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ZAREMBA, SALEM AND THE FRACTAL NATURE OF GHOST DISTRIBUTIONS

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Abstract

Motivated by near-identical graphs of two increasing continuous functions—one related to Zaremba's conjecture and the other due to Salem—we provide an explicit connection between fractals and regular sequences by showing that the graphs of ghost distributions, the distribution functions of measures associated to regular sequences, are sections of self-affine sets. Additionally, we provide a sufficient condition for such measures to be purely singular continuous. As a corollary, and analogous to Salem's strictly increasing singular continuous function, we show that the ghost distributions of the Zaremba sequences are singular continuous.

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1. Motivation

For each integer $k \geqslant 2$, we define the k-Zaremba sequence, z_k, by $z_k(n) := \mathbf{e}_1^T \mathbf{B}_{(n)_k} \mathbf{e}_1$, where $\mathbf{e}_1 := (1\ 0)^T$, the string $(n)_k = i_s \cdots i_1 i_0$ is the base-k expansion of n and $\mathbf{B}_{(n)_k} := \mathbf{B}_{i_s} \cdots \mathbf{B}_{i_1} \mathbf{B}_{i_0}$ where

$$\mathbf{B}_i = \begin{pmatrix} i+1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } i = 0, 1, \ldots, k-1. \tag{1.1}$$

The sequence z_k is a lexicographical enumeration of the denominators of the convergents of all continued fractions of numbers $x \in (0, 1)$ with all partial quotients bounded by k; see Coons [6] for results relating to their power series generating functions. In 1972, Zaremba [22] conjectured that *there is a positive integer k such that z_k takes all positive integer values.* Bourgain and Kontorovich [5] proved that the set of values of z_{50} has full density in $\mathbb{N}$; this was improved by Huang [16] and Jenkinson and Pollicott [17], who proved the analogous result for z_5.

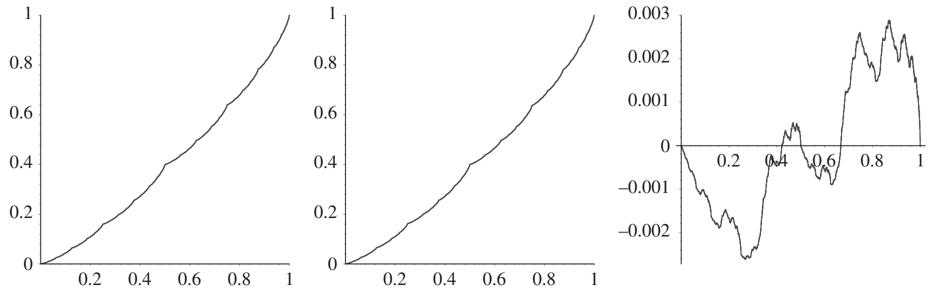

FIGURE 1. The ghost distributions of the 2-regular Salem sequence with $(b_0, b_1) = (2, 3)$ (left) and the Zaremba sequence z_2 (centre), and their pointwise difference, Salem minus Zaremba (right).

Our story starts with trying to gain a better understanding of z_2 by studying the limit

$$Z_2(x) := \lim_{N\to\infty} \frac{1}{2\cdot 4^N} \sum_{m=0}^{2^N x} z_2(2^N + m)$$

over the fundamental intervals $[2^N, 2^{N+1})$ of the normalised partial sums of z_2 for $x \in [0, 1)$. We plotted $Z_2(x)$ and immediately saw a picture we recognised—a strictly increasing singular continuous function whose construction is due to Salem—or so we thought. In fact, these curves are remarkably close to one another, but not the same (see Figure 1). Exact agreement occurs at the points $(0, 0)$, $(1/2, 2/5)$ and $(1, 1)$ by design, and there appear to be two more points of equality, roughly at $x = 5/12$ and $x = 2/3$. The interesting part for us here is that the maximum difference between these two curves is at most 0.003.

One may be interested in why these curves are so close, but not equal. Indeed, as it turns out, and as we shall see later on, the Salem curve lives in the plane, while the Zaremba curve $Z_2(x)$ is the projection of a three-dimensional curve, which is just barely three-dimensional—it is extremely close to being planar.

In the early 1940s, Salem [20] provided a simple direct construction of a family of strictly increasing singular continuous functions from $[0, 1]$ to $[0, 1]$. At the time, this was quite novel—according to Salem, 'functions of this type usually have been obtained by "convolutions" of functions of the Cantor type and the proof that they are singular strictly increasing functions is somewhat difficult'. Recall that a function is singular continuous provided it is continuous and has almost everywhere zero derivative. Here we consider Salem's example, with parameter $\lambda_0 = 2/5$. In this case, we denote Salem's self-affine set by $\mathcal{A}_s$, which is the unique attractor of the iterated function system $\mathcal{S}_s = \{S_0, S_1\} : [0, 1]^2 \to [0, 1]^2$, where

$$S_0\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 2/5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad\text{and}\quad S_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 3/5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}1/2\\2/5\end{pmatrix}. \tag{1.2}$$

Figure 2 shows how the attractor $\mathcal{A}_s$ is developed by iterating the system $\mathcal{S}_s$ starting with the line segment joining $(0, 0)$ and $(1, 1)$ as a seed.

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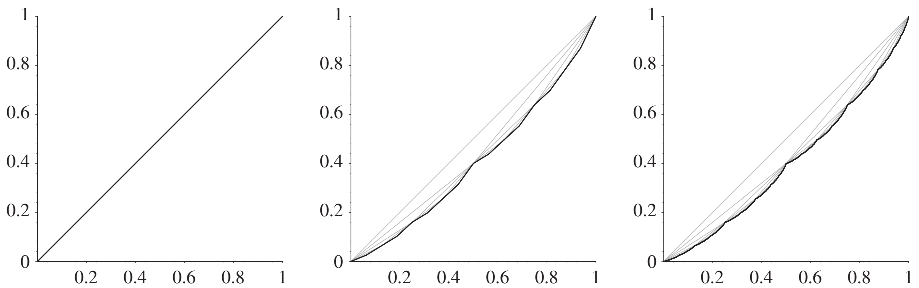

FIGURE 2. The line segment from (0, 0) to (1, 1) after applying $\mathcal{S}_s$, 0 (left), 4 (centre) and 10 (right) times.

