Cardinals, Ordinals, and the Prospects for a Fregean Foundation

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Abstract
There are multiple formal characterizations of the natural numbers available. Despite being inter-derivable, they plausibly codify different possible applications of the naturals – doing basic arithmetic, counting, and ordering – as well as different philosophical conceptions of those numbers: structuralist, cardinal, and ordinal. Some influential philosophers of mathematics have argued for a non-egalitarian attitude according to which one of those characterizations is 'more basic' or 'more fundamental' than the others. This paper addresses two related issues. First, we review some of these non-egalitarian arguments, lay out a laundry list of different, legitimate, notions of relative priority, and suggest that these arguments plausibly employ different such notions. Secondly, we argue that given a metaphysical-cum-epistemological gloss suggested by Frege's foundationalist epistemology, the ordinals are plausibly more basic than the cardinals. This is just one orientation to relative priority one could take, however. Ultimately, we subscribe to an egalitarian attitude towards these formal characterizations: they are, in some sense, equally 'legitimate'.

Introduction

We are all familiar with the natural numbers. They may or may not start with zero, but they continue 1, 2, 3, and so on. There are, however, different formal characterizations of the natural numbers available, and these different characterizations plausibly codify different philosophical conceptions of those numbers: a cardinal conception, an ordinal conception, and a structuralist conception. What's more, various philosophers have defended one of these characterizations as being either in some sense uniquely correct, or else as being more basic or more fundamental than the others. And though much of this discussion has focused on the relative priority of the cardinal and structuralist conceptions, comparatively little attention has been paid the ordinal conception.

The purpose of this paper is four-fold. The first is to lay out these different formal characterizations of the natural numbers and connect them with different available conceptions within the philosophy of mathematics. The second is to briefly survey the arguments

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purporting to establish the relative priority of one of these conceptions. The third is to outline a laundry list of different, but seemingly legitimate, articulations of 'basicness' or 'fundamentality' that one might appeal to, including e.g. metaphysical, epistemological, developmental, and semantic. The ultimate purpose of this paper is to survey the relative priority of the cardinal and ordinal conceptions with respect to one of these potential articulations, based loosely on Gottlob Frege's notion of *proof*. We will tentatively suggest that, given this particular foundationalist orientation, ordinals are plausibly more basic than ordinals.

However, this should not be taken to taken to suggest that ordinals are more basic than cardinals in *every* legitimate sense. As mentioned, there are multiple, and in our view equally legitimate, interpretations of relative 'basicness'. What's more, it is quite plausible that cardinals are more basic than ordinals in some of these other senses. Thus, our aim here is not to completely adjudicate the debate between the aforementioned philosophers, as doing so would require similar examinations for each of the several legitimate senses we lay out. Nevertheless, this raises a couple interesting questions. First, which of the various conceptions of the natural numbers are more basic with respect to any of these potential glosses? Secondly, if in fact one of them is more basic in at least some of these senses, does this lend any credence to the claim that the corresponding conception of the natural numbers is correct, in some sense of 'correct'?

Ultimately, our view is that the various formal characterizations of the natural numbers encode different, but equally legitimate, conceptions of natural number, and indeed different potential empirical *applications* of those numbers. Thus, these formal characterizations can be seen as potential sharpenings, or disambiguations, of our everyday concept of number. Viewed this way, there is little temptation to adjudicate amongst the available conceptions in such a way that one is uniquely correct. Rather, there are simply different *jobs* we need the natural numbers to do – counting, ordering, and doing basic arithmetic – and how we choose to put the natural numbers to use will ultimately depend on want we want to achieve by using them. Obviously, this is something which can change, depending on our interests and goals. Thus, we defend what might be called egalitarianism with respect to the natural numbers: each of the available characterizations is in some sense 'correct', and there is no apparent need to choose one to the exclusion of the others.

The rest of the paper is organized as follows. Section 1 outlines three formal characterizations of the natural numbers and connects them to three corresponding philosophical conceptions. We also briefly provide our motivations for adopting an egalitarian attitude. Section 2 surveys two kinds of non-egalitarian arguments, those based on claims about what is essential to the natural numbers or natural number concepts, and those based on claims about relative basicness, fundamentality, or ground. We then provide a laundry list of different, legitimate glosses of relative 'basicness', 'fundamentality', or 'grounding' that these arguments could be appealing to in Section 3, noting that the considerations adduced plausibly support different interpretations, rather than a single, unequivocal notion. Finally, in Section 4, we sketch one of these potential glosses – a metaphysical/epistemological gloss naturally suggested by Frege's foundationalist epistemology – and argue that, *from this orientation*, ordinals are plausibly more basic than cardinals.

1. Characterizations and Conceptions of the Natural Numbers

How should the natural numbers be formally characterized? There are multiple such characterizations available, it turns out. Perhaps the most familiar is the so-called Dedekind–Peano (DP) Axioms, stated informally as:[1]

(DP1) Zero is a natural number.
(DP2) Each natural number has a unique successor.
(DP3) The successor relation is one-to-one.
(DP4) Zero is not the successor of any natural number.
(DP5) For any property *F*, if *F* holds of zero, and for any natural number *n*, if *F*'s holding of *n* implies that *F* holds of the successor of *n*, then *F* holds of all natural numbers.

Taken together, these characterize the natural numbers as forming an ω-sequence: there is an initial element (zero) followed by an infinite chain of distinct subsequent elements.

A second characterization is given by Crispin Wright[2] via an abstraction principle that he called $N^{=}$, but is now known as Hume's Principle (HP):

$$(HP)\forall F, G.\#[\lambda x.F(x)] = \#\left[\lambda y.G\left(y\right)\right] \leftrightarrow F \approx G$$

Here, '#' is a cardinality-operator mapping a concept φ to a cardinal number n representing the number of objects falling under φ, and '≈'

[1] It is possible to start with one instead of zero.
[2] C. Wright, *Frege's Conception of Numbers as Objects* (Aberdeen University Press, 1983)

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is the equivalence relation of equinumerosity holding between two concepts F and G just in case each F can be mapped to a unique G, and vice versa.[3]

Thus, HP states that two cardinal numbers are identical just in case they number equinumerous concepts. As is well known, it is possible to derive the DP axioms from HP using only suitable definitions and second-order logic, a result now known as Frege's Theorem.[4] One can thus go on to identify the natural numbers with the finite cardinals generated by HP, where the notion of finiteness is defined via Frege's definition of 'ancestor':[5] a natural number will be any cardinal which is an ancestor of zero under the successor relation.

A third, perhaps less familiar characterization of the natural numbers is offered by Øystein Linnebo,[6] via another abstraction principle':[7]

$$(2L\text{-}N)\,\forall n, n'.\forall R, R'. <n, R> = <n', R'> \leftrightarrow <n, R> \sim <n', R'>$$

Here, R and R' are discrete linear orderings, and '$\sim$' is an equivalence relation holding between two object-ordering pairs just in case the first object occurs in the same position with respect to its ordering as the second does with respect to its ordering.

