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Rational maps whose Julia sets are Cantor circles

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Abstract. In this paper, we give a family of rational maps whose Julia sets are Cantor circles and show that every rational map whose Julia set is a Cantor set of circles must be topologically conjugate to one map in this family on their corresponding Julia sets. In particular, we give the specific expressions of some rational maps whose Julia sets are Cantor circles, but they are not topologically conjugate to any McMullen maps on their Julia sets. Moreover, some non-hyperbolic rational maps whose Julia sets are Cantor circles are also constructed.

1. *Introduction*

The study of the topological properties of the Julia sets of rational maps is a central problem in complex dynamics. For each polynomial of degree at least two with a disconnected Julia set, it was proved in [**QY**] that all but countably many components of the Julia set are single points. For rational maps, the Julia sets may exhibit more complex topological structures. Pilgrim and Tan proved that if the Julia set of a hyperbolic (more generally, geometrically finite) rational map is disconnected, then, with the possible exception of finitely many periodic components and their countable collection of preimages, every Julia component is either a point or a Jordan curve [**PT**, Theorem 1.2]. In this paper, we will consider one class of rational maps whose Julia sets possess simple topological structure: each Julia component is a Jordan curve.

A subset of the Riemann sphere $\overline{\mathbb{C}}$ is called a *Cantor set of circles* (sometimes *Cantor circles* in short) if it consists of uncountably many closed Jordan curves homeomorphic to $\mathcal{C} \times \mathbb{S}^1$, where $\mathcal{C}$ is the middle third Cantor set and $\mathbb{S}^1$ is the unit circle. The first example of a rational map whose Julia set is a Cantor set of circles was discovered by McMullen (see [**Mc**, §7]). He showed that if $f(z) = z^2 + \lambda/z^3$ and λ is small enough, then the Julia set of f is a Cantor set of circles. Later, many authors have focused on the following family, which is commonly referred as the *McMullen maps*:

$$g_\eta(z) = z^k + \eta/z^l, \tag{1.1}$$

where $k, l \geq 2$ and $\eta \in \mathbb{C}\backslash\{0\}$ (see [**DLU, St, QWY**] and the references therein). These special rational maps can be viewed as a perturbation of the simple polynomial $g_0(z) = z^k$ if η is small. It is known that when $1/k + 1/l < 1$, there exists a punched neighborhood $\mathcal{M}$ centered at the origin in parameter space, which is called the *McMullen domain*, such that when $\eta \in \mathcal{M}$, then the Julia set of g_η is a Cantor set of circles (see [**Mc**, §7] for $k = 2$, $l = 3$ and [**DLU**, §3] for the general case).

The following three questions arise naturally. (1) Besides McMullen maps, do there exist any other rational maps whose Julia sets are Cantor circles? (2) If the answer to the first question is yes, what do they look like? Or, in other words, can we find specific expressions for them? (3) Can we find all rational maps whose Julia sets are Cantor circles in some sense? This paper will give affirmative answers to these questions.

By quasiconformal surgery, we can obtain many new rational maps after perturbing the immediate super-attracting basin centered at ∞ of g_η into a geometric one. Fix one of them, then this map is not topologically conjugate to g_η on the whole $\overline{\mathbb{C}}$. But they are topologically conjugate to each other on their corresponding Julia sets. In particular, $h_{c,\eta}(z) = (1/z) \circ (z^k + c) \circ (1/z) + (\eta/z^l)$ is an example, where $1/k + 1/l < 1$ and $c, \eta \in \mathbb{C}\backslash\{0\}$ are both small enough. However, these types of rational maps can also be regarded as McMullen maps essentially, which are not what we want to find since they can be obtained by doing a surgery only on the Fatou sets of the genuine McMullen maps. So it will be very interesting to find other types of rational maps with Cantor circles Julia sets which are not topologically conjugate to any McMullen maps on their corresponding Julia sets.

The existence of types of rational maps 'essentially' different from McMullen maps was known previously (see [**HP**, §§1,2]). Here, 'essentially' means there exists no topological conjugacy between the Julia sets of McMullen maps and the rational maps whose Julia sets are Cantor circles. In this paper, we will give specific expressions for these types of rational maps, not only including the cases discussed in [**HP**], but also covering all the rational maps whose Julia sets are Cantor circles 'essentially' (see Theorem 1.2). Let $p \in \{0, 1\}$, $n \geq 2$ be an integer and $d_1, \ldots, d_n$ be n positive integers such that $\sum_{i=1}^{n}(1/d_i) < 1$. We define

$$f_{p,d_1,\ldots,d_n}(z) = z^{(-1)^{n-p}d_1} \prod_{i=1}^{n-1} (z^{d_i+d_{i+1}} - a_i^{d_i+d_{i+1}})^{(-1)^{n-i-p}}, \tag{1.2}$$