REMARK 1.1. We use the term 'singular continuous function' in our paper to directly relate to the cited paper of Salem. Nowadays—especially since we are dealing with increasing singular continuous functions—it is more common to define an increasing singular continuous function as the distribution function of a singular continuous (Lebesgue–Stieltjes) measure.

Motivated by the near-identical pictures in Figure 1, we wondered if the results concerning Salem's curve $\mathcal{A}_s$ hold for the function $Z_2(x)$. In particular, can $Z_2(x)$ be derived from the attractor of an iterated function system? And, is $Z_2(x)$ a strictly increasing singular continuous function?

In this paper we provide positive answers to these questions as well as some generalisations.

2. Results

The Zaremba sequences, z_k, are examples of regular sequences, which are generalisations of automatic sequences. A sequence f is *k-automatic* provided there is a deterministic finite automaton that takes in the base-k expansion of a positive integer n and outputs the value $f(n)$. Automatic sequences can be described in many ways; the one most appropriate for our current purposes is via the k-kernel,

$$\ker_k(f) := \{(f(k^\ell n + r))_{n\geqslant 0} : \ell \geqslant 0, 0 \leqslant r < k^\ell\}.$$

A sequence f is k-automatic if and only if its k-kernel is finite [13, Proposition V.3.3]. It is immediate that an automatic sequence takes only a finite number of values. A natural generalisation to sequences that can be unbounded was given by Allouche and Shallit [1]; a real-valued sequence f is called *k-regular* if the $\mathbb{R}$-vector space $V_k(f) := \langle \ker_k(f)\rangle_{\mathbb{R}}$ is finite-dimensional over $\mathbb{R}$. The set of k-regular sequences admits an algebraic structure; it forms a ring under pointwise addition and convolution.

Let $k \geqslant 2$ be an integer, f be a k-regular sequence and let the set of sequences $\{f = f_1, f_2, \ldots, f_d\}$ be a basis for $V_k(f)$. Set $\mathbf{f}(m) = (f_1(m), f_2(m), \ldots, f_d(m))$ and for each $a \in \{0, \ldots, k-1\}$ let $\mathbf{B}_a$ be the $d \times d$ real matrix such that, for all $m \geqslant 0$, $\mathbf{f}(km + a) = \mathbf{f}(m)\,\mathbf{B}_a$. We call the $\mathbf{B}_a$ *digit matrices* and write $\mathcal{B} := \{\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{k-1}\}$.

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Since the $d \times d$ digit matrices of a regular sequence are formed using a basis of $V_k(f)$, the positive integer d is minimal; we call such a regular sequence d-dimensional. See the seminal paper of Allouche and Shallit [1] for details on existence and the finer definitions. It follows that there is $\mathbf{w} \in \mathbb{R}^{d\times 1}$ such that for each $i \in \{1, \ldots, d\}$ and $n > 0$,

$$f_i(m) = \mathbf{w}^T\, \mathbf{B}_{(m)_k}\mathbf{e}_i = \mathbf{w}^T\, \mathbf{B}_{i_s} \cdots \mathbf{B}_{i_1}\mathbf{B}_{i_0}\mathbf{e}_i,$$

where $\mathbf{e}_i$ is the ith $d \times 1$ elementary column vector, $(m)_k = i_s \cdots i_1 i_0$ is the base-k expansion of m and $\mathbf{B}_{(m)_k} := \mathbf{B}_{i_s} \cdots \mathbf{B}_{i_1}\mathbf{B}_{i_0}$. Set $\mathbf{B} := \sum_{a=0}^{k-1} \mathbf{B}_a$.

Throughout this paper, we let $\rho(\mathbf{M})$ denote the spectral radius of the matrix $\mathbf{M}$ and denote the *joint spectral radius* of a finite set of matrices $\mathcal{M} := \{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_\ell\}$ by

$$\rho^*(\mathcal{M}) = \limsup_{n\to\infty} \max_{1\leqslant i_0, i_1, \ldots, i_{n-1}\leqslant \ell} \|\mathbf{M}_{i_0}\mathbf{M}_{i_1} \cdots \mathbf{M}_{i_{n-1}}\|^{1/n},$$

where $\|\cdot\|$ is any submultiplicative matrix norm—here, we will exclusively use the standard operator norm. This quantity was introduced by Rota and Strang [19] and has a wide range of applications. For an extensive treatment, see Jungers [18]. Set

$$\Sigma(n) := \sum_{m=k^n}^{k^{n+1}-1} f(m) \quad \text{and} \quad \mu_n := \frac{1}{\Sigma(n)} \sum_{m=0}^{k^{n+1}-k^n-1} f(k^n + m)\, \delta_{m/k^n(k-1)},$$

where δ_x denotes the unit Dirac measure at x. We can view $(\mu_n)_{n\in\mathbb{N}_0}$ as a sequence of probability measures on the 1-torus, the latter written as $\mathbb{T} = [0, 1)$ with addition modulo 1. Here, we have simply reinterpreted the (normalised) values of the sequence $(f(m))_{m\geqslant 0}$ between k^n and $k^{n+1} - 1$ as the weights of a pure point probability measure on $\mathbb{T}$ supported on the set $\{m/(k^n(k-1)) : 0 \leqslant m < k^n(k-1)\}$.

With the above notation, Coons *et al.* [7, Theorem 6] recently proved that if f is a nonnegative real-valued k-regular sequence, such that the spectral radius $\rho(\mathbf{B})$ is the unique simple dominant eigenvalue of $\mathbf{B}$ and $\rho^*(\mathcal{B}) < \rho(\mathbf{B})$, then the pure point probability measures μ_n converge weakly to a continuous probability measure μ_f on $\mathbb{T}$. The measure μ_f is called the *ghost measure* of f, and its distribution function is called the *ghost distribution* of f. This measure first appeared in Baake and Coons [2] in relation to Stern's diatomic sequence, but without a prescribed name.

