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Critical exponent in a Stefan problem with kinetic condition

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In this paper we deal with the one-dimensional Stefan problem

$$u_t - u_{xx} = \dot{s}(t)\,\delta(x-s(t)) \quad \text{in } \mathbb{R}\times\mathbb{R}^+, u(x,0) = u_0(x)$$

with kinetic condition $\dot{s}(t) = f(u)$ on the free boundary $F = \{(x,t),\ x = s(t)\}$, where $\delta(x)$ is the Dirac function. We proved in [1] that if $|f(u)| \leqslant M\,\mathrm{e}^{\gamma|u|}$ for some $M > 0$ and $\gamma \in (0, 1/4)$, then there exists a global solution to the above problem; and the solution may blow up in finite time if $f(u) \geqslant C\,\mathrm{e}^{\gamma_1|u|}$ for some γ_1 large. In this paper we obtain the optimal exponent, which turns out to be $\sqrt{2\pi e}$. That is, the above problem has a global solution if $|f(u)| \leqslant M\,\mathrm{e}^{\gamma|u|}$ for some $\gamma \in (0, \sqrt{2\pi e})$, and the solution may blow up in finite time if $f(u) \geqslant C\,\mathrm{e}^{\sqrt{2\pi e}|u|}$.

1 Introduction

In this paper we deal with the following one-dimensional Stefan problem with kinetic condition on the free boundary:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } Q_{1T} \cup Q_{2T}, \\ u^-(s(t),t) = u^+(s(t),t) & \text{on } F, \\ u_x^-(s(t),t) - u_x^+(s(t),t) = \dot{s}(t) & \text{on } F, \\ \dot{s}(t) = f(u), \quad s(0) = b & \text{on } F, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $Q_{1T} = \{(x,t);\ -\infty < x < s(t),\ 0 < t < T\}$, $Q_{2T} = \{(x,t);\ s(t) < x < \infty,\ 0 < t < T\}$, $F = \{(x,t);\ x = s(t)\}$ is the free boundary, $f(u)$ is a locally Lipschitz function, and $u_0 \in C(\mathbb{R})$ is a bounded function.

Problem (1.1) arises in solid combustions [2, 3] and phase transition processes with supercooling or superheating, and has been studied by many authors (see [1–11] and the references therein). Local existence and uniqueness of solutions can be obtained by the fixed point theorem. The global existence of solutions has also been discussed by several authors.

In this paper, we are interested in growth conditions on f so that (1.1) admits a global solution. In this respect the best result, as far as we know, was obtained by the authors [1]. We proved [1] that if $|f(u)| \leqslant M\,\mathrm{e}^{\gamma|u|}$ for some $M > 0$ and $\gamma \in (0, \frac{1}{4})$, there exists a global solution to the problem (1.1); whereas if $f(u) \geqslant \delta\,\mathrm{e}^{\gamma_1|u|}$ for some $\delta > 0$ and γ_1 large, the solution may blow up in finite time. We find that the proof [1] can be refined to get the optimal exponent. We prove in this paper that there exists a critical exponent

$\gamma_0 = \sqrt{2\pi e}$ such that (1.1) has a global solution if $|f(u)| \leqslant M \mathrm{e}^{\gamma|u|}$ for some $\gamma \in (0, \gamma_0)$; and if $f(u) \geqslant C \mathrm{e}^{\gamma_0|u|}$, the solution may blow up in finite time.

This paper is arranged as follows. In §2 we show that there exists a global solution to (1.1) if $|f(u)| \leqslant M \mathrm{e}^{\gamma|u|}$ for some $\gamma \in (0, \gamma_0)$ and $M > 0$. In §3 we give an example to show that the solution may blow up in finite time if $f(u) \geqslant C \mathrm{e}^{\gamma_0|u|}$. The proof in this paper is a refinement of that in our earlier work [1], and hence is somewhat similar.

2 Global existence

In this section we consider the global existence of solutions to (1.1). The local existence, uniqueness, and regularity for solutions of problem (1.1) have been proved [1] by means of the fixed point theorem. To prove the global existence, we need only to establish the following *a priori* estimate:

$$|u(x,t)| \leqslant M \quad \text{for } (x,t) \in \mathbb{R} \times [0,T], \tag{2.1}$$

where M depends only on T, u_0 and f. By the maximum principle we see that $u(x,t)$ attains its maximum on the free boundary F.