where $a_1, \ldots, a_{n-1}$ are $n-1$ small complex numbers satisfying $0 < |a_1| < \cdots < |a_{n-1}| < 1$. In particular, if $n = 2$, then $f_{1,d_1,d_2}(z) = z^{d_2} - a_1^{d_1+d_2}/z^{d_1}$ is the McMullen map that has been well studied by many authors. Moreover, $f_{0,d_1,d_2}(z) = z^{d_1}/(z^{d_1+d_2} - a_1^{d_1+d_2})$ is conformally conjugate to the McMullen map $z \mapsto z^{d_1} + \eta/z^{d_2}$ for some $\eta \neq 0$. The degrees of $f_{p,d_1,\ldots,d_n}$ at 0 and ∞ are d_1 and d_n respectively and $\deg(f_{p,d_1,\ldots,d_n}) = \sum_{i=1}^{n} d_i$. For each element in the family (1.2), it is easy to check that 0 and ∞ belong to the Fatou set of $f_{p,d_1,\ldots,d_n}$. Let D_0 and D_∞ be the Fatou components containing 0 and ∞ respectively. There are four cases:

(1) if $p = 1$ and n is odd, then $f(D_0) = D_0$ and $f(D_\infty) = D_\infty$;
(2) if $p = 1$ and n is even, then $f(D_0) = D_\infty$ and $f(D_\infty) = D_\infty$;
(3) if $p = 0$ and n is odd, then $f(D_0) = D_\infty$ and $f(D_\infty) = D_0$;
(4) if $p = 0$ and n is even, then $f(D_0) = D_0$ and $f(D_\infty) = D_0$.

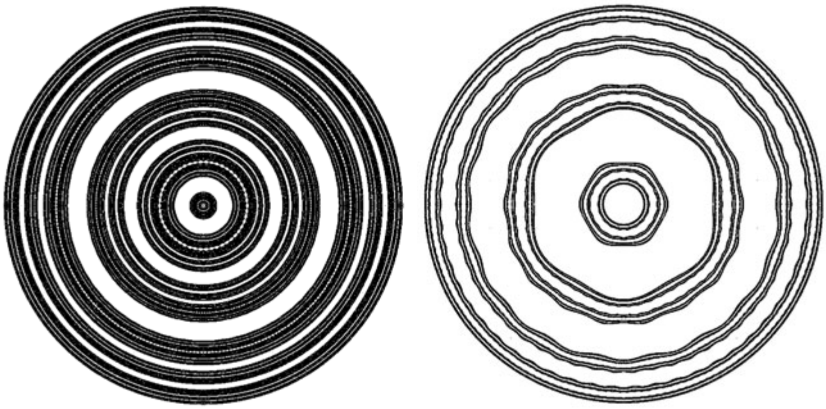

FIGURE 1. The Julia set of $f_{1,5,5,5,5}$ (left picture), which is not topologically conjugate to that of the McMullen map $g_\eta(z) = z^3 + 0.001/z^3$ (right picture). The two Julia sets are both Cantor circles.

Firstly we will find suitable parameters a_i in (1.2), where $1 \le i \le n-1$, such the Julia set of each $f_{p,d_1,\ldots,d_n}$ in the four cases stated above is a Cantor set of circles.

THEOREM 1.1. *For each given $p \in \{0, 1\}$, $n \ge 2$ and $d_1, \ldots, d_n$ satisfying $\sum_{i=1}^{n}(1/d_i) < 1$, there exist suitable parameters a_i, where $1 \le i \le n-1$, such that the Julia set of $f_{p,d_1,\ldots,d_n}$ is a Cantor set of circles.*

The specific value ranges of a_i are given in §2, where $1 \le i \le n-1$ (see (2.1), (2.2) and Theorem 2.5). These rational maps can be seen as the perturbations of z^{d_n} or z^{-d_n} (according to whether $p = 1$ or 0) since each a_i can be arbitrarily small (see Theorem 2.5). Moreover, it will be shown that if $n \ge 3$, then each $f_{p,d_1,\ldots,d_n}$ is not topologically conjugate to any McMullen maps on their corresponding Julia sets (see Theorem 2.7). This means that we have found the specific expressions of rational maps whose Julia sets are Cantor circles which are 'essentially' different from McMullen maps.