To illustrate, consider the Arabic numeral '3' and the Roman numeral 'III'. Since both occur in the third position with respect to their canonical orderings, they name the same natural number, namely three.[8]

3 In modern notation, equinumerosity is defined as follows, where '$\exists!x$' translates as 'there is exactly one x such that …'.

$$\forall F, G\{F \approx G \leftrightarrow \exists R[(\forall x(F(x) \rightarrow \exists!y.R(x, y) \wedge G(y))) \wedge (\forall x.(G(x) \rightarrow \exists!y.R(x, y) \wedge F(y)))]\}$$

The first conjunct on the right-hand side of the biconditional states that R is many-to-one, while the second states that R is one-to-many. Thus, taken together, they state that a bijection holds between the Fs and the Gs.

4 R. Heck, *Frege's Theorem* (Clarendon Press, 2011).

5 G. Frege, *Grundlagen der Arithmetik* (1884).

6 Ø. Linnebo, 'The Individuation of the Natural Numbers' in O. Bueno and Ø. Linnebo (eds), *New Waves in Philosophy of Mathematics* (Palgrave-MacMillan, 2009), 220–238.

7 If left unrestricted, 2L-N is inconsistent, falling to the Burali-Forti paradox. Consequently, Linnebo restricts it to concrete relations among systems of numerals.

8 Here we characterize the (finite) ordinal numbers as applying to individual objects, with respect to a given ordering. It is common in mathematics, however, to define an ordinal to be the order-type of a well-ordering.

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As with HP, Linnebo goes on sketch a derivation of the DP axioms from 2L-N using suitable definitions and a combination of second-order logic and modal logic. He then identifies the natural numbers with finite pairs of numerals and orderings generated by 2L-N.

1.1. Three Conceptions of the Natural Numbers

Thus, we have three formal characterizations on the table. Plausibly, these codify three popular philosophical conceptions of the natural numbers. These are:

i. **The structuralist conception**: The natural numbers are places or positions within an ω-sequence.
ii. **The cardinal conception**: The natural numbers are finite cardinals answering 'how many'-questions.
iii. **The ordinal conception**: The natural numbers are finite ordinals answering questions about the position of objects within an ordering.

Defenders of the structuralist conception include Michael Resnik[9] and Stewart Shapiro.[10] According to these views, any set of objects forming an ω-sequence can play the role of natural numbers, including e.g. strokes in the series |, ||, |||, …, the arabic numerals, the

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This may be because well-orderings are at least one natural way to extend the typical finite orderings used in ordinary ordinal discourse, into the transfinite. In the official foundation for mathematics, Zermelo-Fraenkel set theory, ordinals are identified with pure, transitive sets that are well-ordered under the membership relation. These are typically called *von Neumann ordinals*. And cardinal numbers are identified with certain of the von Neumann ordinals, those that are not equinumerous with any smaller von Neumann ordinal. We will return to these foundational matters in the final section below.

Notice, incidentally, that in the sense of 2L-N, an ordinal is defined in terms of an object with respect to an ordering. So there must be such an object in order to get an ordinal at all. So the smallest ordinal, in that sense, is one (or 'first'). There is no zero ordinal. But there is a zero ordinal, in the mathematical sense. It is the order-type of an empty well-ordering, codified by the the empty set in set theory.

[9] M. Resnik, *Mathematics as a Science of Patterns* (Oxford University Press, 1997).

[10] S. Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford University Press, 1997).

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finite strings on any finite alphabet, in lexical order, and also the finite cardinals and the finite ordinals. The natural number structure is the form common to all of those ω-sequences. Because the DP Axioms characterize such a sequence, they plausibly codify the structuralist conception.

In contrast, Frege,[11] Hale and Wright,[12] and Neil Tennant[13] all defend the cardinal conception, whereby the natural numbers are finite cardinals characteristically answering questions like ‘How many Elmos are on the table?’.[14] Since HP relates the natural numbers to finite concepts that have the same cardinality, it typifies the cardinal conception.

Finally, Linnebo[15] explicitly defends the ordinal conception, which identifies the natural numbers with the finite ordinals, or the sorts of numbers answering questions like ‘In which place did Mary finish the race?’. This is encoded by 2L-N.

1.2. Three Applications of the natural numbers

In addition to codifying different popular conceptions of the natural numbers, the three formal characterizations above also plausibly encode three potential *applications* of these numbers, along with three different notions of number reflected in natural language. The first application is doing basic arithmetic, such as determining the truth of equations like $4 + 3 = 7$. This corresponds to *arithmetic* uses of number expressions like ‘four’ in (1a) and corresponding uses of the the noun ‘number’ in examples like (1b):

(1) a. Four is even.
    b. The number seven is prime.

This use of the natural numbers is most plausibly codified by the DP Axioms, as the latter underwrite the mathematical study of number theory.

[11] G. Frege, *Grundlagen der Arithmetik*.

[12] B. Hale and C. Wright, *The Reason’s Proper Study: Towards a Neo-Fregean Philosophy of Mathematics* (Oxford University Press, 2001).

[13] N. Tennant, *Anti-Realism and Logic: Truth as Eternal* (Clarendon Library of Logic and Philosophy, Oxford University Press, 1987); N. Tennant, ‘On the Necessary Existence of Numbers’, *Nous*, 31 (1997): 307–336.

[14] Though they characterize the natural numbers in different ways.

[15] Linnebo, ‘The Individuation of the Natural Numbers’.

The second application of the natural numbers is counting collections, and corresponds to cardinal uses of 'four' and 'number' such as Frege's (2a,b):[16]

(2) a. Jupiter has four moons.
 b. The number of Jupiter's moons is four.

This application is purportedly codified by HP.

The final application of the natural numbers is locating the position of an individual with respect to some linear ordering, and is reflected in ordinal adjectives like 'fourth' in (3a) and uses of 'number' like (3b).

(3) a. Mary is the fourth contestant.
 b. Mary is contestant number four.

Clearly, this application is captured by 2L-N.

1.3. Egalitarianism and Non-Egalitarianism

We want to highlight that the different number expressions in (1)–(3), as well as the different occurrences of 'number', have different, but arguably related, meanings. For example, 'four' is used as a numeral in (1a), as a cardinal adjective in (2a), and the ordinal adjective 'fourth' in (3a) is a different expression entirely.

Similarly, 'number' plausibly denotes a monadic predicate true of numbers in (1b), a relation holding between a collection and a cardinality in (2b), and a relation holding between a collection and an object within some contextually determined ordering of members of that collection. Despite plausibly having different meanings, it has been argued that the various uses of 'four(th)' and 'number' are semantically related in that they all implicitly or explicitly reference *numerals*.[17]

This largely underwrites our egalitarianism toward the different characterizations above. We start with the observation that the various characterizations of the natural numbers codify different potential applications, and that those applications are reflected in different meanings of number-related expressions. Just as it would be

16 G. Frege, *Grundlagen der Arithmetik*.

17 See Rothstein, *Semantics for Counting and Measuring* (Cambridge University Press, 2017), and Snyder, 'Numbers and Cadinalities: What's Really Wrong with the Easy Argument for Numbers', *Linguistics and Philosophy* 70 (2017): 373–400.

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bizarre to insist that one meaning of 'four' or 'number' is somehow correct to the exclusion of the others, we think it would be just as bizarre to insist that one of the formal characterizations listed above – the DP Axioms, HP, or 2L-N – is somehow correct to the exclusion of the others.

