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Characterization of homogeneous scalar variational problems solvable for all boundary data

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It is known that the condition '*either* $\partial L(F) \neq \emptyset$ *or there exist* $v_1, \dots, v_q \in R^n$ *such that* $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$ *and* $\cap_{i=1}^{q} \partial L(v_i) \neq \emptyset$' characterizes solvability of the problem

$$J(u) = \int_\Omega L(Du)\,\mathrm{d}x \to \min, \qquad u \in W^{1,1}(\Omega), \qquad u|_{\partial\Omega} = f,$$

with $f(\cdot) = \langle F, \cdot \rangle$.

We extend this result to the case of lower semicontinuous integrands $L : R^n \to R$.

We also show that validity of this condition for all $F \in R^n$ is both a necessary and sufficient requirement for solvability of *all minimization problems* with sufficiently regular Ω and f. Moreover, the assumptions on Ω and f can be completely dropped if L has sufficiently fast growth at infinity.

1. Introduction

In this paper we deal with minimization problems

$$J(u) \to \min, \qquad u|_{\partial\Omega} = f, \qquad u \in W^{1,1}(\Omega) \tag{1.1}$$

for integral functionals of the form

$$J(u) = \int_\Omega L(Du(x))\,\mathrm{d}x,$$

where Ω is an open bounded subset of R^n with *Lipschitz boundary*, and $L : R^n \to R$ is a lower semicontinuous function.

A function $u \in W^{1,1}(\Omega)$ is called an *admissible* function for the problem (1.1) if $u|_{\partial\Omega} = f$ and the negative part of the function $L(Du)$ is integrable. In this case, $J(u)$ is well defined but equals $+\infty$ if the positive part of $L(Du)$ is not summable.

We accept the following notations: for a subset A of R^n, the sets $\operatorname{int} A$, $\operatorname{re\,int} A$, $\operatorname{co} A$, and $\operatorname{extr} A$ are, respectively, the interior, the relative interior, the convex hull, and the set of extremum points of A ($a \in \operatorname{extr} A$ if it can not be represented as a convex combination of other points of A). $B(a, \epsilon)$ denotes a ball of radius ϵ centred

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at the point $a \in R^n$, l_a is a linear function with the gradient equal to a everywhere. $\partial L(F)$ denotes the subgradient of L at a point F,

$$\partial L(F) := \{l \in R^n : L(v) - L(F) - \langle l, v - F\rangle \geqslant 0,\ \forall v \in R^n\}.$$

L^{**} is the convexification of L: the epigraph of L^{**} is the convexification of the epigraph of L, that is

$$L^{**}(v_0) := \inf\left\{\sum_{i=1}^{q} c_i L(v_i) : q \in N,\ c_i \geqslant 0,\ v_i \in R^n,\ \sum_{i=1}^{q} c_i = 1,\ \sum_{i=1}^{q} c_i v_i = v_0\right\}.$$

Note that L^{**} exists as a function from R^n to R if and only if L exceeds an affine function everywhere (see lemma 2.1).

Weak and strong convergences of sequences are denoted by $\rightharpoonup$ and $\rightarrow$, respectively.

We will frequently use the following version of the Vitaly covering theorem.

A family G of closed subsets of R^n is said to be a Vitaly cover of a bounded set A if for each $x \in A$ there exists a positive number $r(x) > 0$, a sequence of balls $B(x, \epsilon_k)$ with $\epsilon_k \to 0$, and a sequence $C_k \in G$ such that $x \in C_k$, $C_k \subset B(x, \epsilon_k)$, and $(\operatorname{meas} C_k / \operatorname{meas} B(x, \epsilon_k)) > r(x)$ for all $k \in N$.

The version of the Vitaly covering theorem from [23, p. 109] says that each Vitaly cover of A contains at most a countable subfamily of disjoint sets C_k such that $\operatorname{meas}(A \setminus \cup_k C_k) = 0$.

Problems (1.1) were studied recently in the framework of the existence theory in elasticity: when dealing with homogeneous materials undergoing anti-plane shear deformations

$$(x, y, z) \in R^3 \to (x, y, z + u(x, y)) \in R^3,$$

the problem of minimization of the free energy is of the form (1.1). While the existence results are well known for problems (1.1) with convex integrands (see, for example, [13]), the situation is poorly understood in the case of non-convex problems. Note that active research in the area of non-convex variational problems started since the work of [1], where the first existence results for realistic problems in elasticity were established in the general case (without restrictions on the class of admissible deformations).

Some recent efforts were devoted to the question of the solvability of problems (1.1) under restrictions on integrands motivated by physical reasons (see [2, 19, 22, 24]). In these papers, the solvability problem was treated for particular boundary data. Moreover, the papers [7, 8, 15] indicated conditions on integrands both necessary and sufficient for the problem (1.1) with linear boundary conditions $f = l_F$, $F \in R^n$, to have a solution.

The answer is given by the following theorem.

Theorem 1.1. *The problem (1.1) with a linear boundary condition $f = l_F$ is solvable if and only if*

(C) *either $\partial L(F) \neq \emptyset$, or there exist $v_1, \dots, v_q \in R^n$ ($q \in N$) such that $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$ and $\cap_{i=1}^{q} \partial L(v_i) \neq \emptyset$.*

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Here we state a slightly more general result, since theorem 1.1 was proved in [7, 8, 15] for integrands with superlinear growth at infinity. In [15], the sufficient part was also proved for continuous integrands bounded from below. We also use different terminology in order to formulate the result in terms of L only.

However, a crucial ingredient of the proofs (more precisely of their sufficient parts) is the use of special functions proposed in [7,8,15]. The same functions were in fact used much earlier in [20,21] with the purpose of giving simplified proofs of the relaxation theorems.

If $v_1, \dots, v_q$ are extremum points of a convex compact subset of R^n, and if $F \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$, then the function

$$w_s(x) = \max_{1 \leqslant i \leqslant q} \langle v_i - F, x\rangle - s \tag{1.2}$$

is Lipschitz, $Dw_s(x) \in \{v_i - F : i = 1, \dots, q\}$ a.e., and $w_s|_{\partial P_s} = 0$, where

$$P_s = \{x : \max_{1 \leqslant i \leqslant q} \langle v_i - F, x\rangle \leqslant s\}$$

is a compact set with Lipschitz boundary and non-empty interior.

Note that $P_s = sP_1$.

Since Vitaly covering arguments let us decompose Ω into disjoint sets of the form $y_i + s_i P_1$ and a set of zero measure, we can define u_0 as

$$\langle F, x\rangle + w_{s_i}(x - y_i) \quad \text{for } x \in y_i + s_i P_1.$$

Then $u_0|_{\partial\Omega} = l_F$ on $\partial\Omega$, $u_0 \in W^{1,\infty}(\Omega)$. To prove that u_0 is a solution of problem (1.1), note that if $l \in \cap_{i=1}^{q} \partial L(v_i)$, then for any admissible function $u \in W^{1,1}(\Omega)$, we have

$$\begin{aligned} J(u) - J(u_0) &= \int_\Omega \{L(Du) - L(Du_0) - \langle l, Du - Du_0\rangle\}\,\mathrm{d}x \\ &= \int_\Omega \{L(Du) - L(v_1) - \langle l, Du - v_1\rangle\}\,\mathrm{d}x \\ &\quad - \int_\Omega \{L(Du_0) - L(v_1) - \langle l, Du_0 - v_1\rangle\}\,\mathrm{d}x. \end{aligned}$$

It is easy to see that all the functions $L(v_i) + \langle l, v - v_i\rangle$, $i = 1, \dots, q$, coincide. Then $L(v) - L(v_1) - \langle l, v - v_1\rangle \geqslant 0$ everywhere, with equality in the case $v \in \{v_1, \dots, v_q\}$. Hence the first term is non-negative while the second one equals zero. Therefore, $J(u) - J(u_0) \geqslant 0$.

This proves that the condition $\cap_{i=1}^{q} \partial L(v_i) \neq \emptyset$, with $F \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$, implies solvability of the problem, since we can always find a subset of the set $\{v_1, \dots, v_q\}$ consisting of extremum points of this set. If $\partial L(F) \neq \emptyset$, then the function l_F is a solution. Therefore, each of these two conditions imply solvability of the problem.

These arguments prove the 'sufficient' part of theorem 1.1. The converse will be proved in § 2.

Before explaining what kind of influence these simple arguments had on further developments of solvability theory, let us state the results of this paper.

