\rangle \in \mathbb{S} * \mathbb{T}_\eta \mid \max t = \max \operatorname{dom} s\}$$

is η-directed closed. Let $\rho < \eta$ and let $\{\langle s_i, \check{t}_i\rangle \mid i < \rho\} \subset D$ be a directed set. Let us assume that $\sup \operatorname{dom} s_i$ is a limit ordinal (otherwise, the sequence is fixed on a tail). The condition $\langle s_\star, t_\star\rangle$, where $t_\star = \bigcup t_i$ and $s_\star = (\bigcup s_i)^\frown\langle\{t_\star\}\rangle$ is a condition in D, stronger than s_i for all i.

The claim that $\mathbb{S} * \mathbb{T}_\eta^\rho$ contains a η-closed dense subset (for all $\rho < \mu$), is proved by the same method. For this case, we consider

$$D = \{\langle s, \langle t_\alpha \mid \alpha < \rho\rangle\rangle \mid \forall \alpha < \rho,\ \max t_\alpha = \max \operatorname{dom} s\}.$$

By the same argument, using the fact that the bound on the cardinality of the set $s(\max \operatorname{dom} s)$, for $s \in \mathbb{S}$, is greater than ρ, we conclude that D is dense and η-directed closed in $\mathbb{S} * \mathbb{T}_\eta^\rho$. ⊣

Let us move now toward the proof of 3.1. Let $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ be supercompact cardinals. By using Laver's preparation, we may assume that they are Laver-indestructible, i.e., that for every $n < \omega$ and every κ_n-directed closed forcing $\mathbb{P}$, $\Vdash_\mathbb{P} \check{\kappa}_n$ is supercompact. Let $\mathbb{M} = \prod_{i<\omega} \operatorname{Col}(\kappa_i, <\kappa_{i+1})$ a full support product of Levy collapses.

Lemma 3.7. *After forcing with* $\mathbb{S} \times \mathbb{M}$, *the narrow system property holds at* λ.

Proof. Let $H_S \subseteq \mathbb{S}$, $H_M \subseteq \mathbb{M}$ be mutually generic filters. Let $G = H_S \times H_M$. Let us denote by $H_i \subseteq \operatorname{Col}(\kappa_{i-1}, <\kappa_i)$ be the i-th coordinate of the generic filter H_M ($i > 0$). Let H^i be the generic filters for all the parts of $\mathbb{M}$ except the i-th coordinate, namely $H^i = \langle H_m \mid m \neq i\rangle$.

Let $\mathcal{S} \in V[G]$ be a narrow system on $I \times \eta$, with relations $\mathcal{R}$. Let us assume, towards a contradiction, that $\mathcal{S}$ has no cofinal branch in $V[G]$. Since the set I will play no role later in the proof, we will restrict ourselves to the notation-wise simpler case in which $I = \lambda$. Let $n \geq 2$ be large enough such that $\kappa_{n-2} \geq |\eta \times \mathcal{R}|^+$ in $V^{\mathbb{S}\times\mathbb{M}}$.

Let $W_n = V[H_S][H^n]$. Let us force over W_n with $\mathbb{T}_{\kappa_n}^{\kappa_{n-2}}$. Let $K = \langle K_i \mid i < \kappa_{n-2}\rangle$ be the sequence of pairwise mutually generic filters. We stress that the product, $\mathbb{T}_{\kappa_n}^{\kappa_{n-2}}$, is taken over $V[G]$ and not over W_n.

Fix $\xi < \kappa_{n-2}$. $W_n[K_\xi] \models \kappa_n$ is supercompact since:

(1) $\mathbb{S} * \mathbb{T}_{\kappa_n}^{\kappa_{n-2}}$ contains a dense κ_n-directed closed subset,
(2) $\prod_{n\leq i<\omega} \operatorname{Col}(\kappa_i, <\kappa_{i+1})$ is κ_n-directed closed.
(3) $\prod_{i<n-1} \operatorname{Col}(\kappa_i, <\kappa_{i+1})$ has cardinality κ_{n-1} which is $< \kappa_n$.