Let u be a solution of (1.1). In the sence of distributions u satisfies

$$\begin{cases} u_t - u_{xx} = \dot{s}(t)\,\delta(x - s(t)) & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{2.2}$$

Let

$$K(x,t) = \begin{cases} \dfrac{1}{\sqrt{(4\pi t)}} \mathrm{e}^{-\frac{x^2}{4t}} & t > 0, \\ 0 & t \leqslant 0. \end{cases} \tag{2.3}$$

Then u can be represented by

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} K(x-\xi,t)\,u_0(\xi)\,d\xi + \int_0^t \int_{-\infty}^{\infty} \dot{s}(\tau)\,\delta(\xi - s(\tau))\,K(x-\xi, t-\tau)\,d\xi\,d\tau \\ &= \int_{-\infty}^{\infty} K(x-\xi,t)\,u_0(\xi)\,d\xi + \int_0^t \dot{s}(\tau)\,K(x - s(\tau), t-\tau)\,d\tau. \end{aligned} \tag{2.4}$$

Suppose there exist positive constants $M_0 > 0$ and $\gamma \in (0, \sqrt{2\pi e})$ such that

$$|f(u)| \leqslant M_0 \mathrm{e}^{\gamma|u|} \quad \text{for all } u \in \mathbb{R}. \tag{2.5}$$

To prove (2.1) we argue by contradiction. Suppose for some N large enough,

$$u(s(T),T) \geqslant N. \tag{2.6}$$

In what follows we will use C to denote positive constants which depend only on M_0, γ, and $\|u_0\|_{L^\infty}$, but are independent of T and N.

For any $t \leqslant T$, we define

$$\sigma(t) = \sup\{\alpha \in (\bar{t}, t), \quad |\dot{s}(\alpha)| \cdot |t - \alpha|^{1/2-\delta} > 1\}, \tag{2.7}$$

where $\bar{t} = \max\{t-1, 0\}$,

$$\delta = \frac{1}{4}\left(1 - \frac{\gamma}{\sqrt{2\pi e}}\right). \tag{2.8}$$

If $|\dot{s}(\alpha)|\cdot|t-\alpha|^{1/2-\delta} \leqslant 1$ for all $\alpha \in (\bar{t}, t)$, we define $\sigma(t) = \bar{t}$. By definition we have

$$|\dot{s}(\sigma(t))|\cdot|t-\sigma(t)|^{1/2-\delta} = 1 \quad \text{if } 0 < t-\sigma(t) < 1. \tag{2.9}$$

Let $\alpha_0 = T$. We define α_k inductively by $\alpha_k = \sigma(\alpha_{k-1})$. Since $s(t) \in C^1[0, T]$ we see that there exists $n \geqslant 1$ such that $\alpha_n = 0$.

Lemma 1 *For any $t < T$, we have*

$$|u(s(t), t)| \leqslant C + \beta \log \frac{1}{t-\sigma(t)}, \tag{2.10}$$

where

$$\beta = \frac{1}{2\sqrt{2\pi e}}. \tag{2.11}$$

Proof From (2.4), we have

$$\begin{aligned} u(s(t), t) &= I_1 + I_2 + I_3 \\ &=: \int_{-\infty}^{\infty} \frac{\exp(-(s(t)-\xi)^2/4t)}{\sqrt{4\pi t}} u_0(\xi)\, d\xi \\ &\quad + \left(\int_{\sigma(t)}^{t} + \int_0^{\sigma(t)}\right) \exp\left\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\right\} \frac{\dot{s}(t)}{(4\pi(t-\tau))^{1/2}}\, d\tau, \end{aligned}$$

where $|I_1| \leqslant \sup |u_0(x)|$,

$$|I_2| \leqslant \int_{\sigma(t)}^{t} \frac{(t-\tau)^{-1/2+\delta}}{4\pi(t-\tau))^{1/2}}\, d\tau = \frac{(t-\sigma(t))^{\delta}}{2\sqrt{\pi}\delta} \leqslant \frac{1}{2\sqrt{\pi}\delta} =: C_0. \tag{2.12}$$

