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# Existence and stability of ground-state solutions of a Schrödinger–KdV system

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We consider the coupled Schrödinger–Korteweg–de Vries system

$$\mathrm{i}(u_t + c_1 u_x) + \delta_1 u_{xx} = \alpha uv,$$
$$v_t + c_2 v_x + \delta_2 v_{xxx} + \gamma(v^2)_x = \beta(|u|^2)_x,$$

which arises in various physical contexts as a model for the interaction of long and short nonlinear waves. Ground states of the system are, by definition, minimizers of the energy functional subject to constraints on conserved functionals associated with symmetries of the system. In particular, ground states have a simple time dependence because they propagate via those symmetries. For a range of values of the parameters $\alpha$, $\beta$, $\gamma$, $\delta_i$, $c_i$, we prove the existence and stability of a two-parameter family of ground states associated with a two-parameter family of symmetries.

## 1. Introduction

In this paper we prove existence and stability results for ground-state solutions to the system of equations

$$\left.\begin{aligned} \mathrm{i}(u_t + c_1 u_x) + \delta_1 u_{xx} &= \alpha uv, \\ v_t + c_2 v_x + \delta_2 v_{xxx} + \gamma(v^2)_x &= \beta(|u|^2)_x, \end{aligned}\right\} \tag{1.1}$$

where $u$ is a complex-valued function of the real variables $x$ and $t$, $v$ is a real-valued function of $x$ and $t$, and the constants $c_i$, $\delta_i$, $\alpha$, $\beta$, $\gamma$ are real. We consider here only the pure initial-value problem for (1.1), in which initial data $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$ are posed for $-\infty < x < \infty$, and a solution $(u(x,t), v(x,t))$ is sought for $-\infty < x < \infty$ and $t \geqslant 0$. Well-posedness results for the pure initial-value problem for (1.1) and certain variants have appeared in [7, 21, 34]; we cite below in § 5 the specific results we will need here.

Systems of the form (1.1) appear as models for interactions between long and short waves in a variety of physical settings. For example, Kawahara *et al.* [23] derived (1.1) as a model for the interaction between long gravity waves and capillary waves on the surface of shallow water, in the case when the group velocity of the capillary wave coincides with the velocity of the long wave. In [30, 32], a system of equations is derived for resonant ion-sound/Langmuir-wave interactions in plasmas, which reduces to (1.1) under the assumption that the ion-sound wave is unidirectional. Similarly, one can obtain (1.1) as the unidirectional reduction of a model for the resonant interaction of acoustic and optical modes in a diatomic lattice [38].

In the applications mentioned in the preceding paragraph, all the constants appearing in (1.1) are typically non-zero. On the other hand, system (1.1) with $\delta_2 = \gamma = 0$ was derived in [16, 20] as a model for the interaction between long and short water waves, and appears as well in the plasma physics literature (see, for example, [22,37]). The presence or absence of the terms containing $\delta_2$ and $\gamma$ is determined by the scaling assumptions made in the derivation of the equations. For a discussion of the role of the scaling assumptions in the derivation of equations such as (1.1), the reader may consult [10] or [17].

If $\delta_2 \neq 0$ in (1.1), then, by making appropriate use of the transformations $x \to \theta x$, $t \to \theta t$, $x \to x + t$, $u \to \theta u$, $u \to \bar{u}$ and $u \to \mathrm{e}^{\mathrm{i}(\theta x - \theta^2 t)} u$, where $\theta \in \mathbb{R}$, we can reduce (1.1) to either

$$\left.\begin{aligned} \mathrm{i}u_t + u_{xx} &= -uv, \\ v_t + 2v_{xxx} + 3q(v^2)_x &= -(|u|^2)_x \end{aligned}\right\} \tag{1.2}$$

or

$$\left.\begin{aligned} \mathrm{i}u_t + u_{xx} &= -uv, \\ v_t - 2v_{xxx} + 3q(v^2)_x &= -(|u|^2)_x, \end{aligned}\right\} \tag{1.3}$$

where $q \in \mathbb{R}$. System (1.3) is of the form that arises in [5,30,32]; in these references, one can find, in particular, explicit ground-state solutions. The analysis of (1.3), however, is complicated by the fact that the associated energy functional, analogous to the energy $E(u, v)$ defined below, is not positive-definite. In this paper we only consider (1.2), and we further restrict consideration to the case when $q$ is positive in (1.2). (When (1.2) is used to model interactions between internal and surface-gravity waves in a two-layer fluid, the assumption that $q$ is positive corresponds to the assumption that the ratio of the depth of the upper layer to the depth of the lower layer is less than a certain critical value [17].) For technical reasons, the argument used in this paper to prove existence and stability of ground states for (1.2) does not apply to the case when $q$ is negative. We note, however, that the explicit one-parameter family of ground states for (1.2) given below in (2.8) does continue to negative values of $q$.

We will also have occasion below to consider the case when $\delta_2 = \gamma = 0$ in (1.1). In this case, system (1.1) can be reduced to the form

$$\left.\begin{aligned} \mathrm{i}u_t + u_{xx} &= -uv, \\ v_t &= -(|u|^2)_x. \end{aligned}\right\} \tag{1.4}$$

System (1.4) is of independent mathematical interest because it has been found to have a completely integrable structure. In particular, it has an inverse-scattering transform and explicit $N$-soliton solutions [28,29,37]. (By contrast, equations (1.2) and (1.3) do not have $N$-soliton solutions [9].)

The system (1.2) can be written in Hamiltonian form as

$$(u_t, v_t) = J\delta E(u, v), \tag{1.5}$$

where $J$ is the antisymmetric operator defined by $J(w, z) = (-\frac{1}{2}\mathrm{i}w, z_x)$ and $E(u, v)$, the Hamiltonian functional, is defined by

$$E(u, v) = \int_{-\infty}^{\infty} (|u_x|^2 + v_x^2 - v|u|^2 - qv^3)\,\mathrm{d}x.$$

The notation $\delta E$ in (1.5) refers to the Fréchet derivative, or generalized gradient, of $E$. Since the Hamiltonian $E$ is invariant under time translations, it is a conserved functional for the flow defined by (1.2), i.e. when applied to sufficiently regular solutions $u(x, t)$, $v(x, t)$ of (1.2), $E$ is independent of $t$. There are also two other conserved functionals of (1.2) associated with symmetries, namely,

$$G(u, v) = \int_{-\infty}^{\infty} v^2\,\mathrm{d}x - 2\,\mathrm{Im}\int_{-\infty}^{\infty} u\bar{u}_x\,\mathrm{d}x,$$

which arises from the invariance of (1.2) under space translations $x \to x + \theta$, and

$$H(u) = \int_{-\infty}^{\infty} |u|^2\,\mathrm{d}x,$$

which arises from the invariance of (1.2) under phase shifts $u \to \mathrm{e}^{\mathrm{i}\theta}u$.

Equations (1.4) can also be rewritten in Hamiltonian form as

$$(u_t, v_t) = J\delta K(u, v), \tag{1.6}$$

where $J$ is as above and $K$ is defined by

$$K(u, v) = \int_{-\infty}^{\infty} (|u_x|^2 - v|u|^2)\,\mathrm{d}x.$$

The functionals $G(u, v)$ and $H(u)$ defined above are conserved functionals for (1.4) as well.

Bound-state solutions of (1.2) or (1.4) are, by definition, solutions of the form

$$u(x, t) = \mathrm{e}^{\mathrm{i}\omega t}h(x - ct), \qquad v(x, t) = g(x - ct), \tag{1.7}$$

where $h$ and $g$ are functions that vanish at infinity in some sense (usually $h$ and $g$ are in $H^1(\mathbb{R})$), and $\omega$ and $c$ are real constants. It is easy to see that $u(x, t)$ and $v(x, t)$, as defined in (1.7), are solutions of (1.2) if and only if $(h, g)$ is a critical point for the functional $E(u, v)$, when $u(x)$ and $v(x)$ are varied subject to the constraints that $G(u, v)$ and $H(u)$ be held constant (see § 5 below). If $(h, g)$ is not only a critical point, but, in fact, a global minimizer of the constrained variational problem for $E(u, v)$, then (1.7) is called a ground-state solution of (1.2). The same comments also apply to (1.4), except that the functional being varied in this case is $K(u, v)$. In this paper, our main concern is with ground-state solutions. For a discussion of what is currently known about bound-state solutions of (1.2) in general, see § 2 below.

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The terms 'bound state' and 'ground state' are traditional in the literature concerning the nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} = -u|u|^2. \tag{1.8}$$

Bound-state solutions of (1.8) are solutions of the form $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t}h(x-ct)$, or, equivalently, minimizers of the Hamiltonian functional

$$\int_{-\infty}^{\infty} (|u_x|^2 - \tfrac{1}{2}|u|^4)\,\mathrm{d}x$$

subject to the constraints that $H(u)$ and $\int_{-\infty}^{\infty} u\bar{u}_x\,\mathrm{d}x$ be held constant. It is easy to see that any bound-state solution of (1.8) must have a profile function of the form

$$h(x) = \mathrm{e}^{\mathrm{i}(cx/2+\theta)}\sqrt{2\sigma}\,\mathrm{sech}(\sqrt{\sigma}x + x_0),$$

where $\sigma = \omega - \frac{1}{4}c^2 > 0$, and $x_0, \theta \in \mathbb{R}$. In fact, these bound states are actually ground states [12]. Since $|h(x)|$ decays monotonically to zero as $x$ tends away from $x_0$ to $\infty$ or $-\infty$, bound-state solutions of (1.8) are often called solitary waves. By extension, the term 'solitary wave' is often used to refer to bound-state solutions of equations that are related to (1.8), such as (1.2) or (1.4). This usage, however, is usually eschewed for bound states that are known not to have monotonic profiles, such as the excited bound states known to exist for generalizations of (1.8) to higher dimensions (see, for example, [11]). Since, for system (1.2), we do not know, in general, whether the ground-state solutions we find have profiles that decay monotonically to zero away from a single extremum, we have here avoided calling them solitary waves.

Our main results are as follows. We prove below (see theorem 4.5 and corollary 5.2) that, for a certain range of values of $q$, equation (1.2) has, for every $s > 0$ and $t \in \mathbb{R}$, a non-empty set of ground-state solutions (1.7) with profiles $(h, g)$ satisfying $H(h) = s$ and $G(h, g) = t$. Moreover, for a given pair of values of $s$ and $t$, the set $F_{s,t}$ of profiles of these solutions is stable, in the sense that if $(h, g) \in F_{s,t}$ and a slight perturbation of $(h, g)$ is taken as initial data for (1.2), then the resulting solution of (1.2) can be said to have a profile that remains close to $F_{s,t}$ for all time (see theorem 5.4).

Besides the main results, we also include an existence result for ground-state solutions of (1.2), which is valid for all $q > 0$ (theorem 3.27) and an existence and stability result for ground-state solutions of (1.4) (theorem 5.6). Concerning the latter result, we note that existence of bound-state solutions is obvious, since it is easy to explicitly find all solutions of the equations that result from substituting (1.7) into (1.4) (see lemma 2.2 below). Also, the stability of these solutions has been proved by Laurençot in [24]. However, the method used by Laurençot did not establish whether these bound states were, in fact, ground states.

The results in the present paper are complementary to those contained in an earlier paper of one of us [4], where different techniques were used. In particular, it follows from the results of § 3 of [4] that, for every $q > 0$, we can find, for arbitrary $c > 0$ and arbitrary $\omega \in (\frac{1}{4}c^2, \infty)$, a bound-state solution (1.7) of (1.2) such that $h(x) = \mathrm{e}^{\mathrm{i}cx/2}f(x)$, where $f$ is real valued. Moreover, a stability result for certain sets of such bound states is proved when $\omega$ is near $\frac{1}{4}c^2$. We also note that Chen [14]

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has proved the orbital stability of a two-parameter family of explicit bound-state solutions (see § 2 below) in the special case $q = 2$. Finally, we mention the elegant proof in Ohta [33] of the stability of solitary-wave solutions of the Zakharov system,

$$\left.\begin{aligned} \mathrm{i}u_t + u_{xx} &= -uv, \\ v_{tt} - v_{xx} &= -(|u|^2)_{xx}, \end{aligned}\right\} \tag{1.9}$$

by means of an argument that is related to the arguments used below in § 4.

The proofs below follow the lines of many other proofs of existence and stability of solitary-wave solutions to dispersive equations that have appeared over the last couple of decades. The common elements in these proofs are the reduction of the stability problem to the problem of showing that minimizing sequences of a constrained variational problem are necessarily relatively compact, and the solution of this latter problem by the method of concentration compactness (see [13] for what may be the first example of such a stability proof).

In the present situation, however, application of the concentration compactness method is considerably complicated by the fact that, for a given choice of $q$ in (1.2), we are interested in finding a true two-parameter family of bound-state solutions (parametrized by $c$ and $\omega$). In all the applications of the method to solitary waves that we are aware of, the variational problem has consisted of finding the extremum of a real-valued functional $E(f)$ subject to a single constraint of the form $Q(f) = \lambda$, where $Q$ is another real-valued functional and $\lambda \in \mathbb{R}$ is a constant. This leads to a result concerning a one-parameter family of solitary waves. (In some cases, such as that of the nonlinear Schrödinger equation (1.8) or the Zakharov system (1.9), there at first appear to be two solitary-wave parameters, but it turns out that they are not independent.) Here, on the other hand, we are led to consider a variational problem in which there are not one, but two real-valued constraint functions.

Now, as was already noted in the original papers introducing the concentration compactness method (see, for example, [26, § IV]), the general outline of the method lends itself just as easily to problems in which there are more than one constraint function as to problems with a single constraint functional. But putting the method into practice requires proving the subadditivity of the variational problem with respect to the constraint parameters, and this turns out to be considerably more complicated in the case of two parameters. The task of proving the subadditivity of the relevant two-parameter variational problem will occupy us through most of § 3.

The outline of this paper is as follows. In § 2, we collect some basic facts concerning the properties of bound-state solutions of (1.2) and (1.4). Sections 3 and 4 contain the proof of the relative compactness of minimizing sequences for the variational problems that define ground-state solutions of (1.2) and (1.4). Finally, § 5 discusses the existence and properties of ground-state solutions, including their stability properties.

Notation. We shall denote by $\hat{f}$ the Fourier transform of $f$, defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x.$$

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For $1 \leqslant p \leqslant \infty$, we denote by $L^p = L^p(\mathbb{R})$ the space of all measurable functions $f$ on $\mathbb{R}$ for which the norm $|f|_p$ is finite, where

$$|f|_p = \left(\int_{-\infty}^{\infty} |f|^p \, \mathrm{d}x\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty$$

and $|f|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. Whether we intend the functions in $L^p$ to be real or complex valued will be clear from the context. For $s \geqslant 0$, we denote by $H^s_{\mathbb{C}} = H^s_{\mathbb{C}}(\mathbb{R})$ the Sobolev space of all complex-valued functions $f$ in $L^2$ for which the norm

$$\|f\|_s = \left(\int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2}$$

is finite. We will always view $H^s_{\mathbb{C}}$ as a vector space over the reals, with inner product given by

$$\langle f_1, f_2 \rangle = \mathrm{Re} \int_{-\infty}^{\infty} (1 + |\xi|^2)^s \hat{f}_1 \bar{\hat{f}}_2 \, \mathrm{d}x.$$

The space of all real-valued functions $f$ in $H^s_{\mathbb{C}}$ will be denoted simply by $H^s$. In particular, we use $\|f\|$ to denote the $L^2$ or $H^0$ norm of a function $f$. If $I$ is an open interval in $\mathbb{R}$, we use $H^s(I)$ to denote the set of all functions $f$ on $I$ such that $f\eta \in H^s$ for every smooth function $\eta$ with compact support in $I$. We define the space $X$ to be the Cartesian product $H^1_{\mathbb{C}} \times L^2$, and the space $Y$ to be $H^1_{\mathbb{C}} \times H^1$, each provided with the product norm. Finally, if $T > 0$ and $Z$ is any Banach space, we denote by $\mathcal{C}([0, T], Z)$ the Banach space of continuous maps $f : [0, T] \mapsto Z$, with norm given by

$$\|f\|_{\mathcal{C}([0,T],Z)} = \sup_{t \in [0,T]} \|f(t)\|_Z.$$

The letter $C$ will frequently be used to denote various constants whose actual value is not important for our purposes.

## 2. Bound states

We record here some general results concerning bound-state solutions of (1.2) and related equations. We also include a list of explicit formulae for solutions in a few special cases, for purposes of comparison with the more general solutions we study in later sections.