We are using the indestructibility in the two first items and Lévy-Solovay Theorem in the last one.

https://doi.org/10.1017/jsl.2019.4 Published online by Cambridge University Press

Let $j\colon W_n[K_\xi] \to M$ be a λ-supercompact embedding with crit $j = \kappa_n$. Since $\mathrm{Col}(\kappa_{n-1}, < \kappa_n)$ is κ_n-c.c., after forcing with

$$\mathrm{Col}(\kappa_{n-1}, < j(\kappa_n)) = \mathrm{Col}(\kappa_{n-1}, < \kappa_n) \times \mathrm{Col}(\kappa_{n-1}, [\kappa_n, j(\kappa_n)))$$

we may extend the elementary embedding j to a λ-supercompact elementary embedding $\tilde{j}\colon W_n[H_n][K_\xi] \to M[\tilde{j}(H_n)]$. Since $W_n[H_n] = V[G]$, $\mathcal{S} \in W_n[H_n]$, so $\tilde{j}(\mathcal{S})$ is defined.

Let $L = \langle L_i \mid i < \kappa_{n-2}\rangle$ be a generic filter for $\mathrm{Col}(\kappa_{n-1}, [\kappa_n, j(\kappa_n)))^{\kappa_{n-2}}$. Note that the forcing that adds L is κ_{n-1}-closed over V, the ground model.

Let $\delta = \sup \tilde{j}''\lambda < \tilde{j}(\lambda)$. Let $\leq_i \in \mathcal{R}$ and let

$$b_{i,\epsilon} = \{\langle \alpha, \beta\rangle \mid \langle j(\alpha), \beta\rangle \leq_i \langle \delta, \epsilon\rangle \text{ in } \tilde{j}(\mathcal{S})\}.$$

Since $|\mathcal{R}|, \eta < \kappa_{n-2} <$ crit $\tilde{j}$, for some i, ϵ, $b_{i,\epsilon}$ is a cofinal branch and moreover $\bigcup_{i,\epsilon}\{\alpha \mid \exists \beta,\ \langle \alpha, \beta\rangle \in b_{i,\epsilon}\} = \lambda$.

We say that forcing with $\mathrm{Col}(\kappa_{n-1}, [\kappa_n, j(\kappa_n))) \times \mathbb{T}_{\kappa_n}$ adds a *system of branches* for $\mathcal{S}$. Without loss of generality, all branches are new (and thus, cofinal).

In particular the forcing $\mathrm{Col}(\kappa_{n-1}, [\kappa_n, < j(\kappa_n)))^{\kappa_{n-2}} \times \mathbb{T}_{\kappa_n}^{\kappa_{n-2}}$ introduces κ_{n-2} many distinct realizations for the system of branches $\{\dot{b}_i \mid i \in J\}$. Note that in order to claim that there is no pair of system of branches which are equal we only used the pairwise mutual genericity.

We conclude that in $V[G][H][K][L]$ there are κ_{n-2} different systems of branches, $\{b_i^\alpha \mid \alpha < \kappa_{n-2},\ i \in J\}$. In this model $\kappa_{n-2} \geq |\eta \times \mathcal{R}|^+$ is regular and cf $\lambda \geq \kappa_{n-1}$. Since for every $\alpha < \beta < \kappa_{n-2}$, and every relation $\leq_i \in \mathcal{R}$, b_i^α, b_i^β split at some point below λ, and since there are only κ_{n-2} realizations and only $|\mathcal{R}|$ relations in $\mathcal{R}$, there is $\rho_\star < \lambda$ such that for every $\xi \geq \rho_\star$, and for every α, β, $b_i^\alpha(\xi) \neq b_i^\beta(\xi)$ (where it is possible that only one of them is defined). By the Pigeonhole Principle there are $\alpha, \beta < \kappa_{n-2}$ such that $\langle \rho_\star, \xi\rangle \in b_i^\alpha, b_i^\beta$ for the same ξ, i, because there are only $|\mathcal{R}| \times \eta$ many possibilities for this pair. This is a contradiction to the choice of $\rho_\star$. We conclude that it is impossible that there was not cofinal branch in $\mathcal{S}$ in the ground model, as wanted. ⊣

Let $W = V^{\mathbb{S}\times\mathbb{M}}$. Note that κ is supercompact in W, by the Laver indestructibility of κ.

Theorem 3.8. *There is $\rho < \kappa$ such that forcing with $\mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+1}, < \kappa)$ over W forces the tree property at $\aleph_{\omega+1}$. Further collapsing the new $\aleph_1$ introduces a weak square at $\aleph_{\omega+1}$.*

Proof. Assume otherwise. Let $\mathbb{L}_\rho = \mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+1}, < \kappa)$. For every $\rho < \kappa$, let $\dot{T}_\rho$, be a $\mathbb{L}_\rho$-name for an Aronszajn tree at λ. Since κ is supercompact, there is $j\colon W \to M$ such that ${}^\lambda M \subseteq M$. By our assumption, M models that $\Vdash_{j(\mathbb{L})_\kappa}$ "$j(\dot{T})_\kappa$ is an Aronszajn tree". Let $\delta = \sup j``\lambda < j(\lambda)$, and let $t = \langle \delta, 0\rangle$.