Here, we show that the ghost distribution of a regular sequence can be obtained by a fractal geometric construction. Recall that an *affine contraction* $S : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation of the form $S(x) = T(x) + b$, where T is a linear contraction on $\mathbb{R}^n$ representable as an $n \times n$ matrix and $b \in \mathbb{R}^n$. A finite family of affine contractions $\mathcal{S} = \{S_1, \ldots, S_m\}$, with $m \geqslant 2$, is called an *iterated function system*, and the unique attractor (or invariant set), provided it exists, for an iterated function system of affine contractions is called a *self-affine set*; see Falconer [14, Ch. 9] for details on iterated function systems over $\mathbb{R}^n$. In particular, [14, Ch. 9, Theorem 9.1] guarantees the existence and uniqueness of the self-affine sets we produce in this paper. The following result answers the question of $Z_2(x)$ being an attractor of an iterated function system.

THEOREM 2.1. *Suppose f is a nonnegative k-regular sequence, $\rho(\mathbf{B})$ is the unique simple dominant eigenvalue of* $\mathbf{B}$, *$\rho^*(\mathcal{B}) < \rho(\mathbf{B})$ and there is a real number $c > 0$ such*

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that $\Sigma(n) \sim c \cdot \rho(\mathbf{B})^n$ as n grows. Then the graph of the ghost distribution of f is an affine image of a section of a self-affine set.

In fact, we explicitly describe the self-affine set of Theorem 2.1.

Towards answering our question on the singular continuity of $Z_2(x)$, we consider the generalisations of Salem's example as follows. Let $k \geqslant 2$ be a positive integer. We call a one-dimensional nonnegative regular sequence f a *Salem sequence* and refer to the digit matrices of a Salem sequence simply as *digits*. For example, the Salem attractor $\mathcal{A}_s$ is related to the 2-regular Salem sequence $s = (s(n))_{n\geqslant 0}$ given by $s(n) = 2^{s_0(n)}3^{s_1(n)}$, where $s_0(n)$ and $s_1(n)$ count the number of 0s and 1, respectively, in the binary expansion of n—the careful reader will notice the use of the subscript s in the above discussion of Salem's iterated function system. In a later section, we consider the characterisation of ghost measures of Salem sequences. By modifying the original proof of Salem, we are able to prove the following general result, a corollary of which shows that $Z_2(x)$ is singular continuous.

THEOREM 2.2. *Suppose f is a nonnegative k-regular sequence, $\rho(\mathbf{B})$ is the unique simple dominant eigenvalue of $\mathbf{B}$, $\rho^*(\mathcal{B}) < \rho(\mathbf{B})$ and there is a real number $c > 0$ such that $\Sigma(n) \sim c \cdot \rho(\mathbf{B})^n$ as n grows. If*

$$\prod_{j=0}^{k-1} \|\mathbf{B}_j\|^{1/k} < \frac{\rho}{k}, \tag{2.1}$$

then the ghost measure of f is purely singular continuous.

This paper is organised as follows. In the next section, with the proof of Theorem 2.1 as a goal, we provide an explicit relationship between the solution of a dilation equation and the attractor of an iterated function system of affine contractions (Lemma 3.1). In that section, we use Salem's singular continuous function as a motivating example. In Section 4 we consider 'Salem sequences' as generalisations of Salem's example, and of standard missing-digit sequences, and provide a complete characterisation of their ghost measures. These sequences can be viewed as weighted generalisations of the standard middle-thirds Cantor set—their ghost distributions are generalisations of the Devil's staircase. In Section 5 we apply our results to the Zaremba sequences by showing that any nontrivial Zaremba ghost measure is singular continuous with respect to Lebesgue measure; that is, its distribution function has almost everywhere zero derivative. Finally, in Section 6, we consider some subtleties of our approach.

3. Dilation equations, self-affine sets and ghost distributions

In this section we prove Theorem 2.1. To do this we provide an explicit relationship between the solution of a dilation equation and the attractor of an iterated function system of affine contractions.

Let $\rho = \rho(\mathbf{B})$ be the unique simple dominant eigenvalue of $\mathbf{B}$ and let $\mathbf{v}_\rho$ be the $d \times 1$ eigenvector associated to ρ. We define the vector-valued function $\mathbf{F} : \mathbb{R} \to \mathbb{R}^d$ by

$$\mathbf{F}(x) = \sum_{a=0}^{k-1} \rho^{-1}\,\mathbf{B}_a \cdot \mathbf{F}(kx-a) \quad \text{where } \mathbf{F}(x) = \begin{cases} \mathbf{0} & \text{for } x \leqslant 0 \\ \mathbf{v}_\rho & \text{for } x \geqslant 1. \end{cases} \tag{3.1}$$

Functional equations such as (3.1) are known as *dilation equations* or *two-scale difference equations* in the literature; seminal work on these was done by Daubechies and Lagarias [9, 10] and their relationship to regular sequences was initiated by Dumas [11, 12]. The function $\mathbf{F}$ exists and is unique if $\rho(\mathbf{B}) > \rho^*(\mathcal{B})$. Moreover, the function $\mathbf{F}$ is Hölder continuous with exponent α for any $\alpha < \log_k(\rho/\rho^*)$.

LEMMA 3.1. *Let $k \geqslant 2$ be an integer and f be a k-regular sequence where the spectral radius $\rho = \rho(\mathbf{B})$ is the unique dominant eigenvalue of $\mathbf{B}$ and $\rho(\mathbf{B}) > \rho^*(\mathcal{B})$. Let $\mathbf{F}(x)$ be the unique solution of (3.1). Then the graph*

$$\mathcal{F}_f := \{(x, \mathbf{F}(x)) : x \in [0,1]\} \subset [0,1]^{d+1}$$

of $\mathbf{F}(x)$ is a self-affine set. In particular, $\mathcal{F}_f$ is the attractor of the iterated function system $\mathcal{S}_f = \{S_0, \dots, S_{k-1}\} : [0,1]^{d+1} \to [0,1]^{d+1}$ where, for $j \in \{0, 1, \dots, k-1\}$,

$$S_j \begin{pmatrix} y_0 \\ \vdots \\ y_d \end{pmatrix} = \begin{pmatrix} 1/k & \mathbf{0}^{1\times(k-1)} \\ \mathbf{0}^{(k-1)\times 1} & \rho^{-1}\,\mathbf{B}_j \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_d \end{pmatrix} + \begin{pmatrix} j/k \\ \sum_{a=0}^{j-1} \rho^{-1}\,\mathbf{B}_a \mathbf{v}_\rho \end{pmatrix}.$$