Integrating by parts we have

$$\begin{aligned} &\int_0^{\sigma(t)} \frac{\dot{s}(\tau)}{\sqrt{t-\tau}} \exp\left\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\right\} d\tau \\ &\quad = -2\int_0^{\sigma(t)} \exp\left\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\right\} d\,\frac{s(t)-s(\tau)}{2\sqrt{t-\tau}} \\ &\qquad + \int_0^{\sigma(t)} \frac{s(t)-s(\tau)}{2(t-\tau)^{3/2}} \exp\left\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\right\} d\tau. \end{aligned}$$

The first integral on the right-hand side is bounded. We obtain

$$|I_3| \leqslant C + \frac{1}{2\sqrt{\pi}} \int_0^{\sigma(t)} \frac{1}{t-\tau} z \cdot \exp(-z^2)\, d\tau,$$

where $z = |s(t)-s(\tau)|/2\sqrt{t-\tau}$. Observe that

$$0 \leqslant z \exp(-z^2) \leqslant \frac{1}{\sqrt{2e}} \quad \text{for } z \geqslant 0. \tag{2.13}$$

We obtain

$$|I_3| \leqslant C + \frac{1}{2\sqrt{2\pi e}} \log \frac{1}{t-\sigma(t)}.$$

Lemma 1 is therefore proved. □

Lemma 2 *If* $0 < \alpha_k - \sigma(\alpha_k) < 1$, *we have*

$$|\alpha_{k+1} - \alpha_{k+2}|^{\beta\gamma} \leqslant C|\alpha_k - \alpha_{k+1}|^{1/2-\delta}. \tag{2.14}$$

Proof By (2.9) we have

$$1 = |\dot{s}(\sigma(\alpha_k))| \cdot |\alpha_k - \sigma(\alpha_k)|^{1/2-\delta} = |\dot{s}(\alpha_{k+1})| \cdot |\alpha_k - \alpha_{k+1}|^{1/2-\delta}.$$

Hence by Lemma 1,

$$\begin{aligned} |\alpha_k - \alpha_{k+1}|^{-1/2+\delta} &= |\dot{s}(\alpha_{k+1})| \\ &= |f(u(s(\alpha_{k+1}), \alpha_{k+1}))| \\ &\leqslant M_0 \, e^{\gamma |u(s(\alpha_{k+1}), \alpha_{k+1})|} \\ &\leqslant M_0 \, e^{\gamma (C - \beta \log(\alpha_{k+1} - \sigma(\alpha_{k+1}))} \\ &\leqslant C/(\alpha_{K+1} - \sigma(\alpha_{k+1}))^{\beta\gamma} \\ &= C/(\alpha_{k+1} - \alpha_{k+2})^{\beta\gamma}. \end{aligned}$$

(2.14) follows.

Similar to (2.12), we have

$$\begin{aligned} \int_{\alpha_{k+1}}^{\alpha_k} \dot{s}(\tau) \, K(s(t) - s(\tau), t-\tau) \, d\tau &\leqslant \int_{\alpha_{k+1}}^{\alpha_k} \frac{(\alpha_k - \tau)^{-1/2+\delta}}{\sqrt{4\pi(t-\tau)}} \, d\tau \\ &\leqslant \int_{\alpha_{k+1}}^{\alpha_k} \frac{(\alpha_k - \tau)^{-1/2+\delta}}{\sqrt{4\pi(\alpha_k - \tau)}} \, d\tau \leqslant C_0. \end{aligned}$$

Here by (2.4) we have

$$N \leqslant \sup |u_0(x)| + nC_0. \tag{2.15}$$

Let k_0 be such that

$$\alpha_{k_0} - \alpha_{k_0+1} = \inf\{\alpha_k - \alpha_{k+1}; k = 0, 1, \ldots, n-1\}.$$

By the local existence and regularity of solutions of (1.1) we may suppose $k_0 \neq n-1$, namely, $k_0 \leqslant n-2$, and

$$\alpha_{k_0} - \alpha_{k_0+1} \leqslant \alpha_{k_0+1} - \alpha_{k_0+2}. \tag{2.16}$$