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THEOREM 1.2. *Let $L : R^n \to R$ be a lower semicontinuous function such that $L \geqslant \theta$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$.*

Then all problems of the form (1.1), with $\partial\Omega \in C^2$ of positive curvature and $f \in C^2(\partial\Omega)$, are solvable if and only if, for each $F \in R^n$, condition C holds, where

(C) *either $\partial L(F) \neq \emptyset$, or there exist $v_1, \dots, v_q \in R^n$ $(q \in N)$ such that $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ and $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$.*

Moreover, there exists $M = M(f, \Omega) > 0$ such that each solution u_0 is bounded in $W^{1,\infty}$ norm by M and satisfies the Euler–Lagrange equation

$$\int_\Omega \langle l(x), D\phi(x)\rangle \, \mathrm{d}x = 0 \quad \forall\, \phi \in W_0^{1,1}(\Omega),$$

with $l \in L^\infty(\Omega)$ such that $l(x) \in \partial L(Du_0(x))$ a.e. in Ω.

REMARK 1.3. Condition C still characterizes solvability of all problems (1.1) with $f \in C^2(\partial\Omega)$ and $\partial\Omega \in C^2$ of positive curvature, provided the lower semicontinuous integrand $L : R^n \to R$ has *at least linear growth at infinity:* $L \geqslant \alpha|\cdot| + \beta$, $\alpha > 0$, $\beta \in R$ (see § 5 for the proof). The requirements on Ω and f are optimal in this class of integrands (cf. [18]).

We did not check whether assumptions on growth can be completely dropped, like in theorem 1.1. At least in the case $n = 2$, this is so.

In the case when the growth of L provides certain *a priori* regularity of minimizers, an analogous result holds for arbitrary Ω and f, for which at least one admissible function $u \in W^{1,1}(\Omega)$, with $J(u) < \infty$, exists. However, in this case, we cannot state Lipschitz regularity of solutions.

THEOREM 1.4. *Let $L : R^n \to R$ be a lower semicontinuous function such that $L \geqslant \alpha|\cdot|^p + b$, $\alpha > 0$, $p > n$.*

Then each problem of the form (1.1), for which at least one admissible function $u \in W^{1,1}(\Omega)$ with $J(u) < \infty$ exists, has a solution if and only if, for each $F \in R^n$, condition C holds, where

(C) *either $\partial L(F) \neq \emptyset$, or there exist $v_1, \dots, v_q \in R^n$ $(q \in N)$ such that $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ and $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$.*

REMARK 1.5. Note that the only role of the growth condition $L \geqslant \alpha|\cdot|^p + b$, $\alpha > 0$, $p > n$, is to provide almost everywhere classical differentiability of solutions to the relaxed problem. Hence the result of the theorem holds under any other assumptions on lower semicontinuous integrands $L : R^n \to R$ with superlinear growth, which imply this property of solutions to the relaxed problems.

Note that regularity in minimization problems was studied typically in the context of continuity of solutions and their derivatives (everywhere or on an open set of full measure). However, here we need an intermediate property, which is almost everywhere differentiability in the classical sense. It seems that not too much was known in this direction. In fact, results on continuity of solutions are not sufficient in our situation. Simultaneously, partial regularity of derivatives, which is more than enough for our purposes, was usually treated for elliptic integrands (see, for example, [16]).

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Proofs of theorems 1.2 and 1.4 are further refinements of the above discussed arguments, which were also developed recently in a deeper way in the context of the theory of differential inclusions.

Note that the sufficient part of theorem 1.1 is equivalent to a particular differential inclusion $Du(x) \in \{v_1, \dots, v_q\}$ a.e. in Ω, $u = l_F$ on $\partial\Omega$. When one deals with nonlinear boundary conditions, more complicated differential inclusions should be considered. The typical one is $Du(x) \in \operatorname{extr} U$ for a.e. $x \in \Omega$, $u = f$ on $\partial\Omega$, where $f \in W^{1,\infty}(\Omega)$ and $Df(x) \in U$ for a.e. $x \in \Omega$ (here, U is a compact convex subset of R^n with non-empty interior).

It was observed in [12] that the same functions w_s with $F = Df(x_0)$ (see (1.2)) can be used to perturb a Lipschitz function f by

$$\phi_s(\cdot) := w_s(\cdot - x_0) + f(x_0) + \langle Df(x_0), \cdot - x_0\rangle$$

in such a way that $D\phi_s \in \operatorname{extr} U$ for a.e. x in an open subset $\tilde{\Omega}$ of Ω such that $x_0 + P_{s/2} \subset \tilde{\Omega} \subset x_0 + P_{2s}$ and $\phi_s = f$ on $\partial\tilde{\Omega}$. Refining arguments from [12], one can show that for each $x_0 \in \Omega$, where $Df(x_0)$ exists in the classical sense and $Df(x_0) \in \operatorname{int} U$, such a perturbation exists for all $s > 0$ sufficiently small (see lemma 3.2). Applying Vitaly covering arguments, we can solve the inclusion.

In the case of the paper [12], the authors were dealing with the set-valued function $x \to U(x)$ and solved the inclusion by Baire category arguments following [3].

Note that if U is an n-dimensional proper face of L^{**}, f is a Lipschitz solution of the relaxed problem, and the perturbation ϕ_s is applicable at the point x_0 with $Df(x_0) \in \operatorname{int} U$, then the gradient of the perturbed function stays in the set $\{v : L = L^{**}\}$ on the set of perturbation $\tilde{\Omega}$ (recall that $L = L^{**}$ on the set of extremal points of U, cf. lemma 3.1). Since $L^{**}|_U$ is affine, the perturbed function is still a solution of the relaxed problem.

This argument was used in [28] in order to apply the construction from [12] and to show that the *a priori* assumption of Lipschitz regularity of solutions of the relaxed problems implies solvability of problems (1.1) with integrands L, which coincide with L^{**} everywhere with exception of a finite collection of distinct n-dimensional proper faces of L^{**} and has superlinear growth at infinity. Being more precise, the result was proved under slightly weaker *a priori* assumptions on solutions f of the relaxed problem. It was assumed that for each $k \in N$, the function $f|_{\Omega_k}$ is Lipschitz, where Ω_k, $k \in N$, is a sequence of *open* sets such that $\cup_k \Omega_k = \Omega$.

As already mentioned above, the perturbations ϕ_s can be, in fact, applied at the points of classical differentiability of solutions of the relaxed problems. Thus our observation and Vitaly covering arguments are enough to prove theorem 1.4, which is a characterization result. It is also helpful here to use simple direct arguments constructing a sequence of solutions to the relaxed problem, which converges strongly to a solution of the original problem, instead of Baire category arguments and other techniques from [28] traditionally used for theory of differential inclusions.

Proof of theorem 1.2 needs more subtle arguments, since growth of L does not guarantee almost everywhere classical differentiability of functions, which give finite values to the integral functional. It seems to be a good *open problem* to clarify whether this property holds for solutions of problems (1.1) with convex integrands of superlinear growth in the case of general boundary data.

In case of theorem 1.2, we first prove solvability of the relaxed problem in the class of Lipschitz continuous functions, following arguments introduced first in the context of solvability theory for the Plateau problem (see [18]). Then, careful construction of special perturbations of this solution gives a solution to the original problem in the class of Lipschitz functions. Next, we use a non-smooth analogue of the Euler–Lagrange equation to prove that these solutions are automatically solutions of the boundary value problem (1.1).

We prove theorems 1.1 and 1.4 in §§ 2 and 3, respectively. In § 4 we recall some facts on solvability of problems of the form (1.1) in the class of Lipschitz continuous functions, provided certain regularity on $\partial\Omega$ and f is assumed and L is convex. Here we also prove a non-smooth analogue of the Euler–Lagrange equation. Theorem 1.2, and the result stated in the remark to it, will be proved in § 5.

Note that the situation is much less clear in the case of vectorial ($m > 1$) problems, where recent efforts were focused on problems with quasiaffine quasiconvexifications (see [9, 11, 27]). A number of existence and non-existence results were obtained in [10].

2. Proof of theorem 1.1 and some auxiliary propositions

Recall first some basic facts about convex functions. By the Caratheodory theorem, for each subset A of R^n, we have

$$\operatorname{co} A = \left\{ \sum_{i=1}^{n+1} c_i v_i : c_i \geqslant 0,\ v_i \in A,\ \sum_{i=1}^{n+1} c_i = 1 \right\}.$$

Since the dimension of the epigraph of any lower semicontinuous function $L : R^n \to R$ does not exceed $n+1$, for each $v_0 \in R^n$, we have

$$\begin{aligned} L^{**}(v_0) &:= \inf\left\{ \sum_{i=1}^{q} c_i L(v_i) : q \in N,\ c_i \geqslant 0,\ v_i \in R^n,\ \sum_{i=1}^{q} c_i = 1,\ \sum_{i=1}^{q} c_i v_i = v_0 \right\} \\ &= \inf\left\{ \sum_{i=1}^{n+2} c_i L(v_i) : c_i \geqslant 0,\ v_i \in R^n,\ \sum_{i=1}^{n+2} c_i = 1,\ \sum_{i=1}^{n+2} c_i v_i = v_0 \right\}. \end{aligned}$$

It is also well known that if $L : R^n \to R \cup \{\infty\}$ is a lower semicontinuous convex function, which is bounded in a neighbourhood of v_0, then L is Lipschitz in a smaller neighbourhood. Moreover, $\partial L(v_0) \neq \emptyset$.