For example, let $p = 1$, $n = 4$, $d_1 = d_2 = d_3 = d_4 = 5$ and define

$$f_{1,5,5,5,5}(z) = \frac{(z^{10} - a_1^{10})(z^{10} - a_3^{10})}{z^5(z^{10} - a_2^{10})}, \tag{1.3}$$

where $a_1 = 0.000\,25$, $a_2 = 0.005$ and $a_3 = 0.1$. By a straightforward calculation or using Theorem 2.5 and Remark 2.6, one can show that the Julia set of $f_{1,5,5,5,5}$ is a Cantor set of circles (see Figure 1). The dynamics on the set of Julia components of $f_{1,5,5,5,5}$ is conjugate to the one-sided shift on four symbols $\Sigma_4 := \{0, 1, 2, 3\}^{\mathbb{N}}$ while the set of Julia components of g_η is conjugate to the one-sided shift on only two symbols $\Sigma_2 := \{0, 1\}^{\mathbb{N}}$. This means that $f_{1,5,5,5,5}$ cannot be topologically conjugate to g_η on their corresponding Julia sets.

Note that if the Julia set $J(f)$ of a rational map f is a Cantor set of circles, then there exist no critical points in $J(f)$ since each Julia component is a Jordan closed curve (see Lemma 3.1). This means that every periodic Fatou component of f must be attracting or parabolic. In fact, we have the following theorem.

THEOREM 1.2. *Let f be a rational map whose Julia set is a Cantor set of circles. Then there exist $p \in \{0, 1\}$, positive integers $n \geq 2$, and $d_1, \ldots, d_n$ satisfying $\sum_{i=1}^{n}(1/d_i) < 1$, such that f is topologically conjugate to $f_{p,d_1,\ldots,d_n}$ on their corresponding Julia sets for suitable parameters a_i, where $1 \leq i \leq n-1$.*

Since the dynamics on the Fatou set can be perturbed freely, it follows from Theorem 1.2 that we have found 'all' the possible rational maps whose Julia sets are Cantor circles. A rational map is *hyperbolic* if all critical points are attracted by attracting periodic orbits. For the regularity of the Julia components of $f_{p,d_1,\ldots,d_n}$, it can be shown that each Julia component of $f_{p,d_1,\ldots,d_n}$ is a quasicircle if $f_{p,d_1,\ldots,d_n}$ is hyperbolic (see Corollary 3.3).

If η is small enough, then g_η is hyperbolic (see [**DLU**]). Now we construct some non-hyperbolic rational maps whose Julia sets are Cantor circles. Let $m, n \geq 2$ be two positive integers satisfying $1/m + 1/n < 1$ and $\lambda \in \mathbb{C}\setminus\{0\}$, we define

$$P_\lambda(z) = \frac{(1/n)[(1+z)^n - 1] + \lambda^{m+n} z^{m+n}}{1 - \lambda^{m+n} z^{m+n}}. \tag{1.4}$$

It is straightforward to verify that zero is a parabolic fixed point of P_λ with multiplier one. We then have the following theorem.

THEOREM 1.3. *If $0 < |\lambda| \leq 1/(2^{10m} n^3)$, then P_λ is non-hyperbolic and its Julia set is a Cantor set of circles.*

Inspired by Theorem 1.1, we can construct more non-hyperbolic rational maps whose Julia sets are Cantor circles. For simplicity, for each $n \geq 2$, we only consider the case $d_i = n+1$ for every $1 \leq i \leq n$. For every $n \geq 2$, we define

$$P_n(z) = A_n \frac{(n+1)z^{(-1)^{n+1}(n+1)}}{nz^{n+1}+1} \prod_{i=1}^{n-1} (z^{2n+2} - b_i^{2n+2})^{(-1)^{i-1}} + B_n, \tag{1.5}$$

where $b_1, \ldots, b_{n-1}$ are $n-1$ small complex numbers satisfying $1 > |b_1| > \cdots > |b_{n-1}| > 0$ and

$$A_n = \frac{1}{1+(2n+2)C_n} \prod_{i=1}^{n-1} (1 - b_i^{2n+2})^{(-1)^i}, \quad B_n = \frac{(2n+2)C_n}{1+(2n+2)C_n}$$
$$\text{and} \quad C_n = \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2}}. \tag{1.6}$$

The terms A_n and B_n here can guarantee that $P_n(1) = 1$ and $P_n'(1) = 1$. Namely, 1 is a parabolic fixed point of P_n with multiplier one (see Lemma 5.1).

THEOREM 1.4. *For every $n \geq 2$ and $1 \leq i \leq n-1$, if $|b_i| = s^i$ for $0 < s \leq 1/(25n^2)$, then P_n is non-hyperbolic and its Julia set is a Cantor set of circles.*

It will be seen later that the dynamics of P_n on their Julia sets are conjugate to those of f_n for every $n \geq 2$ $(p = 1)$. One of the differences between their dynamics on the Fatou sets is that the super-attracting basin of f_n at ∞ is replaced by a parabolic basin of P_n.