Nevertheless, we seem to be in the minority on this point since, as a matter of fact, philosophers have tended to take a *non*-egalitarian attitude towards these different characterizations. Indeed, as we will see in the next section, influential philosophers of mathematics have offered a variety of arguments purporting to show that one of these conceptions is either uniquely correct, or else more basic or more fundamental than the others.

2. Two Kinds of Non-Egalitarian Arguments

How might one motivate the claim that one of the characterizations of the natural numbers from the previous section is in some sense better than the others? Non-egalitarians have given a variety of arguments. However, the ones considered here can be grouped into two kinds: (i) those which appeal to what is essential to the natural numbers or natural number concepts, and (ii) those which appeal to the relative basicness, fundamentality, or ground of one conception over the others. We will survey both kinds of arguments.

2.1. Non-Egalitarian Arguments for the Cardinal Conception

The first sort of argument considered here appeals to what is essential to the natural numbers themselves, or to possessing natural number concepts. It is illustrated in the following passage from Bertrand Russell:

> It is obvious to common sense that two finite classes have the same number of terms if they are [equinumerous], but not otherwise. The act of counting consists in establishing a one-one correlation between the set of objects counted and the natural numbers (excluding 0) that are used up in the process. Accordingly common sense concludes that there are as many objects in the set to be counted as there are numbers up to the last number used in the counting …Hence it follows that the last number used in counting a collection is the number of terms in the collection, provided the collection is finite

> …[W]hat we do when we count (say) 10 objects is to show that the set of these objects is [equinumerous] to the set of numbers 1 to 10. The notion of [equinumerosity] is logically presupposed in the operation of counting, and is logically simpler though less familiar. In counting, it is necessary to take the objects counted in a certain order, as first, second, third, etc., but *order is not of the essence of number*: it is an irrelevant addition, an unnecessary complication from the logical point of view. The notion of [equinumerosity] does not demand order.[18]

Here, Russell begins by describing what Paul Benacerraf[19] would later call transitive counting, a procedure used for answering 'how many'-questions by establishing an isomorphism between a collection of objects and an initial segment of numerals standing for natural numbers.[20] For example, transitively counting four Elmos involves (i) isolating the Elmos, (ii) establishing a bijection between the latter and a sequence of numerals '1', '2', …, whose referents are the natural numbers, and (iii) answering the question 'How many Elmos are there?' with the terminal numeral in the sequence. In describing this procedure, Russell appeals to Frege,[21] who, as we have seen, had already shown how to define equinumerosity in purely (second-order) logical terms.

Russell argues that since equinumerosity does not presuppose a notion of order, order is 'an irrelevant addition', logically speaking, to the enterprise. So, if the essence of the natural numbers is somehow tied to their application in determining cardinality, order will not be essential, but equinumerosity will, thus recommending the cardinal conception over the ordinal conception.

But why think that the natural numbers are essentially tied to transitive counting? Bob Hale[22] has recently provided an answer, one which appeals to Frege's (Application) Constraint.[23] Roughly, this

[18] B. Russell, *Introduction to Mathematical Philosophy* (Dover Publications, 1919), 16–17, emphasis added.

[19] P. Benacerraf, 'What Numbers Could Not Be', *The Philosophical Review* 74(1) (1965): 47–73.

[20] Intransitive counting is the mere reciting of numerals, in order, without correlating them with objects.

[21] G. Frege, *Grundlagen der Arithmetik*.

[22] B. Hale, 'Definitions of Numbers and their Applications', in P. Ebert and M. Rossberg (eds), *Abstractionism* (Oxford University Press, 2016).

[23] See also C. Wright, 'Neo-Fregean Foundations for Real Analysis: Some Reflections on Frege's Constraint', *Notre Dame Journal of Formal Logic* 41 (2000): 317–334.

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states that the primary empirical applications of a class of mathematical objects ought be 'built directly into' their formal characterization. Hale argues for Frege's Constraint on the basis of the following contention: while someone equipped with just the DP Axioms, second-order logic, and appropriate definitions could do very basic arithmetic, they would not be able to answer 'how many'-questions like 'How many Elmos are on the table?' with answers like 'Four', and so would fail to possess even a basic grasp of natural number concepts. He concludes:

> …the fact that the natural numbers can be used to count collections of things is no mere accidental feature, *but is essential to them*. And if that is so, then a satisfactory definition of the natural numbers – a characterisation of what they essentially are – should reflect or incorporate that fact. And on the further assumption that elementary arithmetic is intended to be about the natural numbers, it further follows that a fully adequate philosophical account of arithmetic should provide a characterisation of the objects of the theory which not only permits a derivation of its basic laws, but also explains the general possibility of their use in counting.

Since, according to Hale, possessing natural number concepts requires an ability to transitively count, such an ability it is essential to possessing those concepts. Moreover, since a formal characterization of the natural numbers ought to reflect what they are essentially, this application ought to be built directly into that characterization, thus vindicating Frege's Constraint. Moreover, since HP presumably does this but the DP Axioms do not, we thus have a reason for preferring the cardinal conception over the structuralist conception. Of course, the same can be said about 2L-N, as it too fails to encode this essential application.

There are two important questions for Hale's argument. The first is this: Why think that counting is essential to possessing natural number concepts? To be sure, neither the DP Axioms nor 2L-N directly encodes this particular application, nor were they intended to do that. Nevertheless, they do directly encode what we take to be *possible* applications of the natural numbers – doing basic arithmetic and locating the position of an individual amongst some linearly ordered class, respectively. So why think that it is counting, as opposed to one of these other potential applications, that is required for possessing natural number concepts?

It seems that Hale's strategy ultimately relies on kind of intuition-mongering: those who find Hale's thought experiment persuasive may agree that counting is essential to possessing natural number concepts, but clearly this will not include those already attracted to the structuralist or the ordinal conceptions. They presumably would not agree that transitive counting is the most basic application of the natural numbers.

The second question for Hale's argument is this: *Does* HP actually encode this purportedly essential application? It is clear that HP permits certain sorts of answers to such 'how many'-questions, depending on whether collections are equinumerous. For example, it permits the answer 'The same number as the Grovers' if the Elmos and Grovers are equinumerous, or 'Not the same number as the Grovers' if they are not. But this does not constitute transitive counting, since neither answer involves a numeral. We have argued at length elsewhere that while the second-order resources available to Hale afford the capacity for forming numerals, and thus performing the transitive counting procedure, this will not substantiate Hale's intended conclusion, namely that an ability to perform that procedure constitutes a legitimate grasp of finite cardinal concepts, and thus the natural numbers on Hale's construal.[24] If so, then Frege's Constraint will not adjudicate between the cardinal, structuralist, or ordinal conceptions.

2.2. Non-Egalitarian Arguments for the Ordinal Conception

The second sort of non-egalitarian argument appeals to some notion of priority, e.g. relative basicness, fundamentality, or ground. It is illustrated by the following quote from Dummett:

> [Frege] assumed … that the most general application of the natural numbers is to give the cardinality of finite sets. The procedure of counting does not merely establish the cardinality of the set counted: it imposes a particular ordering upon it. It is natural to think this ordering irrelevant, since any two orderings of a finite set will have the same order type; but, if Frege had paid more attention to Cantor's work, he would have understood what it revealed, that *the notion of an ordinal number is more fundamental than that of a cardinal number*. This is true even in the finite case; after all, when we count the strokes of a clock, we are assigning an ordinal number rather than a cardinal. If Frege had

[24] E. Snyder, R. Samuels and S. Shapiro, 'Neologicism, Frege's Constraint, and the Frege-Heck Condition', *Nous* (*forthcoming*).