Recall also a version of the Hahn–Banach theorem. If U is a closed convex subset of R^n and $v_0 \notin \operatorname{int} U$, then there exists $l \in R^n$ such that

$$\langle l, v_0 \rangle \geqslant \langle l, v \rangle \quad \forall\, v \in U.$$

All these facts can be found in any textbook containing chapters on convex analysis (see, for example, [13]).

Before proving theorem 1.1, we state and prove two auxiliary propositions which will be used frequently later on.

Lemma 2.1. *Let $L : R^n \to R$ be a lower semicontinuous function. Then the following assertions are equivalent.*

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(1)

$$L^{**}(\cdot) := \inf\left\{ \sum_{i=1}^{q} c_i L(v_i) : q \in N,\, c_i \geqslant 0,\, \sum_{i=1}^{q} c_i = 1,\, \sum_{i=1}^{q} c_i v_i = \cdot \right\}$$

is a convex continuous function.

(2) *There exist* $l \in R^n$ *and* $c \in R$ *such that*

$$L(v) \geqslant \langle l, v \rangle + c \quad \forall\, v \in R^n.$$

(3) *There exists a point* $F \in R^n$ *such that*

$$\inf\left\{ \sum_{i=1}^{q} c_i L(v_i) : q \in N,\, c_i \geqslant 0,\, v_i \in R^n,\, \sum_{i=1}^{q} c_i = 1,\, \sum_{i=1}^{q} c_i v_i = F \right\} > -\infty.$$

Proof. If $L^{**} : R^n \to R$ is a convex continuous function, then $\partial L^{**}(0) \neq \emptyset$, and, as a consequence, for $l \in \partial L^{**}(0)$, we have

$$L(v) - L^{**}(0) \geqslant L^{**}(v) - L^{**}(0) \geqslant \langle l, v \rangle.$$

Hence, (1) implies (2). The implication (2) $\Rightarrow$ (3) is obvious.

Let us prove the last assertion of the lemma, (3) $\Rightarrow$ (1). Without loss of generality, we can assume that $F = 0$. Consider auxiliary functions $L_k : R^n \to R \cup \{\infty\}$ defined as follows: $L_k = L$ in $\bar{B}(0,k)$, $L_k = \infty$ otherwise. Let also L_k^{**} be convexification of L_k.

Then, for each k, the function $L_k^{**}|_{\bar{B}(0,k)}$ is a lower semicontinuous convex function, which is continuous in $B(0,k)$. Note also that $L_k^{**}(v)$ is a non-increasing sequence for each $v \in R^n$. Moreover, the sequence $L_k^{**}(0)$ is bounded from below. Hence, if $l_k \in \partial L_k^{**}(0)$, then $\sup_k |l_k| < \infty$, and for each limit point l_0 of l_k ($l_0 = \lim_{j\to\infty} l_{k_j}$) and each $v \in R^n$, we have

$$\lim_{k\to\infty} \{L_k^{**}(v) - L_k^{**}(0)\} - \langle l_0, v \rangle = \lim_{j\to\infty} \{L_{k_j}^{**}(v) - L_{k_j}^{**}(0) - \langle l_{k_j}, v \rangle\} \geqslant 0.$$

Since $L_k^{**}(0)$ is bounded from below, the function $L^{**} := \lim_{k\to\infty} L_k^{**}$ is well defined and majorizes $L^{**}(0) + \langle l_0, v \rangle$ everywhere. Note that $L^{**} : R^n \to R$ is convex as a pointwise limit of a non-increasing sequence of convex functions. Since it is also locally bounded, it is continuous.

The proof of lemma 2.1 is complete. □

COROLLARY 2.2. *The lower semicontinuous function* $L : R^n \to R$ *has non-empty subgradient at* $F \in R^n$ *if and only if*

$$\sum_{i=1}^{q} c_i L(v_i) \geqslant L(F)$$

for every $q \in N$, $v_i \in R^n$, $c_i \geqslant 0$ $(i = 1, \ldots, q)$ *such that*

$$\sum_{i=1}^{q} c_i = 1, \qquad \sum_{i=1}^{q} c_i v_i = F.$$

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Proof. Let $l \in \partial L(F)$. Then for all $c_i \geqslant 0$, $v_i \in R^n$ such that

$$\sum_{i=1}^{q} c_i = 1, \quad \sum_{i=1}^{q} c_i v_i = F, \quad q \in N,$$

we obtain

$$\begin{aligned}\sum_{i=1}^{q} c_i L(v_i) - L(F) &= \sum_{i=1}^{q} c_i L(v_i) - L(F) - \sum_{i=1}^{q} c_i \langle l, v_i - F \rangle \\ &= \sum_{i=1}^{q} c_i \{L(v_i) - L(F) - \langle l, v_i - F \rangle\} \geqslant 0.\end{aligned}$$

To prove the converse, note that, by lemma 2.1, L^{**} is a convex continuous function. Since $L(F) = L^{**}(F)$, $\partial L^{**}(F) \neq \emptyset$ and $L \geqslant L^{**}$ everywhere, we infer that $\partial L(F) \neq 0$.

The proof is complete. □

Lemma 2.3. *Let $L : R^n \to R$ be a lower semicontinuous function. Let $v_1, \dots, v_q$ be such points in R^n that*

$$\sum_{i=1}^{q} c_i v_i = F \quad \textit{for some } c_i \geqslant 0,$$

with

$$\sum_{i=1}^{q} c_i = 1.$$

Then, there exists a bounded in $W^{1,\infty}(\Omega)$ sequence u_k such that $u_k|_{\partial\Omega} = l_F$ and

$$J(u_k) \to \sum_{i=1}^{q} c_i L(v_i) \operatorname{meas} \Omega.$$

This lemma is a version of well-known arguments used in relaxation theorems (see [13, ch. 10]). The main difference is that here we have a lower than usual regularity of integrands that forces us to precise construction of the sequence u_k.

Proof. Without loss of generality, we can assume that $c_i > 0$ for all $i \in \{1, \dots, q\}$.

Consider first the case when F has unique representation as a convex combination of $\{v_1, \dots, v_q\}$. In this case, $v_1, \dots, v_q$ are extremum points of a compact convex set. In case $F \in \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$, the claim was proved in § 1, since there we proved the existence of a function u_0 such that $Du_0 \in \{v_1, \dots, v_q\}$, $u_0|_{\partial\Omega} = l_F$. In fact, in this case, we have

$$\int_{\Omega} Du(x) \, dx = \sum_{i=1}^{q} (\bar{c}_i \operatorname{meas} \Omega) v_i = F \operatorname{meas} \Omega, \quad \sum_{i=1}^{q} \bar{c}_i = 1, \quad \bar{c}_i \geqslant 0.$$

Since the representation of F in the form of a convex combination of $v_1, \dots, v_q$ is unique, we obtain that $\bar{c}_i = c_i$ for each i. Hence, defining u_k as u for all $k \in N$, we

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obtain that

$$J(u_k) = \sum_{i=1}^{q} c_i L(v_i) \operatorname{meas} \Omega.$$

Let us now consider the case when $F \notin \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}$. In any case,

$$F \in \operatorname{re} \operatorname{int} \operatorname{co}\{v_1, \dots, v_q\}.$$

Let P be the largest subspace of R^n perpendicular to all vectors $v_i - F$, $i \in \{1, \dots, q\}$. Assume that $\dim P = m$ and $v_{q+1}, \dots, v_{q+1+m}$ are such points in P that $\operatorname{co}\{v_{q+1}, \dots, v_{q+1+m}\}$ has non-empty interior in P and 0 belongs to this interior.