This paper is organized as follows: in §2, we do some estimates and prove Theorem 1.1. In §3, we prove Theorem 1.2. In §4, we show that the Julia set of P_λ is a Cantor set of circles if λ is small enough and prove Theorem 1.3. We will prove Theorem 1.4 in §5 and leave a key lemma to the last section.

Notation. We will use the following notation throughout the paper. Let $\mathbb{C}$ be the complex plane and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. For $r > 0$ and $a \in \mathbb{C}$, let $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ be the Euclidean disk centered at a with radius r. In particular, let $\mathbb{D}_r := \mathbb{D}(0, r)$ be the disk centered at the origin with radius r and $\mathbb{T}_r := \partial\mathbb{D}_r$ be the boundary of $\mathbb{D}_r$. As usual, $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{S}^1 := \mathbb{T}_1$ denote the unit disk and the unit circle, respectively. For $0 < r < R < +\infty$, let $\mathbb{A}_{r,R} := \{z \in \mathbb{C} : r < |z| < R\}$ be the round annulus centered at the origin.

2. *Location of the critical points and the hyperbolic case*

First we give some basic and useful estimations.

LEMMA 2.1. *Let $n \geq 2$ be an integer, $a \in \mathbb{C}\setminus\{0\}$ and $0 < \varepsilon < 1/2$.*

(1) *If $|z - a| \leq \varepsilon|a|$, then $|z^n - a^n| \leq [(1 + \varepsilon)^n - 1]\,|a|^n$.*

(2) *If $|z^n - a^n| \leq \varepsilon|a|^n$, then $|a/z|^n < 1 + 2\varepsilon$ and $|z - ae^{2\pi ij/n}| < \varepsilon|a|$ for some $1 \leq j \leq n$.*

(3) *If $0 < \varepsilon < 1/n$, then $n\varepsilon < (1 + \varepsilon)^n - 1 < 3n\varepsilon$ and $n\varepsilon/3 < 1 - (1 - \varepsilon)^n < n\varepsilon$.*

Proof. Let $z = a(1 + re^{i\theta})$ for $0 \leq r \leq \varepsilon$ and $0 \leq \theta < 2\pi$, then

$$|z^n - a^n| = |(1 + re^{i\theta})^n - 1| \cdot |a|^n \leq [(1 + \varepsilon)^n - 1]\,|a|^n.$$

This proves (1). The first statement in (2) follows from $|a/z|^n \leq 1/(1 - \varepsilon) < 1 + 2\varepsilon$ if $0 < \varepsilon < 1/2$. For the second statement, let $z^n = a^n(1 + re^{i\theta})$ for $0 \leq r \leq \varepsilon$ and $0 \leq \theta < 2\pi$, then $z = ae^{2\pi ij/n}(1 + re^{i\theta})^{1/n}$ for some $1 \leq j \leq n$ and we have

$$|z - ae^{2\pi ij/n}| = |(1 + re^{i\theta})^{1/n} - 1| \cdot |a| \leq [(1 + \varepsilon)^{1/n} - 1] \cdot |a| < \varepsilon|a|$$

if $n \geq 2$. The claim (3) can be proved by applying Lagrange's mean value theorem to $x \mapsto x^n$ on the intervals $[1, 1 + \varepsilon]$ and $[1 - \varepsilon, 1]$ respectively. The proof is complete. □

Fix $n \geq 2$ and let $d_1, \ldots, d_n \geq 2$ be n positive numbers such that $\xi = \sum_{i=1}^{n}(1/d_i) < 1$. We use $K \geq 3$ to denote the maximal number among $d_1, \ldots, d_n$. Let $u_1 = s_1K^{-5}$ and $v_1 = s_1K^{-2}$, where

$$0 < s_1 \leq \min\{K^{-5\xi/(1-\xi)}, K^{5-2K}\} < 1. \tag{2.1}$$

Let $u_0 = s_0^{1+(1/d_n)+2(1-\xi)/3}$, $v_0 = s_0^{(1/d_n)+(1-\xi)/3}$, where

$$0 < s_0 \leq \min\{2^{-(1-\xi)^{-1}[1+(1/d_n)-(2\xi/3)]^{-1}}, (4K)^{-3/(1-\xi)}, K^{-2K[1+(1/d_n)+2(1-\xi)/3]^{-1}}\} < 1. \tag{2.2}$$

For $p \in \{0, 1\}$, let $|a_{n-1,p}| = v_p^{1/d_n}$ and $|a_{i,p}| = u_p^{1/d_{i+1}}|a_{i+1,p}|$ be the $n - 1$ parameters in the family $f_{p,d_1,\ldots,d_n}$, where $1 \leq i \leq n - 2$. Since the cases $p = 0$ and $p = 1$ can be discussed uniformly in general, we use s, u, v and a_i, respectively, to denote s_p, u_p, v_p and $a_{i,p}$ for simplicity when the situation is clear, where $1 \leq i \leq n - 1$.