Recall that a bound-state solution of (1.2) is, by definition, a solution of the form given in (1.7). In what follows, we further require that $h \in H^1_{\mathbb{C}}$ and $g \in H^1$. If we substitute (1.7) into (1.2), we can integrate the second of the resulting two equations, using the fact that $g \in H^1$ to evaluate the constant of integration. We thus see that $(u(x, t), v(x, t))$ is a bound-state solution of (1.2) if and only if $h$ and $g$ satisfy the equations

$$\left.\begin{aligned} h'' - \omega h - \mathrm{i}ch' &= -hg, \\ 2g'' - cg &= -3qg^2 - |h|^2. \end{aligned}\right\} \tag{2.1}$$

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We can further simplify (2.1) by putting $h(x) = \mathrm{e}^{\mathrm{i}cx/2} f(x)$, thus obtaining the system

$$\left.\begin{aligned} f'' - \sigma f &= -fg, \\ 2g'' - cg &= -3qg^2 - |f|^2, \end{aligned}\right\} \tag{2.2}$$

where $\sigma = \omega - \frac{1}{4}c^2$. We can thus consider (2.2) to be the defining equations for bound-state solutions of (1.2).

THEOREM 2.1. *Suppose $(f, g) \in Y$ is a solution of (2.2), in the sense of distributions. Then we have the following.*

(i) $(f, g) \in H^\infty_{\mathbb{C}} \times H^\infty$.

(ii) *If $c > 0$, then either $f$ and $g$ are both identically zero or $g(x) > 0$ for all $x \in \mathbb{R}$.*

(iii) *$f(x) = \varphi(x)\mathrm{e}^{\mathrm{i}\theta_0}$ for $x \in \mathbb{R}$, where $\theta_0$ is a real constant and $\varphi$ is real valued.*

(iv) *If $\sigma > 0$ and $c > 0$, there exist constants $\epsilon_1, \epsilon_2 > 0$ such that $\mathrm{e}^{\epsilon_1|x|} f(x)$ and $\mathrm{e}^{\epsilon_2|x|} g(x)$ are in $L^\infty$.*

*Proof.* For any $s > 0$, define the function $K_s(x)$ by

$$K_s(x) = \frac{1}{2\sqrt{s}} \mathrm{e}^{-\sqrt{s}|x|}.$$

Then $\hat{K}_s(\xi) = (s + \xi^2)^{-1}$, so the operation of convolution with $K_s$ takes $H^s_{\mathbb{C}}$ to $H^{s+2}_{\mathbb{C}}$, and is, in fact, the inverse of the operator $(s - \partial_{xx})$ in the sense that $(s - \partial_{xx})(K_s * f) = f$ for all $f \in H^s_{\mathbb{C}}$. Now we can rewrite (2.2) in the form

$$\left.\begin{aligned} f &= K_{\sigma+a_1} * (fg + a_1 f), \\ g &= K_{c/2+a_2} * (\tfrac{3}{2}qg^2 + \tfrac{1}{2}|f|^2 + a_2 g), \end{aligned}\right\} \tag{2.3}$$

where $a_1$ and $a_2$ are real numbers chosen so that $\sigma + a_1 > 0$ and $\frac{1}{2}c + a_2 > 0$.

From (2.3), statement (i) follows by a standard bootstrap argument. Since $f$ and $g$ are in $H^1_{\mathbb{C}}$, and $H^1_{\mathbb{C}}$ is an algebra, then $g^2$, $|f|^2 = f\bar{f}$ and $fg$ are also in $H^1_{\mathbb{C}}$. Hence (2.3) implies that $f$ and $g$ are in $H^3_{\mathbb{C}}$. But then $g^2$, $|f|^2$ and $fg$ are in $H^3_{\mathbb{C}}$, so (2.3) implies that $f$ and $g$ are in $H^5_{\mathbb{C}}$, and so on.

To prove (ii), observe that if $c > 0$, then we can take $a_2 = 0$ in (2.3). But since $K_{c/2}$ is strictly positive on $\mathbb{R}$ and $g^2 + |f|^2$ is everywhere non-negative, it then follows from the second equation in (2.3) that if either $f$ or $g$ is non-zero on a set of positive measure, then $g(x) > 0$ everywhere.

For (iii), we first observe that, by (i) and the standard uniqueness theory for ordinary differential equations, $f(x)$ and $f'(x)$ cannot both vanish at any point $x \in \mathbb{R}$. Moreover, if the zeros of $f$ accumulate at any point $x \in \mathbb{R}$, then, by Rolle's theorem, the zeros of $\operatorname{Re} f'$ and $\operatorname{Im} f'$ accumulate at $x$ also, leading to the contradictory result that $f(x) = f'(x) = 0$. Therefore, the zeros of $f$ must be isolated.

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Let $x_1$ and $x_2$ be any two consecutive zeros of $f$, where $x_1 < x_2$, and possibly $x_1 = -\infty$ or $x_2 = \infty$ or both. Then we can find infinitely differentiable functions $r$ and $\theta$ on $(x_1, x_2)$, with $r(x) > 0$ on $(x_1, x_2)$ and

$$\lim_{x\to x_1^+} r(x) = \lim_{x\to x_2^-} r(x) = 0,$$

such that, for all $x \in (x_1, x_2)$,

$$f(x) = r(x)\mathrm{e}^{\mathrm{i}\theta(x)}.$$

From the first equation in (2.2), we get

$$\left.\begin{aligned} r'' - \sigma r - r(\theta')^2 &= -rg, \\ 2r'\theta' + r\theta'' &= 0. \end{aligned}\right\} \tag{2.4}$$

Multiplying the second equation in (2.4) by $r(x)$ and integrating, we obtain

$$r^2(x)\theta'(x) = K$$

for all $x \in (x_1, x_2)$, where $K$ is a constant. Now, by (i), $|f'|^2 = (r')^2 + r^2(\theta')^2$ is bounded on $\mathbb{R}$, so $r^2(\theta')^2 = K^2/r^2$ is bounded on $(x_1, x_2)$. But since $r \to 0$ as $x \to x_1$, this implies that $K = 0$ on $(x_1, x_2)$. Hence $\theta$ is constant on $(x_1, x_2)$.

The preceding argument shows that $f(x) = r(x)\mathrm{e}^{\mathrm{i}\theta(x)}$ on $\mathbb{R}$, where $\theta(x)$ is defined and constant on each of the intervals separating the zeros of $r(x)$. Now suppose that $x_0 \in \mathbb{R}$ is such that $r(x_0) = 0$, and define

$$\begin{aligned} \theta^- &= \lim_{x\to x_0^-} \theta(x), & t^- &= \lim_{x\to x_0^-} r'(x), \\ \theta^+ &= \lim_{x\to x_0^+} \theta(x), & t^+ &= \lim_{x\to x_0^+} r'(x). \end{aligned}$$

Then $\mathrm{e}^{\mathrm{i}\theta^-}t^- = f'(x_0) = \mathrm{e}^{\mathrm{i}\theta^+}t^+$, and since $f'(x_0) \neq 0$, both $t^-$ and $t^+$ are non-zero. Therefore, $\mathrm{e}^{\mathrm{i}(\theta^+-\theta^-)} = t^-/t^+ \in \mathbb{R}$, from which it follows that $\mathrm{e}^{\mathrm{i}(\theta^+-\theta^-)}$ is either 1 or $-1$. Hence we can arrange that $f(x) = \varphi(x)\mathrm{e}^{\mathrm{i}\theta_0}$ on both sides of $x_0$, where $\varphi(x)$ is real valued, by taking $\theta_0 = \theta^-$ and defining $\varphi(x) = r(x)$ for $x$ to the left of $x_0$ and $\varphi(x) = r(x)\mathrm{e}^{\mathrm{i}(\theta^+-\theta^-)}$ to the right of $x_0$. Stepping through the intervals between zeros of $r(x)$ one at a time, both rightward and leftward from $x_0$, and iterating this procedure, one obtains the desired result.

To prove (iv), we borrow an argument from the proof of theorem 8.1.1 (iv) of [12]. For each $\epsilon > 0$ and $\eta > 0$, define a function $\zeta$ by $\zeta(x) = \mathrm{e}^{\epsilon|x|/(1+\eta|x|)}$. Multiply the first equation in (2.2) by $\zeta\bar{f}$ and add the result to its complex conjugate to get

$$\mathrm{Re}\int_{-\infty}^{\infty} f'(\zeta\bar{f})'\,\mathrm{d}x + \sigma\int_{-\infty}^{\infty} \zeta|f|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} \zeta g|f|^2\,\mathrm{d}x.$$

Since $\zeta' \leqslant \epsilon\zeta$, we can deduce that

$$\sigma\int_{-\infty}^{\infty} \zeta|f|^2\,\mathrm{d}x \leqslant \int_{-\infty}^{\infty} \zeta g|f|^2\,\mathrm{d}x - \int_{-\infty}^{\infty} \zeta|f'|^2\,\mathrm{d}x + \epsilon\int_{-\infty}^{\infty} \zeta|ff'|\,\mathrm{d}x. \tag{2.5}$$

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Now using the Cauchy–Schwarz inequality with $\epsilon$ chosen to be sufficiently small, we deduce from (2.5) that

$$\int_{-\infty}^{\infty} \zeta|f|^2 \, dx \leqslant C_\epsilon \int_{-\infty}^{\infty} \zeta g|f|^2 \, dx, \tag{2.6}$$

where $C_\epsilon$ does not depend on $\eta$. Since $g \in H^1$, we can find $R > 0$ such that $|g(x)| \leqslant 1/(2C_\epsilon)$ for $|x| \geqslant R$. It then follows from (2.6) that

$$\int_{-\infty}^{\infty} \zeta|f|^2 \, dx \leqslant 2C_\epsilon \int_{|x|\leqslant R} e^{\epsilon|x|} g(x)|f(x)|^2 \, dx,$$

and taking $\eta \to 0$ gives

$$\int_{-\infty}^{\infty} e^{\epsilon|x|}|f(x)|^2 \, dx < \infty. \tag{2.7}$$

Now, since $f \in H^1$, then $f(x) \to 0$ as $|x| \to \infty$ and $f$ is uniformly Lipschitz on $\mathbb{R}$. From these two properties of $f$ and (2.7), it follows easily that $e^{\epsilon_1|x|} f(x)$ is bounded on $\mathbb{R}$ for some $\epsilon_1 \in (0, \epsilon)$ (for details, see the proof of theorem 8.1.7 (iv) in [12]).

The decay estimate for $g$ is obtained in the same way as that for $f$. Multiplying the second equation in (2.2) by $\zeta g$ leads, as above, to the estimate

$$\int_{-\infty}^{\infty} \zeta g^2 \, dx \leqslant C_\epsilon \int_{-\infty}^{\infty} \zeta(g^3 + |f|^2 g) \, dx.$$

Choosing $\epsilon < 2\epsilon_1$, and using the decay result just proved for $f$, we find, as before, that $\int_{-\infty}^{\infty} \zeta g^2 \, dx$ can be bounded by a constant that is independent of $\eta$. Taking $\eta \to 0$ allows us to conclude that

$$\int_{-\infty}^{\infty} e^{\epsilon|x|}|g(x)|^2 \, dx < \infty,$$

and from here the proof proceeds as it did for $f(x)$. □

Funakoshi and Oikawa, in [17], list the following explicit one-parameter families of bound-state solutions to (1.2). For $q \leqslant \frac{2}{3}$, define

$$\left.\begin{aligned} f(x) &= \pm 6B^2\sqrt{2-3q}\,\operatorname{sech}^2(Bx), \\ g(x) &= 6B^2 \operatorname{sech}^2(Bx), \end{aligned}\right\} \tag{2.8}$$

where $B > 0$ is arbitrary. Then $(f, g)$ satisfy (2.2) with $\sigma = 4B^2$ and $c = 8B^2$. If, on the other hand, $q \geqslant \frac{2}{3}$, then we have that

$$\left.\begin{aligned} f(x) &= \pm 6B^2\sqrt{3q-2}\,\operatorname{sech}(Bx)\tanh(Bx), \\ g(x) &= 6B^2 \operatorname{sech}^2(Bx) \end{aligned}\right\} \tag{2.9}$$

is a solution of (2.2) with $\sigma = B^2$ and $c = 2B^2(9q - 2)$. When $q = \frac{2}{3}$, of course, these solutions coincide, with the obvious solution given by

$$f = 0 \quad \text{and} \quad g = \left(\frac{4B^2}{q}\right)\operatorname{sech}^2(Bx),$$

which satisfies (2.2) with $c = 8B^2$ for all $q \neq 0$.

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In [14], Chen considered (1.2) in the special case when $q = 2$, and found a two-parameter family of explicit solutions, given by

$$\left.\begin{aligned} f(x) &= \pm\sqrt{2B^2(c-8B^2)}\,\operatorname{sech}(Bx),\\ g(x) &= 2B^2\operatorname{sech}^2(Bx), \end{aligned}\right\} \tag{2.10}$$

where $B^2 = \sigma$, and $c > 0$ and $\sigma \in (0, \frac{1}{8}c)$ are arbitrary. Then, using the stability theory of [19], he went on to show that if $h(x) = \mathrm{e}^{\mathrm{i}cx/2} f(x)$, $\omega = \sigma + \frac{1}{4}c^2$ and $(u, v)$ is the bound-state solution of (1.2) defined by (2.10) and (1.7), then $(u, v)$ is orbitally stable provided $c \leqslant 1$ and $\sigma \in (0, \frac{1}{12}c)$ (see [14, theorem 2]). Here, orbital stability of $(u, v)$ means that if $F$, the orbit of $(f, g)$, is defined as the set of all $(\tilde{f}, \tilde{g}) \in Y$ such that $\tilde{f}(x) = \mathrm{e}^{\mathrm{i}\theta_0} f(x + x_0)$ and $\tilde{g}(x) = g(x + x_0)$ for some $\theta_0, x_0 \in \mathbb{R}$, then $F$ is stable in the sense of theorem 5.2 below.

In theorem 5.1 below, it is shown that if $(f, g)$ is a solution of (2.2) corresponding to a ground-state solution of (1.2), then, up to a multiplicative constant of absolute value one, $f$ is a positive function on $\mathbb{R}$. Therefore, the bound state given by (2.9) is not a ground state. In fact, in the case $q = 2$, it is not hard to show (see remark 3.18 below) that there is, up to translation and phase shift, a unique ground-state solution of (2.2), and that this solution is given by (2.10). We do not know, however, whether ground states are unique for $q \neq 2$.

In later sections, we will need the following uniqueness results for certain equations related to (2.2).

Lemma 2.2. *Suppose $(f, g) \in X$ is a non-zero solution of the equations*

$$\left.\begin{aligned} f'' + fg &= \lambda f,\\ |f|^2 &= \mu g, \end{aligned}\right\} \tag{2.11}$$

*where $\lambda, \mu \in \mathbb{R}$. Then $\lambda > 0$ and $\mu > 0$, and*

$$f(x) = \mathrm{e}^{\mathrm{i}\theta_0} f_1(x + x_0) \quad \textit{and} \quad g(x) = g_1(x + x_0),$$

*where $\theta_0, x_0 \in \mathbb{R}$ and*

$$\left.\begin{aligned} f_1(x) &= \sqrt{2\lambda\mu}\,\operatorname{sech}(\sqrt{\lambda}x),\\ g_1(x) &= 2\lambda\operatorname{sech}^2(\sqrt{\lambda}x). \end{aligned}\right\} \tag{2.12}$$

Lemma 2.3. *Suppose $g \in H^1$ is a non-zero solution of the equation*

$$-g'' - \tfrac{3}{2}qg^2 = \kappa g, \tag{2.13}$$

*where $\kappa \in \mathbb{R}$. Then $\kappa > 0$ and $g = g_2(x + x_0)$, where $x_0 \in \mathbb{R}$ and*

$$g_2(x) = \frac{\kappa}{q}\operatorname{sech}^2(\tfrac{1}{2}\sqrt{\kappa}x). \tag{2.14}$$

To prove these well-known results, one begins by using a bootstrap argument to establish that any solution must, in fact, be infinitely differentiable. Equation (2.13) can then be integrated twice (after first multiplying by $g'$) to yield (2.14). For equation (2.11), we can argue as in the proof of theorem 2.1 (iii) to show that $f(x) = \mathrm{e}^{\mathrm{i}\theta_0 x}\varphi(x)$, where $\varphi$ is real valued, and then eliminate $g$ to obtain a single equation for $\varphi$, which may be solved by integrating twice. We omit the details.