For each $\delta > 0$, consider the function

$$\phi_{s,\delta}(\cdot) = \max_{1 \leqslant i \leqslant q+1+m} \langle \tilde{v}_i, \cdot \rangle - s,$$

where

$$\tilde{v}_i = v_i - F \quad \text{for } i \in \{1, \dots, q\},$$
$$\tilde{v}_i \in B(v_1, \delta) \cap \operatorname{int} \operatorname{co}\{\tilde{v}_1, \dots, \tilde{v}_q, v_{q+1}, \dots, v_{q+1+m}\} \quad \text{for } i \in \{q+1, \dots, q+1+m\},$$

and the inclusion

$$0 \in \{\tilde{v}_1, \dots, \tilde{v}_q, v_{q+1}, \dots, v_{q+1+m}\}$$

holds. Note that $\tilde{v}_1, \dots, \tilde{v}_{q+1+m}$ are extremum points of a compact convex set. In the set, where $D\phi_{s,\delta} \in \{\tilde{v}_{q+1}, \dots, \tilde{v}_{q+1+m}\}$, we can perturb $\phi_{s,\delta}$ in such a way that the gradient of the perturbation $w_{s,\delta}$ stays in the set $\{\tilde{v}_1, \dots, \tilde{v}_q, v_{q+1}, \dots, v_{q+1+m}\}$. Then $w_{s,\delta} \in W_0^{1,\infty}(P_s)$. Moreover,

$$\frac{\operatorname{meas}\{x \in P_s : Dw_{s,\delta} \notin \{v_1 - F, \dots, v_q - F\}\}}{\operatorname{meas}\{x \in P_s : Dw_{s,\delta} \in \{v_1 - F, \dots, v_q - F\}\}} \to 0$$

as $\delta \to 0$ uniformly with respect to s.

For a $k \in N$, consider the Vitaly covering of Ω by the supports $\Omega_i := x_i + P_{s_i}$ of the functions $\min\{0, w_{s_i,1/k}(\cdot - x_i)\}$ and define $\tilde{u}_k = l_F + w_{s_i,1/k}(\cdot - x_i)$ in Ω_i $(i \in N)$, $\tilde{u}_k = l_F$ otherwise. In this case,

$$\operatorname{meas}\{x \in \Omega : D\tilde{u}_k \notin \{v_1, \dots, v_q\}\} \to 0$$

as $k \to \infty$. Therefore, if, for a subsequence of $\tilde{u}_k$ (not relabelled),

$$c_i^k := \frac{\operatorname{meas}\{x \in \Omega : D\tilde{u}_k = v_i\}}{\operatorname{meas} \Omega} \to \tilde{c}_i, \quad i \in \{1, \dots, q\},$$

then $\sum \tilde{c}_i = 1$, $\sum \tilde{c}_i v_i = F$, and, because of the uniqueness of the representation of F in the form of a convex combination of v_i $(i = 1, \dots, q)$, we infer that $\tilde{c}_i = c_i$. Hence $c_i^k \to c_i$ as $k \to \infty$ for the original sequence c_i^k.

However, we cannot yet assert that

$$J(\tilde{u}_k) \to \sum_{i=1}^{q} c_i L(v_i),$$

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since L can be unbounded in the set $\{F+\delta v_{q+1},\dots,F+\delta v_{q+m+1};\delta\in[0,1]\}$. In order to overcome this difficulty, notice that, for all $\delta>0$ sufficiently small, the vectors $\tilde{v}_i := \delta v_i+F$ $(i=q+1,\dots,m+q+1)$ lie in the interior of the set $\operatorname{co}\{v_1,\dots,v_{q+m+1}\}$. Hence, for all sufficiently large k, the function $\tilde{u}_k$ can be redefined by the procedure described above in each set $\{x\in\Omega : D\tilde{u}_k=\tilde{v}_i\}$, $i\in\{q+1,\dots,m+q+1\}$ (we denote the new function as u_k) in such a way that $Du_k\in\{v_1,\dots,v_{q+m+1}\}$ a.e. on this set and $u_k=\tilde{u}_k$ in the boundary of this set. Since $u_k=\tilde{u}_k$ a.e. on the set $\{x\in\Omega : D\tilde{u}_k(x)\in\{v_1,\dots,v_q\}\}$ and $|L(Du_k)|\leqslant c<\infty$, we infer that

$$J(u_k)\to\sum_{i=1}^{q}c_iL(v_i).$$

The general case can be reduced to the one discussed above (which is the case when F has unique representation in the form of a convex combination of $v_1,\dots,v_q$). We can assume, without loss of generality, that $v_i\neq F$, $c_i>0$, for all $i\in\{1,\dots,q\}$.

For $q=2$, we can assert that there exists a sequence of piecewise affine functions u_k such that $u_k|_{\partial\Omega}=l_F$, $\operatorname{meas}\{x\in\Omega : Du_k=v_i\}\to c_i\operatorname{meas}\Omega$ $(i=1,2)$, and $J(u_k)\to\sum c_iL(v_i)\operatorname{meas}\Omega$, since F has unique representation in the form of a convex combination of v_1, v_2.

Let this claim be valid for $q=s$. To prove it for $q=s+1$, consider vectors $\tilde{v}_1,\dots,\tilde{v}_s$ such that $\tilde{v}_i=v_i$ for $i\leqslant s-1$, $\tilde{v}_s=(c_sv_s+c_{s+1}v_{s+1})/(c_s+c_{s+1})$. Then

$$F=\sum_{i=1}^{s}\tilde{c}_i\tilde{v}_i,$$

where $\tilde{c}_i=c_i$ for $i\leqslant s-1$ and $\tilde{c}_s=c_s+c_{s+1}$. By the induction assumption, there exists a sequence of piecewise affine functions u_k such that

$$\begin{gathered}u_k|_{\partial\Omega}=l_F,\\ \operatorname{meas}\{x\in\Omega : Du_k(x)=\tilde{v}_i\}\to\tilde{c}_i\operatorname{meas}\Omega\quad(i=1,\dots,s),\\ J(u_k)\to\sum\tilde{c}_iL(\tilde{v}_i)\operatorname{meas}\Omega.\end{gathered}$$

For each $k\in N$, assume $\Omega_k:=\operatorname{int}\{x\in\Omega : Du_k(x)=v_s\}$. We can find a sequence $u_j^k\in W^{1,\infty}(\Omega)$ such that $u_j^k=u_k$ in $\Omega\setminus\Omega_k$, $\|u_j^k\|_{W^{1,\infty}(\Omega)}\leqslant c<\infty$, and

$$\operatorname{meas}\{x\in\Omega_k : Du_j^k=v_i\}\to\frac{c_i}{\tilde{c}_s}\operatorname{meas}\Omega_k,\quad i=s,\quad s+1,\quad j\to\infty,$$

$$J(u_j^k;\Omega_k)\to\sum_{i=s}^{s+1}\frac{c_i}{\tilde{c}_s}L(v_i)\operatorname{meas}\Omega_k=\sum_{i=s}^{s+1}c_iL(v_i)\operatorname{meas}\Omega,\quad j\to\infty.$$

Then, for an appropriate subsequence $w_k:=u_{j(k)}^k$ $(k\to\infty)$, we obtain

$$J(w_k)\to\sum_{i=1}^{s+1}c_iL(v_i)\operatorname{meas}\Omega.$$

Proof of the lemma is complete. □

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Proof of theorem 1.1. Sufficiency of the condition *either* $\partial L(F) \neq \emptyset$*, or there exist* $v_1, \dots, v_q \in R^n$ *such that* $F \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$ *and* $\cap_{i=1}^{q} \partial L(v_i) = \emptyset$ for solvability of the problem

$$\int_{\Omega} L(Du)\,\mathrm{d}x \to \min, \qquad u|_{\partial\Omega} = l_F, \qquad u \in W^{1,1}(\Omega)$$

has been proved in § 1.

In order to prove its necessity, first note that

$$\inf\left\{\sum_{i=1}^{q} c_i L(v_i) : q \in N,\ c_i \geqslant 0,\ v_i \in R^n,\ \sum_{i=1}^{q} c_i = 1,\ \sum_{i=1}^{q} c_i v_i = F\right\} > -\infty. \quad (2.1)$$

In fact, lemma 2.3 implies that, for each collection c_i, v_i, $i = 1, \dots, q$, from (2.1),

$$\inf\{J(u) : u|_{\partial\Omega} = l_F,\ u \in W^{1,\infty}\} \leqslant \sum_{i=1}^{q} c_i L(v_i). \quad (2.2)$$

Since solvability of problem (1.1) implies the inequality

$$\inf\{J(u) : u|_{\partial\Omega} = l_F,\ u \in W^{1,\infty}\} > -\infty,$$

we infer that (2.1) holds.

By lemma 2.1, we infer that L^{**} is a convex continuous function. Moreover, if u_0 is a solution of problem (1.1), then (2.1), (2.2) imply that $J(u_0) \leqslant L^{**}(F)\operatorname{meas}\Omega$.

Let $l \in \partial L^{**}(F)$. For each admissible u, we have

$$\int_{\Omega} \langle l, Du - F\rangle\,\mathrm{d}x = 0.$$

Then

$$J(u) - L^{**}(F)\operatorname{meas}\Omega = \int_{\Omega} \{L(Du) - L^{**}(F) - \langle l, Du - F\rangle\}\,\mathrm{d}x \geqslant 0.$$

Hence $J(u_0) = L^{**}(F)\operatorname{meas}\Omega$.