By Lemma 2,

$$|\alpha_{k_0+1} - \alpha_{k_0+2}|^{\beta\gamma} \leqslant C|\alpha_{k_0} - \alpha_{k_0+1}|^{\delta} \cdot |\alpha_{k_0} - \alpha_{k_0+1}|^{1/2-2\delta}.$$

By (2.16) we obtain

$$|\alpha_{k_0+1} - \alpha_{k_0+2}|^{\beta\gamma} \leqslant C|\alpha_{k_0} - \alpha_{k_0+1}|^{\delta} \cdot |\alpha_{k_0+1} - \alpha_{k_0+2}|^{1/2-2\delta}.$$

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Note that $\beta\gamma = \frac{1}{2} - 2\delta$, we obtain

$$|\alpha_{k_0} - \alpha_{k_0+1}| \geqslant C^{1/\delta}.$$

Hence

$$T = \sum_{k=0}^{n-1} (\alpha_k - \alpha_{k+1}) \geqslant nC^{-1/\delta},$$

i.e. $n \leqslant C^{1/\delta}T$. By (2.15) we obtain

$$N \leqslant \sup |u_0(x)| + C_0 C^{1/\delta} T. \tag{2.17}$$

We have thus proved

Theorem 1 *Suppose* (2.5) *holds. Then* (1.1) *has a global solution u which satisfies*

$$|u(x,t)| \leqslant C(1+t), \tag{2.18}$$

where C depends only on γ, M_0, *and* $\sup |u_0(x)|$.

Remark It is easy to see from the above proof that Theorem 1 still holds if (2.5) is replaced by

$$|f(u)| \leqslant C\,\mathrm{e}^{\sqrt{2\pi e}|u|}/(1+|u|)^{1+\epsilon} \tag{2.19}$$

for some $\epsilon > 0$. Indeed, one needs only to replace $|t-\alpha|^{1/2-\delta}$ by $|t-\alpha|^{1/2} \log^{1+\epsilon/2}(t-\alpha)$ in (2.7). Then a slight modification of the above argument still gives (2.18).

3 A blow-up example

In this section we show that the exponent $\sqrt{2\pi e}$ in the last section is optimal for the global existence. We will construct $f(u)$ with $f(u) \leqslant C\,\mathrm{e}^{\sqrt{(2\pi e)}|u|}$ such that the solution u of (1.1) blows up in finite time. The example given here is actually the same as in our earlier work [1].

Let $s(t) \in C^2[0,T)$ be given; we consider the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } Q_{1T} \cup Q_{2T}, \\ u^-(s(t),t) = u^+(s(t),t) & \text{on } F, \\ u_x^-(s(t),t) - u_x^+(s(t),t) = \dot{s}(t) & \text{on } F, \\ u(x,0) = u \text{ and } u(x,t) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.1}$$

Problem (3.1) is uniquely solvable. In the sense of distributions the solution of (3.1) is equivalent to the solution of

$$\begin{cases} u_t - u_{xx} = \dot{s}(t)\,\delta(x - s(t)) & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = 0, \end{cases} \tag{3.2}$$

where $\delta(x)$ is the Dirac function. By (2.4) the solution of (3.2) is given by

$$u(x,t) = \int_0^t \frac{\dot{s}(t-\tau)}{\sqrt{4\pi\tau}} \mathrm{e}^{-\frac{|x-s(t-\tau)|^2}{4\tau}}\, d\tau. \tag{3.3}$$

Let $s(t) = -\sqrt{2(T-t)}$. Denote $\epsilon = T-t$. We have

$$\dot{s}(t-\tau) = \frac{1}{\sqrt{2(\epsilon+\tau)}}, \quad \ddot{s}(t-\tau) = \frac{1}{[2(\epsilon+\tau)]^{3/2}}. \tag{3.4}$$