Work in M. For every $\alpha < \lambda$, pick a condition $p_\alpha = \langle c_\alpha, q_\alpha\rangle$ such that

$$\exists \zeta < j(\kappa^{+\omega}),\ p_\alpha \Vdash_{j(\mathbb{L}_\kappa)} \langle j(\alpha), \zeta\rangle \leq_{j(\dot{T})_\kappa} \check{t}.$$

Let us denote this ζ by ζ_α. We may pick the conditions p_α in a way that q_α is a decreasing sequence. Since λ is regular and $|\mathrm{Col}(\omega, \kappa^{+\omega})| = \kappa^{+\omega} < \lambda$, there is a cofinal set $I \subseteq \lambda$, $n < \omega$ and $c_\star \in \mathrm{Col}(\omega, \kappa^{+\omega})$ such that for every $\alpha \in I$, $c_\alpha = c_\star$ and $\zeta_\alpha < j(\kappa^{+n})$.

By elementarity, for every $\alpha, \beta \in I$, there are $\gamma, \gamma' < \kappa^{+n}$, $\rho < \kappa$ and $p \in \mathbb{L}_\rho$ such that $p \Vdash_{\mathbb{L}_\rho} \langle \alpha, \gamma \rangle \leq_{\dot{T}_\rho} \langle \beta, \gamma' \rangle$.

This defines a narrow system in W: The domain of the system is $I \times \kappa^{+n}$. The indices set is $\bigcup_{\rho<\kappa} \mathbb{L}_\rho \times \{\rho\}$. $\langle \alpha, \xi \rangle \leq_{p,\rho} \langle \beta, \zeta \rangle$ iff $p \Vdash_{\mathbb{L}_\rho} \langle \alpha, \xi \rangle \leq_{\dot{T}_\rho} \langle \beta, \zeta \rangle$.

By the narrow system property there is a cofinal branch in W. Namely there are $\rho < \kappa$, $p \in \mathbb{L}_\rho$, and $\gamma < \kappa^{+n}$ such that for every $\alpha, \beta \in I$, $p \Vdash_{\mathbb{L}_\rho} \langle \alpha, \gamma \rangle \leq \langle \beta, \gamma \rangle$.

This proves that the tree property holds at $\aleph_{\omega+1}$ in the generic extension.

For the last claim, note that after collapsing $\aleph_1$, for every $\gamma < \aleph_{\omega+1}$ either cf $\gamma = \omega$ or $\mathcal{C}_\gamma \neq \emptyset$. Thus, one can complete the partial square to a full $\square_{\aleph_\omega, <\aleph_\omega}$ by adding cofinal ω-sequences. ⊣

§4. **Indestructible tree property.** In this section we will build three models in which the tree property at a successor of singular cardinal is indestructible under certain class of forcing notions. We start by building a model in which the tree property holds at $\aleph_{\omega^2+1}$ and it is indestructible under any forcing $\mathbb{P}$ of cardinality less than $\aleph_{\omega^2}$. Similarly, we will construct a model for the tree property at $\aleph_{\omega+1}$ in which the tree property still holds after any σ-closed forcing of cardinality $< \aleph_\omega$.

We remark that we do not know whether it is possible to force the tree property at $\aleph_{\omega+1}$ to be indestructible under any $\aleph_{\omega+1}$-closed forcing notions.

4.1. Indestructible tree property for $\aleph_{\omega^2+1}$. In this subsection, we will show that in Sinapova's model for the tree property at $\aleph_{\omega^2+1}$ [7] (without the failure of SCH), the tree property is indestructible under small forcings. We start with some simple observations:

LEMMA 4.1. *Let λ be a cardinal such that the tree property holds at λ^+ and it is indestructible by any forcing of the form* Col(ω, ρ) *for $\rho < \lambda$. Then the tree property at λ^+ is indestructible by any forcing of size $< \lambda$. Moreover, it is enough to assume that for every $\rho < \lambda$ there is $\rho \leq \rho' < \lambda$ such that* Col(ω, ρ') *forces the tree property at λ^+.*