Let

$$P_l := \{v \in R^n : L(v) - L^{**}(F) - \langle l, v - F\rangle = 0\}.$$

Since $J(u_0) = L^{**}(F)\operatorname{meas}\Omega$, and the expression defining P_l is non-negative everywhere, we infer that $Du_0(x) \in P_l$ for a.a. $x \in \Omega$.

It is obvious that P_l is a closed set. Moreover, we claim that $F \in \operatorname{int}\operatorname{co} P_l$ if $L(F) \neq L^{**}(F)$. Otherwise, by the Hahn–Banach theorem, there exists $a \in R^n$ such that $\langle F, a\rangle \geqslant \langle v, a\rangle$ for all $v \in \operatorname{co} P_l$. Then $\langle F, a\rangle \geqslant \langle Du_0, a\rangle$ a.e. on Ω. Since

$$\int_{\Omega} \langle Du_0, a\rangle\,\mathrm{d}x = \langle F, a\rangle \operatorname{meas}\Omega,$$

we infer that $Du_0 \in \{v \in R^n : \langle v - F, a\rangle = 0\}$ a.e. on Ω. As a consequence,

$$\frac{\partial(u_0 - l_F)}{\partial a} = \langle Du_0 - F, a\rangle = 0$$

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a.e. on Ω. Since $u_0 = l_F$ in $\partial\Omega$, we infer that $u_0 = l_F$ a.e. on Ω. Hence $F \in P_l$, and, as a consequence, $L(F) = L^{**}(F)$. This is a contradiction.

We have proved that either $F \in \operatorname{int}\operatorname{co} P_l$ or $L(F) = L^{**}(F)$. In the first case, there exist $v_1, \dots, v_q \in P_l$ such that $F \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$. It is obvious that in this case $l \in \partial L(v_i)$ for every $i \in \{1, \dots, q\}$. Hence $\cap_{i=1}^{q} \partial L(v_i) \neq \emptyset$. This completes the proof of theorem 1.1. □

3. Proof of theorem 1.4

In § 1 we already mentioned the following standard fact, a short proof of which we include for convenience of a reader.

LEMMA 3.1. *Let $L : R^n \to R$ be a lower semicontinuous function such that $L(v) \geqslant \theta(v)$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$. Let $F \in R^n$ and $l \in \partial L^{**}(F)$. Also let*

$$P_l := \{v \in R^n : L^{**}(v) - L^{**}(F) - \langle l, v - F\rangle = 0\}.$$

Then $L = L^{**}$ in the set of extremum points of P_l.

Proof. Assume that $v_0 \in \operatorname{extr} P_l$. By the Caratheodory theorem, there exists $c_i^k \geqslant 0$, $v_i^k \in R^n$ $(i = 1, \dots, n+2)$ such that

$$\sum_{i=1}^{n+2} c_i^k = 1, \quad \sum_{i=1}^{n+2} c_i^k v_i^k = v_0 \quad \text{and} \quad \sum_{i=1}^{n+2} c_i^k L(v_i^k) \to L^{**}(v_0) \quad \text{as } k \to \infty.$$

We can assume also that $c_i^k \to c_i$ and either $v_i^k \to v_i$ or $|v_i^k| \to \infty$ as $k \to \infty$. Since $c_i^k |v_i^k| \to 0$ in the case $|v_i^k| \to \infty$ (recall that $\theta(v)/|v| \to \infty$ as $|v| \to \infty$), we obtain that for all $i \in \{1, \dots, n+2\}$ such that $c_i > 0$, the convergence $v_i^k \to v_i$ holds and $\sum c_i v_i = v_0$. Because of lower semicontinuity of L, we have

$$\sum c_i L(v_i) = L^{**}(v_0).$$

Since $L(v) - L^{**}(v_0) - \langle l, v - v_0\rangle \geqslant 0$ everywhere, we infer that $L(v_i) - L^{**}(v_0) - \langle l, v_i - v_0\rangle = 0$ for each v_i. Then $v_i \in P_l$ for each i. Because $v_0 \in \operatorname{extr} P_l$, we obtain that $v_i = v_0$ for all i under consideration. Hence $L(v_0) = L^{**}(v_0)$.

The proof is complete. □

LEMMA 3.2. *Let $u_0 \in C(B(x_0, r))$ be differentiable at x_0 in the classical sense. Let U be a convex compact subset in R^n, and let $v_1, \dots, v_q \in \operatorname{extr} U$ be such that*

$$Du_0(x_0) \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}.$$

Then, for all $s > 0$ sufficiently small, the function

$$\phi_s(\cdot) := w_s(\cdot - x_0) + u_0(x_0) + \langle Du_0(x_0), \cdot - x_0\rangle,$$

where $w_s(x) := \max_{1 \leqslant i \leqslant q}\langle v_i - Du_0(x_0), x\rangle - s$, has the properties

$$\phi_s < u_0, \quad x \in x_0 + P_{s/2},$$
$$\phi_s > u_0, \quad x \in x_0 + \partial P_{2s},$$

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where

$$P_s = \{x \in R^n : \max_{1\leqslant i\leqslant q} \langle v_i - Du_0(x_0), x\rangle \leqslant s\}.$$

Proof. The proof is straightforward. We have

$$\begin{aligned} u_0(x) - \phi_s(x) &= u_0(x) - \langle Du_0(x_0), x - x_0\rangle - u_0(x_0) - w_s(x - x_0) \\ &= o(|x - x_0|) - w_0(x - x_0) + s. \end{aligned}$$

Since $|w_0(\cdot - x_0)| \leqslant \frac{1}{2}s$ inside $x_0 + P_{s/2}$, we obtain that $u_0 - \phi_s > 0$ inside $x_0 + P_{s/2}$ if $s > 0$ is sufficiently small.

Since $w_0(x - x_0) = 2s$ for $x \in x_0 + P_{2s}$, we infer that $u_0 - \phi_s < 0$ in ∂P_{2s} if $s > 0$ is sufficiently small.

The proof is complete. □

Proof of theorem 1.4. By theorem 1.1, solvability of all problems (1.1) with linear boundary data implies that *for each* $F \in R^n$, *either* $\partial L(F) \neq \emptyset$ *or there exist* $v_1, \dots, v_q \in R^n$ *such that* $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ *and* $F \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$. We need to prove that this condition suffices for solvability of all problems (1.1) with boundary data f admitting at least one function $u \in W^{1,1}(\Omega)$ such that $J(u) < \infty$.

First note that the function $L^{**} : R^n \to R$ is a continuous convex function satisfying the growth condition $L^{**} \geqslant \alpha|\cdot|^p + b$, $\alpha > 0$, $p > n$ (cf. lemma 2.1).

Let Ω and f be of the type described above. Let u_0 be a solution of the problem

$$\int_\Omega L^{**}(Du(x))\,\mathrm{d}x \to \min, \qquad u|_{\partial\Omega} = f, \qquad u \in W^{1,1}(\Omega). \tag{3.1}$$

We will construct a solution $\tilde{u}$ of problem (3.1), for which the inclusion $D\tilde{u}(x) \in \{v : L(v) = L^{**}(v)\}$ holds almost everywhere in Ω, as a limit in $W^{1,1}$ norm of a sequence of perturbations of u_0, each of which is also a solution of problem (3.1). Note that in this case $\tilde{u}$ is automatically a solution to the original problem (1.1).

Let $\tilde{\Omega}$ be the set of those points $x \in \Omega$, where u_0 is differentiable in the classical sense and $L(Du_0(x)) \neq L^{**}(Du_0(x))$. Note that u is differentiable in the classical sense almost everywhere in Ω, since $u \in W^{1,p}(\Omega)$ with $p > n$ (cf. [14, p. 234]).

Let $x_0 \in \tilde{\Omega}$. There exist $v_1, \dots, v_q$, which are extremum points of a compact convex set, and $l \in R^n$ such that $Du_0(x_0) \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$, $l \in \cap_i \partial L(v_i)$. Note that $L^{**}(\cdot) = L(v_1) + \langle l, \cdot - v_1\rangle$ in $\operatorname{co}\{v_1, \dots, v_q\}$.

Since problems (3.1) with the integrands L^{**} and $L^{**} + \langle l, \cdot\rangle + c$ have the same solutions, we can assume, without loss of generality, that $L^{**} = 0$ in $U := \operatorname{co}\{v_1, \dots, v_q\}$ and $L^{**} \geqslant 0$ otherwise.