Let $F(t,\tau) = 1/4\tau|s(t)-s(t-\tau)|^2 = 1/2\tau|\sqrt{\epsilon+\tau}-\sqrt{\epsilon}|^2$, then

$$F(t,\tau) = \frac{1}{2}\frac{\sqrt{\epsilon+\tau}-\sqrt{\epsilon}}{\sqrt{\epsilon+\tau}+\sqrt{\epsilon}} < \frac{1}{2}, \tag{3.5}$$

$$\frac{\partial}{\partial t}F(t,\tau) = \frac{1}{2\tau}(\sqrt{\epsilon+\tau}-\sqrt{\epsilon})\left(\frac{1}{\sqrt{\epsilon}}-\frac{1}{\sqrt{\epsilon+\tau}}\right) = \frac{F(t,\tau)}{\sqrt{\epsilon}\sqrt{\epsilon+\tau}}. \tag{3.6}$$

From (3.3) we therefore obtain

$$\begin{aligned} u(s(t),t) &\geqslant \frac{e^{-1/2}}{\sqrt{4\pi}}\int_0^t \frac{\dot{s}(t-\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau \\ &= \frac{1}{2\sqrt{2\pi e}}\int_0^t \frac{1}{\sqrt{\tau}\,\sqrt{\epsilon+\tau}}\,\mathrm{d}\tau \\ &= \frac{1}{2\sqrt{2\pi e}}\log\left(\sqrt{1+\frac{t}{\epsilon}}+\sqrt{\frac{t}{\epsilon}}\right) \\ &\geqslant \frac{1}{2\sqrt{2\pi e}}\log\left(1+\frac{t}{\epsilon}\right) = \frac{1}{2\sqrt{2\pi e}}\log\frac{T}{T-t}. \end{aligned} \tag{3.7}$$

Hence $u(s(t),t)\to\infty$ as $t\to T$. Next we show that $u(s(t),t)$ is strictly increasing for $t\in(0,T)$. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}u(s(t),t) &= \int_0^t \frac{\ddot{s}(t-\tau)}{\sqrt{4\pi\tau}}\mathrm{e}^{-F(t,\tau)}\,\mathrm{d}\tau - \int_0^t \frac{\dot{s}(t-\tau)}{\sqrt{4\pi\tau}}F_t(t,\tau)\,\mathrm{e}^{-F(t,\tau)}\,\mathrm{d}\tau \\ &\quad + \frac{\dot{s}(0)}{\sqrt{4\pi t}}\mathrm{e}^{-|s(t)-s(0)|^2/4t} =: I_1 - I_2 + I_3, \end{aligned} \tag{3.8}$$

where $I_3 > 0$. To show $\frac{\mathrm{d}}{\mathrm{d}t}u(s(t),t) \geqslant 0$, it suffices to verify $I_1 \geqslant I_2$. By (3.4)–(3.6) we have

$$I_1 = \frac{1}{4\sqrt{2\pi}}\int_0^t \frac{1}{\tau^{1/2}(\epsilon+\tau)^{3/2}}\exp\left[-\frac{1}{2}\frac{\sqrt{\epsilon+\tau}-\sqrt{\epsilon}}{\sqrt{\epsilon+\tau}+\sqrt{\epsilon}}\right)\mathrm{d}\tau,$$

$$I_2 = \frac{1}{4\sqrt{2\pi}}\int_0^t \frac{1}{(\epsilon\tau)^{1/2}(\epsilon+\tau)}\frac{\sqrt{\epsilon+\tau}-\sqrt{\epsilon}}{\sqrt{\epsilon+\tau}+\sqrt{\epsilon}}\exp\left(-\frac{1}{2}\frac{\sqrt{e+t}-\sqrt{\epsilon}}{\sqrt{\epsilon+\tau}+\sqrt{\epsilon}}\right)\mathrm{d}\tau.$$

Let $\tau = \epsilon\alpha$; then we need only to verify that

$$\begin{aligned} &\int_0^{t/\epsilon} \frac{1}{\alpha^{1/2}(1+\alpha)^{3/2}}\exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\right]\mathrm{d}\alpha \\ &\qquad \geqslant \int_0^{t/\epsilon} \frac{1}{\alpha^{1/2}(1+\alpha)}\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\right]\mathrm{d}\alpha. \end{aligned}$$