## 3. The reduced variational problem

In this section we consider the problem of finding

$$I(s,t)=\inf\{E(f,g):(f,g)\in Y,\ \|f\|^2=s \text{ and } \|g\|^2=t\},\tag{3.1}$$

where $s,t>0$. Our approach will be to split the functional $E$ into two parts and consider the variational problem associated with each part. Define $K:X\to\mathbb{R}$ by

$$K(f,g)=\int_{-\infty}^{\infty}(|f'|^2-g|f|^2)\,\mathrm{d}x,$$

and $J:H^1\to\mathbb{R}$ by

$$J(g)=\int_{-\infty}^{\infty}((g')^2-qg^3)\,\mathrm{d}x.$$

Then

$$E(f,g)=K(f,g)+J(g).$$

Hence, if we define $M:H^1\to\mathbb{R}$ by

$$M(g)=\inf\{K(f,g):f\in H^1_{\mathbb{C}} \text{ and } \|f\|=1\},\tag{3.2}$$

then

$$I(s,t)=\inf\{sM(g)+J(g):g\in H^1 \text{ and } \|g\|^2=t\}.\tag{3.3}$$

This expression for $I(s,t)$ suggests analysing the subsidiary variational problems defined by

$$\begin{aligned}I_1(s,t)&=\inf\{K(f,g):(f,g)\in X,\ \|f\|^2=s \text{ and } \|g\|^2=t\}\\&=\inf\{sM(g):g\in H^1 \text{ and } \|g\|^2=t\}\end{aligned}\tag{3.4}$$

and

$$I_2(t)=\inf\{J(g):g\in H^1 \text{ and } \|g\|^2=t\}.\tag{3.5}$$

Lemma 3.1. *If $(f,g)\in X$, then $(|f|,g)\in X$ also, and $K(|f|,g)\leqslant K(f,g)$.*

*Proof.* What has to be proved is that if $f\in H^1_{\mathbb{C}}$, then $F(x)=|f(x)|$ is in $H^1$, with $\|F\|_1\leqslant\|f\|_1$. We do not prove this elementary fact here, but remark that a proof can be given, which, by working with $\hat f$ and $\hat F$ instead of $f$ and $F$, avoids the annoying question of the differentiability of $F$ at points where $F=0$. Such a proof is easily constructed by adapting the proof of lemma 3.4 in [3]. □

Lemma 3.2. *For all $s,t\geqslant 0$, $I_1(s,t)$ and $I_2(t)$ are finite.*

*Proof.* Let $(f,g)\in X$ with $\|f\|^2=s$ and $\|g\|^2=t$. Then, from the Cauchy–Schwarz inequality and the Sobolev embedding theorem, we have

$$\left|\int_{-\infty}^{\infty}g|f|^2\,\mathrm{d}x\right|\leqslant C\|f\|_1\|f\|\|g\|\leqslant\int_{-\infty}^{\infty}|f'|^2\,\mathrm{d}x+Cs(1+t)$$

and

$$\left|\int_{-\infty}^{\infty}g^3\,\mathrm{d}x\right|\leqslant C\|g\|_1\|g\|^2\leqslant\int_{-\infty}^{\infty}(g')^2\,\mathrm{d}x+Cs^2.$$

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Hence $I_1(s,t) \geqslant -Cs(1+t) > -\infty$ and $I_2(t) \geqslant -Cs^2 > -\infty$. ☐

Lemma 3.3. *For all $s,t > 0$, we have $I_1(s,t) < 0$ and $I_2(t) < 0$. Also, $I_1(s,0) = 0$ for all $s \geqslant 0$, $I_1(0,t) = 0$ for all $t \geqslant 0$ and $I_2(0) = 0$.*

*Proof.* When $s,t > 0$, we can choose $(f,g) \in X$ such that

$$\|f\|^2 = s, \qquad \|g\|^2 = t, \qquad \int_{-\infty}^{\infty} g|f|^2 \,\mathrm{d}x > 0, \qquad \int g^3 \,\mathrm{d}x > 0.$$

Then, for each $\theta > 0$, the functions $f_\theta(x) = \theta^{1/2} f(\theta x)$ and $g_\theta(x) = \theta^{1/2} g(\theta x)$ satisfy $\|f_\theta\|^2 = s$, $\|g_\theta\|^2 = t$,

$$K(f_\theta, g_\theta) = \theta^2 \int_{-\infty}^{\infty} |f'|^2 \,\mathrm{d}x - \theta^{1/2} \int_{-\infty}^{\infty} g|f|^2 \,\mathrm{d}x$$

and

$$J(g_\theta) = \theta^2 \int_{-\infty}^{\infty} (g')^2 \,\mathrm{d}x - \theta^{1/2} \int_{-\infty}^{\infty} g^3 \,\mathrm{d}x.$$

Hence, by taking $\theta$ sufficiently small, we get $K(f_\theta, g_\theta) < 0$ and $J(g_\theta) < 0$, proving that $I_1(s,t) < 0$ and $I_2(t) < 0$.

If $s \geqslant 0$, then, choosing any $f \in H^1$ with $\|f\| = s$ and defining $f_\theta$ as in the preceding paragraph, we get

$$K(f_\theta, 0) = \theta^2 \int_{-\infty}^{\infty} |f'|^2 \,\mathrm{d}x \geqslant I_1(s,0) \geqslant 0.$$

Then, by letting $\theta$ tend to zero, we see that $I_1(s,0) = 0$.

Finally, the equalities $I_1(0,t) = 0$ and $I_2(0) = 0$ are obvious. ☐

Lemma 3.4. *Suppose $\sigma > 0$, and define a map $g \to g^*$ from $H^1$ onto $H^1$ by*

$$g^*(x) = \sigma^{2/3} g(\sigma^{1/3} x).$$

*Then, for each $g \in H^1$,*

$$M(g^*) = \sigma^{2/3} M(g) \tag{3.6}$$

*and*

$$J(g^*) = \sigma^{5/3} J(g). \tag{3.7}$$

*Proof.* A simple change of variables in the integral proves (3.7). To prove (3.6), for each $f \in H^1_{\mathbb{C}}$ such that $\|f\| = 1$, define $\tilde{f}$ by

$$\tilde{f}(x) = \sigma^{1/6} f(\sigma^{1/3} x).$$

Then $\|\tilde{f}\| = 1$ and $K(\tilde{f}, g^*) = \sigma^{2/3} K(f,g)$, whence (3.6) follows by taking infima on both sides. ☐

Lemma 3.5. *For all $s,t \geqslant 0$, we have*

$$I_1(s,t) = st^{2/3} I_1(1,1) \tag{3.8}$$

*and*

$$I_2(t) = t^{5/3} I_2(1) \tag{3.9}$$

*Proof.* We may assume $s, t > 0$. Let $(f, g) \in X$ be such that $\|f\|^2 = s$ and $\|g\|^2 = t$, and let $\tilde{f}$ and $g^*$ be as defined in lemma 3.4 and its proof, with $\sigma = t^{-1}$. Define $z = s^{-1/2}\tilde{f}$. Then $\|z\|^2 = 1$, $\|g^*\|^2 = 1$,

$$K(f, g) = st^{2/3} K(z, g^*), \tag{3.10}$$

and

$$J(g) = t^{5/3} J(g^*). \tag{3.11}$$

The equality (3.8) follows by taking the infimum of both sides of (3.10) with respect to $f$ and $g$, while (3.9) follows by taking the infimum of both sides of (3.11) with respect to $g$. □

LEMMA 3.6. *Suppose $s_1, t_1, s_2, t_2 > 0$. If $t_1/t_2 = s_1/s_2 = \sigma$, then*

$$I(s_1, t_1) = \sigma^{5/3} I(s_2, t_2).$$

*Proof.* For $g \in H^1$, let $g^*$ be as defined in lemma 3.4. Then

$$\begin{aligned} I(s_1, t_1) &= \inf\{s_1 M(g^*) + J(g^*) : g^* \in H^1 \text{ and } \|g^*\|^2 = t_1\} \\ &= \inf\{\sigma^{5/3}(s_2 M(g) + J(g)) : g \in H^1 \text{ and } \|g\|^2 = t_2\} \\ &= \sigma^{5/3} I(s_2, t_2). \end{aligned}$$

□

LEMMA 3.7. *Let $s_1, s_2, t_1, t_2 \geqslant 0$, and suppose that $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $s_1 + t_1 > 0$ and $s_2 + t_2 > 0$. Then*

$$I_1(s_1 + s_2, t_1 + t_2) < I_1(s_1, t_1) + I_1(s_2, t_2). \tag{3.12}$$

*Also, if $t_1, t_2 > 0$, then*

$$I_2(t_1 + t_2) < I_2(t_1) + I_2(t_2). \tag{3.13}$$

*Proof.* To prove (3.12), we consider three cases: when $s_1 = 0$; when $t_1 = 0$; and when neither $s_1$ nor $t_1$ is 0. In the first case, we must have $s_2 > 0$ and $t_1 > 0$, so

$$s_2(t_1 + t_2)^{2/3} > s_2 t_2^{2/3}.$$

Since $I_1(1, 1) < 0$ and $I_1(s_1, t_1) = 0$ by lemma 3.3, multiplying both sides by $I(1, 1)$ and using lemma 3.5 gives the desired inequality. Similarly, in the second case, we must have $s_1 > 0$ and $t_2 > 0$, so

$$(s_1 + s_2)(t_1 + t_2)^{2/3} > s_1 t_1^{2/3} + s_2 t_2^{2/3},$$

and again multiplying by $I_1(1, 1)$ gives the desired inequality. Finally, in the third case, when $s_1 > 0$ and $t_1 > 0$, we must have either $s_2 > 0$ or $t_2 > 0$. If $s_2 > 0$, then

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we write

$$\begin{aligned}(s_1+s_2)(t_1+t_2)^{2/3} &= s_1(t_1+t_2)^{2/3}+s_2(t_1+t_2)^{2/3}\\ &> s_1(t_1+t_2)^{2/3}+s_2t_2^{2/3}\\ &\geqslant s_1t_1^{2/3}+s_2t_2^{2/3}.\end{aligned}$$

If $t_2>0$, we can write the same string of inequalities, with the penultimate expression replaced by $s_1t_1^{2/3}+s_2(t_1+t_2)^{2/3}$. In either case, we have established that

$$(s_1+s_2)(t_1+t_2)^{2/3} > s_1t_1^{2/3}+s_2t_2^{2/3},$$

which, when multiplied by $I_1(1,1)<0$, gives the desired result.

To prove (3.13), we merely observe that

$$(t_1+t_2)^{5/3} > t_1^{5/3}+t_2^{5/3}$$

for $t_1,t_2>0$, and apply lemmas 3.3 and 3.5. □

The next result, which we state here without proof, is taken from [15, lemma 2.4]. For a proof, see [26, lemma I.1].

LEMMA 3.8. *Suppose $p,r\in[1,\infty)$, $\{f_n\}$ is a bounded sequence in $L^r$ and $\{f_n'\}$ is bounded in $L^p$. If, for some $\omega>0$,*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-\omega}^{y+\omega}|f_n|^r\,\mathrm{d}x=0,$$

*then, for every $s>r$,*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f_n|^s\,\mathrm{d}x=0.$$

We will now prove the existence of minimizing pairs for problems (3.4) and (3.5). Actually, we accomplish somewhat more: using the method of concentration compactness [25,26], we show that, in fact, every minimizing sequence for these variational problems has a subsequence that converges, after suitable translations, to a solution of the problem. From this property of minimizing sequences, there easily follow stability results for the evolution equations (1.2) and (1.4) (see theorems 5.4 and 5.6 below).

Let us first consider minimizing sequences for (3.4), which are, by definition, sequences $\{(f_n,g_n)\}$ in $X$ satisfying

$$\lim_{n\to\infty}\|f_n\|^2=s,\qquad \lim_{n\to\infty}\|g_n\|^2=t,\qquad \lim_{n\to\infty}K(f_n,g_n)=I_1(s,t).$$

(Note that we do not require the elements $(f_n,g_n)$ of a minimizing sequence to satisfy exactly the constraints in (3.4). This convention will be useful later, in the proof of theorem 5.4.) To each such sequence, we associate a sequence of non-decreasing functions $Q_n(\omega)$, defined for $\omega>0$ by

$$Q_n(\omega)=\sup_{y\in\mathbb{R}}\int_{y-\omega}^{y+\omega}(|f_n|^2(x)+g_n^2(x))\,\mathrm{d}x.$$

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Since $\|f_n\|$ and $\|g_n\|$ remain bounded, then $\{Q_n\}$ comprises a uniformly bounded sequence of non-decreasing functions on $[0,\infty)$. A standard argument then implies that $\{Q_n\}$ must have a subsequence, which we denote again by $\{Q_n\}$, which converges pointwise and uniformly on compact sets to a non-decreasing limit function on $[0,\infty)$. Let $Q$ be this limit function, and define

$$\alpha = \lim_{\omega\to\infty} Q(\omega). \tag{3.14}$$

From the assumption that $\|f_n\|^2 + \|g_n\|^2 \to s+t$, it follows that $0 \leqq \alpha \leqq s+t$. The concentration-compactness method distinguishes three cases: $\alpha = s+t$, called the case of *compactness*; $\alpha = 0$, called the case of *vanishing*; and $0 < \alpha < s+t$, called the case of *dichotomy*. Our goal is to show that, for minimizing sequences of (3.4), only the case of compactness can occur. It will follow, by a standard argument, that every minimizing sequence is relatively compact, after suitable translations (cf. theorem 3.12 below). Later, we will show that this compactness property is also enjoyed by problem (3.1).

LEMMA 3.9. *Suppose $s,t \geqslant 0$. If $\{(f_n, g_n)\}$ is a minimizing sequence for $I_1(s,t)$, then $\{(f_n,g_n)\}$ is bounded in $X$.*

*Proof.* From standard Sobolev embedding and interpolation theorems, we have

$$\left|\int_{-\infty}^{\infty} g_n|f_n|^2\,\mathrm{d}x\right| \leqslant |f_n|_4^2\|g_n\| \leqslant C\|f_n\|_1^{1/2}\|f_n\|^{3/2}\|g_n\|.$$

But, for a minimizing sequence, $\|f_n\|$ and $\|g_n\|$ stay bounded, so it follows that

$$\left|\int_{-\infty}^{\infty} g_n|f_n|^2\,\mathrm{d}x\right| \leqslant C\|f_n\|_1^{1/2},$$

where $C$ is independent of $n$. Hence, since $\{K(f_n,g_n)\}$ is a bounded sequence, we obtain

$$\|f_n\|_1^2 = K(f_n,g_n) + \int_{-\infty}^{\infty} g_n|f_n|^2\,\mathrm{d}x + \|f_n\|^2 \leqslant C(1+\|f_n\|_1^{1/2}),$$

from which it follows that $\|f_n\|_1$ is bounded. Therefore,

$$\|(f_n,g_n)\|_X^2 = \|f_n\|_1^2 + \|g_n\|^2 \leqslant C,$$

and we are done. □

LEMMA 3.10. *Suppose $s,t > 0$, and let $\{(f_n,g_n)\}$ be any minimizing sequence for $I_1(s,t)$. Let $\alpha$ be as defined in (3.14). Then there exist numbers $s_1 \in [0,s]$ and $t_1 \in [0,t]$ such that*

$$s_1 + t_1 = \alpha \tag{3.15}$$

*and*

$$I_1(s_1,t_1) + I_1(s-s_1, t-t_1) \leqslant I_1(s,t). \tag{3.16}$$

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*Proof.* Let $\epsilon$ be an arbitrary positive number. From the definition of $\alpha$, it follows that, for $\omega$ sufficiently large, we have $\alpha-\epsilon < Q(\omega) \leqq Q(2\omega) \leqq \alpha$. By taking $\omega$ larger if necessary, we may also assume that $1/\omega < \epsilon$. Now, according to the definition of $Q$, we can choose $N$ so large that, for every $n \geqslant N$,

$$\alpha - \epsilon < Q_n(\omega) \leqslant Q_n(2\omega) \leqq \alpha + \epsilon. \tag{3.17}$$

Hence, for each $n \geqq N$, we can find $y_n$ such that

$$\int_{y_n-\omega}^{y_n+\omega} (|f_n|^2 + g_n^2)\,\mathrm{d}x > \alpha - \epsilon \quad \text{and} \quad \int_{y_n-2\omega}^{y_n+2\omega} (|f_n|^2 + g_n^2)\,\mathrm{d}x < \alpha + \epsilon. \tag{3.18}$$

Now choose smooth functions $p$ and $r$ on $\mathbb{R}$ such that

$$p(x) = \begin{cases} 1 & \text{for } x \in [-1,1], \\ 0 & \text{for } x \notin [-2,2], \end{cases}$$

$$r(x) = \begin{cases} 1 & \text{for } x \notin [-2,2], \\ 0 & \text{for } x \in [-1,1], \end{cases}$$

and $p^2(x) + r^2(x) = 1$ for all $x \in \mathbb{R}$. Define $p_\omega(x) = p(x/\omega)$ and $r_\omega(x) = r(x/\omega)$, and let

$$(\varphi_n(x), h_n(x)) = (p_\omega(x-y_n)f_n(x), p_\omega(x-y_n)g_n(x))$$

and

$$(l_n(x), j_n(x)) = (r_\omega(x-y_n)f_n(x), r_\omega(x-y_n)g_n(x)).$$

From lemma 3.9, it follows that the sequences $\{\varphi_n\}$, $\{h_n\}$, $\{l_n\}$ and $\{j_n\}$ are bounded in $L^2$. So, by passing to subsequences, we may assume that there exist $s_1 \in [0,s]$ and $t_1 \in [0,t]$ such that

$$\int_{-\infty}^{\infty} |\varphi_n|^2\,\mathrm{d}x \to s_1 \quad \text{and} \quad \int_{-\infty}^{\infty} h_n^2\,\mathrm{d}x \to t_1,$$

whence it follows also that

$$\int_{-\infty}^{\infty} |l_n|^2\,\mathrm{d}x \to s - s_1 \quad \text{and} \quad \int_{-\infty}^{\infty} j_n^2\,\mathrm{d}x \to t - t_1.$$