By lemma 3.2, for all sufficiently small $s > 0$, the function

$$\phi_s := \langle Du_0(x_0), \cdot - x_0\rangle + u_0(x_0) + w_s(\cdot - x_0),$$

where

$$w_s(x) = \max_{1\leqslant i\leqslant q} \langle v_i - Du_0(x_0), x\rangle - s,$$

satisfies the inequalities

$$\left.\begin{aligned} \phi_s &< u_0, \quad x \in x_0 + P_{s/2}, \\ \phi_s &> u_0, \quad x \in x_0 + \partial P_{2s} \end{aligned}\right\} \tag{3.2}$$

with $P_\sigma := \{x \in R^n : \max_{1\leqslant i\leqslant q} \langle v_i - Du_0(x_0), x\rangle \leqslant \sigma\}$.

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Hence the function u_1, which is equal to u_0 outside the set $x_0 + P_{2s}$ and to $\min\{\phi_s, u_0\}$ inside this set, is defined as an element of $W^{1,p}(\Omega)$.

At the same time, if $\Omega' := \{x \in \Omega : u_1 \neq u_0\}$, then $Du_1 \in \operatorname{extr} U$ a.e. in $\bar{\Omega}'$ and, as a consequence, we have $L^{**}(Du_1) = 0$ a.e. in $\bar{\Omega}'$. Hence

$$\int_{\bar{\Omega}'} L^{**}(Du_1)\,\mathrm{d}x = \int_{\bar{\Omega}'} L^{**}(Du_0)\,\mathrm{d}x.$$

Since sets of the form $\bar{\Omega}'$ gives a Vitaly cover of $\tilde{\Omega}$ (cf. (3.2)), by the Vitaly covering theorem we can cover $\tilde{\Omega}$ by disjoint closed sets $\bar{\Omega}_j$, $j \in N$, and a set of zero measure such that, for each $j \in N$, there exists a function $\psi_j \in W_0^{1,\infty}(\Omega_j)$ such that $Du_0 + D\psi_j \in \{v \in R^n : L(v) = L^{**}(v)\}$ a.e. in $\bar{\Omega}_j$, and

$$\int_{\bar{\Omega}_j} L^{**}(Du_0)\,\mathrm{d}x = \int_{\bar{\Omega}_j} L^{**}(Du_0 + D\psi_j)\,\mathrm{d}x.$$

Define u_i as $u_0 + \psi_j$ in $\bar{\Omega}_j$, $j \leqslant i$, and as u_0 otherwise. Then u_i is a sequence of solutions of problem (3.1). Note that this sequence converges strongly in $W^{1,1}(\Omega)$. Indeed, in view of the growth conditions on L, we have

$$\begin{aligned}\|Du_k - Du_l\|_{L_1} &= \|Du_k - Du_l\|_{L_1(\cup_{k\leqslant j\leqslant l}\bar{\Omega}_j)}\\ &\leqslant 2\int_{\cup_{k\leqslant j\leqslant l}\bar{\Omega}_j} (c_1 L^{**}(Du_0) + c_2)\,\mathrm{d}x \to 0 \quad \text{as } k, l \to \infty.\end{aligned}$$

Therefore, the function $\tilde{u}$, which is the limit of u_i in $W^{1,1}(\Omega)$, is also a solution of problem (3.1). We also have $\operatorname{meas}\{x \in \Omega : L(Du_i) \neq L^{**}(Du_i)\} \to 0$ and, as a consequence, $D\tilde{u} \in \{v : L(v) = L^{**}(v)\}$ a.e. in Ω. Hence $\tilde{u}$ is a solution of the original problem (1.1).

The proof of the theorem is complete. □

4. Some auxiliary facts related to solvability of boundary value minimization problems with convex integrands and validity of the Euler–Lagrange equation for their solutions

In this section we recall some standard facts about solvability of problems (1.1) with convex integrands. These facts have been established in the context of solvability theory for the Plateau problem (see [18]). We also prove a version of the Euler–Lagrange equation, which is valid for all Lipschitz minimizers of problems (1.1) with convex integrands.

Recall that boundary data f is said to satisfy *the bounded slope condition* if there exists $M > 0$ such that, for each point $x_0 \in \partial\Omega$, we can find l_1, $l_2 \in R^n$ such that $|l_1|$, $|l_2| \leqslant M$ and

$$\langle l_1, x - x_0\rangle + f(x_0) \leqslant f(x) \leqslant \langle l_2, x - x_0\rangle + f(x_0) \quad \forall\, x \in \partial\Omega.$$

For the proof of the following theorem, see, for example, [18].

THEOREM 4.1. *Let $L : R^n \to R$ be a convex continuous function. Let boundary data f satisfy the bounded slope condition with $M > 0$. Then there exists a solution u_0 of the problem $J(u) \to \min$, $u|_{\partial\Omega} = f$ in the class of Lipschitz functions. Moreover, u_0 can be chosen satisfying the inequality $\|Du_0\|_{L^\infty} \leqslant M$.*

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REMARK 4.2. Let Ω be a convex domain with $\partial\Omega \in C^2$ of positive curvature and let $f \in C^2(\partial\Omega)$. Then f satisfies the bounded slope condition with certain $M > 0$.

Lipschitz minimizers always satisfy a non-smooth version of the Euler–Lagrange equation.

THEOREM 4.3. *Let $L : R^n \to R$ be a continuous convex function. Let $u_0 \in W^{1,\infty}(\Omega)$ be a local minimizer of the functional J : $J(u_0) \leqslant J(u_0 + \phi)$ for all $\phi \in C_0^1(\Omega)$ with $\|\phi\|_{C^1} \leqslant \epsilon$, $\epsilon > 0$.*

Then there exists a function $l \in L^\infty(\Omega)$ such that $l(x) \in \partial L(Du_0(x))$ a.e. in Ω and

$$\int_\Omega \langle l(x), D\phi(x)\rangle \, \mathrm{d}x = 0 \quad \forall\, \phi \in W_0^{1,1}(\Omega).$$

Moreover, u_0 is a solution of problem (1.1) with $f = u_0|_{\partial\Omega}$.

Proof. Let $M > \|u_0\|_{W^{1,\infty}(\Omega)}$. Define L^M to be equal to L for $|v| \leqslant M + 1$ and to ∞ for $|v| > M + 1$.

Let

$$K := \sup\{|l| : l \in \partial L^M(v), |v| \leqslant M\} + \sup\{|L^M(v)| : |v| \leqslant M\}.$$

Consider convexification G^{**} of the function $G := \min\{L^M, K|v| + K(M + 1)\}$.

Because of lower semicontinuity of G, by lemma 2.1, we infer continuity and convexity of G^{**}. Note that $G^{**} = L$ for $|v| < M$. Indeed, for these v we have $|L(v)| \leqslant K \leqslant K|v| + K(M + 1)$. Then $G = L$ for $|v| < M$. Moreover, for each $v_0 \in B(0, M)$ and each $l \in \partial L^M(v_0)$, we have

$$\begin{aligned} L(v_0) + \langle l, v - v_0\rangle &\leqslant L^M(v), \\ |L(v_0) + \langle l, v - v_0\rangle| &\leqslant K + KM + K|v|. \end{aligned}$$

Hence $l \in \partial G(v_0)$ and, as a consequence, $\partial G(v_0) = \partial L^M(v_0) \neq \emptyset$. Then, by corollary 2.2, we get $L = G = G^{**}$ in $B(0, M)$.

Since $G^{**} = L$ in $B(0, M)$, the function u_0 is a local minimizer for the integral functional with the integrand F, where $F(x, v) := G^{**}(v) + |v - Du_0(x)|^2$. In this case, u_0 is automatically a solution of the minimization problem. To prove this, note that for each non-trivial $\phi \in W_0^{1,2}(\Omega)$, the function

$$I(\epsilon) := \int_\Omega \{G^{**}(Du_0(x) + \epsilon D\phi(x)) + \tfrac{1}{2}(\epsilon D\phi(x))^2 - G^{**}(Du_0(x))\} \, \mathrm{d}x$$

is a convex function of ϵ and $I(0) = 0$. Moreover, for $\epsilon > 0$ sufficiently small, we have $I(\epsilon) \geqslant I(0) = 0$, since u_0 is a local minimizer. Because of strict convexity of I, we infer that $I(\epsilon) > 0$ everywhere. Since $\phi \in W_0^{1,2}(\Omega)$ is arbitrary, we obtain that u_0 is unique global minimizer in $W^{1,2}(\Omega)$.