PROOF. Let $\mathbb{P}$ be a forcing notion of cardinality $< \lambda$. Let $\mu = |\mathbb{P}|$. Col(ω, ρ) adds a generic filter for $\mathbb{P}$. Let $G \subseteq \mathbb{P}$ be a generic filter. The quotient forcing Col$(\omega, \rho)/G$ has cardinality at most ρ and therefore it does not add a cofinal branch to any λ^+-Aronszajn tree. Since the tree property holds after forcing with Col(ω, ρ) and the forcing Col$(\omega, \rho)/G$ does not add a branch to Aronszajn tree – the tree property holds in $V[G]$ as well. ⊣

THEOREM 4.2. *Let $\kappa = \kappa_0 < \kappa_1 < \cdots$ be a sequence of ω supercompact cardinals. Let $\mu = \sup \kappa_n$ and $\lambda = \mu^+$. There is a generic extension in which $\kappa = \aleph_{\omega^2}$, $\lambda = \aleph_{\omega^2+1}$ and for every $\rho < \mu$, the tree property holds after forcing with* Col(ω, ρ).

In order to prove this theorem, we will work with Sinapova's model for the tree property at $\aleph_{\omega^2+1}$ from [7]. We will not need to violate SCH at this point, so the proof is somewhat simpler at some points.

The main idea behind the indestructibility is that one can define a projection $f : \mathbb{P} \times \text{Col}(\omega, \rho) \to \mathbb{P}_n$ that shifts the Prikry sequence by n steps to the left, where $\mathbb{P}_n$ is a "shifted" version of the forcing $\mathbb{P}$ which forces the tree property as well. This way, we can analyze the sets that were added by a forcing of the form Col(ω, ρ) simply by shifting the first element of the Prikry sequence to be above ρ.

https://doi.org/10.1017/jsl.2019.4 Published online by Cambridge University Press

We start with a well known fact:

LEMMA 4.3. *Let* $\mathbb{M} = \prod_{n<\omega} \mathrm{Col}(\kappa_n, < \kappa_{n+1})$*—a full support product of Levy collapses. In* $V^{\mathbb{M}}$ *the narrow branch property holds at* λ^+.

The proof is similar to the proof of Lemma 3.7 and appears in [5].

Work in $V^{\mathbb{M}}$. The cardinal $\kappa = \kappa_0$ is still supercompact, by the Laver indestructibility. Let $\mathcal{U}$ be a normal measure on $P_\kappa\lambda$ in $V^{\mathbb{M}}$. Let $\mathcal{U}_n$ be the projection of $\mathcal{U}$ to $P_\kappa\kappa_n$ for $n < \omega$.

Let $j_n\colon W \to N_n \cong \mathrm{Ult}(W, \mathcal{U}_n)$ be the elementary embedding derived from $\mathcal{U}_n$. Let us construct an N_n-generic filter H_n for the forcing $\mathrm{Col}(\kappa^{+\omega+2}, < j(\kappa))^{N_n}$. This is possible by the standard arguments: the forcing notion $\mathrm{Col}(\kappa^{+\omega+2}, < j(\kappa))^{N_n}$ is κ^{+n+1}-closed in W and has only κ^{+n+1}-dense subsets in N_n (as counted by $V^{\mathbb{M}}$).

Let us define the main forcing notion $\mathbb{P}$:

A condition $p \in \mathbb{P}$ has the following form

$$p = \langle d_0, a_0, c_0, \dots, a_{n-1}, c_{n-1}, A_n, C_n, \dots \rangle$$

where,

(1) $a_i \in P_\kappa\kappa^{+i}$ and $A_i \in \mathcal{U}_i$. Let $\rho_i = a_i \cap \kappa$ if $i < n$ and $\rho_i = \kappa$ otherwise.
(2) $d_0 \in \mathrm{Col}(\omega, \rho_0^{+\omega})$ if $\rho_0 < \kappa$ and otherwise $d_0 \in \mathrm{Col}(\omega, \kappa)$.
(3) $c_i \in \mathrm{Col}(\rho_i^{+\omega+2}, < \rho_{i+1})$.
(4) $C_i\colon A_i \to W$ such that $C_i(a) \in \mathrm{Col}((a \cap \kappa)^{+\omega+2}, < \kappa)$ for every $a \in A_i$ and $[C_i]_{\mathcal{U}_i} \in H_i$.

n is called the length of p and we denote $\mathrm{len}(p) = n$.

A condition p is stronger than q ($p \leq q$) if:

(1) $\mathrm{len}(p) \geq \mathrm{len}(q)$.
(2) $d_0^p \leq d_0^q$.
(3) $a_i^p = a_i^q$ and $c_i^p \leq c_i^q$ for every $i < \mathrm{len}(q)$.
(4) $a_i^p \in A_i^q$ and $c_i^p \leq C_i^q(a_i)$ for $\mathrm{len}(q) \leq i < \mathrm{len}(p)$.
(5) $A_i^p \subseteq A_i^q$ for $i \geq \mathrm{len}(p)$.
(6) $C_i^p(a) \leq C_i^q(a)$ for every $a \in A_i^p$.