LEMMA 2.2.

(1) $u^{2/K} \leq K^{-4}$.

(2) *If $1 \leq j \leq i \leq n - 1$, then $|a_j/a_i| \leq u^{(i-j)/K}$.*

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(3) *If* $p = 1$, *then:*
 (3a) $(s/|a_1|)^{d_1} < su/(2v) = sK^{-3}/2$; *and*
 (3b) $(|a_1|/s)^{d_1} v/2 > K$.
(4) *If* $p = 0$, *then:*
 (4a) $2Ku/v < s$ *and* $1/(2Kv) > (2/s)^{1/d_n}$;
 (4b) $(s/|a_1|)^{d_1} < sv/2 < u^{1/2}/2$; *and*
 (4c) $(|a_1|/s)^{d_1} u/(2v) > (2/s)^{1/d_n}$.

Proof. (1) From (2.1) and (2.2), we have $s_1 \le K^{5-2K}$ and $s_0 \le K^{-2K\{1+(1/d_n)+2(1-\xi)/3\}^{-1}}$. This means that $u_1^{2/K} = (s_1K^{-5})^{2/K} \le K^{-4}$ and $u_0^{2/K} \le K^{-4}$.

(2) If $j = i$, then (2) is trivial. Suppose that $1 \le j < i \le n-1$, then

$$|a_j/a_i| = u^{(1/d_{j+1})+\cdots+(1/d_i)} \le u^{(i-j)/K}$$

since $K \ge d_i$ for $1 \le i \le n$. This proves (2).

(3) If $p = 1$, then $u = sK^{-5}$ and $v = sK^{-2}$. Since $s \le K^{-5\xi/(1-\xi)}$, we have $s^{1-\xi}K^{5\xi} \le 1$, so

$$s^{1-(1/d_1)} s^{-[(1/d_2)+\cdots+(1/d_n)]} K^{5[(1/d_2)+\cdots+(1/d_{n-1})]+(2/d_n)} 2^{1/d_1} K^{3/d_1} < 1.$$

This is equivalent to $s^{1-(1/d_1)} 2^{1/d_1} K^{3/d_1}/|a_1| < 1$ since

$$|a_1| = u^{(1/d_2)+\cdots+(1/d_{n-1})} v^{1/d_n} = s^{(1/d_2)+\cdots+(1/d_n)} / K^{5[(1/d_2)+\cdots+(1/d_{n-1})]+(2/d_n)}.$$

So we have $(s/|a_1|)^{d_1} < su/(2v) = sK^{-3}/2$ and (3a) is proved. Moreover, (3b) can be derived from (3a) directly since $(|a_1|/s)^{d_1} > 2K^3/s = 2K/v$.

(4) If $p = 0$, then

$$u = s^{1+(1/d_n)+2(1-\xi)/3}, \qquad v = s^{(1/d_n)+(1-\xi)/3}.$$

From (2.2), we know $4Ks^{(1-\xi)/3} \le 1$, which means $2Ku/v = 2Ks^{1+[(1-\xi)/3]} < s$. Note that $2^{1+(1/d_n)} Ks^{(1-\xi)/3} < 1$, which is equivalent to $1/(2Kv) > (2/s)^{1/d_n}$. This ends the proof of (4a).

From (2.2), we know that

$$\begin{aligned}
1 &\ge 2s^{(1-\xi)[1+(1/d_n)-2\xi/3]} > 2^{1/d_1} s^{(1-\xi)[1+(1/d_n)-2\xi/3]}\\
&= \frac{2^{1/d_1} s^{1-(1/d_1)}}{s^{[(1/d_2)+\cdots+(1/d_{n-1})]+(1/d_n)[(1/d_1)+\cdots+(1/d_n)]+2\xi(1-\xi)/3}}\\
&> \frac{2^{1/d_1} s^{1-(1/d_1)}}{s^{[(1/d_2)+\cdots+(1/d_{n-1})]+(1/d_n)[(1/d_1)+\cdots+(1/d_n)]+[(1-\xi)/3]\{(1/d_1)+2[(1/d_2)+\cdots+(1/d_{n-1})]+(1/d_n)\}}}\\
&= s^{1-(1/d_1)}(2/v)^{1/d_1}/|a_1|.
\end{aligned}$$

This means that $(s/|a_1|)^{d_1} < sv/2 = u^{1/2} s^{[1+(1/d_n)]/2}/2 < u^{1/2}/2$. So (4b) holds.