The proof reduces to finding a function $l_M \in L^\infty(\Omega; R^n)$ such that $l_M(x) \in \partial_v F(x, Du_0(x))$ for a.e. $x \in \Omega$ and

$$\int_\Omega \langle l_M(x), D\phi(x)\rangle \, \mathrm{d}x = 0 \quad \forall\, \phi \in C_0^1(\Omega). \tag{4.1}$$

In fact, since for a.e. $x \in \Omega$, the identity $\partial_v F(x, Du_0(x)) = \partial L^M(Du_0(x))$ holds, we obtain that $l_M(x) \in \partial L^M(Du_0(x))$ a.e. in Ω. Note that, for each $v \in R^n$, the identity $\partial L(v) = \cap_M \partial L^M(v)$ holds. Note also that, since $\cup_{x\in\Omega}\partial L^M(Du_0(x))$ is a non-increasing sequence of bounded sets, we infer that all functions l_M are equibounded in L^∞. Then, by the Banach–Mazur theorem (see, for example, [13, ch. 1, § 1]), there exists a sequence

$$\tilde{l}_k := \sum_{i=M_k+1}^{M_{k+1}} c_i l_i,$$

with $M_k \to \infty$, $c_i \geqslant 0$ such that

$$\sum_{i=M_k+1}^{M_{k+1}} c_i = 1$$

and $\tilde{l}_k \to l_0$ in L_1. Since $\tilde{l}_k(x) \in \partial L^{M_{k+1}}(Du_0(x))$ for a.a. $x \in \Omega$, we obtain that $l_0(x) \in \cap_M \partial L^M(Du_0(x)) = \partial L(Du_0(x))$ a.e. in Ω. It is also clear that (4.1) holds with l_0 instead of l_M. This proves that (4.1) implies the first claim of theorem 4.3. The second claim is an immediate consequence of the first one.

In order to prove (4.1), notice that in the case $F(x,\cdot) \in C^1$ for a.e. $x \in \Omega$, the identity (4.1) holds with $l_M(x) = F_v(x, Du_0(x))$. The general case can be reduced to this one by approximation arguments.

Consider functions $F^\epsilon : \Omega \times R^n \to R$ such that, for each $x_0 \in \Omega$, $v_0 \in R^n$,

$$F^\epsilon(x_0, v_0) = \int_{R^n} F(x_0, v) * \rho_\epsilon(v - v_0)\,\mathrm{d}v,$$

where $\rho \geqslant 0$ is a usual mollifying kernel, i.e. ρ is smooth with the support in the unit ball, $\int_{R^n} \rho = 1$, and $\rho_\epsilon = \epsilon^{-n}\rho(x/\epsilon)$.

It is easy to see that F^ϵ is convex in v and $F^\epsilon(x,\cdot) \in C^\infty$ for a.e. $x \in \Omega$. Moreover,

$$A_1|v|^2 + B_1 \leqslant F^\epsilon(x, v) \leqslant A_2|v|^2 + B_2, \quad \epsilon \in]0,1], \quad A_2 \geqslant A_1 > 0,$$

and, for a.e. $x \in \Omega$, the family $F^\epsilon(x,\cdot)$ converges to $F(x,\cdot)$ uniformly in each compact set.

Since each problem $J^\epsilon \to \min$, $u|_{\partial\Omega} = f$, $u \in W^{1,2}$ has a solution u^ϵ, we infer that u^ϵ, $\epsilon \in]0,1]$, form a relatively compact set in the weak topology of $W^{1,2}$. It is obvious also that $J^\epsilon(u^\epsilon) \to J(u_0)$. Then, because of lower semicontinuity of convex functionals with respect to weak convergence in $W^{1,2}$, we infer that

$$J(u_0) = \lim_{\epsilon\to 0} J^\epsilon(u^\epsilon) \geqslant J(\tilde{u})$$

for each limit function $\tilde{u}$ of u^ϵ ($u^{\epsilon_k} \to \tilde{u}$ in L_2 for some $\epsilon_k \to 0$), see, for example, [25].

Since u_0 is the unique solution of the original problem, we infer that $u^\epsilon \to u_0$ in L_2. Then $u^\epsilon \rightharpoonup u_0$ in $W^{1,2}$, where $\rightharpoonup$ denotes the weak convergence. For strictly convex functionals, convergences $u^\epsilon \rightharpoonup u_0$ in $W^{1,2}$, $J^\epsilon(u^\epsilon) \to J(u_0)$ imply strong convergence of u^ϵ to u_0 in $W^{1,2}$ (see [25] for a simple proof, and [26] for the characterization of this property of integral functionals in terms of integrands).

For each $\epsilon > 0$, we have

$$\int_{\Omega} \langle F_v^{\epsilon}(x, Du^{\epsilon}(x)), D\phi(x)\rangle \,\mathrm{d}x = 0 \quad \forall\, \phi \in C_0^1(\Omega). \tag{4.2}$$

Locally, uniform convergence of $F^{\epsilon_k}(x,\cdot)$ to $F(x,\cdot)$ for an $x \in \Omega$ implies that, for each sequence $v_k \in R^n$ such that $\epsilon_k \to 0$, $v_k \to v_0$, $F_v^{\epsilon_k}(x, v_k) \to \tilde{l}$, the inclusion $\tilde{l} \in \partial_v F(x, v_0)$ holds (see, for example, [25]).

In view of the convergence $\|u^{\epsilon} - u_0\|_{W^{1,2}} \to 0$, $\epsilon \to 0$, we obtain strong in L^1 norm convergence of $F_v^{\epsilon_k}(\cdot, Du^{\epsilon_k}(\cdot))$ to $l \in L^1$ such that $l(x) \in \partial_v F(x, Du_0(x))$ a.e. in Ω. Being the L^1 limit of $F_v^{\epsilon_k}(\cdot, Du^{\epsilon_k}(\cdot))$, the function $l(\cdot)$ automatically satisfies (4.1) (cf. (4.2)).

The proof is complete. □

5. Proof of theorem 1.2

In this section we give proof to the last result of this paper, theorem 1.2.

Proof of theorem 1.2. Due to theorem 1.1, solvability of all problems (1.1) with linear boundary conditions and a fixed Ω already implies that *for each* $F \in R^n$, *either* $\partial L(F) \neq \emptyset$ *or there exist* $v_1, \dots, v_q \in R^n$ *such that* $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ *and* $F \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$

To prove the converse assertion of the theorem, fix Ω with $\partial\Omega \in C^2$ of positive curvature and $f \in C^2(\partial\Omega)$.

By lemma 2.1, $L^{**} : R^n \to R$ is a continuous convex function. It is clear also that $L^{**} \geqslant \theta$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$.

By theorem 4.1 and the remark to it, there is a solution $u_0 \in W^{1,\infty}(\Omega)$ of the relaxed problem

$$J^{**}(u) \to \min, \qquad u|_{\partial\Omega} = f \tag{5.1}$$

in the class $u \in W^{1,\infty}(\Omega)$.

We can also prove that u_0 is a solution of problem (5.1) in $W^{1,1}(\Omega)$. In fact, by theorem 4.3, there exists $l \in L^{\infty}(\Omega)$ such that $l(x) \in \partial L^{**}(Du_0(x))$ for a.e. $x \in \Omega$ and

$$\int_{\Omega} \langle l(x), D\phi(x)\rangle \,\mathrm{d}x = 0 \quad \forall\, \phi \in W_0^{1,1}(\Omega).$$

If $u|_{\partial\Omega} = u_0|_{\partial\Omega} = f$, then we obtain

$$J^{**}(u) - J^{**}(u_0) = \int_{\Omega} \{L^{**}(Du) - L^{**}(Du_0) - \langle l(x), Du - Du_0\rangle\} \,\mathrm{d}x.$$

Since the expression in the brackets is non-negative almost everywhere in Ω, we obtain that u_0 is a solution in $W^{1,1}$. Note also that in the case where $\operatorname{ess\,sup}|Du|$ is sufficiently large, the expression in the brackets is positive in a set of positive measure since L^{**} has superlinear growth at infinity, and, as a consequence, $J^{**}(u) - J^{**}(u_0) > 0$. Therefore, all solutions to problem (5.1) in $W^{1,1}(\Omega)$ are bounded in $W^{1,\infty}(\Omega)$ by $M = M(f, \Omega) > 0$.

Let u_0 be such a solution. By Rademacher's theorem (cf. [14, p. 81]), u_0 has a classical derivative almost everywhere in Ω. Let $\tilde{\Omega}$ be the set, where Du_0 exists in the

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classical sense and $L(Du_0(x_0)) \neq L^{**}(Du_0(x_0))$. There exists $M_1 > 0$ such that, for each point $x \in \tilde{\Omega}$, there exist $v_1, \dots, v_q \in B(0, M_1)$ with $Du_0(x) \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$ and $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$ (as a consequence, L^{**} is affine on $\operatorname{co}\{v_1, \dots, v_q\}$). Indeed, because of superlinear growth of L^{**} at infinity, the union of those compact convex sets intersecting $B(0, M)$, where L^{**} is affine, is a bounded set.