For the proof of Theorem 4.2, we will also need to consider the following shifted version of $\mathbb{P}$. For every $s < \omega$, we define the forcing $\mathbb{P}_s$.

A condition $p \in \mathbb{P}_s$ has the following form

$$p = \langle d_0, a_0, c_0, \dots, a_{n-1}, c_{n-1}, A_n, C_n, \dots \rangle$$

where,

(1) $a_i \in P_\kappa\kappa^{+i+s}$ and $A_i \in \mathcal{U}_{i+s}$. Let $\rho_i = a_i \cap \kappa$ if $i < n$ and $\rho_i = \kappa$ otherwise.
(2) $d_0 \in \mathrm{Col}(\omega, \rho_0^{+\omega})$ if $\rho_0 < \kappa$ and otherwise $d_0 \in \mathrm{Col}(\omega, \kappa)$.
(3) $c_i \in \mathrm{Col}(\rho_i^{+\omega+2}, < \rho_{i+1})$.
(4) $C_i\colon P_\kappa\kappa^{+i+s} \to W$ such that $C_i(a) \in \mathrm{Col}((a \cap \kappa)^{+\omega+2}, < \kappa)$ for every $a \in A_i$ and $[C_i]_{\mathcal{U}_{i+s}} \in H_{i+s}$.

We order the conditions in the same way as we did for $\mathbb{P}$. Note that $\mathbb{P}_0 = \mathbb{P}$.

THEOREM 4.4 (Sinapova). *For every* $s < \omega$, $\mathbb{P}_s$ *forces that* $\lambda = \aleph_{\omega^2+1}$ *and the tree property holds in* λ.

PROOF. We will give a sketch of the proof. We will show that the claim holds for $s = 0$. The argument for general s is the same, notation wise more complicated.

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Let $p \in \mathbb{P}$ be a condition and let $\dot{T}$ be a name for a λ-Aronszajn tree. Let n be the length of p. Let $j\colon V \to M$ be a λ-supercompact embedding, with a critical point κ which is compatible with $\mathcal{U}_n$ (namely $\mathcal{U}_n$ is the $P_\kappa\kappa^{+n}$ measure which is derived from j).

In M, let us look at the forcing $j(\mathbb{P})$ below a condition $q \leq j(p)$ of length $n+1$ such that $a^q_n = j''\kappa^{+n}$. In other words, q is an extension of $j(p)$ that forces that the $n+1$-th element of the diagonal Prikry sequence is $j''\kappa^{+n}$. The forcing $j(\mathbb{P})/q$ preserves λ as a regular cardinal and realizes $j(\dot{T})$ to be a $j(\lambda)$-Aronszajn tree.

Let us denote $\delta = \sup j``\lambda < j(\lambda)$ and let us look at the name of a partial branch $\{\langle j(\alpha), \zeta_\alpha\rangle \mid M^{j(\mathbb{P})} \models \langle j(\alpha), \zeta_\alpha\rangle \leq_{j(\dot{T})} \langle \delta, 0\rangle\}$.

Using the Prikry property, we may find a direct extension of q, $q^\star$, such that for every $\alpha < \lambda$ the value of $k < \omega$ such that $\zeta_\alpha < j(\kappa^{+k})$ is determined by $q^\star$ up to forcing with the first n lower parts of $j(\mathbb{P})$. Since there are less than λ many possible values for the first n coordinates of the conditions below $q^\star$, there is a cofinal subset of λ, I, a natural number $n_\star < \omega$ large enough and a fixed lower part $a_\star$ of length $n+1$ such that

$$I = \{\alpha < \lambda \mid \exists r \leq q^\star, \mathrm{stem}(r) = a_\star, r \Vdash \exists \zeta < j(\kappa^{+n_\star}), \langle j(\alpha), \zeta\rangle \leq \langle \delta, 0\rangle\}.$$

In particular, for every $\alpha, \beta \in I$, M thinks that there is an extension of $j(p)$, $q^{\star\star}$ of length $n+1$ and ordinals $\zeta, \zeta' < j(\kappa^{+n_\star})$ such that $q^{\star\star} \Vdash \langle j(\alpha), \zeta\rangle \leq_{j(\dot{T})} \langle j(\beta), \zeta'\rangle$. Reflecting this to V we conclude that for every $\alpha, \beta \in I$ there is a condition $q' \leq p$ with stem of length $n+1$ and $\zeta, \zeta' < \kappa^{+n_\star}$ such that $q' \Vdash \langle \alpha, \zeta\rangle \leq_{\dot{T}} \langle \beta, \zeta'\rangle$.