The proof of (4c) is similar to (4b). We just need to note that

$$\begin{aligned}
1 &\ge 2s^{(1-\xi)[1+(1/d_n)-2(\xi/3)]} > 2^{(1/d_1)[1+(1/d_n)]} s^{(1-\xi)[1+(1/d_n)-2(\xi/3)]}\\
&> (s/|a_1|)(2v/u)^{1/d_1}(2/s)^{1/d_1 d_n}.
\end{aligned}$$

This means that $(|a_1|/s)^{d_1} u/(2v) > (2/s)^{1/d_n}$. □

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In the following, we use f to denote $f_{p,d_1,\ldots,d_n}$ for simplicity. Note that 0 and ∞ are critical points of f with multiplicity d_1 and d_n respectively, and the degree of f is $\sum_{i=1}^n d_i$. Denoting $D_i = d_i + d_{i+1}$, we have $5 \le D_i \le 2K$, where $1 \le i \le n-1$. Besides 0 and ∞, the rest of the $\sum_{i=1}^{n-1} D_i$ critical points of f are the solutions of

$$(-1)^p z\frac{f'(z)}{f(z)} = \sum_{i=1}^{n-1} \frac{(-1)^{n-i} D_i z^{D_i}}{z^{D_i} - a_i^{D_i}} + (-1)^n d_1 = 0. \tag{2.3}$$

For $1 \le i \le n-1$, let $\widetilde{CP}_i := \{\widetilde{w}_{i,j} = r_i a_i \exp(\pi i((2j-1)/D_i)) : 1 \le j \le D_i\}$ be the collection of D_i points lying on the circle $\mathbb{T}_{r_i|a_i|}$ uniformly, where $r_i = \sqrt[D_i]{d_i/d_{i+1}}$. The following lemma shows that the $\sum_{i=1}^{n-1} D_i$ *free* critical points of f are very 'close' to $\bigcup_{i=1}^{n-1} \widetilde{CP}_i$.

LEMMA 2.3. *For every $\widetilde{w}_{i,j} \in \widetilde{CP}_i$, where $1 \le i \le n-1$ and $1 \le j \le D_i$, there exists $w_{i,j}$, which is a solution of (2.3), such that $|w_{i,j} - \widetilde{w}_{i,j}| < u^{2/K}|a_i|$. Moreover, $w_{i_1,j_1} = w_{i_2,j_2}$ if and only if $(i_1, j_1) = (i_2, j_2)$.*

Proof. Note that the right side of equation (2.3) is equivalent to

$$(-1)^{n-i}\left(\frac{D_i z^{D_i}}{z^{D_i} - a_i^{D_i}} - d_i\right) + G_i(z) = 0, \tag{2.4}$$

where

$$G_i(z) = \sum_{1\le j\le n-1, j\ne i} \frac{(-1)^{n-j} D_j z^{D_j}}{z^{D_j} - a_j^{D_j}} + (-1)^n d_1 + (-1)^{n-i} d_i. \tag{2.5}$$

After multiplying both sides of (2.4) by $(z^{D_i} - a_i^{D_i})/d_{i+1}$, where $1 \le i \le n-1$, we have

$$(-1)^{n-i}(z^{D_i} + d_i a_i^{D_i}/d_{i+1}) + (z^{D_i} - a_i^{D_i}) G_i(z)/d_{i+1} = 0. \tag{2.6}$$

Let $\Omega_i = \{z : |z^{D_i} + d_i a_i^{D_i}/d_{i+1}| \le \varepsilon |a_i|^{D_i}\}$, where $\varepsilon = u^{2/K}$ and $1 \le i \le n-1$. For every $z \in \Omega_i$, since $\varepsilon \le K^{-4}$ by Lemma 2.2(1), we have

$$K^{-1} < d_i/d_{i+1} - \varepsilon \le |z/a_i|^{D_i} \le d_i/d_{i+1} + \varepsilon < K - 1 < K. \tag{2.7}$$

This means that

$$K^{-1} < |a_i/z|^{D_i} < K \quad \text{and therefore } K^{-1} < |a_i/z|^5 < K. \tag{2.8}$$

If $1 \le j < i$ and $z \in \Omega_i$, we have

$$|a_j/z|^{D_i} \le |a_i/z|^{D_i} |a_{i-1}/a_i|^{D_i} < K u^{1+d_{i+1}/d_i} < 1. \tag{2.9}$$

Therefore, $|a_j/z| < 1$. By a similar argument it can be shown that $|z/a_j| < 1$ if $i < j \le n-1$ and $z \in \Omega_i$. If $1 \le j < i$, by Lemma 2.2(1) and (2) and (2.8), we have

$$|a_j/z|^{D_j} \le |a_i/z|^5 |a_j/a_i|^5 < K\varepsilon^{5(i-j)/2} \le K^{-9}. \tag{2.10}$$