Therefore, for each $x_0 \in \tilde{\Omega}$, we can isolate extremum points v_i, $i \in \{1, \dots, q\}$, of a compact convex set such that $v_1, \dots, v_q \in B(0, M_1)$, $Du_0(x_0) \in \operatorname{int}\operatorname{co}\{v_1, \dots, v_q\}$, and $\cap_i \partial L(v_i) \neq \emptyset$. Let w_s be functions from (1.2), with $F = Du_0(x_0)$. By lemma 3.2, we have that, for all sufficiently small $s > 0$, the function

$$\phi_s := u_0(x_0) + \langle Du_0(x_0), \cdot - x_0 \rangle + w_s(\cdot - x_0)$$

has properties

$$\begin{aligned} \phi_s < u_0, \quad & x \in x_0 + P_{s/2}, \\ \phi > u_0, \quad & x \in x_0 + \partial P_{2s}, \end{aligned}$$

where $P_\sigma = \{x \in R^n : \max_{1 \leqslant i \leqslant q} \langle v_i - Du_0(x_0), x \rangle \leqslant \sigma\}$.

Hence, we can define a perturbation u_1 of u_0 as follows:

$$u_1 = \begin{cases} u_0 & \text{for } x \in (\Omega \setminus \{x_0 + P_{2s}\}), \\ \min\{\phi_s, u_0\} & \text{otherwise.} \end{cases}$$

Let $\Omega' := \{x \in \Omega : u_1 < u_0\}$. Then Ω' is a proper open subset of Ω, $x_0 \in \Omega'$, and $Du_1(x) \in \{v_1, \dots, v_q\}$ a.e. in $\bar{\Omega}'$.

We can repeat arguments from the proof to theorem 1.4 in order to obtain

$$\int_{\bar{\Omega}'} L^{**}(Du_1)\,\mathrm{d}x = \int_{\bar{\Omega}'} L^{**}(Du_0)\,\mathrm{d}x. \tag{5.2}$$

In fact, if $l \in \cap_i \partial L(v_i)$, then

$$\begin{aligned} \int_{\bar{\Omega}'} \{L^{**}(Du_1) - L^{**}(Du_0)\}\,\mathrm{d}x = {} & \int_{\bar{\Omega}'} \{L^{**}(Du_1) - \langle l, Du_1 \rangle - L(v_1)\}\,\mathrm{d}x \\ & - \int_{\bar{\Omega}'} \{L^{**}(Du_0) - \langle l, Du_0 \rangle - L(v_1)\}\,\mathrm{d}x, \end{aligned} \tag{5.3}$$

where the first integral vanishes and the second one is non-negative. Hence the left-hand side in (5.3) does not exceed zero. The converse inequality is obvious, since u_0 is a solution to (5.1).

Since (5.2) is established, by Vitaly covering arguments, we can isolate at most a countable family of disjoint closed sets $\bar{\Omega}_i$ and corresponding functions $\psi_i \in W_0^{1,\infty}(\Omega_i)$ such that $\operatorname{meas}(\tilde{\Omega} \setminus \cup_i \bar{\Omega}_i) = 0$,$(D\psi_i + Du_0) \in \{v : L = L^{**}\}$ a.e. in $\bar{\Omega}_i$, $\|D\psi_i + Du_0\|_{L^\infty(\bar{\Omega}_i)} \leqslant M_1$, and

$$\int_{\bar{\Omega}_i} L^{**}(Du_0)\,\mathrm{d}x = \int_{\bar{\Omega}_i} L^{**}(Du_1)\,\mathrm{d}x$$

for each $i \in N$.

Therefore, if $\tilde{u} = u_0 + \psi_i$ in $\bar{\Omega}_i$, and $\tilde{u} = u_0$ in $\Omega \setminus \cup \bar{\Omega}_i$, then $\tilde{u} \in W^{1,\infty}(\Omega)$, $\|D\tilde{u}\|_{L^\infty} \leqslant M_1$, and $J^{**}(\tilde{u}) = J^{**}(u_0)$. In particular, $\tilde{u}$ is also a solution of the

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relaxed problem. Since $D\tilde{u} \in \{v : L = L^{**}\}$ a.e., we infer that $\tilde{u}$ is a solution of the problem

$$J(u) \to \min, \qquad u|_{\partial\Omega} = f, \qquad u \in W^{1,1}(\Omega).$$

Since $J(\tilde{u}) = J^{**}(\tilde{u})$, $J^{**}(u) > J^{**}(\tilde{u})$ for all admissible for problem (5.1), functions $u \in W^{1,1}(\Omega)$ with sufficiently large $W^{1,\infty}(\Omega)$ norm and $L \geqslant L^{**}$ everywhere, we infer that all solutions to the original problem (1.1) are bounded in $W^{1,\infty}$ norm.

The proof of the theorem is complete. □

Proof of remark 1.3. By the assumptions, we have $L \geqslant \alpha|\cdot| + \beta$, with $\alpha > 0$ and $\beta \in R$. Then lemma 2.1 implies continuity and convexity of L^{**}. Moreover, $L^{**} \geqslant \alpha|\cdot| + \beta$.

Fix Ω with $\partial\Omega \in C^2$ of positive curvature and $f \in C^2(\partial\Omega)$. To prove solvability of problem (1.1), we can combine arguments from the proofs of theorems 1.2 and 1.4.

By theorems 4.1 and 4.3, the relaxed problem has a solution $u_0 \in W_0^{1,\infty}(\Omega)$. Let $\tilde{\Omega}$ be the set, where Du_0 exists in the classical sense and $L(Du_0(x_0)) \neq L^{**}(Du_0(x_0))$.

Arguing as in the proof of the above theorem, we can isolate at most a countable family of disjoint closed sets $\bar{\Omega}_i$ and corresponding functions $\psi_i \in W_0^{1,\infty}(\Omega_i)$ such that $\text{meas}(\tilde{\Omega} \setminus \cup_i \bar{\Omega}_i) = 0$, $(D\psi_i + Du_0) \in \{v : L = L^{**}\}$ a.e. in $\bar{\Omega}_i$, and

$$\int_{\bar{\Omega}_i} L^{**}(Du_0)\,\mathrm{d}x = \int_{\bar{\Omega}_i} L^{**}(Du_0 + D\psi_i)\,\mathrm{d}x$$

for each $i \in N$.

Define $u_i := u_0 + \phi_j$ in each Ω_j, with $j \leqslant i$, $u_i = u_0$ otherwise. Then u_i is a sequence of solutions of the relaxed problem. Since $|\cdot| \leqslant c_1 L^{**}(\cdot) + c_2$, with certain c_2, $c_1 > 0$, we obtain

$$\|D(u_0 + \psi_j)\|_{L^1(\bar{\Omega}_j)} \leqslant \int_{\bar{\Omega}_j} (c_1 L^{**}(D(u_0 + \psi_j)) + c_2)\,\mathrm{d}x = \int_{\bar{\Omega}_j} (c_1 L^{**}(Du_0) + c_2)\,\mathrm{d}x.$$

This estimate is enough to assert strong convergence of u_i in $W^{1,1}$ norm. Then the limit function $\tilde{u}$ is still a solution of the relaxed problem and $L(D\tilde{u}) = L^{**}(D\tilde{u})$ a.e. in Ω. Therefore, the function $\tilde{u}$ is a solution of the original problem (1.1). □

6. Note added in proof

While this paper was under review, I was informed by colleagues of several similar and some new results in the field. The idea of using a.e. classical differentiability in order to perturb minimizers of the relaxed problem to obtain a solution of the original problem was suggested independently by Zagatti in [28]. Moreover, he then improved both the proofs and the results of his work (see [28]; S. Zagatti, personal communication), again independently, the result of theorem 1.2 by the method described in this paper. I thank Professor Del Maso for bringing the last reference to my attention.

A year later, Celada and Perrota [6] again rediscovered, independently of all other work in the field, that a.e. classical differentiability of solutions of the relaxed problems is key for establishing the attainment results in the original minimization problem if applying the perturbations w_s; see (1.2). They also independently

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proved theorem 1.2 by the same method as in this paper (S. Zagatti, personal communication). Moreover, they proved a.e. classical differentiability of minimizers in problems with p-growth following some suggestions of Šverák and using the estimates from [17]; see [6]. This result allowed them to extend the attainment results to the case of integrands with p-growth. More recently, a number of integrands with lower-order terms, which can be treated by the construction, were indicated by them in [4–6].

Note also that all the authors of the papers [4–6, 28] developed the research programme of their supervisor, Professor Cellina. In fact, the first version of theorem 1.1 is the content of the papers [7, 8] (the paper [15], theorem 2.12 of which contains equivalent results, was submitted later). Another important observation that the perturbations w_s can be applied to Lipschitz functions was noticed by Professor De Blasi and Professor Pianigiani in [12]. Lemma 3.2 of this work is only an improvement towards the use of classical differentiability. Note also that the set where $\theta_s < u_0$ can be selected with the boundary having zero measure, which reduces the situation considered in [12] to the one treated in [3].

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