This defines a narrow system on $I \times \kappa^{+n_\star}$, indexed by the stems of length $n+1$ which are stems of some condition which is stronger than p. By the narrow system property, there is a cofinal branch. So there is $I' \subseteq I$, a stem $s_\star$ and an ordinal $\zeta_\star < \kappa^{+n_\star}$ such that for every $\alpha < \beta$ in I' there is a condition q with stem $s_\star$ forcing $\langle \alpha, \zeta_\star\rangle \leq_{\dot{T}} \langle \beta, \zeta_\star\rangle$.

Next we will build inductively a sequence of conditions $\langle p_\alpha \mid \alpha \in I' \setminus \rho\rangle$ (for some $\rho < \lambda$), such that for every $\alpha < \beta$,

$$p_\alpha \wedge p_\beta \Vdash \langle \alpha, \zeta_\star\rangle \leq_{\dot{T}} \langle \beta, \zeta_\star\rangle.$$

The construction is done by induction on $m < \omega$, where at each step we define $p_\alpha \restriction m$ in a way that for all α, β (except a bounded segment) there is a condition q with $q \restriction m = p_\alpha \restriction m \wedge p_\beta \restriction m$ such that

$$q \Vdash \langle \alpha, \zeta_\star\rangle \leq_{\dot{T}} \langle \beta, \zeta_\star\rangle.$$

Extending $p_\alpha \restriction m$ to $p_\alpha \restriction (m+1)$ is done by defining a narrow system corresponding to the possible extension and using the branch in order to define the relevant value for all $\alpha \in I'$ above the first level that the branch meets.

Eventually, we obtain a sequence of conditions $\{p_\alpha \mid \alpha \in I' \setminus \rho\}$, for some $\rho < \lambda$, $p_\alpha \leq p$. Using the chain condition of the forcing $\mathbb{P}$ we conclude that there is an extension of p that forces that for unbounded many ordinals $\alpha < \lambda$, p_α will be in the generic filter. But then $\{\langle \alpha, \zeta_\star\rangle \mid p_\alpha \in G\}$ is a cofinal branch in $\dot{T}$ (where G is the generic filter for $\mathbb{P}$). ⊣

In order to show the indestructibility, we need to show that there is a simple connection between the different shifts of the forcing:

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LEMMA 4.5. *Let* $p \in \mathbb{P}$, $\operatorname{len}(p) = n + 1$, $n \geq 1$ *and let* $f \in \operatorname{Col}(\omega, \rho_n^{+\omega})$. *There is a condition* $q \in \mathbb{P}_n$, *of length one such that* $\rho_0^q = \rho_n^p$, *such that* $\mathbb{P}_n/q \cong (\mathbb{P}/p) \times \left(\operatorname{Col}(\omega, \rho_n^{+\omega})/f\right)$.

PROOF. Let $\eta = \left(\rho_n^p\right)^{+\omega}$.

The forcing $\mathbb{P}/p$ is the product $\mathbb{C}/p^{<n} \times \mathbb{P}^{\geq n}/p^{\geq n}$ where

$$\mathbb{C} = \operatorname{Col}(\omega, \left(\rho_0^p\right)^{+\omega}) \times \prod_{i<n} \operatorname{Col}(\left(\rho_i^p\right)^{+\omega+2}, < \rho_{i+1}^p)$$

and $\mathbb{P}^{\geq n}$ is the set of the n-upper part of the conditions of $\mathbb{P}$. More precisely, a condition $s \in \mathbb{P}^{\geq n}$ is an ω-sequence of the form

$$s = \langle a_n^s, c_n^s, \dots, a_{l-1}^s, c_{l-1}^s, A_l^s, C_l^s, \dots \rangle,$$

where $l \geq n$ and $a_i^s, c_i^s, A_i^s, C_i^s$ are as in the definition of $\mathbb{P}$ (in particular, $a_i^s \in P_\kappa \kappa^{+i}$).