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Let $\;(\alpha)$ denote the integrand on the left-hand side, and $\psi(\alpha)$ denote the one on the right-hand side. If $t \leqslant 2(1+\sqrt{2})\,\epsilon$, we have $\;(\alpha) \geqslant \psi(\alpha)$, and so we are through. If $t > 2(1+\sqrt{2})\,\epsilon$, it suffices to verify

$$\int_{2(1+\sqrt{2})}^{t/\epsilon} [\psi(\alpha) - \;(\alpha)]\,\mathrm{d}\alpha \leqslant \int_0^{2(1+\sqrt{2})} [\psi(\alpha) - \psi(\alpha)]\,\mathrm{d}\alpha. \tag{3.9}$$

Set $\alpha = 1/\alpha'$; then left-hand side of (3.9) is bounded by

$$\int_{2(1+\sqrt{2})}^{\infty} \frac{\alpha - 2\sqrt{1+\alpha}}{\alpha^{1/2}(1+\alpha)^{3/2}(1+\sqrt{1+\alpha})} \exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\right]\mathrm{d}\alpha$$

$$= \int_0^{(\sqrt{2}-1)/2} \frac{1-2\sqrt{\alpha}\sqrt{1+\alpha}}{\alpha^{1/2}(1+\alpha)^{3/2}(\sqrt{\alpha}+\sqrt{1+\alpha})} \exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-\sqrt{\alpha}}{\sqrt{1+\alpha}+\sqrt{\alpha}}\right]\mathrm{d}\alpha.$$

Hence it suffices to verify that for $\alpha \in (0, \frac{1}{2}(\sqrt{2}-1))$,

$$\frac{1-2\sqrt{\alpha}\sqrt{1+\alpha}}{\alpha^{1/2}(1+\alpha)^{3/2}(\sqrt{\alpha}+\sqrt{1+\alpha})} \exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-\sqrt{\alpha}}{\sqrt{1+\alpha}+\sqrt{\alpha}}\right]$$

$$\leqslant \;(\alpha) - \psi(\alpha)$$

$$= \frac{2\sqrt{1+\alpha}-\alpha}{\alpha^{1/2}(1+\alpha)^{3/2}(1+\sqrt{1+\alpha})} \exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\right]. \tag{3.10}$$

Note that for $\alpha \in (0, \frac{1}{2}(\sqrt{2}-1))$,

$$\exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-\sqrt{\alpha}}{\sqrt{1+\alpha}+\sqrt{\alpha}}\right] \leqslant \exp\left[-\frac{1}{2}\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\right].$$

Hence (3.10) follows from the inequality

$$\frac{1-2\sqrt{\alpha}\sqrt{1+\alpha}}{\sqrt{\alpha}+\sqrt{1+\alpha}} \leqslant \frac{2\sqrt{1+\alpha}-\alpha}{1+\sqrt{1+\alpha}}.$$

The last inequality holds for $\alpha \in (0, \frac{1}{2}(\sqrt{2}-1))$. Hence $(\mathrm{d}/\mathrm{d}t)\,u(s(t),t)$ is strictly increasing.

Now we can define $f(u): \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(u(s(t),t)) = \dot{s}(t) = \frac{1}{\sqrt{2(T-t)}}. \tag{3.11}$$

$f(u)$ is obviously positive and increasing in $(0,T)$. For the function $f(u)$ defined above, we therefore conclude by uniqueness that the solution $u(x,t)$ of (1.1) blows up at time $t = T$.

From (3.7) it is easy to see that $f(u)$ is of exponential growth as $u \to +\infty$ with

$$f(u) \leqslant \frac{1}{\sqrt{2T}} \mathrm{e}^{\sqrt{2\pi e}|u|}. \tag{3.12}$$

Moreover, $\lim_{u\to 0} f(u) = 1/\sqrt{2T}$.

4 Conclusion

Theorem 1 shows that the solution $u(\cdot, t)$ exists globally and grows linearly under the assumption (2.5) or (2.19). On the other hand, the example constructed above shows that $u(\cdot, t)$ may blow up in finite time if $f(u) \geqslant C e^{\sqrt{2\pi e}|u|}$. Hence $\sqrt{2\pi e}$ is the critical exponent for the global existence. Notice that by choosing T large enough, the coefficient $1/\sqrt{2T}$ on the right-hand side of (3.12) may be as small as we want, which suggests that the global existence depends on the growth rate rather than the magnitude of f itself. By the above construction it is easy to see that for some f, the problem (1.1) admits a global solution which grows superlinearly.

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