Now

$$s_1 + t_1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} (|\varphi_n|^2 + h_n^2)\,\mathrm{d}x = \lim_{n\to\infty} \int_{-\infty}^{\infty} p_\omega^2(|f_n|^2 + g_n^2)\,\mathrm{d}x.$$

Here and below we have suppressed the arguments of $p_\omega$ and $r_\omega$ for brevity of notation. From (3.18) it follows that, for every $n \in \mathbb{N}$,

$$\alpha - \epsilon < \int_{-\infty}^{\infty} p_\omega^2(|f_n|^2 + g_n^2)\,\mathrm{d}x < \alpha + \epsilon.$$

Hence

$$|(s_1 + t_1) - \alpha| < \epsilon.$$

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Next observe that

$$|p_\omega'|_\infty + |r_\omega'|_\infty \leqq \frac{1}{\omega}(|p'|_\infty + |r'|_\infty) \leqslant \frac{C}{\omega},$$

and, by lemma 3.9, $\|f_n\|_1 \leqslant C$, where $C$ denotes constants that are independent of $\omega$ and $n$. Hence

$$K(\varphi_n, h_n) \leqslant \int_{-\infty}^{\infty} (p_\omega^2|f_n'|^2 - p_\omega^2 g_n|f_n|^2)\,\mathrm{d}x + \int_{-\infty}^{\infty} (p_\omega^2 - p_\omega^3) g_n|f_n|^2\,\mathrm{d}x + \frac{C}{\omega} \tag{3.19}$$

and

$$K(l_n, j_n) \leqslant \int_{-\infty}^{\infty} (r_\omega^2|f_n'|^2 - r_\omega^2 g_n|f_n|^2)\,\mathrm{d}x + \int_{-\infty}^{\infty} (r_\omega^2 - r_\omega^3) g_n|f_n|^2\,\mathrm{d}x + \frac{C}{\omega}. \tag{3.20}$$

On the other hand, from (3.18), we get

$$\begin{aligned}\left|\int_{-\infty}^{\infty} ((p_\omega^2 - p_\omega^3) + (r_\omega^2 - r_\omega^3)) g_n|f_n|^2\,\mathrm{d}x\right| &\leqslant 2|f_n|_\infty \int_{\omega\leqq|x-y_n|\leqq 2\omega} (|f_n|^2 + g_n^2)\,\mathrm{d}x \\ &\leqslant C\epsilon.\end{aligned}$$

Therefore, adding (3.19) and (3.20) and using $p_\omega^2 + r_\omega^2 = 1$, we get

$$K(\varphi_n, h_n) + K(l_n, j_n) \leqslant K(f_n, g_n) + C\left(\epsilon + \frac{1}{\omega}\right) \leqslant K(f_n, g_n) + C\epsilon. \tag{3.21}$$

For any given value of $\epsilon$, each of the terms in (3.21) is bounded independently of $n$, so, by passing to subsequences, we may assume that $K(\varphi_n, h_n) \to K_1$ and $K(l_n, j_n) \to K_2$, where

$$K_1 + K_2 \leqslant I_1(s, t) + C\epsilon.$$

Combining the results of the preceding paragraphs, and recalling that $\epsilon$ can be taken arbitrarily small and $\omega$ arbitrarily large, we see that, for every $k \in \mathbb{N}$, we can find sequences

$$\{(\varphi_n^{(k)}, h_n^{(k)})\} \quad \text{and} \quad \{(l_n^{(k)}, j_n^{(k)})\} \quad \text{in } X$$

such that

$$\begin{aligned}\|\varphi_n^{(k)}\|^2 &\to s_1(k),\\ \|h_n^{(k)}\|^2 &\to t_1(k),\\ \|l_n^{(k)}\|^2 &\to s - s_1(k),\\ \|j_n^{(k)}\|^2 &\to t - t_1(k)\end{aligned}$$

and

$$\begin{aligned}K(\varphi_n^{(k)}, h_n^{(k)}) &\to K_1(k),\\ K(l_n^{(k)}, j_n^{(k)}) &\to K_2(k),\end{aligned}$$

where $s_1(k) \in [0, s]$, $t_1(k) \in [0, t]$,

$$|s_1(k) + t_1(k) - \alpha| \leqslant \frac{1}{k} \tag{3.22}$$

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and

$$K_1(k) + K_2(k) \leqslant I_1(s,t) + \frac{1}{k}. \tag{3.23}$$

By passing to subsequences, we may assume that $s_1(k)$, $t_1(k)$, $K_1(k)$ and $K_2(k)$ converge to numbers $s_1 \in [0,s]$, $t_1 \in [0,t]$, $K_1$ and $K_2$. Moreover, by redefining $(\varphi_n, g_n)$ and $(h_n, j_n)$ as the diagonal subsequences

$$(\varphi_n, g_n) = (\varphi_n^{(n)}, g_n^{(n)}) \quad \text{and} \quad (h_n, j_n) = (h_n^{(n)}, j_n^{(n)}),$$

we may assume that

$$\begin{aligned}\|\varphi_n\|^2 &\to s_1,\\ \|h_n\|^2 &\to t_1,\\ \|l_n\|^2 &\to s - s_1,\\ \|j_n\|^2 &\to t - t_1\end{aligned}$$

and

$$\begin{aligned}K(\varphi_n, g_n) &\to K_1,\\ K(h_n, j_n) &\to K_2.\end{aligned}$$

Now letting $k \to \infty$ in (3.22) gives (3.15), and similarly (3.23) will imply (3.16), provided we can show that

$$K_1 \geqslant I_1(s_1, t_1) \tag{3.24}$$

and

$$K_2 \geqslant I_1(s - s_1, t - t_1). \tag{3.25}$$

To prove (3.24), we consider three cases: (i) $s_1 > 0$ and $t_1 > 0$; (ii) $s_1 = 0$; and (iii) $t_1 = 0$. In case (i), for $n$ sufficiently large, we have $\|\varphi_n\| > 0$ and $\|h_n\| > 0$, so we may define

$$\beta_n = \frac{\sqrt{s_1}}{\|\varphi_n\|} \quad \text{and} \quad \theta_n = \frac{\sqrt{t_1}}{\|h_n\|}.$$

Then $\|\beta_n \varphi_n\|^2 = s_1$ and $\|\theta_n h_n\|^2 = t_1$, so

$$K(\beta_n \varphi_n, \theta_n h_n) \geqslant I_1(s_1, t_1).$$

But since $\beta_n$ and $\theta_n$ approach 1 as $n \to \infty$, we have $K(\beta_n \varphi_n, \theta_n h_n) \to K_1$, from which (3.24) follows. In case $s_1 = 0$, we have $\|\varphi_n\| \to 0$, so

$$\left| \int_{-\infty}^{\infty} h_n |\varphi_n|^2 \, \mathrm{d}x \right| \leqslant \|\varphi_n\|_1 \|\varphi_n\| \|h_n\| \to 0, \tag{3.26}$$

whence

$$K_1 = \lim_{n\to\infty} K(\varphi_n, h_n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} (|\varphi_n'|^2 - h_n |\varphi_n|^2) \, \mathrm{d}x \geqslant 0. \tag{3.27}$$

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Since $I_1(s_1,t_1) = I_1(0,t_1) = 0$, this proves (3.24) in case (ii). Finally, if $t_1 = 0$, then $\|h_n\| \to 0$, so (3.26) and (3.27) again hold, which proves (3.24) in this case since $I_1(s_1,0) = 0$. Therefore, equation (3.24) has been proved in all cases. The proof of (3.25) is similar, with $s - s_1$ and $t - t_1$ playing the roles of $s_1$ and $t_1$. □

LEMMA 3.11. *Suppose $s,t > 0$, and let $\{(f_n,g_n)\}$ be any minimizing sequence for $I_1(s,t)$. If $\alpha$ is as defined in (3.14), then $\alpha = s+t$.*

*Proof.* First we show that $\alpha \neq 0$. If $\alpha = 0$, then

$$\sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} |f_n|^q \,\mathrm{d}x \to 0$$

for every $\omega > 0$, so lemma 3.8 implies that $f_n \to 0$ in $L^4$. But then, since

$$\left|\int_{-\infty}^{\infty} g_n|f_n|^2 \,\mathrm{d}x\right| \leqslant |f_n|_4^{1/2}\|g_n\|$$

and $\|g_n\|$ stays bounded, we have that

$$\int_{-\infty}^{\infty} g_n|f_n|^2 \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$I_1(s,t) = \lim_{n\to\infty} K(f_n,g_n) \geqq \liminf_{n\to\infty} \int_{-\infty}^{\infty} |f_n'|^2 \,\mathrm{d}x \geqq 0,$$

which contradicts lemma 3.3.

It remains then to show that $\alpha$ cannot lie in $(0, s+t)$. Suppose to the contrary that $0 < \alpha < s+t$. Let $s_1$ and $t_1$ be as defined in lemma 3.10, and let $s_2 = s - s_1$, $t_2 = t - t_1$. Then (3.15) implies both that $s_1 + t_1 = \alpha > 0$ and $s_2 + t_2 = (s+t) - \alpha > 0$. Since $s_1 + s_2 = s > 0$ and $t_1 + t_2 = t > 0$, we conclude from lemma 3.7 that (3.12) holds. But this contradicts (3.16). □

THEOREM 3.12. *Let $s,t > 0$, and let $\{(f_n,g_n)\}$ be any minimizing sequence for $I_1(s,t)$. Then there is a subsequence $\{(f_{n_k},g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

*converges strongly in $X$ to some $(f,g)$. The pair $(f,g)$ is a minimizer for $I_1(s,t)$, i.e. $\|f\|^2 = s$, $\|g\|^2 = t$ and $K(f,g) = sM(g) = I_1(s,t)$.*

*Proof.* The proof is a variation on that of the fundamental lemma I.1 (i) of [25]. For any minimizing sequence $\{(f_n,g_n)\}$ of $I_1(s,t)$, define $\alpha$ as in (3.14), and let $\{(f_n,g_n)\}$ continue to denote the subsequence associated with $\alpha$. From lemma 3.11, we have that $\alpha = s+t$. Hence there exists $\omega_0$ such that, for $n$ sufficiently large, $Q_n(\omega_0) > \frac{1}{2}(s+t)$. For such $n$, we choose $y_n$ such that

$$\int_{y_n-\omega_0}^{y_n+\omega_0} (|f_n|^2 + g_n^2) \,\mathrm{d}x > \tfrac{1}{2}(s+t).$$

Now let $\sigma$ be an arbitrary number in the interval $(\frac{1}{2}(s+t), s+t)$. Then we can find $\omega_1$ such that, for $n$ sufficiently large, $Q_n(\omega_1) > \sigma$, and so we can choose $\tilde{y}_n$ such that

$$\int_{\tilde{y}_n-\omega_1}^{\tilde{y}_n+\omega_1} (|f_n|^2 + g_n^2)\,\mathrm{d}x > \sigma.$$

Since

$$\int_{-\infty}^{\infty} (|f_n|^2 + g_n^2)\,\mathrm{d}x \to s+t \quad \text{as } n \to \infty,$$

it follows that, for large $n$, the intervals $[\tilde{y}_n-\omega_1, \tilde{y}_n+\omega_1]$ and $[y_n-\omega_0, y_n+\omega_0]$ must overlap. Therefore, defining $\omega = 2\omega_1+\omega_0$, we have that, for $n$ sufficiently large,

$$[\tilde{y}_n-\omega_1, \tilde{y}_n+\omega_1] \subset [y_n-\omega, y_n+\omega].$$

Hence

$$\int_{y_n-\omega}^{y_n+\omega} (|f_n|^2 + g_n^2)\,\mathrm{d}x > \sigma.$$

In particular, we may take $\sigma = s+t-1/k$, and thus we have shown that, for every $k \in \mathbb{N}$, there exists $\omega_k \in \mathbb{R}$ such that, for all sufficiently large $n$,

$$\int_{y_n-\omega_k}^{y_n+\omega_k} (|f_n|^2 + g_n^2)\,\mathrm{d}x > s+t-\frac{1}{k}. \tag{3.28}$$

Let us now define $w_n(x) = f_n(x+y_n)$ and $z_n(x) = g_n(x+y_n)$. Then, by (3.28), for every $k \in \mathbb{N}$, we have

$$\int_{-\omega_k}^{\omega_k} (|w_n|^2 + z_n^2)\,\mathrm{d}x > s+t-\frac{1}{k}, \tag{3.29}$$

provided $n$ is sufficiently large. Now, by lemma 3.9, $\{(w_n, z_n)\}$ is bounded in $X$, so there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges weakly in $X$ to a limit $(f, g) \in X$. By Fatou's lemma, $\|f\|^2 \leqq s$ and $\|g\|^2 \leqq t$. For each $k \in \mathbb{N}$, the inclusion of $H^1(-\omega_k, \omega_k)$ into $L^2(-\omega_k, \omega_k)$ is compact, so, by passing to a subsequence, we may assume that $w_n \to f$ strongly in $L^2(-\omega_k, \omega_k)$. Furthermore, by using a diagonalization argument, we may assume that a single subsequence of $\{w_n\}$ has been chosen which has this property for every $k$. Now

$$\limsup_{n\to\infty} \int_{-\omega_k}^{\omega_k} z_n^2\,\mathrm{d}x \leqslant t,$$

so taking $n \to \infty$ in (3.29) gives

$$\int_{-\infty}^{\infty} |f|^2\,\mathrm{d}x \geqslant \int_{-\omega_k}^{\omega_k} |f|^2\,\mathrm{d}x = \lim_{n\to\infty} \int_{-\omega_k}^{\omega_k} |w_n|^2\,\mathrm{d}x \geqslant s-\frac{1}{k}.$$

Since $\|f\|^2 \leqslant s$ and $k \in \mathbb{N}$ is arbitrary, this implies that $\|f\|^2 = s$. Hence $w_n \to f$ strongly in $L^2$.

Next, observe that

$$\int_{-\infty}^{\infty} (z_n|w_n|^2 - g|f|^2)\,\mathrm{d}x = \int_{-\infty}^{\infty} z_n(|w_n|^2 - |f|^2)\,\mathrm{d}x + \int_{-\infty}^{\infty} (z_n - g)|f|^2\,\mathrm{d}x, \tag{3.30}$$

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and consider separately the behaviour of the integrals on the right-hand side as $n \to \infty$. For the first integral, we have

$$\left| \int_{-\infty}^{\infty} z_n(|w_n|^2 - |f|^2)\,\mathrm{d}x \right| \leqslant \|z_n\| \|w_n - f\|(\|w_n\|_1 + \|f\|_1),$$

and the right-hand side goes to zero since $\{(w_n, z_n)\}$ is bounded in $X$, $f$ is in $H^1$ and $w_n \to f$ in $L^2$. The second integral on the right-hand side of (3.30) converges to zero because $f^2 \in L^2$ and $z_n$ converges to $g$ weakly in $L^2$. It follows then from (3.30) that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} z_n|w_n|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} g|f|^2\,\mathrm{d}x. \tag{3.31}$$

Since, by Fatou's lemma,

$$\int_{-\infty}^{\infty} |f'|^2\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{-\infty}^{\infty} |w_n'|^2\,\mathrm{d}x,$$

it follows that

$$I_1(s,t) = \lim_{n\to\infty} K(w_n, z_n) \geqslant \int_{-\infty}^{\infty} (|f'|^2 - g|f|^2)\,\mathrm{d}x = K(f,g). \tag{3.32}$$

We now claim that $\|g\|^2 = t$. To see this, first observe that lemma 3.3 and (3.32) imply that

$$\int_{-\infty}^{\infty} g|f|^2\,\mathrm{d}x > 0. \tag{3.33}$$

In particular, equation (3.33) gives that $\|g\| \neq 0$. So $0 < \|g\|^2 \leqq t$, and we can define $\eta \geqslant 1$ by $\eta = \sqrt{t}/\|g\|$. Then $\|\eta g\|^2 = t$, so, by (3.32),

$$\begin{aligned} I_1(s,t) &\leqq K(f, \eta g) \\ &= K(f,g) + (1-\eta) \int_{-\infty}^{\infty} g|f|^2\,\mathrm{d}x \\ &\leqq I_1(s,t) + (1-\eta) \int_{-\infty}^{\infty} g|f|^2\,\mathrm{d}x. \end{aligned}$$

But then (3.33) implies that $(1-\eta) \geqq 0$, so $\eta = 1$ and $\|g\|^2 = t$, as was claimed.