Note that Lemma 3.1 is a two-way correspondence, providing a way to go back and forth between iterated function systems of contractions and solutions of dilation equations. As an example, here, as in the first section, we consider Salem's singular continuous function, with parameter $\lambda_0 = 2/5$, and denote Salem's self-affine set by $\mathcal{A}_s$, which is the unique attractor of the iterated function system $\mathcal{S}_s$ from (1.2). We can interpret the attractor $\mathcal{A}_s$ as the restriction to $[0,1]$ of the graph of the function $F_s : \mathbb{R} \to \mathbb{R}$ satisfying

$$F_s(x) = \frac{2}{5} \cdot F_s(2x) + \frac{3}{5} \cdot F_s(2x-1) \quad \text{where } F_s(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x \geqslant 1. \end{cases} \tag{3.2}$$

If F_s is the unique solution to the dilation equation (3.2), then we claim that the set $\{(x, F_s(x)) : x \in [0,1]\}$ is the attractor $\mathcal{A}_s$ of Salem's iterated function system $\mathcal{S}_s$. To see this, we write $\mathcal{F}_s := \{(x, F_s(x)) : x \in [0,1]\}$, and show that $\mathcal{S}(\mathcal{F}_s) = \mathcal{F}_s$. As a first step, we note that $(0,0) \in \mathcal{F}_s$ so that this set is not trivially empty. Second, we take $(x)_2 = .x_1x_2\cdots \in [0,1]$ and consider $(x, F_s(x))$. Now,

$$S_0 \begin{pmatrix} x \\ F_s(x) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2/5 \end{pmatrix} \begin{pmatrix} x \\ F_s(x) \end{pmatrix} = \begin{pmatrix} x/2 \\ (2/5)F_s(x) \end{pmatrix} = \begin{pmatrix} .0x_1x_2\cdots \\ (2/5)F_s(x) \end{pmatrix} = \begin{pmatrix} y \\ (2/5)F_s(2y) \end{pmatrix},$$

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where $y := .0x_1x_2\cdots$, so $y \in [0, 1/2]$ and $x = 2y$. Thus, $2y - 1 \leqslant 0$, so (3.2) gives

$$\tfrac{2}{5} \cdot F_s(2y) = \tfrac{2}{5} \cdot F_s(2y) + \tfrac{3}{5} \cdot F_s(2y-1) = F_s(y),$$

and so $S_0(\mathcal{F}_s) \subseteq \mathcal{F}_s$. Similarly,

$$\begin{aligned} S_1\begin{pmatrix} x \\ F_s(x) \end{pmatrix} &= \begin{pmatrix} 1/2 & 0 \\ 0 & 3/5 \end{pmatrix}\begin{pmatrix} x \\ F_s(x) \end{pmatrix} + \begin{pmatrix} 1/2 \\ 2/5 \end{pmatrix} \\ &= \begin{pmatrix} .1x_1x_2\cdots \\ (3/5)F_s(x) + 2/5 \end{pmatrix} = \begin{pmatrix} w \\ (3/5)F_s(2w-1) + 2/5 \end{pmatrix}, \end{aligned}$$

where $w := .1x_1x_2\cdots$, so $w \in [1/2, 1]$ and $x = 2w - 1$. Thus, $2w \geqslant 1$, so (3.2) gives

$$\tfrac{2}{5} + \tfrac{3}{5} \cdot F_s(2w-1) = \tfrac{2}{5} \cdot F_s(2w) + \tfrac{3}{5} \cdot F_s(2w-1) = F_s(w),$$

and so $S_1(\mathcal{F}_s) \subseteq \mathcal{F}_s$. Thus, $\mathcal{S}_s(\mathcal{F}_s) \subseteq \mathcal{F}_s$. The reverse inclusion follows by using the above equalities backwards.

The proof of the lemma follows similarly.

PROOF OF LEMMA 3.1. Let $(x)_k = .x_1x_2\cdots \in [0, 1]$ and consider $(x, \mathbf{F}(x))$. For $j \in \{0, 1\ldots, k-1\}$, we have

$$\begin{aligned} S_j\begin{pmatrix} x \\ \mathbf{F}(x) \end{pmatrix} &= \begin{pmatrix} 1/k & \mathbf{0}^{1\times(k-1)} \\ \mathbf{0}^{(k-1)\times 1} & \rho^{-1}\,\mathbf{B}_j \end{pmatrix}\begin{pmatrix} x \\ \mathbf{F}(x) \end{pmatrix} + \begin{pmatrix} j/k \\ \sum_{a=0}^{j-1}\rho^{-1}\,\mathbf{B}_a\mathbf{v}_\rho \end{pmatrix} \\ &= \begin{pmatrix} x/k + j/k \\ \rho^{-1}\,\mathbf{B}_j\mathbf{F}(x) + \sum_{a=0}^{j-1}\rho^{-1}\,\mathbf{B}_a\mathbf{v}_\rho \end{pmatrix} \\ &= \begin{pmatrix} .jx_1x_2\cdots \\ \rho^{-1}\,\mathbf{B}_j\mathbf{F}(k(.jx_1x_2\cdots) - j) + \sum_{a=0}^{j-1}\rho^{-1}\,\mathbf{B}_a\mathbf{v}_\rho \end{pmatrix}. \end{aligned}$$

To finish the proof, it is enough to show that

$$\mathbf{F}(.jx_1x_2\cdots) = \rho^{-1}\,\mathbf{B}_j\mathbf{F}(k(.jx_1x_2\cdots) - j) + \sum_{a=0}^{j-1}\rho^{-1}\,\mathbf{B}_a\mathbf{v}_\rho.$$