The conditions $p^{\geq n} \in \mathbb{P}^{\geq n}$, $p^{<n} \in \mathbb{C}$ are defined as follows:

$$p^{<n} = \langle d_0^p, c_0^p, \dots, c_{n-1}^p \rangle,$$

$$p^{\geq n} = \langle a_n^p, c_n^p, A_{n+1}^p, C_{n+1}^p, \dots, A_l^p, C_l^p, \dots \rangle.$$

Clearly, $|\mathbb{C}| \leq \eta$ and thus $(\mathbb{C}/p^{<n}) \times \left(\operatorname{Col}(\omega, \eta)/f\right) \cong \operatorname{Col}(\omega, \eta)$. Let us fix an isomorphism $\pi_0 \colon \operatorname{Col}(\omega, \eta) \to (\mathbb{C}/p^{<n}) \times \left(\operatorname{Col}(\omega, \eta)/f\right)$. Note that $\pi_0(\emptyset) = (p^{<n}, f)$.

Let $q \in \mathbb{P}_n$ be the condition $\langle \emptyset \rangle^\frown p^{\geq n}$.

By the definition of $\mathbb{P}_n$ and $\mathbb{P}^{\geq n}$,

$$\mathbb{P}_n/q \cong \operatorname{Col}(\omega, \eta) \times \left(\mathbb{P}^{\geq n}/p^{\geq n}\right).$$

Combining this with the isomorphism π_0, we obtain the isomorphism:

$$\mathbb{P}_n/q \cong (\mathbb{P}/p) \times \left(\operatorname{Col}(\omega, \eta)/f\right). \qquad \dashv$$

THEOREM 4.6. $\mathbb{P}$ *forces the tree property at* $\aleph_{\omega^2+1}$ *to be indestructible by any forcing of size* $< \aleph_{\omega^2}$.

PROOF. Is it enough to show that it is the case for $\operatorname{Col}(\omega, \aleph_{\omega \cdot n})$. Recall that $\aleph_{\omega \cdot n} = \rho_n^{+\omega}$ so we are in the situation of Lemma 4.5. This means that after forcing with $\operatorname{Col}(\omega, \aleph_{\omega \cdot n})$ the tree property holds, as the iteration is isomorphic to the forcing notion $\mathbb{P}_n$ below some condition. $\dashv$

4.2. Indestructible tree property for $\aleph_{\omega+1}$ under small σ-closed forcings. Let us construct a model very similar to Section 4.1, in which we have the tree property at $\aleph_{\omega+1}$ and it will be indestructible under any σ-closed forcing of cardinality $< \aleph_\omega$. The additional restriction on the forcing notions (namely that the forcing is σ-closed), implies that those forcing notions cannot collapse ω_1.

THEOREM 4.7. *It is consistent, relative to the existence of* ω *many supercompact cardinals, that the tree property holds at* $\aleph_{\omega+1}$ *and it is indestructible under any* σ*-closed forcing of cardinality* $< \aleph_\omega$.

PROOF. We will start with a model of the narrow system property at $\kappa^{+\omega+1}$ for κ a supercompact cardinal. This can be obtained, for example, by forcing with the product of the Levy collapses between the supercompact cardinals as in Lemma 4.3. Let

$\mathcal{U}_0$ be a normal ultrafilter on κ generated from a $\kappa^{+\omega+1}$-supercompact elementary embedding, $j \colon V \to M$.

Let us show that for every $n < \omega$, there is a large set $A_n \in \mathcal{U}_0$ such that for every $\rho \in A_n$, forcing with $\mathbb{L}_\rho = \mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+1}, \kappa^{+n})$ forces the tree property at $\kappa^{+\omega+1}$.

Assume that this is not the case and let $\dot{T}_\rho$ be a counter example for every bad choice of ρ, for a fixed $n < \omega$. Since the set of bad choices is in $\mathcal{U}_0$, κ is a bad choice of ordinal in M. Let us force with $j(\mathbb{L})_\kappa$, and let $M[H]$ be the generic extension. Let $T = j(\dot{T})^H_\kappa$ be an Aronszajn tree at $j(\kappa^{+\omega+1})$. Let $\delta = \sup j``\kappa^{+\omega+1}$ and for every $\alpha < \kappa^{+\omega+1}$ let $\beta_\alpha < j(\kappa^{+\omega})$ be the element in the level $j(\alpha)$ below $\langle \delta, 0 \rangle$.

Using the same arguments as in the proof of Theorem 3.8, there is a cofinal set $I \subseteq \kappa^{+\omega+1}$, a decreasing sequence of conditions $q_\alpha \in \mathrm{Col}(\kappa^{+\omega+1}, j(\kappa)^{+n})$, a condition $p \in \mathrm{Col}(\omega, \kappa^{+\omega})$ and a natural number $N < \omega$ such that for every $\alpha \in I$ there is $\beta < j(\kappa^{+N})$ such that $(p, q_\alpha) \Vdash \langle j(\alpha), \beta \rangle \leq_T \langle \delta, 0 \rangle$.