It follows that $\{z_n\}$ converges strongly to $g$, and that $(f,g)$ is a minimizer for $I_1(s,t)$. To complete the proof of the lemma, it remains only to observe that since equality holds in (3.32), then

$$\int_{-\infty}^{\infty} |w_n'|^2\,\mathrm{d}x \to \int_{-\infty}^{\infty} |f'|^2\,\mathrm{d}x \quad \text{as } n \to \infty,$$

and therefore $w_n$ converges to $f$ strongly in $H^1$. □

The variational problem in (3.5) can also be solved by the method of concentration compactness, and indeed this has already been done in several places in the literature (see, for example, [1, theorem 2.9]). However, in the results above, we have already done most of the work involved in the proof, so, for the reader's

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convenience, we sketch here the remainder of the proof. Assuming $t > 0$, one lets $\{g_n\}$ be any minimizing sequence for $I_2(t)$, and defines

$$\tilde{Q}_n(\omega) = \sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} g_n^2(x)\,\mathrm{d}x.$$

Again, we may assume that $\tilde{Q}_n$ converges pointwise to a non-decreasing function $\tilde{Q}$ on $[0,\infty)$, and we define

$$\tilde{\alpha} = \lim_{\omega\to\infty} \tilde{Q}(\omega).$$

The same arguments as in the proofs of lemmas 3.9 and 3.10 show that $\|g_n\|_1$ remains bounded, and that

$$I_2(\tilde{\alpha}) + I_2(t-\tilde{\alpha}) \leqslant I_2(t).$$

But it then follows from (3.13) that $\tilde{\alpha} \notin (0,t)$, and, as before, we see from lemma 3.8 that $\tilde{\alpha} \neq 0$. Hence $\tilde{\alpha} = t$, and, using the same argument as in the proof of theorem 3.12, we deduce the following result.

THEOREM 3.13. *Let $t > 0$, and let $\{g_n\}$ be any minimizing sequence for $I_2(t)$. Then there is a subsequence $\{g_{n_k}\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$g_{n_k}(\cdot + y_k)$$

*converges strongly in $H^1$ to some $g \in H^1$. The limit $g$ is a minimizer for $I_2(t)$, i.e. $\|g\|^2 = t$ and $J(g) = I_2(t)$.*

As consequences of theorems 3.12 and 3.13, we obtain explicit values for the constant $I_1(1,1)$ and $I_2(1)$.

COROLLARY 3.14. *For every $s, t \geqslant 0$,*

$$I_1(s,t) = A_1 s t^{2/3},$$

*where $A_1 = I_1(1,1) = -(\frac{3}{16})^{2/3}$.*

*Proof.* We may assume $s, t > 0$. Let $(f,g) \in X$ be a minimizer for $I_1(s,t)$, whose existence is guaranteed by theorem 3.12. Then $f$ and $g$ satisfy the Lagrange multiplier equations (2.11), in which $\lambda$ and $\mu$ are the multipliers. Therefore, up to a phase factor and a translation, $f = f_1$ and $g = g_1$, where $f_1$ and $g_1$ are given in (2.12).

To determine the values of $\lambda$ and $\mu$, we substitute $f_1$ and $g_1$ into the constraint equations $\|f\|^2 = s$ and $\|g\|^2 = t$. Using the formula

$$\int_{-\infty}^{\infty} \operatorname{sech}^{2n}(x)\,\mathrm{d}x = \frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(\frac{1}{2}(2n+1))}, \tag{3.34}$$

one finds that $\lambda = (\frac{3}{16}t)^{2/3}$, $\mu = s(12t)^{-1/3}$ and

$$K(f_1,g_1) = -4\lambda^{3/2}\mu = -s(\tfrac{3}{16}t)^{2/3}.$$

Since $I_1(s,t) = K(f_1,g_1)$, this completes the proof. □

COROLLARY 3.15. *For every* $t \geqslant 0$,

$$I_2(t) = A_2 t^{5/3},$$

*where* $A_2 = I_2(1) = -\frac{8}{5}(\frac{3}{8})^{5/3} q^{4/3}$.

*Proof.* We may assume $t > 0$. Let $g$ be a minimizer for $I_2(t)$, whose existence is guaranteed by theorem 3.13. Then $g$ satisfies the Lagrange multiplier equation (2.13), in which $\kappa$ is the multiplier. Therefore, up to translation, $g = g_2$, where $g_2$ is given in (2.14). From $\|g_2\|^2 = t$ and (3.34), we deduce that

$$\kappa = (\tfrac{3}{8} q^2 t)^{2/3}.$$

The statement of the corollary then follows from the substitution of the formulae for $g_2(x)$ and $\kappa$ into the expression

$$I_2(t) = J(g_2) = \int_{-\infty}^{\infty} ((g_2')^2 - q g_2^3)\,\mathrm{d}x,$$

and using again (3.34). □

LEMMA 3.16. *Suppose* $s, t > 0$. *Let* $(f_1, g_1)$ *be a minimizer for* $I_1(s,t)$ *and let* $g_2$ *be a minimizer for* $I_2(t)$. *Then*

$$M(g_2) = A_3 t^{2/3},$$

*where*

$$A_3 = \frac{-2 \cdot 3^{2/3} q^{1/3}}{q + 8 + \sqrt{q^2 + 16q}}. \tag{3.35}$$

*Proof.* The proof of (3.35) depends on being able to find explicitly the minimizing function $f$ for $K(f, g_2)$ on the constraint set $\{\|f\| = 1\}$. The Lagrange multiplier equation for this variational problem is

$$-f'' - f g_2 = \lambda f, \tag{3.36}$$

so we see that the minimizer $f$ is an eigenfunction for the Schrödinger operator $\mathcal{L} = -\mathrm{d}^2/\mathrm{d}x^2 - g_2$ with potential $g_2$, and the Lagrange multiplier $\lambda$ is the eigenvalue corresponding to $f$. Further, multiplying (3.36) by $f$ and integrating over $\mathbb{R}$, we see that the constant $C$ being sought is actually the same as the least or ground-state eigenvalue $\lambda$, so that $f$ is a ground-state eigenfunction.

Now, $g_2(x) = a\,\mathrm{sech}^2(bx)$, where $a$ and $b$ are constants, and for such potentials, with arbitrary positive values of $a$ and $b$, the complete solution of the spectral problem for $\mathcal{L}$ is well known (see, for example, [31, p. 768]). It turns out that the ground-state eigenfunction is a constant multiple of $\mathrm{sech}^p(bx)$, where

$$p = \sqrt{\left(\frac{a}{b^2}\right) + \frac{1}{4}} - \frac{1}{2}, \tag{3.37}$$

and the corresponding eigenvalue is

$$\lambda = -b^2 p^2. \tag{3.38}$$

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In the proof of corollary 3.15, we saw that the particular values of $a$ and $b$ corresponding to our potential $g_2$ are $a = \kappa/q$ and $b = \frac{1}{2}\sqrt{\kappa}$, where $\kappa = (\frac{3}{8}q^2 t)^{2/3}$. Using these values to compute $p$ and $\lambda$ from (3.37) and (3.38), we obtain the asserted value for $A_3 = \lambda/t^{2/3}$. □

COROLLARY 3.17. *For $s, t \geqslant 0$, we have*

$$A_1 s t^{2/3} + A_2 t^{5/3} \leqslant I(s,t) \tag{3.39}$$

*and*

$$I(s,t) \leqslant A_3 s t^{2/3} + A_2 t^{5/3}. \tag{3.40}$$

*Proof.* From (3.3), we have

$$I_1(s,t) + I_2(t) \leqslant I(s,t),$$

which, in view of corollaries 3.14 and 3.15, yields (3.39). To prove (3.40), let $g_2$ be as in lemma 3.16. Then lemma 3.16 and (3.3) give

$$I(s,t) \leqslant sM(g_2) + J(g_2) \leqslant A_3 s t^{2/3} + A_2 t^{5/3}.$$

□

REMARK 3.18. The case when $q = 2$ is special, because then the function $g_1$ defined in corollary 3.14 coincides with the function $g_2$ defined in corollary 3.15. It follows that, in this case, $A_1 = A_3$, and hence

$$I(s,t) = A_1 s t^{2/3} + A_2 t^{5/3}.$$

Moreover, the pair $(f_1, g_1)$ defined in corollary 3.14 is an explicit minimizer for the problem (3.1). In fact, it follows from the uniqueness of the solutions of (3.4) and (3.5) that $(f_1, g_1)$ is the unique minimizer for (3.1) (up to a translation in $x$ and a multiplication of $f_1$ by a constant of absolute value 1). This is the case analysed by Chen in [14].

Our next goal is to investigate the subadditivity of $I(s,t)$. The preceding corollary and remark suggest the strategy of comparing $I(s,t)$ with a function of the type $At^{5/3} + Bst^{2/3}$, which, as was seen in the proof of lemma 3.7, is subadditive when $A$ and $B$ are negative constants. The next few lemmas are devoted to showing that $I(s,t)$ is close enough to a function of this type to inherit the property of subadditivity.

LEMMA 3.19. *Suppose $s, t \geqslant 0$. Then we can find a sequence $\{g_n^{s,t}\}$ in $H^1$ such that $\lim_{n\to\infty} M(g_n^{s,t}) = M_0(s,t)$ and $\lim_{n\to\infty} J(g_n^{s,t}) = J_0(s,t)$ exist and satisfy*

(i) $sM_0(s,t) + J_0(s,t) = I(s,t)$;

(ii) $A_1 s t^{2/3} \leqslant sM_0(s,t) \leqslant A_3 s t^{2/3}$; *and*

(iii) $A_2 t^{5/3} \leqslant J_0(s,t) \leqslant A_2 t^{5/3} + (A_3 - A_1) s t^{2/3}$.

*Proof.* Let $\{g_n^{s,t}\}$ be any minimizing sequence for $I(s,t)$ in the strict sense, i.e. a sequence in $H^1$ such that $\|g_n^{s,t}\|^2 = t$ and

$$\lim_{n\to\infty}(sM(g_n^{s,t}) + J(g_n^{s,t})) = I(s,t). \tag{3.41}$$

Since $\{M(g_n^{s,t})\}$ and $\{J(g_n^{s,t})\}$ are bounded sequences of real numbers, by passing to a subsequence, we may assume that the limits $M_0(s,t)$ and $J_0(s,t)$ exist as defined above. Then (i) follows immediately from (3.41).

Next, observe that corollaries 3.14 and 3.15 imply that

$$A_1 s t^{2/3} \leqslant sM_0(s,t) \tag{3.42}$$

and

$$A_2 t^{5/3} \leqslant J_0(s,t). \tag{3.43}$$

From (i), (3.40) and (3.42), we get

$$A_1 s t^{2/3} + J_0(s,t) \leqslant A_3 s t^{2/3} + A_2 t^{5/3},$$

which implies the upper bound in (iii). From (i), (3.40) and (3.43), we get

$$sM_0(s,t) + A_2 t^{5/3} \leqslant A_3 s t^{2/3} + A_2 t^{5/3},$$

which implies the upper bound in (ii). □

REMARK 3.20. As defined above in lemma 3.19, the quantities $M_0(s,t)$ and $J_0(s,t)$ could depend on the choice of the minimizing sequence $\{g_n^{s,t}\}$, as well as on $s$ and $t$. This ambiguity of notation will not affect the validity of the statements that follow.

LEMMA 3.21. *Suppose $s_1, s_2, t_1, t_2 \geqslant 0$ with $s_2 t_1 > s_1 t_2$. Then*

$$t_2^{5/3} J_0(s_1,t_1) \leqslant t_1^{5/3} J_0(s_2,t_2) \tag{3.44}$$

*and*

$$t_2^{2/3} M_0(s_1,t_1) \geqslant t_1^{2/3} M_0(s_2,t_2). \tag{3.45}$$

*Proof.* The inequalities are obvious when $t_2 = 0$, so we may assume that $t_2 > 0$, and hence also $t_1 > 0$. Let $\sigma = t_1/t_2$, and for any $g \in H^1$ define $g^*$ as in lemma 3.4. Then, for all $n \in \mathbb{N}$, $\|(g_n^{s_2,t_2})^*\|^2 = t_1$, so, by (3.3), lemma 3.4 and lemma 3.19(i), we have

$$\begin{aligned} s_1 M_0(s_1,t_1) + J_0(s_1,t_1) &= I(s_1,t_1) \\ &= \inf\{s_1 M(g) + J(g) : \|g\|^2 = t_1\} \\ &\leqslant s_1 M((g_n^{s_2,t_2})^*) + J((g_n^{s_2,t_2})^*) \\ &= s_1 \sigma^{2/3} M(g_n^{s_2,t_2}) + \sigma^{5/3} J(g_n^{s_2,t_2}). \end{aligned}$$

Taking $n \to \infty$ then gives

$$s_1 M_0(s_1,t_1) + J_0(s_1,t_1) \leqslant s_1 \sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2). \tag{3.46}$$

Similarly, we obtain

$$s_2 M_0(s_2,t_2) + J_0(s_2,t_2) \leqslant s_2\sigma^{-2/3}M_0(s_1,t_1) + \sigma^{-5/3}J_0(s_1,t_1). \tag{3.47}$$

Multiplying (3.46) by $s_2$ and (3.47) by $s_1\sigma^{2/3}$, and adding the results, we obtain

$$\sigma^{-5/3}J_0(s_1,t_1)(s_2\sigma - s_1) \leqslant J_0(s_2,t_2)(s_2\sigma - s_1).$$

Since $s_2\sigma - s_1 > 0$, this implies (3.44). Similarly, multiplying (3.47) by $\sigma^{5/3}$, adding to (3.47) and rearranging, we obtain

$$\sigma^{2/3}M_0(s_2,t_2)(s_2\sigma - s_1) \leqslant M_0(s_1,t_1)(s_2\sigma - s_1),$$

which implies (3.45). □

LEMMA 3.22. *Suppose* $s_1, s_2, t_1, t_2 > 0$. *Let* $\eta = t_1/t_2$.

(i) *If*

$$\eta > |A_1/A_3|^{3/2} - 1, \tag{3.48}$$

*then*

$$(1 + 1/\eta)^{2/3}M_0(s_1,t_1) < M_0(s_2,t_2). \tag{3.49}$$

(ii) *Let* $\alpha(\eta) = ((1+\eta)^{2/3} - 1)\eta^{-5/3}$. *If*

$$\alpha(\eta) > |A_1/A_3| - 1, \tag{3.50}$$

*then*

$$s_2[(1+\eta)^{2/3}-1]M_0(s_2,t_2) < J_0(s_1,t_1)+J_0(s_2,t_2)-(1+\eta)^{5/3}J_0(s_2,t_2). \tag{3.51}$$

*Proof.* Since $s_1 > 0$, we can use lemma 3.19 (ii) to write

$$(1+1/\eta)^{2/3}M_0(s_1,t_1) \leqslant (1+1/\eta)^{2/3}A_3 t_1^{2/3} = (t_1+t_2)^{2/3}A_3$$

and

$$A_1 t_2^{2/3} \leqslant M(s_2,t_2).$$

Combining these inequalities with (3.48), we obtain (3.49). This proves (i).

To prove (ii), use lemma 3.19 (ii) to write

$$s_2[(1+\eta)^{2/3} - 1]M_0(s_2,t_2) \leqslant s_2[(1+\eta)^{2/3} - 1]A_3 t_2^{2/3},$$

and use lemma 3.19 (iii) to write

$$\begin{aligned} J_0(s_1,t_1) + J_0(s_2,t_2) - (1+\eta)^{5/3}J_0(s_2,t_2) \\ &\geqslant J_0(s_1,t_1) - \eta^{5/3}J_0(s_2,t_2) \\ &\geqslant A_2 t_1^{5/3} - \eta^{5/3}(A_2 t_2^{5/3} + s_2(A_3 - A_1)t_2^{2/3}) \\ &= -s_2\eta^{5/3}|A_3 - A_1|t_2^{2/3}. \end{aligned}$$

Also, equation (3.50) implies that

$$A_3((1-\eta)^{2/3} - 1) < |A_3 - A_1|\eta^{5/3}.$$

Combining these inequalities gives (3.51). □

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Now define $\eta_1(q) = (|A_1/A_3|^{3/2} - 1)$, and define $\eta_2(q)$ to be the value of $\eta$ for which the right- and left-hand sides of (3.50) are equal. (When the right-hand side is zero, we can take $\eta_2(q) = \infty$.) If $\eta_2(q) > \eta_1(q)$, then any positive real number $\eta$ satisfies at least one of the inequalities (3.48) or (3.50). Analysis of the functions $\eta_1$ and $\eta_2$ shows that there does exist a non-empty interval $(q_1, q_2)$ of values of $q$ for which the inequality $\eta_2(q) > \eta_1(q)$ is valid. In fact, when $q = 2$, one has $A_1 = A_3$ (see remark 3.18), so $\eta_1(2) = 0$, while $\eta_2(2) = \infty$. Therefore, the interval $(q_1, q_2)$ contains at least a neighbourhood of $q = 2$. On the other hand, as $q \to 0$ or $q \to \infty$, one has $\eta_1(q) \to \infty$ and $\eta_2(q) \to 0$, so the interval $(q_1, q_2)$ is bounded above and bounded away from zero.

