Bull. Aust. Math. Soc. **106** (2022), 170–173
doi:10.1017/S000497272200051X

STOCHASTIC MODELLING AND STATISTICAL ANALYSIS OF SPATIAL AND LONG-RANGE DEPENDENT DATA

RAVINDI NANAYAKKARA

(Received 20 April 2022; first published online 10 June 2022)

2020 *Mathematics subject classification*: primary 60G60; secondary 62F12, 60G22, 62P12.

Keywords and phrases: spherical random fields, multifractality, multifractionality, Rényi function, cosmic microwave background radiation, asymptotic normality, cyclic long-memory.

In general, observations are assumed to be independent in the theory of statistics. However, in many natural phenomena, the independence assumption is often an approximation and not valid for real data. Thus, random processes that model dependent observations have received enormous attention in the last two decades. They have been applied to temporal and spatial data from diverse disciplines such as cosmology, geoscience, finance, genomics, embryology, hydrology and telecommunications. It is important to study statistical models with dependence, in particular, the long-range dependence and spatial dependence of such data for time series and spatial analysis.

According to the type of dependence structure, different types of stochastic processes have been introduced. This thesis mainly studies long-range dependent stochastic processes. The main objective of this thesis is to study and develop stochastic models and statistical methods for spatial and temporal long-range dependent data. These data are modelled as realisations of spherical random fields and functional time series. In the literature, isotropic Gaussian spherical random fields are used as the standard stochastic model, describing various astrophysical, cosmological and environmental data [7]. The spectral methods are of great importance in studying and investigating second-order random fields with homogeneous and isotropic properties (see [9]).

In the first part of this thesis, we mainly consider random fields defined on the unit sphere and their cosmological applications. The first and the second directions of research are carried out with the main motivation of investigating the non-Gaussianity and other anomalies in the spatial data. The implemented methodologies are applied to the cosmic microwave background radiation (CMB) data from the Planck mission. The concepts of multifractality and multifractionality play an important role in studying the

Thesis submitted to La Trobe University in September 2021; degree approved on 9 February 2022; principal supervisor Andriy Olenko, co-supervisor Philip Broadbridge.

properties of the underlying random fields. Although multifractal and multifractional approaches have been extensively used in the one-dimensional case, their applications in the multi-dimensional case or on manifolds are less developed. This thesis extends the applications of multifractal and multifractional theory to the case of spherical random fields.

The third research direction studied is the statistical inference of long-range dependent data with specific consideration of cyclic long-memory processes. It is well known that spectral densities of these processes have singularities at nonzero frequencies, which can play an important role in limit theorems (see [8]). The inferential statistics are developed for functional time series with spectral singularities at nonzero frequencies. In particular, simultaneous estimators of the two parameters of cyclic long-range dependent time series are obtained.

First, the multifractal behaviour of spherical random fields is studied. The Rényi function plays a key role in the analysis of multifractal random fields. It quantifies the variation of surface or trajectory characteristics along with the change in the image resolution or scale. In the literature, there are three multifractal models for spherical random fields where the Rényi function is known explicitly [6]. In this study, the existing multifractal models are investigated and several new theoretical models and numerical multifractality studies are presented. The new multifractal models for spherical random fields are developed so that the Rényi functions and the multifractal spectra can be computed explicitly. For all the models considered, the specific formulae of the multifractal spectra are derived. The behaviour of the Rényi functions and multifractal spectra and the dependence of the Rényi functions on the scaling parameter of these models are investigated for all the cases.

Then, the developed methodology is applied to analyse the CMB data from the Planck mission. Numerical studies of the Rényi function are carried out extensively for the CMB data from different windows of the CMB sky sphere. The empirical Rényi functions are computed to verify the consistency of the actual CMB data with the theoretical models considered. The existing and the newly developed multifractal models are fitted to actual CMB data to check for any presence of non-Gaussianity in the CMB data. The results suggest that a very minor multifractality of the CMB data may exist for the currently available resolution.