We can now compute $\mathbf{F}(.jx_1x_2\cdots)$ using (3.1). Since $y := .jx_1x_2\cdots \in [0, 1]$, we have $ky - a \geqslant 1$ for $a \in \{0, \ldots, j-1\}$, and $ky - a \leqslant 0$ for $a \in \{j+1, \ldots, k-1\}$. Thus, for this y, using the definition of $\mathbf{F}(x)$ outside of $(0, 1)$, we have $\mathbf{F}(ky - a) = \mathbf{F}(1) = \mathbf{v}_\rho$, for $a \in \{0, \ldots, j-1\}$, and $\mathbf{F}(ky - a) = \mathbf{0}$, for $a \in \{j+1, \ldots, k-1\}$. Thus,

$$\begin{aligned} \mathbf{F}(y) &= \sum_{a=0}^{k-1}\rho^{-1}\,\mathbf{B}_a\mathbf{F}(ky - a) \\ &= \sum_{a=j+1}^{k-1}\rho^{-1}\,\mathbf{B}_a\mathbf{F}(ky - a) + \rho^{-1}\,\mathbf{B}_j\mathbf{F}(ky - j) + \sum_{a=0}^{j-1}\rho^{-1}\,\mathbf{B}_a\mathbf{F}(ky - a) \end{aligned}$$

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$$= \mathbf{0} + \rho^{-1}\,\mathbf{B}_j\mathbf{F}(ky - j) + \sum_{a=0}^{j-1} \rho^{-1}\,\mathbf{B}_a\mathbf{F}(1)$$

$$= \rho^{-1}\,\mathbf{B}_j\mathbf{F}(ky - j) + \sum_{a=0}^{j-1} \rho^{-1}\,\mathbf{B}_a\mathbf{v}_\rho,$$

which shows that $\mathcal{S}_f(\mathcal{F}_f) \subseteq \mathcal{F}_f$. As above, the reverse inclusion follows by using the above equalities in the reverse direction. □

It turns out, for Salem's sequence s, that the ghost distribution of s is the attractor $\mathcal{A}_s$, which itself is precisely the solution F_s, of the dilation equation (3.2). In general, the relationship between the ghost distribution and the solution of the dilation equation is only slightly more complicated. A recent result, recorded below, makes this relationship between the regular sequence, the dilation equation and the attractor explicit; see Theorem 5 and its proof in Coons *et al.* [7].

THEOREM 3.2 (Coons, Evans and Mañibo). *Suppose f is a nonnegative k-regular sequence, $\rho(\mathbf{B})$ is the unique simple dominant eigenvalue of $\mathbf{B}$, $\rho^*(\mathcal{B}) < \rho(\mathbf{B})$ and there is a real number $c > 0$ such that $\Sigma(n) \sim c \cdot \rho(\mathbf{B})^n$ as n grows. If $\mathbf{F}(x)$ is the solution to (3.1), then*

$$\mu_f([0, x]) = \frac{\mathbf{e}_1^T\Big(\mathbf{F}\Big(\dfrac{1 + (k-1)x}{k}\Big) - \mathbf{F}\Big(\dfrac{1}{k}\Big)\Big)}{\mathbf{e}_1^T\Big(\mathbf{F}(1) - \mathbf{F}\Big(\dfrac{1}{k}\Big)\Big)}. \tag{3.3}$$

We now prove Theorem 2.1 by combining Theorem 3.2 with Lemma 3.1.

PROOF OF THEOREM 2.1. We have two cases depending on the distinctness of the digit matrices of f. Firstly, if all of the digit matrices are equal, then the sequence f is constant between powers of k, and so the ghost measure is just standard Lebesgue measure, whose distribution function is clearly self-affine. Secondly, suppose the digit matrices of f are not all equal. Combining the result of Lemma 3.1—that the solution of the related dilation equation is self-affine—we obtain the desired conclusion, noting that the relationship in (3.3) shows that the ghost distribution is the affine image of a section of the solution of a dilation equation. □

Figure 3 provides an illustration of Theorem 2.1 for the Zaremba sequence, z_2. Here, the corresponding functions $S_j\colon [0, 1]^3 \to [0, 1]^3$ are

$$S_0\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad S_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 1/4 \\ 0 & 1/4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 3/8 \\ 1/4 \end{pmatrix},$$

where the (normalised) digit matrices $\rho^{-1}\mathbf{B}_j$ appear in the lower block of the linear part of S_j.

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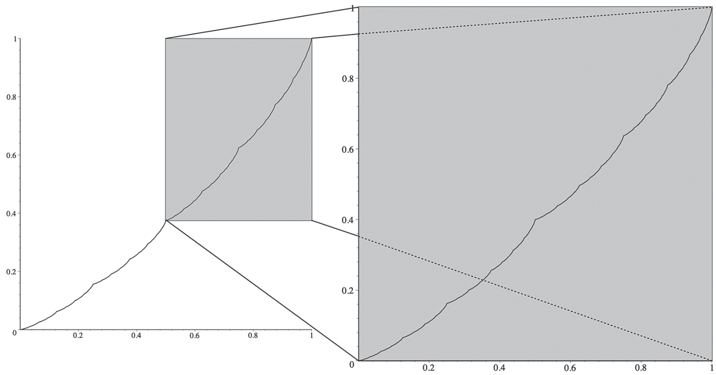

FIGURE 3. The function $Z_2(x)$ (right) is the affine image of a section of the 2-Zaremba attractor (left).

4. Characterisation of ghost distributions of Salem sequences

Here we characterise the spectral type of the ghost measures of one-dimensional regular sequences, that is, the Salem sequences.

It is clear that if the digits (nonnegative real numbers) of a nonzero Salem sequence are all equal then the resulting ghost measure is precisely Lebesgue measure. The result is quite different when the digits are not all equal. Indeed, Salem [20] showed that the ghost distribution of a positive 2-regular Salem sequence whose two digits are not equal is singular continuous; Salem's result and proof have also appeared in Billingsley [4, page 407]. The main result of this section is the following proposition.

PROPOSITION 4.1. *Let f be a k-regular Salem sequence with digits $b_0, b_1, \dots, b_{k-1}$ that are not all equal.*

(a) *If only b_0 is nonzero, then the ghost measure is the zero measure.*
(b) *If only one of the digits is nonzero and $b_0 = 0$, then the ghost measure is pure point.*
(c) *If more than one of the digits is nonzero, then the ghost measure is singular continuous.*

PROOF. Parts (a) and (b) are straightforward. If b_0 is the only nonzero digit, then since all base expansions of positive integers start with a digit other than zero, the Salem sequence is the zero sequence, and so its measure is the zero measure; this proves (a). If only one of the digits is nonzero, say b_j with $j \neq 0$, then the ghost measure is supported on the single number in $[0, 1]$ whose k-ary expansion is $0.jjjjj\cdots$, so that the ghost measure is equal to the delta measure $\delta_{j/(k-1)}$; this proves (b).