We can now prove that $I(s, t)$ is subadditive, at least when $q \in (q_1, q_2)$.

THEOREM 3.23. *Suppose that $q \in (q_1, q_2)$. Let $s_1, s_2, t_1, t_2 \geqslant 0$, and suppose that $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $s_1 + t_1 > 0$ and $s_2 + t_2 > 0$. Then*

$$I(s_1 + s_2, t_1 + t_2) < I(s_1, t_1) + I(s_2, t_2). \tag{3.52}$$

*Proof.* We may assume, without loss of generality, that $s_2t_1 \geqslant s_1t_2$. If $s_2t_1 = s_1t_2$, then our assumptions imply that $s_1$, $s_2$, $t_1$ and $t_2$ must all be positive, and since $(t_1 + t_2)/t_2 = (s_1 + s_2)/s_2$, we can write

$$\begin{aligned} I(s_1 + s_2, t_1 + t_2) &= \left(\frac{t_1 + t_2}{t_2}\right)^{5/3} I(s_2, t_2) = \left(1 + \frac{t_1}{t_2}\right)^{5/3} I(s_2, t_2) \\ &< \left[1 + \left(\frac{t_1}{t_2}\right)^{5/3}\right] I(s_2, t_2) = I(s_2, t_2) + I(s_1, t_1). \end{aligned}$$

Here we have twice used lemma 3.6, and have also used the fact that $I(s_2, t_2) < 0$, which is a consequence of lemma 3.17.

We may therefore assume that $s_2t_1 > s_1t_2$, and, in particular, that $s_2 > 0$ and $t_1 > 0$. For now, we assume also that $t_2 > 0$, and we define $\eta = t_1/t_2$. From our hypothesis on $q$, we know that $\eta$ satisfies either (3.48) or (3.50); we consider the two cases separately.

In the case when (3.48) holds, define $\sigma = 1 + 1/\eta$ and $h_n(x) = \sigma^{2/3} g_n^{s_1,t_1}(\sigma^{1/3}x)$. By passing to a subsequence if necessary, we may assume that $J(h_n)$ and $M(h_n)$ converge as $n \to \infty$. Then, using lemma 3.4 and (3.44), we get

$$\begin{aligned} \lim_{n\to\infty} J(h_n) &= \sigma^{5/3} J_0(s_1, t_1) \\ &\leqslant J_0(s_1, t_1) + \left(\frac{t_2}{t_1}\right)^{5/3} J_0(s_1, t_1) \\ &\leqslant J_0(s_1, t_1) + J_0(s_2, t_2). \end{aligned} \tag{3.53}$$

Next, suppose that $s_1 > 0$. Then, from lemma 3.4 and (3.49), we have

$$\begin{aligned} (s_1 + s_2) \lim_{n\to\infty} M(h_n) &= (s_1 + s_2)\sigma^{2/3} M_0(s_1, t_1) \\ &\leqslant s_1 M_0(s_1, t_1) + s_2\sigma^{2/3} M_0(s_1, t_1) \\ &< s_1 M_0(s_1, t_1) + s_2 M_0(s_2, t_2). \end{aligned} \tag{3.54}$$

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Now, since $\|h_n\|^2 = t_1 + t_2$, we get from (3.53) and (3.54) that

$$\begin{aligned} I(s_1+s_2, t_1+t_2) &\leqq (s_1+s_2) \lim_{n\to\infty} M(h_n) + \lim_{n\to\infty} J(h_n) \\ &< s_1 M_0(s_1,t_1) + s_2 M_0(s_2,t_2) + J_0(s_1,t_1) + J_0(s_2,t_2) \\ &= I(s_1,t_1) + I(s_2,t_2), \end{aligned}$$

as desired.

If, on the other hand, $s_1 = 0$, then we cannot use the above argument, since (3.54) does not hold. Instead, we use corollary 3.17 and (3.48) to write

$$\begin{aligned} I(0+s_2, t_1+t_2) &\leqslant A_3 s_2 (t_1+t_2)^{2/3} + A_2 (t_1+t_2)^{5/3} \\ &\leqslant A_3 s_2 (1+1/\eta)^{2/3} t_2^{2/3} + A_2 t_1^{5/3} + A_2 t_2^{5/3} \\ &< A_1 s_2 t_2^{2/3} + A_2 t_2^{5/3} + A_2 t_1^{5/3} \\ &\leqslant I(s_2,t_2) + I_2(t_1) \\ &= I(s_2,t_2) + I(0,t_1), \end{aligned}$$

which again gives (3.52).

In the case when (3.50) holds, we define

$$j_n(x) = \sigma^{2/3} g_n^{s_2,t_2}(\sigma^{1/3} x), \quad \text{where } \sigma = 1+\eta.$$

Again, we may assume that $M(j_n)$ and $J(j_n)$ converge, and since $\|j_n\|^2 = t_1 + t_2$, we have

$$I(s_1+s_2, t_1+t_2) \leqslant (s_1+s_2) \lim_{n\to\infty} M(j_n) + \lim_{n\to\infty} J(j_n).$$

It follows from lemma 3.4 that

$$I(s_1+s_2, t_1+t_2) \leqslant (s_1+s_2)\sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2).$$

Now, from (3.45), we have

$$\sigma^{2/3} M_0(s_2,t_2) < \eta^{2/3} M_0(s_2,t_2) \leqslant M_0(s_1,t_1),$$

so

$$I(s_1+s_2, t_1+t_2) < s_1 M_0(s_1,t_1) + s_2 \sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2).$$

Also, from (3.51), we have

$$s_2 \sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2) < s_2 M_0(s_2,t_2) + J_0(s_1,t_1) + J_0(s_2,t_2).$$

Combining the last two inequalities, we get (3.52).

Finally, it remains to consider the case when $t_2 = 0$, which implies $I(s_2,t_2) = 0$ by corollary 3.17. If $s_1 > 0$, then $M_0(s_1,t_1) < 0$ by lemma 3.19 (ii), so, letting $h_n = g_n^{s_1,t_1}$, we have

$$\begin{aligned} I(s_1+s_2, t_1) &\leqslant (s_1+s_2) \lim_{n\to\infty} M(h_n) + \lim_{n\to\infty} J(h_n) \\ &= (s_1+s_2) M_0(s_1,t_1) + J_0(s_1,t_1) \\ &< s_1 M_0(s_1,t_1) + J_0(s_1,t_1) \\ &= I(s_1,t_1) \\ &= I(s_1,t_1) + I(s_2,t_2). \end{aligned}$$

If, on the other hand, $s_1 = 0$, then we use corollary 3.17 to write

$$I(s_2, t_1) \leqslant A_3 s_2 t_1^{2/3} + A_2 t_1^{5/3} < A_2 t_1^{5/3} = I_2(t_1) = I(0, t_1) = I(0, t_1) + I(s_2, 0),$$

and we are done. □

LEMMA 3.24. *Suppose $s, t > 0$. If $\{(f_n, g_n)\}$ is a minimizing sequence for $I(s, t)$, then $\{(f_n, g_n)\}$ is bounded in $Y$.*

*Proof.* For a minimizing sequence, $\|f_n\|$ and $\|g_n\|$ stay bounded, so that, as in the proof of lemma 3.9, we have that

$$\left|\int_{-\infty}^{\infty} g_n |f_n|^2 \, dx\right| \leqslant C \|f_n\|_1^{1/2},$$

where $C$ is independent of $n$. Also, Sobolev embedding and interpolation theorems give

$$\left|\int_{-\infty}^{\infty} g_n^3 \, dx\right| \leqslant |g_n|_3^3 \leqslant C\|g_n\|_{1/6}^3 \leqslant C\|g_n\|_1^{1/2}\|g_n\|^{5/2} \leqslant C\|g_n\|_1^{1/2}.$$

Hence

$$\begin{aligned}
\|(f_n, g_n)\|_Y^2 &= \|f_n\|_1^2 + \|g_n\|_1^2 \\
&= E(f_n, g_n) + \int_{-\infty}^{\infty} g_n |f_n|^2 \, dx + \int_{-\infty}^{\infty} g_n^3 \, dx + \|f_n\|^2 + \|g_n\|^2 \\
&\leqslant C(1 + \|f_n\|_1^{1/2} + \|g_n\|_1^{1/2}) \\
&\leqslant C(1 + \|(f_n, g_n)\|_Y^{1/2}),
\end{aligned}$$

from which the desired conclusion follows. □

Now we establish the relative compactness, up to translations, of minimizing sequences for $I(s, t)$. The idea again is to use the method of concentration compactness. Let $\{(f_n, g_n)\}$ be a minimizing sequence for $I(s, t)$, and let $P_n(\omega)$ be the sequence of non-decreasing functions defined for $\omega > 0$ by

$$P_n(\omega) = \sup_{y \in \mathbb{R}} \int_{y-\omega}^{y+\omega} (|f_n|^2(x) + g_n^2(x)) \, dx.$$

Then $\{P_n\}$ has a pointwise convergent subsequence on $[0, \infty)$, which we denote again by $\{P_n\}$. Let $P$ be the non-decreasing function to which $P_n$ converges, and define

$$\alpha_0 = \lim_{\omega \to \infty} P(\omega). \tag{3.55}$$

Then, as was true for $\alpha$ in (3.14), we have $0 \leqq \alpha_0 \leqq s + t$.

LEMMA 3.25. *Suppose $s, t > 0$, and let $\{(f_n, g_n)\}$ be any minimizing sequence for $I(s, t)$. Let $\alpha_0$ be as defined in (3.55). Then there exist numbers $s_1 \in [0, s]$ and $t_1 \in [0, t]$ such that*

$$s_1 + t_1 = \alpha_0 \tag{3.56}$$

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*and*

$$I(s_1,t_1)+I(s-s_1,t-t_1)\leqslant I(s,t). \tag{3.57}$$

*Proof.* As in the proof of lemma 3.10, we can define sequences $\{(\varphi_n,h_n)\}$ and $\{(l_n,j_n)\}$ in $Y$ such that $\|\varphi_n\|^2\to s_1$, $\|h_n\|^2\to t_1$, $\|l_n\|^2\to s-s_1$, $\|j_n\|^2\to t-t_1$, $E(\varphi_n,h_n)\to E_1$ and $E(l_n,j_n)\to E_2$, where $s_1\in[0,s]$ and $t_1\in[0,t]$ satisfy (3.56) and

$$E_1+E_2\leqslant I(s,t).$$

The only change that has to be made is that in place of the estimates (3.19), (3.20) and (3.21) for the functional $K$, we must put similarly obtained estimates for the functional $E$.

To complete the proof of the lemma, it only remains to show that $E_1\geqslant I(s_1,t_1)$ and $E_2\geqslant I(s-s_1,t-t_1)$. We need only prove the first of these inequalities, since the proof of the second is similar. As in the proof of (3.24), we consider separately the three cases when $s_1>0$ and $t_1>0$, when $s_1=0$ and $t_1>0$ and when $t_1=0$. When $s_1>0$ and $t_1>0$, we use the same argument as was used in this case for (3.24). When $s_1=0$, then $\|\varphi_n\|\to 0$, so (3.26) is established by the same proof as before. Then we have, as in (3.27),

$$E_1=\lim_{n\to\infty}E(\varphi_n,h_n)=\lim_{n\to\infty}(K(\varphi_n,h_n)+J(h_n))\geqslant\liminf_{n\to\infty}J(h_n).$$

Also, since $\|h_n\|>0$ for $n$ large, we can put $\theta_n=\sqrt{t_1}/\|h_n\|$, and we have

$$I(0,t_1)=J(t_1)\leqslant J(\theta_n h_n)\leqslant\liminf_{n\to\infty}J(h_n),$$

since $\theta_n\to 1$. Therefore, $E_1\geqslant I(0,t_1)$. Finally, if $t_1=0$, then $\|h_n\|\to 0$, so (3.26) still holds, and, moreover,

$$\left|\int_{-\infty}^{\infty}h_n^3\,\mathrm{d}x\right|\leqslant\|h_n\|_1\|h_n\|^2\to 0.$$

Therefore,

$$E_1=\lim_{n\to\infty}\int_{-\infty}^{\infty}(|\varphi_n'|^2-h_n|\varphi_n|^2+(h_n')^2-h_n^3)\,\mathrm{d}x\geqslant 0=I(s_1,0).$$

□

THEOREM 3.26. *Suppose $q\in(q_1,q_2)$, and let $s,t>0$. Then every minimizing sequence $\{(f_n,g_n)\}$ for $I(s,t)$ is relatively compact in $Y$ up to translations, i.e. there is a subsequence $\{(f_{n_k},g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(f_{n_k}(\cdot+y_k),g_{n_k}(\cdot+y_k))$$

*converges strongly in $Y$ to some $(f,g)$, which is a minimizer for $I(s,t)$.*

*Proof.* If $\alpha_0=0$, then, as in the proof of lemma 3.11, we get $|f_n|_4\to 0$ and $|g_n|_3\to 0$ as $n\to\infty$, whence

$$I(s,t)=\lim_{n\to\infty}E(f_n,g_n)\geqslant\liminf_{n\to\infty}\int(|f_n'|^2+(g_n')^2)\,\mathrm{d}x\geqslant 0,$$

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contradicting corollary 3.17. Hence $\alpha_0 > 0$. On the other hand, if $\alpha_0 \in (0, s+t)$, then it follows from theorem 3.23 that

$$I(s,t) < I(s_1,t_1) + I(s-s_1, t-t_1),$$

which contradicts (3.57). Therefore, we must have $\alpha_0 = s+t$.

It now follows, as in the proof of theorem 3.12, that we can find real numbers $\{y_n\}$ such that, if $w_n(x) = f_n(x+y_n)$ and $z_n(x) = g_n(x+y_n)$, then, for every $k \in \mathbb{N}$, there exists $\omega_k \in \mathbb{R}$ such that

$$\int_{-\omega_k}^{\omega_k} (|w_n|^2 + z_n^2)\,\mathrm{d}x > s + t - \frac{1}{k}, \tag{3.58}$$

provided $n$ is sufficiently large (cf. (3.29)). Since the sequence $\{(w_n, z_n)\}$ is bounded in $Y$, there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges weakly in $Y$ to a limit $(f, g)$. Then Fatou's lemma implies that

$$\|f\|^2 + \|g\|^2 \leqslant \liminf_{n\to\infty} \int_{-\infty}^{\infty} (|w_n|^2 + z_n^2)\,\mathrm{d}x = s+t.$$

Moreover, for fixed $k$, $(w_n, z_n)$ converges weakly in $H^1(-\omega_k, \omega_k) \times H^1(-\omega_k, \omega_k)$ to $(f, g)$, and therefore has a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges strongly to $(f, g)$ in $L^2(-\omega_k, \omega_k) \times L^2(-\omega_k, \omega_k)$. By a diagonalization argument, we may assume that the subsequence has this property for every $k$ simultaneously. It then follows from (3.58) that

$$\int_{-\infty}^{\infty} (|f|^2 + g^2)\,\mathrm{d}x \geqq \int_{-\omega_k}^{\omega_k} (|f|^2 + g^2)\,\mathrm{d}x \geqq s + t - \frac{1}{k}.$$

Since $k$ was arbitrary, we get

$$\int_{-\infty}^{\infty} (|f|^2 + g^2)\,\mathrm{d}x = s+t,$$

which implies that $(w_n, z_n)$ converges strongly to $(f, g)$ in $L^2 \times L^2$.