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> understood this, he would therefore have characterised the natural numbers as finite ordinals rather than as finite cardinals.[25]

Here, Dummett is plausibly appealing to Frege's Constraint, as his primary concern is with 'the most general application of the natural numbers'. Thus, he agrees with Hale that a formal characterization of the natural numbers ought to capture their primary empirical applications (see especially chapter 23 of Dummett's *Frege: Philosophy of Mathematics*). However, he disagrees with Hale regarding what that primary application is: ordering rather than counting. He also disagrees with Russell regarding the status of ordering within (transitive) counting: whereas Russell saw the need to order objects as 'an irrelevant addition … from the logical point of view', Dummett instead insists that the transitive counting procedure involves assigning an ordinal rather than cardinal number.[26]

This suggests that, according to Dummett, the primary application relevant to satisfying Frege's Constraint is not (transitive) counting, but rather locating the position of an object with respect to a given linear ordering. And this, of course, is precisely what Linnebo's 2L-N does:[27] it identifies natural numbers on the basis of whether numerals, and thus individuals associated with those numerals, occupy the same positions with respect to their canonical linear orderings.

This, however, raises a question similar to that broached above for Hale's account: Why think that ordering, as opposed to counting or doing basic arithmetic, is the *primary* empirical application of the

25 M. Dummett, *Frege: Philosophy of Mathematics* (Duckworth, 1991), 293, emphasis added.

26 It is curious that earlier in the same book, Dummett (*Frege: Philosophy of Mathematics*, 53), gives pride of place to the notion of *cardinal*:

> …what is constitutive of the number 3 is not its position in any progression whatever, or even in some particular progression, …but something more fundamental than any of these: the fact that, if certain objects are counted 'One, two, three' or, equally, 'Nought, one, two', then there are 3 of them. The point is so simple that it needs a sophisticated intellect to overlook it; and it shows Frege to have been right, as against Dedekind, to have made the use of the natural numbers as finite cardinals intrinsic to their characterisation.

Perhaps the proper exegetical conclusion to draw is that, for Dummett, the notion of ordinal is more fundamental than that of cardinal, but that the natural numbers are, after all, cardinal numbers.

27 Linnebo, 'The Individuation of the Natural Numbers'.

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natural numbers? Dummett suggestion seems to be that because ordering is necessary for counting, the former is more fundamental than the latter. But 'more fundamental' in what sense – epistemologically, metaphysically, cognitively, …? (Stay tuned, Section 3.) Also, unlike transitive counting, doing basic arithmetic does not appear to involve ordering collections of objects. So why think that ordering is more fundamental, in whatever sense is intended, than doing basic arithmetic?

Linnebo's argument for the relative priority of 2L-N does not rely on how the natural numbers are actually used. Rather, he puts forth a number of broadly empirical observations purporting to establish the relative priority of ordinal conception to the cardinal conception. In Linnebo's words (emphasis added):

> How are the natural numbers individuated? The views found in the literature naturally fall into two types: those that take the natural numbers to be individuated as cardinal numbers, and those that take them to be individuated as ordinal numbers. According to the former type of view, the natural numbers are individuated by the cardinalities of the concepts or the collections that they number.
>
> According to the competing view, the natural numbers are individuated by their ordinal properties, that is, by their position in the natural number sequence. For instance, *our most fundamental way of thinking* of the number 5 is as the fifth element of this sequence.

More specifically, Linnebo argues that if the cardinal conception were correct, then numbers like zero would not be 'special', since zero is just another cardinal. Yet zero was only admitted into actual mathematical practice relatively late in its history. Similar considerations apply to various infinites, of course: 'If our fundamental conception of a natural number had been of the form $\#F$, then infinite cardinals should have been much more obvious and natural than they in fact were'. Furthermore, Linnebo argues that the cardinal conception is inherently flawed because it, via HP, assumes that reference to numbers is modulated via definite descriptions, i.e. phrases of the form 'the number of *F*s'. However, Linnebo argues that this is mistaken: 'My claim is that the descriptions cannot serve as a fundamental mode of reference to numbers because they have an internal semantic articulation which presupposes some more basic form of reference to numbers'.

Such considerations lead Linnebo to the following conclusion:

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> I grant that the cardinal conception provides one possible way of thinking and talking about the natural numbers. But I deny that this is how we actually single out the natural numbers for reference *in our most basic arithmetical thought and reasoning*.

The (added) emphasis here is on 'most basic', or above 'most fundamental'. As with Dummett's argument, the relevant question here is: 'most basic' or 'most fundamental' in what sense? The last quote suggests a cognitive gloss: ordinals are prior to cardinals in our actual thought and reasoning. But how? Also, why think that historical considerations, e.g. that zero and the various infinities emerged relatively late in the history of mathematical practice, or for that matter questions of reference, have any bearing on numerical cognition?

3. Possible Interpretations of Relative Basicness: A Laundry List

Talk of relative basicness and fundamentality cover a multitude of different number-related issues. In this section, we develop an incomplete list of possible articulations of relative 'basicness' or 'fundamentality'. It will emerge that the various considerations mentioned in Section 2 appealing to these notions plausibly support different possible articulations. If so, then the right question is not whether the ordinal or cardinal conception is more basic or more fundamental *full stop*, but rather which of these various articulations is best supported by each conception. Thus, the upshot of this section is that, when making claims about priority, one needs to be very careful to articulate what precisely the question is.

In order to articulate relative 'basicness' or 'fundamentality', two important questions need to be addressed. At bottom, both notions involve some sort of relation of priority. Thus, the first question is about the intended relata: What are the *entities* we are ordering in terms of relative priority? There are different legitimate possibilities. Are we asking about kinds of *numbers*: ordinals and cardinals. Or kinds of *concepts*: the concept of cardinal, the concept of ordinal. Or kinds of *propositions* or *facts*: propositions or facts about cardinals, propositions or facts about ordinals. Or are we asking about *linguistic* kinds: various expressions, or their meanings?

In some contexts, perhaps, it makes sense to elide these distinctions, or to run them together. But in other contexts, e.g. in psychology and linguistics, it is decidedly unwise to ignore the differences between the kinds of relata.

The second question is about the relation itself: what kind of priority are we talking about? Again, there are different possibilities.

3.1. Temporal Priority Relations

i. **Temporal Priority**: X is *temporally prior* to Y if x precedes Y in time.

We can further distinguish between various possibilities by taking into account different time-scales. For example, we get one notion of priority by considering the time-scale of a species.

ia. **Phylogenetic Priority**: X is *phylogentically prior* to Y if X precedes Y in the development of a species.

Relevant priority questions here might include those in (4):

(4) a. Which came first in the evolution of humans, walking or talking?
  b. Which came first, Australopithecus or Neanderthal?

We get a different temporal priority relation by instead focusing on the timescales of particular organisms:

ib. **Ontogenetic Priority**: X is *ontologenetically prior* to Y if X precedes Y in the development of an organism.