Similarly, if $i < j \le n-1$, we have

$$|z/a_j|^{D_j} \le |z/a_i|^5 |a_i/a_j|^5 < K\varepsilon^{5(j-i)/2} \le K^{-9}. \tag{2.11}$$

By definition, we have

$$\sum_{1\le j<i}(-1)^{n-j}D_j+(-1)^n d_1+(-1)^{n-i}d_i=0. \tag{2.12}$$

From (2.5), (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
&|G_i(z)|\\
&=\left|\sum_{1\le j<i}\frac{(-1)^{n-j}D_j}{1-(a_j/z)^{D_j}}+\sum_{i<j\le n-1}\frac{(-1)^{n-j-1}D_j(z/a_j)^{D_j}}{1-(z/a_j)^{D_j}}+(-1)^n d_1+(-1)^{n-i}d_i\right|\\
&\le 2K\left|\sum_{1\le j<i}\frac{(-1)^{n-j}(a_j/z)^{D_j}}{1-(a_j/z)^{D_j}}+\sum_{i<j\le n-1}\frac{(-1)^{n-j-1}(z/a_j)^{D_j}}{1-(z/a_j)^{D_j}}\right|\\
&<\frac{4K^2}{1-K^{-9}}\sum_{k=1}^{n-1}\varepsilon^{5k/2}<\frac{4K^2}{1-K^{-9}}\frac{\varepsilon^{5/2}}{1-\varepsilon^{5/2}}<5K^2\varepsilon^{5/2}
\end{aligned}$$

since $\varepsilon^{5/2}\le K^{-10}$. This means that if $z\in\Omega_i$, we have

$$|z^{D_i}-a_i^{D_i}|\cdot|G_i(z)|/d_{i+1}<3K^3\varepsilon^{5/2}|a_i|^{D_i}<\varepsilon|a_i|^{D_i} \tag{2.13}$$

by (2.7) and Lemma 2.2(1).

From (2.6) and by Rouché's theorem, there exists a solution $w_{i,j}$ of (2.3) such that $w_{i,j}\in\Omega_i$ for every $1\le j\le D_i$. In particular, $|w_{i,j}-\widetilde{w}_{i,j}|<\varepsilon|a_i|$ by the second statement of Lemma 2.1(2). Note that for $1\le i\le n-2$, we have

$$|a_{i+1}|-|a_i|-2\varepsilon|a_i|-2\varepsilon|a_{i+1}|>|a_{i+1}|[1-2\varepsilon-(1+2\varepsilon)K^{-2}]>0. \tag{2.14}$$

By Lemma 2.2(1) and $r_i=\sqrt[D_i]{d_i/d_{i+1}}\le(K/2)^{1/5}$, we have,

$$\frac{r_i|a_i|\sin(\pi/D_i)}{\varepsilon|a_i|}\ge K^4\left(\frac{2}{K}\right)^{1/5}\cdot\frac{2}{\pi}\cdot\frac{\pi}{2K}>K^2>1. \tag{2.15}$$

This means that $w_{i_1,j_1}=w_{i_2,j_2}$ if and only if $(i_1,j_1)=(i_2,j_2)$. The proof is complete. □

For $1\le i\le n-1$, let $CP_i:=\{w_{i,j}:1\le j\le D_i\}$ be the collection of D_i free critical points of f which lie close to the circle $\mathbb{T}_{r_i|a_i|}$ and denote $CV_i=f(CP_i)$.

LEMMA 2.4. *For every $1\le i\le n-1$, there exists an annular neighborhood A_i containing $CP_i\cup\mathbb{T}_{r_i|a_i|}\cup\mathbb{T}_{|a_i|}$, such that:*

(1) *if $p=1$, then $f(\overline{A}_i)\subset\mathbb{D}_s$ for odd $n-i$ and $f(\overline{A}_i)\subset\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_K$ for even $n-i$. In particular, the set of critical values of f satisfies $\bigcup_{i=1}^{n-1}CV_i\subset\mathbb{D}_s\cup\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_K$. The disks $\overline{\mathbb{D}}_s$ and $\overline{\mathbb{C}}\setminus\mathbb{D}_K$ lie in the Fatou set of f and $f^{-1}(\mathbb{A}_{s,K})\subset\mathbb{A}_{s,K}$;*