Now we have that

$$\int_{-\infty}^{\infty} z_n |w_n|^2\,\mathrm{d}x \to \int_{-\infty}^{\infty} g|f|^2\,\mathrm{d}x \quad \text{as } n \to \infty,$$

by the same argument used to establish (3.31), or by an even simpler argument that uses the strong convergence of $z_n$ to $g$ in $L^2$. Moreover,

$$|z_n - g|_3 \leqslant C\|z_n - g\|_1^{1/6} \|z_n - g\|^{5/6} \leqslant C\|z_n - g\|^{5/6},$$

so

$$\int_{-\infty}^{\infty} z_n^3\,\mathrm{d}x \to \int_{-\infty}^{\infty} g^3\,\mathrm{d}x.$$

Therefore, by another application of Fatou's lemma, we get

$$I(s,t) = \lim_{n\to\infty} E(w_n, z_n) \geqq E(f, g), \tag{3.59}$$

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whence $E(f,g)=I(s,t)$. Thus $(f,g)$ is a minimizer for the variational problem (3.1). Finally, since equality holds in (3.59), then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}(|w_n'|^2+(z_n'))\,\mathrm{d}x=\int_{-\infty}^{\infty}(|f'|^2+(g')^2)\,\mathrm{d}x,$$

so $(w_n,z_n)$ converges strongly to $(f,g)$ in $Y$. □

For each $s>0$ and $t>0$, define $G_{s,t}$ to be the set of solutions to the variational problem (3.1), that is,

$$G_{s,t}=\{(f,g)\in Y : E(f,g)=I(s,t),\ \|f\|^2=s \text{ and } \|g\|^2=t\}.$$

As a consequence of theorem 3.26, we have that $G_{s,t}$ is non-empty for all $s,t>0$, provided $q\in(q_1,q_2)$. As will be seen below in § 5, this translates into an existence result for ground-state solutions of (1.2).

We next present a somewhat weaker version of theorem 3.26 that is valid for all $q>0$. For $\gamma>0$, define $Q_\gamma : Y\to\mathbb{R}$ by

$$Q_\gamma(f,g)=\int_{-\infty}^{\infty}(|f|^2+\gamma g^2)\,\mathrm{d}x,$$

and for each $\beta>0$, define

$$R(\beta,\gamma)=\inf\{E(f,g):(f,g)\in Y \text{ and } Q_\gamma(f,g)=\beta\}. \tag{3.60}$$

THEOREM 3.27. *Suppose $q>0$ and let $\beta,\gamma>0$. Then every minimizing sequence $\{(f_n,g_n)\}$ for $R(\beta,\gamma)$ is relatively compact in $Y$ up to translations, i.e. there is a subsequence $\{(f_{n_k},g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(f_{n_k}(\cdot+y_k),g_{n_k}(\cdot+y_k))$$

*converges strongly in $Y$ to some $(f,g)$, which is a minimizer for $R(\beta,\gamma)$.*

*Proof.* This theorem follows from the proof of theorem 2.1 in [2]. First note that, if we decompose $f$ into its real and imaginary parts as $f=\eta+\mathrm{i}\theta$, and define $z:\mathbb{R}\to\mathbb{R}^3$ by $z=(\eta,\theta,g)$, then, in the notation of [2], we have

$$E(f,g)=\int_{-\infty}^{\infty}(\tfrac{1}{2}\langle z,Lz\rangle-N(z))\,\mathrm{d}x$$

and

$$Q_\gamma(f,g)=\int_{-\infty}^{\infty}\tfrac{1}{2}\langle z,Dz\rangle\,\mathrm{d}x,$$

where $Lz=-2z_{xx}$, $N(z)=g(\eta^2+\theta^2+qg^2)$ and $Dz=2(\eta,\theta,\gamma g)$. Also, in the notation of [2], we have $\sigma_0=0$. Therefore, the variational problem (3.60) is the same as the problem that defines $I_\beta$ in [2], and $R(\beta,\gamma)=I_\beta$. It is easily verified that $L$, $N$ and $D$ satisfy the conditions in [2, § 2]. To check that $I_\beta<0$ for all $\beta>0$, we can either use the identity

$$R(\beta,\gamma)=\inf\{I(s,t):s>0,\ t>0 \text{ and } s+\gamma t=\beta\} \tag{3.61}$$

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in conjunction with (3.17), or use [2, theorem 2.2]. Therefore, all the hypotheses of [2, theorem 2.1] are verified, and we conclude from the proof of that Theorem that every minimizing sequence for $R(\beta,\gamma)$ is relatively compact in $Y$ up to translations. □

To compare the results in theorems 3.26 and 3.27, let us consider the sets

$$Q_{\beta,\gamma} = \left\{(f,g) \in Y : E(f,g) = R(\beta,\gamma) \text{ and } \int_{-\infty}^{\infty} (|f|^2 + \gamma g^2)\,\mathrm{d}x = \beta\right\}$$

of solutions to problem (3.60). A consequence of theorem 3.27 is that $Q_{\beta,\gamma}$ is non-empty for all $\beta,\gamma > 0$, regardless of the value of $q > 0$. In particular, from (3.61), it follows that if $Q_{\beta,\gamma}$ is non-empty, then so is $G_{s,t}$, for some values of $s$ and $t$ satisfying $s + \gamma t = \beta$. One drawback, however, is that we do not know whether the sets $Q_{\beta,\gamma}$ constitute a true two-parameter family of disjoint sets. In particular, it is not clear whether every pair $s,t > 0$ corresponds to a pair $\beta$, $\gamma$ such that $Q_{\beta,\gamma} \subseteq G_{s,t}$. A related drawback to theorem 3.27 is that it does not lend itself as easily as does theorem 3.26 to a result on ground-state solutions of (1.2) (see remark 4.6 below).

## 4. The full variational problem

We consider the problem of finding, for any $s > 0$ and $t \in \mathbb{R}$,

$$W(s,t) = \inf\{E(h,g) : (h,g) \in Y,\ H(h) = s \text{ and } G(h,g) = t\}. \tag{4.1}$$

Following our usual convention, we define a minimizing sequence for $W(s,t)$ to be a sequence $(h_n, g_n)$ in $Y$ such that $H(h_n) \to s$, $G(h_n,g_n) \to t$ and $E(h_n,g_n) \to W(s,t)$ as $n \to \infty$.

LEMMA 4.1. *Suppose $s > 0$ and $t \in \mathbb{R}$. If $\{(h_n,g_n)\}$ is a minimizing sequence for $W(s,t)$, then $\{(h_n,g_n)\}$ is bounded in $Y$.*

*Proof.* For a minimizing sequence, $\|h_n\| = \sqrt{H(h_n)}$ stays bounded, and since

$$\|g_n\|^2 = G(h_n,g_n) + 2\,\mathrm{Im}\int_{-\infty}^{\infty} h_n(\bar{h}_n)_x\,\mathrm{d}x,$$

it follows that $\|g_n\|^2 \leqslant C(1 + \|h_n\|_1)$, where $C$ is independent of $n$. Arguing as in the proofs of lemmas 3.9 and 3.24, we deduce that

$$\left|\int_{-\infty}^{\infty} g_n|h_n|^2\,\mathrm{d}x\right| \leqslant C\|h_n\|_1^{1/2}\|g_n\| \leqslant C(1 + \|h_n\|_1)$$

and

$$\left|\int_{-\infty}^{\infty} g_n^3\,\mathrm{d}x\right| \leqslant C\|g_n\|_1^{1/2}\|g_n\|^{5/2} \leqslant C\|g_n\|_1^{1/2}(1 + \|h_n\|_1^{5/4}).$$

Hence, as in the proof of lemma 3.24, we get

$$\|(h_n,g_n)\|_Y^2 \leqslant C(1 + \|h_n\|_1 + \|g_n\|_1^{1/2}(1 + \|h_n\|_1^{5/4})) \leqslant C(1 + \|(h_n,g_n)\|_Y^{7/4}),$$

which is sufficient to bound $\|(h_n,g_n)\|_Y$. □

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LEMMA 4.2. *Suppose $k, \theta \in \mathbb{R}$ and $h \in H^1_{\mathbb{C}}$. If $f(x) = \mathrm{e}^{\mathrm{i}(kx+\theta)} h(x)$, then*

$$E(f,g) = E(h,g) + k^2 H(h) - 2k \operatorname{Im} \int_{-\infty}^{\infty} h\bar{h}_x \, \mathrm{d}x$$

*and*

$$G(f,g) = G(h,g) + 2kH(h).$$

We omit the proof, which is elementary.

Now we can establish a relation between problems (4.1) and (3.1).

LEMMA 4.3. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b = b(a) = (a-t)/(2s)$ for $a \geqslant 0$. Then*

$$W(s,t) = \inf_{a \geqslant 0} \{ I(s,a) + b(a)^2 s \} \tag{4.2}$$

*and*

$$W(s,t) < I(s,0) + b(0)^2 s. \tag{4.3}$$

*Proof.* First, suppose $a \geqslant 0$, and let $(h,g) \in Y$ be given such that $\|h\|^2 = s$ and $\|g\|^2 = a$. Let

$$b = b(a) \quad \text{and} \quad c = \operatorname{Im} \int_{-\infty}^{\infty} h\bar{h}_x \, \mathrm{d}x,$$

and put $f(x) = \mathrm{e}^{\mathrm{i}kx} h(x)$ with $k = (c/s) - b$. Then, from lemma 4.2, we deduce that

$$E(f,g) = E(h,g) - \frac{c^2}{s} + b^2 s \leqslant E(h,g) + b^2 s$$

and

$$G(f,g) = \|g\|^2 - 2bs = t.$$

Since $H(f) = s$, we conclude that

$$W(s,t) \leqslant E(f,g) \leqslant E(h,g) + b^2 s.$$

Taking the infimum over $h$ and $g$ gives

$$W(s,t) \leqslant I(s,a) + b^2 s,$$

and now taking the infimum over $a$ gives

$$W(s,t) \leqslant \inf_{a \geqslant 0} \{ I(s,a) + b(a)^2 s \}. \tag{4.4}$$

Next, suppose $(h,g) \in Y$ is given such that $H(h) = s$ and $G(h,g) = t$. Define

$$a = t + 2 \operatorname{Im} \int_{-\infty}^{\infty} h\bar{h}_x \, \mathrm{d}x \quad \text{and} \quad f(x) = \mathrm{e}^{\mathrm{i}bx} h(x),$$

where $b = b(a)$. Then, by lemma 4.2,

$$E(f,g) = E(h,g) + b^2 s - b(a-t) = E(h,g) - b^2 s,$$

and since $\|f\|^2 = s$ and $\|g\|^2 = a$, we have $a \geqslant 0$ and $I(s,a) \leqslant E(f,g)$. Hence

$$E(h,g) \geqslant I(s,a) + b^2 s \geqslant \inf_{a \geqslant 0}\{I(s,a) + b(a)^2 s\},$$

and taking the infimum over $h$ and $g$ gives

$$W(s,t) \geqslant \inf_{a \geqslant 0}\{I(s,a) + b(a)^2 s\}. \tag{4.5}$$

Combining (4.4) and (4.5) gives (4.2).

To prove (4.3), we see from (4.4) that it suffices to show there exists $a > 0$ for which $I(s,a) + b(a)^2 s < I(s,0) + b(0)^2 s$, or

$$I(s,a) < \frac{a(2t-a)}{4s}.$$

For $a > 0$ sufficiently small, we have $a(2t-a)/(4s) > -Ca$, where we can take

$$C = \begin{cases} |t|/s & \text{if } t < 0, \\ 1 & \text{if } t = 0, \\ 0 & \text{if } t > 0. \end{cases}$$

On the other hand, from (3.40), we have

$$I(s,a) \leqslant A_3 s a^{2/3} + A_2 a^{5/3} \leqslant A_3 s a^{2/3}.$$

Choosing $a > 0$ so small that $|A_3| s a^{2/3} > Ca$, we obtain the desired result. □

LEMMA 4.4. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b(a) = (a-t)/(2s)$ for $a \geqslant 0$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then there exist a positive number $a$ and a subsequence $\{(h_{n_k}, g_{n_k})\}$ such that $\{(e^{ib(a)x} h_{n_k}, g_{n_k})\}$ is a minimizing sequence for $I(s,a)$, and*

$$W(s,t) = I(s,a) + b(a)^2 s. \tag{4.6}$$

*Proof.* For each $n \in \mathbb{N}$, define $a_n \geqslant 0$ by

$$a_n = \int_{-\infty}^{\infty} g_n^2 \, dx = G(h_n, g_n) + 2\,\mathrm{Im} \int_{-\infty}^{\infty} h_n (\bar{h}_n)_x \, dx.$$

Then $a_n$ remains bounded by lemma 4.1, so, by passing to a subsequence, we may assume that $a_n$ converges to a limit $a \geqslant 0$. Let $b = b(a)$, and define $f_n(x) = e^{ibx} h_n(x)$. Then

$$\begin{aligned} \lim_{n\to\infty} E(f_n, g_n) &= \lim_{n\to\infty} (E(h_n, g_n) + b^2 H(h_n) - b(a_n - G(h_n, g_n))) \\ &= W(s,t) + b^2 s - b(a-t) \\ &= W(s,t) - b^2 s \leqslant I(s,a), \end{aligned} \tag{4.7}$$

where we have used lemmas 4.2 and 4.3.

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Next we claim that

$$\lim_{n\to\infty} E(f_n, g_n) \geqslant I(s,a). \tag{4.8}$$

For $a > 0$, we prove (4.8) by defining

$$\beta_n = \frac{\sqrt{s}}{\|f_n\|} \quad \text{and} \quad \theta_n = \frac{\sqrt{a}}{\|g_n\|},$$

so that $\beta_n \to 1$ and $\theta_n \to 1$ as $n \to \infty$, and observing that

$$\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} E(\beta_n f_n, \theta_n g_n),$$

while $E(\beta_n f_n, \theta_n g_n) \geqslant I(s,a)$ for all $n$. For $a = 0$, we have $\|g_n\| \to 0$, and since $\|g_n\|_1$ and $\|f_n\|_1$ remain bounded by lemma 4.1, it follows as in the proofs of lemmas 3.10 and 3.25 that

$$\int_{-\infty}^{\infty} g_n^3 \,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{-\infty}^{\infty} g_n |f_n|^2 \,\mathrm{d}x \to 0.$$

Therefore,

$$\lim_{n\to\infty} E(f_n, g_n) \geqslant 0 = I(s,0),$$

as desired.

It now follows from (4.7) and (4.8) that (4.6) holds, and that $E(f_n, g_n) \to I(s,a)$, which shows that $\{(f_n, g_n)\}$ is a minimizing sequence for $I(s,a)$. Finally, equations (4.6) and (4.3) imply that $a > 0$. ☐

THEOREM 4.5. *Suppose $q \in (q_1, q_2)$, and let $s > 0$ and $t \in \mathbb{R}$ be given. Then every minimizing sequence $\{(h_n, g_n)\}$ for $W(s,t)$ is relatively compact in $Y$ up to translations, i.e. there is a subsequence $\{(h_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(h_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

*converges strongly in $Y$ to some $(h,g)$, which is a minimizer for $W(s,t)$.*

*Proof.* By lemma 4.4, given a minimizing sequence $\{(h_n, g_n)\}$ for $W(s,t)$, we may assume, on passing to a subsequence, that $\{\mathrm{e}^{\mathrm{i}bx} h_n(x), g_n(x)\}$ is a minimizing sequence for $I(s,a)$, where $a > 0$, $b = b(a)$ and (4.6) holds. Then theorem 3.26 allows us to conclude, again after passing to a subsequence, that there exist numbers $y_n$ such that

$$(\mathrm{e}^{\mathrm{i}b(x+y_n)} h_n(x + y_n), g_n(x + y_n))$$

converges in $Y$ to some $(f,g)$ that minimizes $I(s,a)$. By passing to a subsequence yet again, we may assume that $\mathrm{e}^{\mathrm{i}by_n} \to \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$. We then have that

$$(h_n(\cdot + y_n), g_n(\cdot + y_n)) \to (h,g) \quad \text{in } Y,$$

where $h(x) = \mathrm{e}^{-\mathrm{i}(bx+\theta)} f(x)$. Now lemma 4.2 gives

$$\begin{aligned} I(s,a) = E(f,g) &= E(h,g) + b^2 H(h) - 2b \operatorname{Im} \int_{-\infty}^{\infty} h \bar{h}_x \,\mathrm{d}x \\ &= E(h,g) + b^2 s + b(G(h,g) - \|g\|^2) \\ &= E(h,g) - b^2 s. \end{aligned} \tag{4.9}$$

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From (4.6) and (4.9), we get $E(h,g) = W(s,t)$, so $(h,g)$ is a minimizer for $W(s,t)$. □

As a consequence of theorem 4.5, we can now assert the existence of a two-parameter family of ground-state solutions of (1.2), when $q \in (q_1, q_2)$. For $s > 0$ and $t \in \mathbb{R}$, define

$$F_{s,t} = \{(h,g) \in Y : E(h,g) = W(s,t),\ H(h) = s \text{ and } G(h,g) = t\}.$$

From theorem 4.5, we see, in particular, that $F_{s,t}$ is non-empty. In the next section we will see that $F_{s,t}$ is also stable.