Relevant priority questions here might include those in (5):

(5) a. Which develops first in an embryo, limbs or teeth?
  b. Which comes first in a child's numerical cognitive development, counting or doing basic arithmetic?

As (5b) suggests, research within developmental psychology establishes claims about ontogenetic priority. For example, facts about the developmental sequence can be understood this way: roughly, children first learn to intransitively count, then transitively count, and then, typically much later on, manipulate arithmetic symbols to solve basic arithmetic equations such as $3 + 2 = 5$.

Finally, we get a different notion of temporal priority by considering the time-scale of a culture or society.

ic. **Cultural-Historical Priority**: X is *cultural-historically prior* to Y if X precedes Y in the development of a culture or society.

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Again, relevant priority questions here might include (6a,b).

(6) a. Which came first in American history, the Women's Rights Movement or the Civil Rights Movement?
b. Which came first in the history of mathematics, zero or one?

As (6a) suggests, Linnebo's argument from 'special numbers'[28] – if the cardinal conception were correct, then zero or various infinities would not be special numbers, contrary to fact – is most plausibly understood as establishing, if successful, the Cultural-Historical priority of the ordinal conception.

3.2. Dependency Priority Relations

We can also think of priority in terms of dependency-relations.

ii. **Dependence Priority**: X is *dependency prior* to Y if Y depends on X.

Again, there are several possibilities, resulting from different notions of dependency. For example, one results from taking it to be that of cause and effect.

iia. **Causal Priority**: X is *causally dependent* on Y if Y is a cause of X.

Relevant priority questions here might include those in (7).

(7) a. What causes the flu?
b. What caused the extinction of the dinosaurs?

This overlaps with our first category, as causes typically precede their effects in time. As above, this distinction makes sense for number concepts and number words, but not, we suppose, for numbers *per se*. As abstract objects, numbers do not cause anything.

Applied to cognitive development, we can ask if cardinal number *concepts* are causally implicated in the acquisition of ordinal number concepts, whether it is the other way around, or whether the two are causally independent.

There may be some non-causal dependency relations. For example, it might be that various concepts are in some ways arranged in a non-causal priority hierarchy. Perhaps one kind of concept in some way

[28] Linnebo, 'The Individuation of the Natural Numbers'.

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grounds another kind of concept. For example, if concepts have compositional structure, then one kind of concept might be more basic than another because the former is a constituent of the latter. This leads to a different kind of dependency priority.

iib. **Conceptual Grounding**: *X conceptually depends on (or is conceptually grounded by) Y* if *X* is part of *Y* in the conceptual hierarchy.

Hence, potentially relevant questions here might include those in (8).

(8) a. Does the concept man ground the concept bachelor?
b. Does the concept cardinal number ground the concept ordinal number?

As (8b) suggests, with this notion one may ask whether cardinal concepts are grounded in ordinal concepts, or vice versa, or whether neither is conceptually prior to the other. This is one possible interpretation Linnebo's conclusion from above, repeated here for convenience:

> I grant that the cardinal conception provides one possible way of thinking and talking about the natural numbers. But I deny that this is how we actually single out the natural numbers for reference *in our most basic arithmetical thought and reasoning*.

On this interpretation, cardinal concepts are in some sense part of ordinal concepts within the conceptual hierarchy.

3.3. Semantic Priority Relations

It also possible to distinguish different kinds of semantic priority relations. For example, consider polymorphic expressions, or expressions which take on different semantic types, and thus meanings, in different syntactic environments. Familiar examples here include conjunctions like 'and', names such as 'Mary', and indeed number expressions like 'four'. It is commonly assumed among linguists that these expressions take on different meanings via type-shifting, i.e. shifting from the *basic* type to a different type through the application of a certain type-shifting principle.[29] This naturally suggests a kind of semantic priority relation between meanings of polymorphic expressions:

[29] For classic discussions of type-shifting with respect to conjunctions and names, see B. Partee and M. Rooth, 'Generalized Conjunction and

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iiia. **Semantic Priority**: For any polymorphic expression α and meanings m and m' of α , m is *semantically prior* to m' if m' is derived from m via type-shifting.

Relevant priority questions here might include those in (9).

(9) a. Which meaning of 'Mary' is more basic, its referential or non-referential meaning?
 b. Which meaning of 'four' is more basic, its referential or non-referential meaning?

Questions like (9a) are potentially important for recent debates within both linguistics and the philosophy of language. For example, some have argued based on examples like (10) that names are actually predicates rather than singular terms:[30]

(10) a. I saw three Marys at the mall.
 b. Every Mary I know is dating a John.

On the other hand, if one of these meaning is the result of type-shifting, then there is hardly any motivation to claim that names are *exclusively* singular terms or predicative count nouns.[31] Nevertheless, it would remain a genuinely interesting question whether the

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Type Ambiguity', in R. Bauerle, C. Schwarze and A. von Stechow (eds), *Meaning, Use, and Interpretation of Language* (De Gruyter, 1983), 361–383; B. Partee, 'Ambiguous Pseudoclefts with Unambiguous *Be*', in S. Bergman, J. Choe and J. McDonough (eds) *Proceedings of the Northwestern Linguistics Society* 16 (GLSA, 1986). And, for discussions relevant to number expressions, see e.g. B. Geurts, 'Take "Five", in S. Vogleer, and L. Tasmowski (eds), *Non-Definiteness and Plurality* (Benjamins, 2006), 311–329; E. Snyder, 'Numbers and Cardinalities: What's Really Wrong with the Easy Argument?', *Linguistics and Philosophy* 40 (2017): 373–400.

30 See, for example, O. Matushansky, 'Why Rose is the Rose: On the Use of Definite Articles in Proper Names', *Empirical Issues in Syntax and Semantics*, 6 (2006): 285–307; and D. G. Fara, 'Names are predicates' *Philosophical Review* 124(1) (2015): 59–117.

31 Especially considering that names take on a interesting variety of uses *beyond* those witnessed in (10). Consider those in (i), for instance.

(i) a. He was part of the Obama election team.
 b. Let's Skype tomorrow.
 c. How much Rover is on the road?

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predicative meaning is *semantically prior to* the referential meaning, or vice versa.

Similarly, questions like (9b) are important for the philosophy of mathematics. After all, as Frege[32] originally observed, 'four' appears to have both referential (11a) and non-referential uses (11b):

(11) a. The number of Jupiter's moons is four.
b. Jupiter has four moons.

Some influential philosophers have suggested that number expressions are exclusively singular terms, while others more recently have argued that they are exclusively non-referential expressions.[33] On the other hand, if meanings appropriate for either use arise instead via type-shifting, as others have argued,[34] then there would appear to be hardly any linguistic motivation for thinking that the meaning of 'four' must be *exclusively* one or the other. Nevertheless, it would remain a genuinely interesting question of which of these meanings is *semantically prior to* the other. In which direction does the type-shifting go?

One could also, perhaps, define a *metasemantic* notion of semantic priority, one which appeals to relative priority with respect to how the referents of expressions are determined. Indeed, something like the following is presupposed in Linnebo's argument from definite descriptions:

iiib. **Metasemantic Priority**: A mode of referring X is *metasemantically prior* to Y if X involves less 'semantic articulation' than Y.