Case (c) splits into two cases. The first is also straightforward—the case where one of the digits is zero. If one of the digits is zero and at least two are not zero, then the ghost measure is supported on an uncountable set of measure zero: the set of numbers in $[0, 1]$ whose k-ary expansions contain only the k-ary digits j with $b_j \neq 0$.

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Thus, we can suppose that none of the b_j are zero. Using Theorem 2.1, to show that the ghost measure is singular continuous, it is enough to show that the associated Salem attractor, or solution to the related dilation equation, provided for in Lemma 3.1 is singular continuous, that is, the attractor of the iterated function system $\mathcal{S} := \{S_0, \ldots, S_{k-1}\}$ with

$$S_j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/k & 0 \\ 0 & b_j/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} j/k \\ \sum_{a=0}^{j-1} b_a/b \end{pmatrix},$$

where $b := b_0 + b_1 + \cdots + b_{k-1}$. We write the points of the attractor as $(x, S(x))$, and note that $S(x)$ is the solution to the dilation equation

$$S(x) = \frac{1}{b} \sum_{a=0}^{k-1} b_a \cdot S(kx - a) \quad \text{where } S(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x \geqslant 1. \end{cases}$$

We use an old argument of Salem involving simply normal numbers. Recall that a real number x is simply normal to the base k, provided each k-ary digit occurs in its base expansion with frequency $1/k$, and that Lebesgue-almost all numbers in $[0, 1]$ are simply normal to the base k. Let $x \in [0, 1]$ be a simply normal number to the base k with base-k expansion $(x)_k = 0.x_1x_2x_3 \cdots$. Note that for such x and any $j \in \{0, 1, \ldots, k-1\}$, the number of $x_i = j$ with i up to n is $n/k + o(n)$. Now, for each $n \geqslant 0$, set

$$y_n = x + \frac{b_{n+1}}{k^{n+1}} \quad \text{where } b_{n+1} = \begin{cases} 1 & \text{if } x_{n+1} = 0 \\ -1 & \text{otherwise.} \end{cases}$$

The first n digits in the k-ary expansion of y_n agree with those of x, thus

$$\begin{aligned} \left| \frac{S(x) - S(y_n)}{x - y_n} \right| &< k^{n+1} (b_{x_1}/b)(b_{x_2}/b) \cdots (b_{x_n}/b) \\ &\leqslant \left(\frac{k}{b} \prod_{j=0}^{k-1} b_j^{1/k} \right)^n \cdot k \cdot \left(\prod_{j=0}^{k-1} b_j \right)^{|r(n)|}, \end{aligned}$$

where $|r(n)| = o(n)$ as $n \to \infty$. Also, since S is continuous and increasing, the derivative of S exists almost everywhere. To finish our proof, it is enough to invoke the arithmetic-geometric mean inequality, to see that

$$\frac{k}{b} \prod_{j=0}^{k-1} b_j^{1/k} < \frac{k}{b} \left(\frac{b_0 + b_1 + \cdots + b_{k-1}}{k} \right) = 1,$$

so that $|S(x) - S(y_n)|/|x - y_n| \to 0$ as $n \to \infty$. □

The careful reader acquainted with so-called ‘missing-digit’ sets, such as the middle-third Cantor set, may recognise that the ghost measures of Salem sequences are precisely the generalisation of standard mass distributions supported on missing-digit

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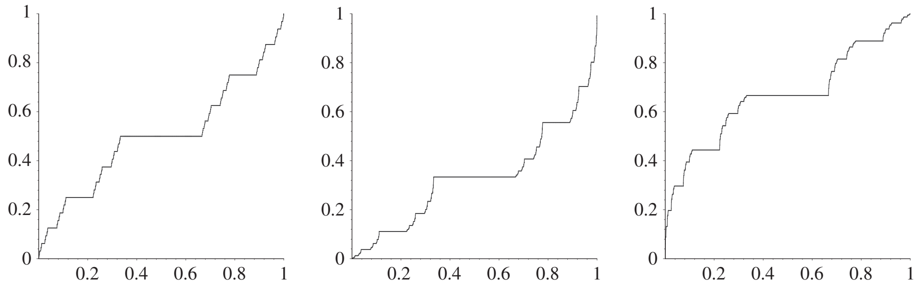

FIGURE 4. The Devil's staircase (left)—the ghost distribution of the standard Cantor middle-thirds set—along with the ghost distributions of the 3-regular Salem sequences with digits (b_0, b_1, b_2) equal to $(1, 0, 2)$ (middle) and $(2, 0, 1)$ (right).

sets. Ghost measures of Salem sequences are self-similar, that is, they satisfy the relation

$$\mu = \sum_{j=0}^{k-1} \frac{1}{k}(\mu \circ T_j^{-1}),$$

where $T_j \colon [0, 1] \to [0, 1]$ is the affine contraction $T_j(x) = (b_j/b)x + \sum_{a=0}^{j-1}(b_a/b)$; compare [14, Section 17.3]. For example, the standard Cantor measure, supported on real numbers $x \in [0, 1]$ with a ternary expansion not containing the digit 1, is the ghost measure of the 3-regular Salem sequence with $(b_0, b_1, b_2) = (1, 0, 1)$. Indeed, it follows from self-similarity that the measure μ must be spectrally pure, that is, it is either pure point, purely absolutely continuous or purely singular continuous; see Varju [21] for the case of Bernoulli convolutions. Unlike standard missing-digit distributions, corresponding to choosing only the digits 0 or 1 for our Salem sequence construction, in our context we can choose different weights, which then, by Proposition 4.1, give new singular continuous distributions; see Figure 4 for the standard Cantor distribution (the Devil's staircase) and two Salem sequence variants.