Reflecting this back to V, we conclude that for every $\alpha, \alpha' \in I$:

$$\exists \beta, \beta' < \kappa^{+N},\ \rho < \kappa,\ p \in \mathbb{L}_\rho \text{ such that } p \Vdash_{\mathbb{L}_\rho} \langle \alpha, \beta \rangle \leq_{T_\rho} \langle \alpha', \beta' \rangle.$$

This gives us a narrow system, similar to the one in the proof of Theorem 3.8. A branch through this system provides us an ordinal ρ which was a bad choice, a condition $r \in \mathbb{L}_\rho$, a cofinal set $J \subseteq I$ and for all $\alpha \in J$ an ordinal $\beta_\alpha < \kappa^{+N}$ such that for all $\alpha, \alpha' \in J$,

$$r \Vdash_{\mathbb{L}_\rho} \langle \alpha, \beta_\alpha \rangle, \langle \alpha', \beta_{\alpha'} \rangle \text{ are compatible.}$$

This is a contradiction to the fact that this $\dot{T}_\rho$ was a name for an λ-Aronszajn tree.

Let $A = \bigcap_{n<\omega} A_n$ and let $\rho \in A$. Forcing with $\mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+1}, \kappa)$ forces the tree property. For every small σ-closed forcing notion $\mathbb{Q}$ there is n such that $\mathrm{Col}(\rho^{+\omega+1}, \kappa) * \mathbb{Q}$ is a regular subforcing of $\mathrm{Col}(\rho^{+\omega+1}, \kappa^{+n})$ and since the tree property holds after this forcing and since the quotient is small and thus cannot add branches to Aronszajn trees - we are done. $\dashv$

§5. Open questions. In Section 4.1 we proved that the tree property at $\aleph_{\omega^2+1}$ can be made indestructible under any small forcing poset.

QUESTION 5.1. *Is it consistent that the tree property at $\aleph_{\omega+1}$ is indestructible under any forcing of cardinality $< \aleph_\omega$?*

On the other hand, one can ask whether it is possible to extend the results of Theorem 3.1.

QUESTION 5.2. *Is it consistent that the tree property holds at $\aleph_{\omega+1}$ but there is a small forcing (of cardinality $< \aleph_\omega$), that does not collapse cardinals and adds an $\aleph_{\omega+1}$-Aronszajn tree?*

Note that in all the currently known models for the tree property at $\aleph_{\omega+1}$, adding a single Cohen real does not add an Aronszajn tree at $\aleph_{\omega+1}$. So we ask the following stronger version of Question 5.2:

QUESTION 5.3. *Is it consistent that the tree property holds at $\aleph_{\omega+1}$ but adding a Cohen real adds an $\aleph_{\omega+1}$-Aronszajn tree?*

This question is particularly interesting when we assume that $\aleph_\omega$ is strong limit since then adding a Cohen real cannot add a weak square for $\aleph_\omega$, assuming that there is no weak square in the ground model.

Acknowledgments. We would like to thank the anonymous referee for improving the readability and accuracy of this article.

REFERENCES

[1] S. Ben-David and S. Shelah, *Souslin trees and successors of singular cardinals*. ***Annals of Pure and Applied Logic***, vol. 30 (1986), no. 3, pp. 207–217.

[2] J. Cummings, M. Foreman, and M. Magidor, *Squares, scales and stationary reflection*. ***Journal of Mathematical Logic***, vol. 1 (2001), no. 1, pp. 35–98.

[3] C. Lambie-Hanson, *Squares and narrow systems*, this Journal, vol. 82 (2017), no. 3, pp. 834–859.

[4] M. Magidor and S. Shelah, *The tree property at successors of singular cardinals*. ***Archive for Mathematical Logic***, vol. 35 (1996), no. 5–6, pp. 385–404.

[5] I. Neeman, *The tree property up to* $\aleph_{\omega+1}$, this Journal, vol. 79 (2014), no. 2, pp. 429–459.

[6] A. Rinot, *A cofinality-preserving small forcing may introduce a special Aronszajn tree*, ***Archive for Mathematical Logic***. vol. 48 (2009), no. 8, pp. 817–823.

[7] D. Sinapova, *The tree property and the failure of the singular cardinal hypothesis at* $\aleph_{\omega^2}$, this Journal, vol. 77 (2012), no. 3, pp. 934–946.

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