As we saw, Linnebo argues that definite descriptions cannot 'serve as a fundamental mode of reference to numbers because they have an internal semantic articulation which presupposes some more basic

32 Frege, *Grundlagen der Arithmetik*.

33 For example, Wright (*Frege's Conception of Numbers as Objects* (Aberdeen University Press, 1983)) is plausibly understood as defending the former view, while Hofweber (*Ontology and the Ambitions of Metaphysics* (Oxford University Press, 2016)) and Moltmann ('Reference to Numbers in Natural Language', *Philosophical Studies* 162 (2013): 499–536) have recently defended the latter.

34 See e.g. C. Kennedy, 'A Scalar Semantics for Scalar Readings of Number Words' in I. Caponigro and C. Cecchetto (eds) *From Grammar to Meaning: the Spontaneous Logicality of Language* (Cambridge University Press, 2013), 172–200; Snyder, 'Numbers and Cardinalities: What's Really Wrong with the Easy Argument?'.

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form of reference to numbers'. The idea seems to be that in order to understand a definite description like 'the tallest man in the room', one must first be able to understand its semantic components, whereas numerals 'are semantically simple expressions with no internal semantic articulation'. It is not entirely evident to us what this notion of 'semantic articulation' amounts to. Nevertheless, it is clearly meant to establish names over definite descriptions as fundamental modes of reference, and thus some kind of priority relation between different modes of reference.

The priority relations articulated above are largely empirical; most are in the province of developmental psychology or natural language semantics. We now turn to more *a priori* matters.

3.4. Metaphysical Priority Relations

Metaphysicians have articulated a number of priority relations:

iv. **Metaphysical Priority**: X is *metaphysically prior to* Y if X constitutes or grounds or realizes or supervenes on or is more fundamental than or … Y .

In contrast with some of the above distinctions, these relations are typically synchronic. Moreover, metaphysicians have proposed competing notions of *ground*. For some, e.g. Fine[35] and Sider,[36] grounding is a relation among *facts*. For others, e.g. Schaffer,[37] grounding is relation holding between objects.[38]

Thus, one question we could ask is whether facts about cardinals are grounded in facts about ordinals, or vice versa. Or are both grounded in other kinds of facts? Another kind of question we could ask is whether cardinal numbers are grounded in ordinal numbers, or vice versa.

[35] K. Fine, 'Guide to Ground', in F. Correia and B. Schnieder (eds) *Metaphysical Grounding* (Cambridge University Press, 2012), 37–80.

[36] T. Sider, *Writing the Book of the World* (Oxford University Press, 2013).

[37] J. Schaffer, 'On What Grounds What', in D. Manley, D. J. Chalmers and R. Wasserman (eds), *Metametaphysics: New Essays on the Foundations of Ontology* (Oxford University Press, 2009), 347–383.

[38] Though Sider (*Writing the Book of the World*) also speaks of entities as being fundamental, or as being more fundamental than others.

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It is possible, though far from certain, that Dummett[39] had one of these notions in mind when charging Frege with missing the notion of ordinal. To repeat part of the quote from above:

> The procedure of counting does not merely establish the cardinality of the set counted: it imposes a particular ordering upon it. It is natural to think this ordering irrelevant, since any two orderings of a finite set will have the same order type; but, if Frege had paid more attention to Cantor's work, he would have understood what it revealed, that *the notion of an ordinal number is more fundamental than that of a cardinal number* …If Frege had understood this, he would therefore have characterised the natural numbers as finite ordinals rather than as finite cardinals.

Given Dummett's remark about ignoring Cantor's work, it is possible that Dummett is appealing here to how the ordinals are defined within modern mathematics, as order-types of well-ordered sets – ultimately as certain sets, the von Neumann ordinals. According to Cantor, we get from an ordinal to a cardinal by ignoring the order, focusing only on the size of the collection. So, one possible interpretation of Dummett's argument is that because facts about ordinals ground facts about cardinals, Frege should have began with the finite ordinals and defined the cardinals from those.

3.5. Epistemic Priority Relations

The final sort of priority relation considered here stems from foundationalist epistemologies, whereby propositions form a kind of justificatory hierarchy, with the base of the hierarchy forming the foundation – the ultimate justification for knowledge.

v. **Epistemic Priority**: X is epistemically prior to Y if X justifies Y within a foundationalist epistemology.

Thus, potentially relevant priority questions here include those in (12):

(12) a. Does the claim that it's raining and I'm carrying my umbrella justify the claim that it's raining?
b. Does the claim that Rover lives next door justify the claim that at least one dog lives next door?

Similarly, we can ask whether propositions about ordinals are properly (or canonically, or foundationally) justified in terms of

[39] Dummett, *Frege: Philosophy of Mathematics*.

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propositions about cardinals, or vice versa. Or is neither kind of proposition Properly justified in terms of the other?

4. Cardinals, Ordinals, and the Prospects for a Fregean Foundation

To summarize our conclusions up to this point, we have seen that there are formal characterizations of the natural numbers available, and that these various characterizations correspond to different available conceptions of those numbers: a cardinal conception, an ordinal conception, and a structuralist conception. Moreover, we have seen that defenders of some of those conceptions have offered various non-egalitarian arguments purporting to show that one of them is either uniquely correct, or else more basic or more fundamental than the others. And we have seen that there are different plausible articulations of 'more basic' or 'more fundamental' available; the aforementioned non-egalitarian considerations plausibly support different articulations of those notions.

In this concluding section, we develop one metaphysical/epistemic foundationalist framework, inspired by (one reading of) Frege's central works, and show how our question about the relative basicness of cardinals and ordinals plays out under this interpretation. Perhaps this is of independent historical or philosophical interest.

To reemphasize one of our primary conclusions, our arguments here should not be understood as supporting the claim that one of these conceptions is more basic than the other *full stop*. Rather, they should be taken to show that given these particular interpretations of relative 'basicness', and given Frege's foundationalist orientation, there are good reasons for thinking that ordinals are more basic than cardinals. Nevertheless, as above, we maintain that both corresponding conceptions – the cardinal and ordinal – are equally legitimate, in keeping with our overall egalitarian attitude.

4.1. Proof and Justification in Frege

At the beginning of the *Grundlagen*, Frege observes that 'it is in the nature of mathematics to prefer proof, where proof is possible',[40] noting that 'Euclid gives proofs of many things which anyone would concede him without question'. Frege goes on to tell us *why*

[40] Frege, *Grundlagen der Arithmetik*, §2.

it is that it is that mathematicians 'prefer proof, where proof is possible':

> The aim of proof is, in fact, not merely to place the truth of the proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is unmoveable, …there remains the further question, what is it that supports it so securely?

To unpack the metaphor, Frege believed that true or at least knowable propositions have dependence relations to one another. These relations are objective, in the sense that it is not a matter of how some person or other comes to discover or believe a given proposition, or even of how some person or other comes to know the proposition.

Rather, it is a matter of what the truth of the proposition *rests upon*:

> …we are concerned here not with the way in which [the laws of number] are discovered but with the kind of ground on which their proof rests; or in Leibniz's words, 'the question here is not one of the history of our discoveries, which is different in different men, but of the connection and natural order of truths, which is always the same'[41]

Like Bolzano's ground-consequence relation,[42] Frege's dependency relation is asymmetric: if proposition A depends on proposition B, then B does not depend on A. It follows that the relation is not reflexive: no proposition grounds itself. Presumably, the relation is transitive.