It is interesting to note that the Cantor distribution was one of the first examples of a singular continuous function, while Salem's contribution was 'to give simple, direct constructions of strictly increasing singular functions'. While at the time these seemed to be very different ideas, our definition of the Salem sequence shows in fact that these are just two examples of the same phenomenon. These functions seem to arise in many contexts, for example, in the study of Riesz products (see Benedetto, Bernstein and Konstantinidis [3, Figure 3]).

REMARK 4.2. Note that the Zaremba ghost measure is not self-similar, so, in particular, it is not the ghost measure of any Salem sequence. There are a host of ways to see this, the easiest of which is that the matrices in the affine maps in (1.1) do not share a nontrivial invariant subspace, so do not commute, and hence are not simultaneously diagonalisable.

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5. Singular continuity of Zaremba's ghost distributions

In this section we prove Theorem 2.2, and as a corollary that $Z_k(x)$ is singular continuous for every $k \geqslant 2$.

PROOF OF THEOREM 2.2. Since f is nonnegative, its ghost distribution and the related attractor are both increasing functions on $[0, 1]$, and so of bounded variation. Thus, they are almost everywhere differentiable. Additionally, since by Theorem 2.1 the ghost distribution of f is an affine section of the related attractor, to prove that the ghost distribution of f is a singular continuous function, it is enough to prove that the first two coordinates of the related attractor give the graph of a singular continuous function. We note that the attractor is continuous since $\rho = \rho(\mathbf{B}) > \rho^*(\mathcal{B})$.

Now, denote by $(x, S(x))$ the points of the curve defined by the first two coordinates of the solution of the related dilation equation or attractor. As in the proof of Theorem 4.1 above, we let $x \in [0, 1]$ be a simply normal number with base-k expansion

$$(x)_k = 0.x_1x_2x_3\cdots.$$

We remind the reader that, for such x and any $j \in \{0, 1, \ldots, k-1\}$, the number of $x_i = j$ with i up to n is $n/k + o(n)$. Now, for each $n \geqslant 0$, set

$$y_n = x + \frac{b_{n+1}}{k^{n+1}} \quad \text{where } b_{n+1} = \begin{cases} 1 & \text{if } x_{n+1} = 0 \\ -1 & \text{otherwise.} \end{cases}$$

The first n bits in the binary expansion of y_n agree with those of x, thus

$$\begin{aligned} \left|\frac{S(x) - S(y_n)}{x - y_n}\right| &< k^{n+1} \cdot |\mathbf{e}_1^T(\rho^{-1}\mathbf{B}_{x_1})(\rho^{-1}\mathbf{B}_{x_2})\cdots(\rho^{-1}\mathbf{B}_{x_n})\mathbf{v}_\rho| \\ &< (k\rho^{-1})^n \cdot \|\mathbf{B}_{x_1}\mathbf{B}_{x_2}\cdots\mathbf{B}_{x_n}\| \cdot k \cdot \|\mathbf{e}_1^T\| \cdot \|\mathbf{v}_\rho\| \\ &\leqslant \Big(k\rho^{-1}\prod_{j=0}^{k-1}\|\mathbf{B}_j\|^{1/k}\Big)^n \cdot k \cdot \|\mathbf{v}_\rho\|\Big(\prod_{j=0}^{k-1}\|\mathbf{B}_j\|\Big)^{|r(n)|}, \end{aligned}$$

where $|r(n)| = o(n)$ as $n \to \infty$, and, as it does throughout this paper, $\|\cdot\|$ denotes the standard Euclidean norm for a vector and the induced operator norm for a matrix. But since we have assumed that (2.1) holds, there is a constant $a < 1$ such that

$$\left|\frac{S(x) - S(y_n)}{x - y_n}\right| < c^n \cdot k \cdot \|\mathbf{v}\|\Big(\prod_{j=0}^{k-1}\|\mathbf{B}_j\|\Big)^{|r(n)|} = o(1).$$

Since the derivative of $S(x)$ exists almost everywhere, by the above, it is zero for Lebesgue-almost all $x \in [0, 1]$; that is, $S(x)$, and therefore $\mu_{z_k}([0, x])$, is singular continuous. □

Recall that

$$\rho = \rho(\mathbf{B}) = \rho(\mathbf{B}_0 + \mathbf{B}_1 + \cdots + \mathbf{B}_{k-1}) \leqslant \|\mathbf{B}_0 + \mathbf{B}_1 + \cdots + \mathbf{B}_{k-1}\|.$$

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Thus, using submultiplicativity of the operator norm for matrices, if (2.1) holds, it implies that

$$\|\mathbf{B}_0\mathbf{B}_1\cdots\mathbf{B}_{k-1}\|^{1/k} < \frac{1}{k}\cdot\|\mathbf{B}_0+\mathbf{B}_1+\cdots+\mathbf{B}_{k-1}\|,$$

which is exactly the arithmetic-geometric mean inequality for the operator norm of matrices. That is, (2.1) is a strong version of the arithmetic-geometric mean inequality (recall that our assumptions imply that the $\mathbf{B}_i$ are not all equal). As we shall see, in the case of the Zaremba sequence z_k such an inequality holds.

COROLLARY 5.1. *The ghost measure of the Zaremba sequence z_k is singular continuous.*

PROOF OF COROLLARY 5.1. A result of Coons *et al.* [7, Theorem 6] shows that the ghost measure of z_k is continuous. Now, since the matrices

$$\mathbf{B}_i = \begin{pmatrix} i+1 & 1 \\ 1 & 0 \end{pmatrix}$$

for the Zaremba sequence are all symmetric and nonnegative, the operator norm is equal to the largest eigenvalue. Thus,

$$\|\mathbf{B}_i\| = \frac{i+1+\sqrt{(i+1)^2+4}}{2} \quad\text{and}\quad \rho = \frac{k(k+1)+k\sqrt{(k+1)^2+16}}{4}.$$

Since $k \geqslant 2$,

$$\prod_{i=0}^{k-1}\|\mathbf{B}_i\|^{1/k} = \frac{1}{2}\prod_{i=1}^{k}(i+\sqrt{i^2+4})^{1/k} < \frac{(k+1)+\sqrt{(k+1)^2+16}}{4} = \frac{\rho}{k}.$$