REMARK 4.6. It is natural to ask whether theorem 3.27, which is valid for all $q > 0$, can be used to establish a result on ground-state solutions similar to theorem 4.5. In fact, although lemma 4.4 is valid for all $q > 0$, it turns out that one can not obtain a compactness result for minimizing sequences of $W(s,t)$ from theorem 3.27 without a finer knowledge of the function $I(s,a)$. We do not pursue this topic here, and limit ourselves to stating an extra assumption that would lead to such a result. Suppose it could be shown that (4.6) uniquely defines $a$ as a function of $s$ and $t$. Then the above arguments allow us to deduce the following from theorem 3.27: if $(s_0, t_0)$ is such that, for some $\beta, \gamma > 0$,

$$I(s_0, a(s_0, t_0)) = \inf\{I(s,a) : s \geqslant 0,\ a \geqslant 0 \text{ and } s + \gamma a = \beta\},$$

then every minimizing sequence for $W(s_0, t_0)$ is relatively compact in $Y$ up to translations. Moreover, the set of minimizers for $W(s_0, t_0)$ is stable, in the sense of theorem 5.4 below.

## 5. Ground-state solutions

We begin this section with a couple of results showing that the qualitative description of bound states in theorem 2.1 can be improved when the solutions in question are ground states.

THEOREM 5.1. *Suppose $s, t > 0$. If $(f,g) \in G_{s,t}$, then there exist $\sigma > 0$ and $c > 0$ such that (2.2) holds. Moreover, $g(x) > 0$ for all $x \in \mathbb{R}$, and there exist $\theta \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \varphi(x)\mathrm{e}^{\mathrm{i}\theta}$ and $\varphi(x) > 0$ for all $x \in \mathbb{R}$.*

*Proof.* If $(f,g) \in G_{s,t}$, then, by the Lagrange-multiplier principle (cf. [27, theorem 7.7.2]), $(f,g)$ is a solution of the Euler–Lagrange equation

$$\delta E(f,g) = \lambda \delta H(f,g) + \mu \delta H_1(f,g), \tag{5.1}$$

where $H$, $H_1$ are defined as operators on $Y$ by $H(f,g) = \|f\|^2$ and $H_1(f,g) = \|g\|^2$, $\delta$ denotes the Fréchet derivative and $\lambda, \mu \in \mathbb{R}$ are the Lagrange multipliers. Computing the Fréchet derivatives involved, we see that (5.1) becomes

$$\left.\begin{aligned} -f'' - gf &= \lambda f, \\ -2g'' - 3qg^2 - |f|^2 &= 2\mu g, \end{aligned}\right\} \tag{5.2}$$

which is (2.2) with $\sigma = -\lambda$ and $c = -2\mu$.

We claim that $\lambda < 0$ and $\mu < 0$. To see this, multiply the first equation in (5.2) by $\bar{f}$ and integrate over $\mathbb{R}$ to obtain that

$$\lambda s = K(f,g), \tag{5.3}$$

and multiply the second equation in (5.2) by $g$ and integrate over $\mathbb{R}$ to obtain that

$$\mu t = \int_{-\infty}^{\infty} ((g')^2 - \tfrac{1}{2}g|f|^2 - \tfrac{3}{2}qg^3)\,\mathrm{d}x \leqslant \tfrac{1}{2}K(f,g) + \tfrac{3}{2}J(g). \tag{5.4}$$

Now, from $I(s,t) = E(f,g)$, it follows that $K(f,g) = sM(g)$, and from the proof of parts (ii) and (iv) of lemma 3.19, we see that $M(g) < 0$ and $J(g) < 0$. Therefore, equations (5.3) and (5.4) imply that $\lambda < 0$ and $\mu < 0$.

We have now proved that $(f,g)$ satisfies (2.2) with $\sigma > 0$ and $c > 0$. The remaining assertions of the theorem then follow from theorem 2.1, except for the positivity of $\varphi$. To prove this, let $w = |\varphi|$ and observe that since $K(\varphi,g) = K(w,g) = sM(g)$ by lemma 3.1, then $(\varphi,g)$ and $(w,g)$ are both in $G_{s,t}$. Hence, as shown above, we have

$$\left.\begin{aligned} -\varphi'' - g\varphi = \lambda\varphi,\\ -w'' - gw = \lambda w,\end{aligned}\right\} \tag{5.5}$$

where $\lambda = M(g)$. Multiplying the first equation in (5.5) by $w$ and the second by $\varphi$ and adding, we see that the Wronskian $W = \varphi w' - \varphi' w$ is constant. But since $W \to 0$ as $x \to \infty$ by theorem 2.1, we must have $W(x) = 0$ for all $x \in \mathbb{R}$. Hence $\varphi$ and $w$ are linearly dependent, so $\varphi$ must be of one sign on $\mathbb{R}$ and, by changing the value of $\theta$ if necessary, we may assume that $\varphi(x) \geqslant 0$ on $\mathbb{R}$. Finally, since $\sigma = -\lambda > 0$, system (5.5) implies that $K_\sigma * (g\varphi) = \varphi$, where $K_\sigma$ is as defined in the proof of theorem 2.1. It follows that $\varphi > 0$ on $\mathbb{R}$. □

Corollary 5.2. *Suppose $s > 0$ and $t \in \mathbb{R}$. If $(h,g) \in F_{s,t}$, then there exist $c > 0$ and $\omega > \frac{1}{4}c^2$ such that (2.1) holds. Moreover, $g(x) > 0$ for all $x \in \mathbb{R}$, and there exist $\theta, b \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $h(x) = \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}bx}\varphi(x)$ and $\varphi(x) > 0$ for all $x \in \mathbb{R}$.*

*Proof.* If $(h,g) \in F_{s,t}$, then, as in the proof of theorem 5.1, we have the Lagrange multiplier equation

$$\delta E(h,g) = \lambda\delta H(h,g) + \mu\delta G(h,g). \tag{5.6}$$

Computation of the Fréchet derivatives shows that (5.6) is equivalent to (2.1), with $\omega = -\lambda$ and $c = -2\mu$.

On the other hand, the sequence $\{(h_n,g_n)\}$ defined by $(h_n,g_n) = (h,g)$ for all $n \in \mathbb{N}$ is a minimizing sequence for $W(s,t)$, so, from lemma 4.4, it follows that $(\mathrm{e}^{\mathrm{i}bx}h(x), g(x)) \in G_{s,a}$, where $a > 0$ and $b \in \mathbb{R}$. Letting $f(x) = \mathrm{e}^{\mathrm{i}bx}h(x)$, we then have, from theorem 5.1, that $(f,g)$ satisfies (2.2) for some $\sigma > 0$ and some $c > 0$. Substituting $f(x) = \mathrm{e}^{\mathrm{i}bx}h(x)$ into (2.2) and comparing with (2.1), we see that $b = -\frac{1}{2}c$ and $\omega = \sigma + b^2 = \sigma + \frac{1}{4}c^2$. Therefore, $\omega > \frac{1}{4}c^2$. The remaining assertions of the corollary follow immediately from theorem 5.1. □

Next we show that the set $F_{s,t}$ is stable with regard to the flow generated by system (1.2). Concerning the well posedness of (1.2), a variety of results have appeared,

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showing that (1.2) can be posed, at least locally in time, in Sobolev spaces of low order [7, 34]. For our purposes, the following result, due to Guo and Miao [21], is most convenient because it is set in the energy space $Y$.

THEOREM 5.3. *Assume $q \neq 0$ in (1.2). Suppose $(\varphi, \psi) \in Y$. Then, for every $T > 0$, system (1.2) has a unique solution $(u, v) \in \mathcal{C}([0, T], Y)$ satisfying*

$$(u(x, 0), v(x, 0)) = (\varphi(x), \psi(x)).$$

*The map $(\varphi, \psi) \mapsto (u, v)$ is a locally Lipschitz map from $Y$ to $\mathcal{C}([0, T], Y)$. Moreover, $E(u(\cdot, t), v(\cdot, t))$, $G(u(\cdot, t), v(\cdot, t))$ and $H(u(\cdot, t))$ are independent of $t \in [0, T]$.*

In particular, we note that the regularity result in theorem 5.3 is enough to allow one to prove the invariance of the functionals $E$, $G$ and $H$ along the solutions being considered. This may be done in the usual way, by first establishing the invariance of the functionals for smooth solutions, and then using the continuity of solutions with respect to their initial data to extend the result to solutions in $\mathcal{C}([0, T], Y)$. We omit the details of this argument.

THEOREM 5.4. *Suppose $s > 0$ and $t \in \mathbb{R}$. For every $\epsilon > 0$, there exists $\delta > 0$ with the following property. Suppose $(\varphi, \psi) \in Y$ and*

$$\inf_{(h,g) \in F_{s,t}} \|(\varphi, \psi) - (h, g)\|_Y < \delta,$$

*and let $(u(x, t), v(x, t))$ be the unique solution of (1.2) with*

$$(u(x, 0), v(x, 0)) = (\varphi(x), \psi(x)),$$

*guaranteed by theorem 5.3 to exist in $\mathcal{C}([0, T], Y)$ for every $T > 0$. Then*

$$\inf_{(h,g) \in F_{s,t}} \|(u(\cdot, t), v(\cdot, t)) - (h, g)\|_Y < \epsilon$$

*for all $t \geqslant 0$.*

*Proof.* Suppose that $F_{s,t}$ is not stable. Then there exists $\epsilon > 0$ such that, for every $n \in \mathbb{N}$, we can find $(\varphi_n, \psi_n) \in Y$ and $t_n \geqslant 0$ such that

$$\inf_{(h,g) \in F_{s,t}} \|(\varphi_n, \psi_n) - (h, g)\|_Y < \frac{1}{n} \tag{5.7}$$

and

$$\inf_{(h,g) \in F_{s,t}} \|(u_n(\cdot, t_n), v_n(\cdot, t_n)) - (h, g)\|_Y \geqslant \epsilon, \tag{5.8}$$

where $(u_n(x, t), v_n(x, t))$ solves (1.2) with initial data

$$(u_n(x, 0), v_n(x, 0)) = (\varphi_n(x), \psi_n(x)).$$

For brevity, let us denote $u_n(\cdot, t_n)$ by $\Phi_n$ and $v_n(\cdot, t_n)$ by $\Psi_n$.

From (5.7), it follows that

$$\lim_{n \to \infty} E(\varphi_n, \psi_n) = W(s, t), \qquad \lim_{n \to \infty} H(\varphi_n) = s, \qquad \lim_{n \to \infty} G(\varphi_n, \psi_n) = t.$$

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By theorem 5.3, this implies that

$$\lim_{n\to\infty} E(\Phi_n,\Psi_n)=W(s,t), \qquad \lim_{n\to\infty} H(\Phi_n)=s, \qquad \lim_{n\to\infty} G(\Phi_n,\Psi_n)=t.$$

Therefore, $\{(\Phi_n,\Psi_n)\}$ is a minimizing sequence for $W(s,t)$.

Now, by theorem 4.5, there exists a subsequence $\{(\Phi_{n_k},\Psi_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that $(\Phi_{n_k}(\cdot+y_k),\Psi_{n_k}(\cdot+y_k))$ converges strongly in $Y$ to some $(h_0,g_0)\in F_{s,t}$. In particular, there exists $k$ large enough that

$$\|(\Phi_{n_k}(\cdot+y_k),\Psi_{n_k}(\cdot+y_k))-(h_0,g_0)\|_Y<\epsilon.$$

But this implies

$$\|(\Phi_{n_k},\Psi_{n_k})-(h_0(\cdot-y_k),g_0(\cdot-y_k))\|_Y<\epsilon,$$

and the invariance under translations of the functionals $E$, $H$ and $G$ implies that $(h_0(\cdot-y_k),g_0(\cdot-y_k))$ is also in $F_{s,t}$. Therefore,

$$\inf_{(h,g)\in F_{s,t}}\|(\Phi_{n_k},\Psi_{n_k})-(h,g)\|_Y<\epsilon,$$

contradicting (5.8). □

We conclude with a result on the ground-state solutions of (1.4). By definition, $(u(x,t),v(x,t))$ is a bound-state solution of (1.4) if $u$ and $v$ are of the form given by (1.7). Equivalently, $h$ and $g$ in (1.7) must satisfy the equations

$$\left.\begin{aligned} h''-\omega h-\mathrm{i}ch'&=-hg,\\ cg&=|h|^2,\end{aligned}\right\} \tag{5.9}$$

which are the Euler–Lagrange equations for the variational problem

$$W_1(s,t)=\inf\{K(h,g):(h,g)\in X,\ H(h)=s \text{ and } G(h,g)=t\}. \tag{5.10}$$

If we put $h(x)=\mathrm{e}^{\mathrm{i}cx/2}f(x)$ in (5.9), we obtain the system

$$\left.\begin{aligned} f''-\sigma f&=-fg,\\ cg&=|f|^2,\end{aligned}\right\} \tag{5.11}$$

where $\sigma=\omega-\frac{1}{4}c^2$. From lemma 2.2, we see that the only solutions of (2.2) are given by $f(x)=\mathrm{e}^{\mathrm{i}\theta_0}f_1(x+x_0)$, $g(x)=g_1(x+x_0)$, where $\theta_0,x_0\in\mathbb{R}$, and $f_1$, $g_1$ are as given in (2.12) with $\lambda=\sigma>0$ and $\mu=c>0$. Therefore, these are all the bound-state solutions of (1.4).

Well-posedness results for (1.4) have appeared in [6, 8, 24, 35, 36]. The following result is a consequence of proposition 1.3 in [18].

Theorem 5.5. *For every $T>0$ and every $(u_0,v_0)\in X$, there is a unique solution $(u(x,t),v(x,t))$ to (1.4) in $\mathcal{C}([0,T],X)$ such that $(u(x,0),v(x,0))=(u_0,v_0)$. Moreover, the map from $(u_0,v_0)$ to $(u,v)$ is a continuous map from $X$ to $\mathcal{C}([0,T],X)$, and we have*

$$K(u(\cdot,t),v(\cdot,t))=K(u_0,v_0)$$

*for all $t\in[0,T]$.*

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In [24], Laurençot established a stability result for bound-state solutions of (1.4). Here we recover Laurençot's stability result (see theorem 5.6 (iii)), and we also obtain the additional fact that the bound-state solutions of (1.4) are, in fact, ground states. That is, any critical point for the variational problem (5.10) is actually a global minimizer or, in other words, an element of the set

$$F^1_{s,t} = \{(h,g) \in X : K(h,g) = W_1(s,t),\ H(h) = s \text{ and } G(h,g) = t\}$$

for some $s > 0$ and $t \in \mathbb{R}$.

THEOREM 5.6. *Suppose $s > 0$ and $t \in \mathbb{R}$. Then we have the following.*

(i) *Every minimizing sequence $\{(h_n, g_n)\}$ for $W_1(s,t)$ is relatively compact in $X$ up to translations, i.e. there is a subsequence $\{(h_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(h_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

*converges strongly in $X$ to some $(h,g)$, which is a minimizer for $W_1(s,t)$.*

(ii) *In particular, $F^1_{s,t}$ is non-empty, and consists of all pairs $(f,g)$ with*

$$f(x) = \mathrm{e}^{\mathrm{i}\theta_0} f_1(x + x_0) \quad \textit{and} \quad g(x) = f_1(x + x_0),$$

*where $\theta_0, x_0 \in \mathbb{R}$ and $f_1$, $g_1$ are as given in (2.12) with $\lambda = (\frac{3}{16}t)^{2/3}$ and $\mu = s(12t)^{-1/3}$.*

(iii) *$F^1_{s,t}$ is stable, in the sense that, for every $\epsilon > 0$, there exists $\delta > 0$ with the following property. Suppose $(\varphi, \psi) \in X$ and*

$$\inf_{(h,g)\in F^1_{s,t}} \|(\varphi,\psi) - (h,g)\|_X < \delta,$$

*and let $(u(x,t), v(x,t))$ be the unique solution of (1.4) with*

$$(u(x,0), v(x,0)) = (\varphi(x), \psi(x)),$$

*guaranteed by theorem 5.5 to exist in $\mathcal{C}([0,T], X)$ for every $T > 0$. Then*

$$\inf_{(h,g)\in F^1_{s,t}} \|(u(\cdot,t), v(\cdot,t)) - (h,g)\|_X < \epsilon$$

*for all $t \geqslant 0$.*

*Proof.* To prove (i), we need make only minor modifications to the proof of theorem 4.5. In fact, the statements and proofs of lemmas 4.1, 4.2, 4.3 and 4.4 continue to be valid if we replace throughout $E$ by $K$, $W$ by $W_1$ and $I$ by $I_1$, except that we can use (3.10) instead of (3.40) at the end of lemma 4.4. The statement and proof of theorem 4.5 also remain valid once the same modifications are made, except that we use theorem 3.12 instead of theorem 3.26.

Since every ground state in $F^1_{s,t}$ is also a bound state, statement (ii) follows from (i) and the remarks concerning bound states that were made after (5.11).

Finally, the proof of (iii) is identical to that of theorem 5.4, once the obvious modifications are made. □

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