Frege's account of the notions of *analyticity* and *a priority* are formulated in terms of these dependency relations:

> [T]hese distinctions between *a priori* and *a posteriori*, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement. … When …a proposition is called *a posteriori* or analytic in my sense, this is not a judgement about the conditions, psychological, physiological, and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgement about the way in which some other man has come …to believe it true; rather it is a judgement about the ultimate

[41] Frege, *Grundlagen der Arithmetik*, §17; G.W. Leibniz, et al., '*Nouveaux Essais Sur l'Entendement Humain: Avantpropos et Premier Livre*' (Belin, 1885), §9.

[42] B. Bolzano, 'Theory of Science', trans. by R. George (University of Berkeley, 1837).

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> ground upon which rests the justification for holding it to be true.
>
> This means that the question is removed from the sphere of psychology, and assigned, if the truth concerned is a mathematical one, to the sphere of mathematics. The problem becomes …that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one …If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some general science, then the proposition is a synthetic one. For a truth to be *a posteriori*, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general …But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is *a priori*.[43]

It seems to us that despite the use of terms like 'proof' and 'justification' here, Frege's relation of dependence is as much metaphysical as it is epistemic. In terms of the taxonomy of the previous section, it is just as much at home under a metaphysical gloss as under an epistemological gloss.

For one thing, Frege is explicit that his dependency relation here has nothing to do with how individual people come to believe or even know propositions. Presumably, it is also not a matter of *whether* we know, for example, that $3 + 4 = 7$, or that every number has a successor. Those propositions were known long before the foundational work began.

Moreover, for most of us, this knowledge need not, and in fact did not, go via the proposed founding definitions. Frege's dependency relationship thus seems to require a distinction between the state of knowing, or the state of being justified, and the ultimate or objective *ground* or *justification* of a proposition. His foundational framework concerns the latter.

4.2. Proper Foundational Knowledge

Let us call knowledge that is based on objective grounding relations among the known propositions *proper foundational knowledge*.

43 G. Frege, *Grundlagen der Arithmetik*, 1884, §3.

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In Aristotelian terms, if one has proper foundational knowledge of a mathematical proposition *p*, then one has an (or the) *explanation* of *why p* is true.[44]

When we have proper foundational knowledge, and are aware that we do, we then know *why* the boulder cannot be moved. Crucially for Frege, proper foundational knowledge is needed to determine whether a given proposition is analytic or *a priori*, since those notions concern the proposition's metaphysical-cum-epistemic pedigree. One of the purposes of Frege's logicism was to demonstrate that arithmetic and analysis are analytic, in his sense of that term.

Robin Jeshion,[45] sums up the aims of Frege's logicism as follows:[46]

> Euclidean rationale: Frege thought that primitive truths of mathematics have two properties. (i) They are *selbstversẗandlich*: foundationally secure, yet are not grounded on any other truth, and, as such, do not stand in need of proof. (ii) And they are self-evident: clearly grasping them is a sufficient and compelling basis for recognizing their truth. He also thought that the relations of epistemic justification in science mirrors the natural ordering of truths: in particular, what is self-evident is *selbstversẗandlich*. Finding many propositions of arithmetic non-self-evident, Frege concluded that they stand in need of proof.

The relevant notion of 'self-evidence' here is not obviousness, not a mere subjective feeling of certainty. Reliance on obviousness would smack of the psychologism that Frege so vehemently opposed. Moreover, there are obvious propositions, such as $2 + 3 = 5$, that are not self-evident. Frege emphasized that one must *reason* one's way to this knowledge.

44 Aristotle, (*Physics*, Chapter 3): 'Men do not think they know a thing unless they have grasped the 'why' of it'. So proper foundational knowledge is of-a-piece with what Jaegwon Kim calls 'explanatory knowledge' – knowledge why – as opposed to mere descriptive knowledge – knowledge that (J. Kim, 'Explanatory Knowledge and Metaphysical Dependence', *Philosophical Issues* 5 (1994), 51). However, Joshua Schechter (p.c.) suggested that what we call 'proper foundational knowledge' may not be a special kind of knowledge at all: it is to know *and* to have an (or the) explanation for what one knows. That would make the Fregean hierarchy purely metaphysical, and in no way epistemic. There is no need to settle this matter of classification/exegesis here.

45 R. Jeshion, 'Frege's Notions of Self-Evidence', *Mind* 110(440) (2001), 944.

46 R. Jeshion, 'Frege: Evidence for Self-Evidence', *Mind* 113(449) (2004): 131–138.

Conversely, there are, or at least could be, self-evident propositions that, at least at first, are not obvious. Even before he learned of Russell's Paradox, Frege conceded that his Basic Law V is not obvious. He wrote that he had 'never concealed' from himself Basic Law V's 'lack of self-evidence which the others possess, and which must properly be demanded of a law of logic'.[47] In motivating his own system, he writes:

> If we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based. A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not yet expressly enunciated …Yet I hold that it is a law of pure logic.[48]

In retrospect, this is a most ironic passage.

For present purposes, the most compelling question is how one manages to obtain proper foundational knowledge. How do we figure out *that* a given proposition is to be justified, or explained, in terms of another one? Jeshion,[49] puts the issue well, with respect to the foregoing interpretation of Frege:

> As he was acutely aware of the possibility of errors resulting from conceptual understanding, Frege regarded reliance on obviousness as insufficient for identifying …primitive truths. As Frege noted, we are not given concepts 'in their pure form' (Frege 1884, vii). Our partial or incorrect understanding of concepts results in mistaken judgements. Such errors are not recognized 'from within' as, perhaps, are mistakes from inattention, sloppiness, or haste in judgement. And they are not remedied merely (!) by exercising control on one's thought, as, perhaps, are the others. The mistakes in question sometimes occur even when exercising tightest control on our intellection.

To focus on the case at hand, how do we determine whether propositions about cardinals are based on propositions about ordinals, or the other way around? Or perhaps both are based on something else equally, or, implausibly, they have different bases that are not related to each other?

[47] G. Frege, *Grundgesetze der Arithmetik* vol. II (Olms, 1903), 253.
[48] G. Frege, *Grundgesetze der Arithmetik*, vol. I (H. Pohle, 1893), vii.
[49] Jeshion, 'Frege's Notions of Self-Evidence', 967.

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4.3. Holism and Systematization

Apparently, Frege did not say in much detail about how we properly come to know the starting points, nor about how we know that putative starting points *are* starting points. Nor do we have much guidance on how to determine, in a given case, just what is based on what, in the indicated hierarchy. So we must get speculative.