An application of Theorem 2.2 finishes the proof. □

Note that (2.1) does not hold in general. It is not even true for strictly positive symmetric matrices. For a counterexample, note that

$$\left\|\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\|^{1/2}\left\|\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right\|^{1/2} = (3+\sqrt{5})^{1/2} > 2.288 > \frac{5+\sqrt{17}}{4} = \frac{1}{2}\cdot\rho\left(\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\right).$$

Under some natural assumptions on the matrices $\mathbf{B}_i$, one can associate a Lyapunov exponent $\chi_{\mathcal{B}}$ to the matrix semigroup $\langle\mathcal{B}\rangle$ given by $\chi_{\mathcal{B}} = \lim_{n\to\infty} n^{-1}\log\|\mathbf{B}_{i_{n-1}}\cdots\mathbf{B}_{i_0}\|$, which, by the seminal result of Furstenberg and Kesten [15], exists and is constant for almost every sequence $(i_n)_{n\geqslant 0} \in \{0,\ldots,k-1\}^{\mathbb{N}}$. Trivially, $\mathrm{e}^{\chi_{\mathcal{B}}} \leqslant \rho^*$. Another sufficient condition for the singularity of the ghost measure is that $\chi_{\mathcal{B}} < \log(\rho/k)$. We note, however, that Lyapunov exponents are difficult to compute in general—the conditions in Theorem 2.2 are comparatively easier to verify for concrete examples.

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6. Concluding remarks

In this paper, we connected the ghost measure of a regular sequence to an attractor of an iterated function system of affine contractions. We then proved that certain ghost measures are singular continuous by showing that the related attractor is a singular continuous curve. Our results rested on the set of given digit matrices satisfying a strong version of the arithmetic-geometric mean inequality (2.1). This is used to show that, in the fundamental region $[k^m, k^{m+1}r)$, most values of a regular sequence are smaller than the average value of the sequence. This means that there are only a small number of values that are pushing up the average. In fact, it is not hard to show that the maximal values of a regular sequence have density zero in each fundamental region.

The following proposition is a generalisation of a result of Coons and Spiegelhofer [8, Proposition 2.3.20] originally proved in the special case where f is the Stern sequence.

PROPOSITION 6.1. *Suppose f is a nonnegative k-regular sequence, $\rho(\mathbf{B})$ is the unique simple dominant eigenvalue of $\mathbf{B}$, $\rho/k < \rho^*(\mathcal{B}) < \rho(\mathbf{B})$ and there is a real number $c > 0$ such that $\Sigma(n) \sim c \cdot \rho(\mathbf{B})^n$ as n grows. For $m \geqslant 0$, let g_m be the function defined on* $[0, 1]$ *by*

$$g_m(x) = \frac{1}{(\rho^*)^m} \cdot f(k^m + \lfloor k^m(k-1)x \rfloor).$$

Then the sequence $\{g_m(x)\}_{m\geqslant 0}$ of functions converges to zero for λ-almost all x in $[0, 1]$.

PROOF. By assumption, the sum of f over the interval $[k^m, k^{m+1})$ is asymptotic to $c_1\rho^m$ for some $c_1 > 0$. Set $M_m := \max_{n\in[k^m,k^{m+1})} f(n)$. To prove the proposition, we need to show that there are exponentially few integers n in $[k^m, k^{m+1})$ such that $f(n) \geqslant \varepsilon M_m$, for any given $\varepsilon > 0$. By the nonnegativity of f, the number N of such integers satisfies $N\varepsilon M_m \leqslant c_1\rho^m$, therefore $N \leqslant c_1\rho^m/(M_m\varepsilon) \ll (\rho/\rho^*)^m/\varepsilon$. Since $\rho^* > \rho/k$, there are exponentially few integers n such that $f(n)$ is large; in particular, there is a $K < 1$ such that $\lambda(\{x \in [0, 1] : g_m(x) \geqslant \varepsilon\}) \leqslant K^m/\varepsilon$. It follows that

$$\begin{aligned}\lambda(\{x \in [0,1] : \exists m \geqslant M \text{ such that } g_m(x) \geqslant \varepsilon\}) &= \lambda\Big(\bigcup_{m\geqslant M}\{x \in [0,1] : g_m(x) \geqslant \varepsilon\}\Big)\\ &\leqslant \sum_{m\geqslant M} \lambda(\{x \in [0,1] : g_m(x) \geqslant \varepsilon\}) \leqslant \frac{1}{\varepsilon}\sum_{m\geqslant M} K^m = \frac{1}{\varepsilon}\cdot\frac{K^M}{1-K}.\end{aligned}$$

Thus,

$$\lambda(\{x \in [0,1] : g_m(x) < \varepsilon \text{ for all } m \geqslant M\}) \geqslant 1 - \frac{1}{\varepsilon}\cdot\frac{K^M}{1-K},$$

so that

$$1 = \lambda\Big(\bigcup_{M\geqslant 1}\{x \in [0,1] : g_m(x) < \varepsilon \text{ for all } m \geqslant M\}\Big) = \lambda(A_\varepsilon),$$

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where $A_\varepsilon = \{x \in [0,1] : \exists M \geqslant 1 \text{such that } g_m(x) < \varepsilon \text{ for all } m \geqslant M\}$. Therefore,

$$\lambda(\{x \in [0,1] : g_m(x) \to 0 \text{ as } m \to \infty\}) = \lambda\Big(\bigcap_{\varepsilon>0} A_\varepsilon\Big) = \lambda\Big(\bigcap_{n\geqslant 1} A_{1/n}\Big) = 1. \qquad \square$$

The following corollary is immediate.

COROLLARY 6.2. *Suppose* $\rho(\mathbf{B})$ *is the unique simple dominant eigenvalue of* $\mathbf{B}$ *and* $\rho/k < \rho^*(\mathcal{B}) < \rho(\mathbf{B})$. *Then for Lebesgue-almost all* $(x)_2 = 0.x_1x_2x_3 \cdots \in [0,1]$, *we have* $\lim_{n\to\infty}(\rho^*)^{-n} \cdot \|\mathbf{B}_{x_1}\mathbf{B}_{x_2}\cdots\mathbf{B}_{x_n}\| = 0$.

Unfortunately, Corollary 6.2 is not strong enough to replace (2.1).

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