Second, the multifractional behaviour of spherical random fields is studied. Two approaches are developed to investigate the multifractionality of these random fields. The Generalized Multifractional Brownian Motion is used as the main stochastic model for the CMB data. Its Hölder regularity can vary considerably. The Hölder exponent is used to measure the roughness of random fields in a rigorous mathematical way [3]. Traditionally, the method of quadratic variations is often used for the estimation of the pointwise Hölder exponents. The pointwise Hölder exponent values are estimated for one- and two-dimensional regions of the CMB temperature intensities by considering the HEALPix ring and nested ordering schemes respectively. Then, the developed multifractional approaches are employed for the statistical analysis of CMB temperature intensities, detection of potential CMB anomalies in inpainted CMB maps and for comparison of the obtained results with the other methods available in the literature. The results suggest that the estimated pointwise Hölder exponents substantially change from location to location in the CMB sky sphere. Therefore, the analysis suggests that the CMB temperature intensities are multifractional. The suggested approach is used to find CMB anomalies, which are mostly located in the galactic region of the CMB sky sphere. The identified CMB temperature anisotropies are consistent with the TMASK of unreliable CMB temperature intensities.

Finally, the asymptotic behaviour of simultaneous estimators of cyclic long-memory processes is studied. Although the parameter estimation of cyclic long-memory processes has resulted in a proliferation of research, the simultaneous estimation of both long-memory and singularity parameters is a challenging problem. Notably, less is known regarding the inferential statistics of the proposed estimators. The publication [1] proposed a semiparametric estimation method for the simultaneous estimation of long-memory and singularity location parameters by using the Generalized Filtered Method of Moments approach. In this study, the investigations of these simultaneous estimators are continued and their asymptotic properties are derived. A wide class of Gegenbauer-type semiparametric models with spectral singularities is studied. The asymptotic normality and consistency of the Generalized Filtered Method of Moments simultaneous estimators are proved. Also, novel adjusted simultaneous estimators are proposed for cyclic long-memory processes and their properties are investigated. The methodology includes wavelet transformations as a particular case. Numerical examinations for Meyer, Shannon and Mexican hat wavelets and substantial simulation studies are carried out to exemplify the theoretical findings.

All models are implemented and numerically studied by using the software Maple 2019.0, Python version 3.9.4 and R version 4.0.3. The implemented computing techniques are freely available online as source codes. The results obtained and the computing techniques developed in this thesis may be applied to other spherical, geoscience, directional, environmental and medical imaging data.

The main results of the thesis have been published in the papers [2, 4, 5].

Acknowledgements

This work was supported by a La Trobe University Postgraduate Research Scholarship and a La Trobe University Full Fee Research Scholarship. This research was partially supported under the Australian Research Council's Discovery Projects funding scheme (project number DP160101366). This research includes computations using the Linux computational cluster Gadi of the National Computational Infrastructure (NCI), which is supported by the Australian Government and La Trobe University.

References

[1] H. Alomari, A. Ayache, M. Fradon and A. Olenko, 'Estimation of cyclic long-memory parameters', *Scand. J. Stat.* **47**(1) (2020), 104–133.

[2] A. Ayache, M. Fradon, R. Nanayakkara and A. Olenko, 'Asymptotic normality of simultaneous estimators of cyclic long-memory processes', *Electron. J. Stat.* **16**(1) (2022), 84–115.

[3] A. Ayache and J. Véhel, 'On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion', *Stochastic Process. Appl.* **111**(1) (2004), 119–156.
[4] P. Broadbridge, R. Nanayakkara and A. Olenko, 'On multifractionality of spherical random fields with cosmological applications', *ANZIAM J.*, to appear.
[5] N. Leonenko, R. Nanayakkara and A. Olenko, 'Analysis of spherical monofractal and multifractal random fields', *Stoch. Environ. Res. Risk. Assess.* **35**(3) (2021), 681–701.
[6] N. Leonenko and N.-R. Shieh, 'Rényi function for multifractal random fields', *Fractals* **21**(2) (2013), Article no. 1350009.
[7] D. Marinucci and G. Peccati, *Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications* (Cambridge University Press, New York, 2011).
[8] A. Olenko. 'Limit theorems for weighted functionals of cyclical long-range dependent random fields', *Stoch. Anal. Appl.* **31**(2) (2013), 199–213.
[9] M. Yadrenko, *Spectral Theory of Random Fields* (Optimization Software, New York, 1983).

RAVINDI NANAYAKKARA, Department of Mathematical and Physical Sciences,
La Trobe University, Melbourne, Victoria 3086, Australia
e-mail: D.Nanayakkara@latrobe.edu.au