Frege,[50] praised the goal of *organizing* mathematical knowledge, presumably in a way that reflects the objective grounding relations:

> The essence of mathematics has to be defined from [a] kernel of truths, and until we have learnt what these primitive truths are, we cannot be clear about the nature of mathematics. If we assume that we have succeeded in discovering these primitive truths, and that mathematics has been developed from them, then it will appear as a system of truths that are connected to one another by logical inference.
>
> Euclid had an inclination of this idea of a system; but he failed to realize it and it almost seems as if at the present time we were further from this goal than ever. We see mathematicians each pursuing their own work on some fragment of the subject, but these fragments do not fit together into a system; indeed, the idea of a system seems almost to have been lost. And yet the striving for a system is a justified one. We cannot long remain content with the fragmentation that prevails at present. Order can only be created by a system …
>
> … we must avoid such expressions as 'a moment's reflection shows that' or 'as we can easily see'. We must put the moment's reflection into words so that we can see what inferences it consists of and what premises it makes use of. In mathematics we must never rest content with the fact that something is obvious or that we are convinced of something, but we must seek to obtain a clear insight into the network of inferences that support our conviction. Only in this way can we discover what the primitive truths are.

There is no doubt that Frege's framework is ultimately a foundationalist enterprise. Nevertheless, Tyler Burge[51] suggests that at some

[50] G. Frege, 'Logik in der Mathematik', *Nachgelassene Schriften* (1914), 205.

[51] T. Burge, 'Frege on Knowing the Foundation', *Mind* 107(426) (1998), 328.

level, Frege's methodology is holistic, writing that 'in arguing for his logic [Frege] made use of methods that were explicitly pragmatic and contextualist …':

> In *Basic Laws* (1893) we find Frege recommending to those who are sceptical of his logical system that they get to know it from the inside. He thinks that familiarity with the proofs themselves will engender more confidence in his basic principles … In the Introduction to *Basic Laws*, Frege repeatedly appeals to advantages, to simplicity, and to the power of his axioms in producing proofs of widely recognized mathematical principles, as recommendations of his logical axioms.

Jeshion also sounds a holistic theme.[52] According to her reading, we come to know that a given proposition is *selbstversẗandlich*, at the bottom of the hierarchy, by examining its role in a carefully worked out scheme of knowledge. She has Frege

> advocating the sane view that what seems obvious may require proof and that obviousness needs supplementation by systematization. To identify a proposition as not needing proof … we need to systematize our knowledge and see whether the proposition can fulfill the role of an axiom within an ideal Euclidean system of mathematics. It does so by being fruitful, by enabling the derivation of all known mathematical knowledge and by affording means of generating more. It must also satisfy the traditional rationalist goals of surveyability, simplicity, economy, and unificatory power.

The idea here seems to be that a good systematization counts as at least defeasible, *prima facie* evidence for the foundational relationships. It tells us, or might tell us, what is grounded, or 'justified', or explained in terms of what.

As suggested by the reference to Euclid, it has to be a systematization for *all* of mathematics, if not all of science. The systematization will suggest inferential relationships between the various propositions and concepts. If all has gone well, those will reflect the correct foundational relationships.

To repeat part of passage quoted earlier, if 'we find everything in order, then we have accurate knowledge of the grounds upon which each individual theorem is based'.[53]

52 Jeshion, 'Frege's Notions of Self-Evidence', 969.
53 Frege, *Grundgesetze der Arithmetik*, vii.

4.4. Piecemeal Definitions

A bit later, Frege[54] insists that piecemeal definitions are unacceptable in the foundational work. This happens when one provides a definition for a certain restricted domain, and then extends the definition to a wider domain. For example, we might first define 'square root' with respect to the positive integers, and then go on to apply that same definition to larger domains, the integers or real numbers, for example.

One problem with piecemeal definitions is that previously proved theorems might be invalidated in the extended case. For example, when thinking of the natural numbers, we conclude/prove that each number has at most one square root. But we have to revise that when considering the integers. The number four has two square roots in the integers, 2 and −2.

More important, piecemeal definitions fail to take into account the full range of cases, and so violate Frege's insistence that functions be defined on *all* objects.

This is clearly related to the theme that the foundation must be comprehensive, applying to all of mathematics, if not all of science.

So, if a systematization is to have a notion of cardinal number and/or a notion of ordinal number, it should provide a single account, a single definition of *all* cardinals, finite or infinite, and/ or a single definition of *all* ordinals, finite or infinite. The system itself will adjudicate the proper inferential, or basing questions between those.

As we saw earlier, Dummett took Frege's own foundational work to task for not having a notion of ordinal number, just a notion of cardinal number. But perhaps a Fregean could insist that there is no serious scientific need for a notion of ordinal number – in which case, there is no need to ponder the grounds of propositions about ordinal numbers. But such a case has not been made, and we have no idea how one might make it.

In any case, the sad fact of history is that Frege's own attempt at systematization failed, due to the inconsistency of Basic Law V. A logicist successor, Whitehead and Russell,[55] failed to be comprehensive (although it, too, lacked a notion of ordinal number).

[54] Ibid., §61.

[55] A.N. Whitehead and B. Russell, *Principia mathematica*, 3 (1910/ 1913).

4.5. So Ordinals are Foundationally Basic – or are they?

Today, however, there is in place another attempt at a systematization for all of mathematics, namely Zermelo–Fraenkel set theory – albeit without logicist goals. As mentioned, ordinals are defined within set theory to be certain sets, the so-called von Neumann ordinals (transitive pure sets that are well-ordered by the membership relation). And cardinals are defined to be certain von Neumann ordinals (those that are not equinumerous with any smaller ordinal).

So, putting the pieces together, this is prima facie evidence that ordinal number is more basic than cardinal number. It is defined first, in the prevalent systematization. Cardinals are defined in terms of ordinals – indeed, cardinals just *are* certain ordinals. So propositions about cardinals just are propositions about certain ordinals.

But this conclusion depends on the metaphysical and epistemic details of the Fregean foundational hierarchy, Frege's insistence on comprehensiveness, and the particular foundation in place today. It would seem that this broadly Fregean perspective does leave it open that there might be *another* systematization that first defines (finite and infinite) cardinal numbers and then defines (finite and infinite) ordinals in terms of those. But we have no idea how *that* development might go. It is indeed natural to follow Cantor and think of a cardinal as derived from an ordinal by ignoring the order.

Alternatively, one could maintain the broad outlines of the current set-theoretic foundation, but refuse to identify cardinal numbers with (some of the) von Neumann ordinals. We presume that at least some contemporary neologicists would be content to just introduce cardinal numbers on the Zermelo-Fraenkel foundation with Hume's Principle. Or one might just paraphrase propositions about cardinality in terms of Hume's Principle.[56] We do not know if this rather slight modification would make for a better foundation, based on whatever criteria are appropriate in this holistic enterprise.[57] Relatedly, there are other proposed systematizations for mathematics currently available. Some of those are based on category theory. We will not speculate as to how the debate over cardinals and ordinals plays out in those cases.

In general, there is a central issue of how to adjudicate these priority questions, from this broadly Fregean perspective, when there is more than one systematization of mathematics in the offing. Recall

[56] Thanks to Gil Sagi for these suggestions.

[57] See S. Shapiro, 'We Hold These Truths to be Self-Evident: But What Do We Mean by That?', *Review of Symbolic Logic* 2 (2009): 175–207.

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that the Fregean foundational relations are supposed to be *objective*, not depending on individual whims and preferences concerning how to organize things. The underlying assumption is that there will be, in the end, at most one foundational system. And it is supposed to reflect the genuine jusitificatory/explantory relations between the propositions. We will not speculate any further on how these priority questions play out when there is more than one systematization for mathematics available.

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