(2) *if $p=0$, then $f(\overline{A}_i)\subset\mathbb{D}_s$ for even $n-i$ and $f(\overline{A}_i)\subset\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_M$ for odd $n-i$, where $M=(2/s)^{1/d_n}$. In particular, the set of critical values of f satisfies $\bigcup_{i=1}^{n-1}CV_i\subset\mathbb{D}_s\cup\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}_M$. The disks $\overline{\mathbb{D}}_s$ and $\overline{\mathbb{C}}\setminus\mathbb{D}_M$ lie in the Fatou set of f and $f^{-1}(\overline{\mathbb{A}}_{s,M})\subset\mathbb{A}_{s,M}$.*

Proof. Let $\varepsilon=u^{2/K}\le K^{-4}$ be the number that appeared in Lemma 2.3. For every $1\le i\le n-1$, define the annulus

$$A_i=\{z:(\min\{r_i,1\}-2\varepsilon)|a_i|<|z|<(\max\{r_i,1\}+2\varepsilon)|a_i|\}, \tag{2.16}$$

where $r_i = \sqrt[D_i]{d_i/d_{i+1}}$. Obviously, $A_i \supset CP_i \cup \mathbb{T}_{r_i|a_i|} \cup \mathbb{T}_{|a_i|}$. By the definition, we have

$$(2/K)^{1/D_i} \le \min\{r_i, 1\} \le \max\{r_i, 1\} \le (K/2)^{1/D_i}. \tag{2.17}$$

If $z \in \overline{A}_i$, we have

$$|a_i/z| \le \frac{1}{(2/K)^{1/D_i} - 2\varepsilon} \le \frac{(K/2)^{1/D_i}}{1 - 2K^{-4}(K/2)^{1/5}} < (K/2)^{1/D_i}(1 + 4/K^{19/5}) \tag{2.18}$$

and

$$|z/a_i| \le (K/2)^{1/D_i} + 2\varepsilon \le (K/2)^{1/D_i} + 2/K^4 < (K/2)^{1/D_i}(1 + 1/K^3). \tag{2.19}$$

This means that

$$|a_i/z|^5 < (K/2)^{5/D_i}(1 + 4/K^{19/5})^5 < (K/2)e^{20/K^{19/5}} < (K/2)e^{20/3^{19/5}} < 7K/10 \tag{2.20}$$

and also

$$|z/a_i|^5 < (K/2)^{5/D_i}(1 + 1/K^3)^5 < (K/2)e^{5/K^3} < (K/2)e^{5/27} < 7K/10. \tag{2.21}$$

Moreover, similar to the argument of (2.20) and (2.21), we have

$$|a_i/z|^{d_i} + |z/a_i|^{d_{i+1}} < 7K/5. \tag{2.22}$$

Recall that $|a_i/a_{i+1}|^{d_{i+1}} = u$ for every $1 \le i \le n-2$ and $|a_{n-1}|^{d_n} = v$. Letting $1 \le i_1 \le i_2 \le n-1$ and $p \in \{0, 1\}$, we have

$$\begin{aligned}
\prod_{j=i_1}^{i_2} |a_j|^{(-1)^{n-j-p}D_j} &= |a_{i_1}|^{(-1)^{n-i_1-p}d_{i_1}} |a_{i_2}|^{(-1)^{n-i_2-p}d_{i_2+1}} \prod_{j=i_1}^{i_2-1} \left|\frac{a_j}{a_{j+1}}\right|^{(-1)^{n-j-p}d_{j+1}} \\
&= |a_{i_1}|^{(-1)^{n-i_1-p}d_{i_1}} |a_{i_2}|^{(-1)^{n-i_2-p}d_{i_2+1}} u^{[(-1)^{n-i_1-p}-(-1)^{n-i_2-p}]/2} \\
&= \begin{cases} (|a_1|^{d_1}u/v)^{(-1)^p} & \text{if } i_1 = 1 \text{ and } i_2 = n-1 \text{ is even}, \\ (|a_1|^{-d_1}/v)^{(-1)^p} & \text{if } i_1 = 1 \text{ and } i_2 = n-1 \text{ is odd}. \end{cases}
\end{aligned} \tag{2.23}$$

By (1.2) and the second equation of (2.23), we have

$$\begin{aligned}
&|f(z)| \\
&= |z^{D_i} - a_i^{D_i}|^{(-1)^{n-i-p}} |z|^{(-1)^{n-p}d_1} \prod_{j=1}^{i-1} |z|^{(-1)^{n-j-p}D_j} \prod_{j=i+1}^{n-1} |a_j|^{(-1)^{n-j-p}D_j} \cdot Q_i(z) \\
&= |1 - (z/a_i)^{D_i}|^{(-1)^{n-i-p}} |z/a_i|^{(-1)^{n-i-p+1}d_i} \\
&\quad \times |a_{n-1}|^{(-1)^{1-p}d_n} u^{[(-1)^{n-i-p}-(-1)^{1-p}]/2} \cdot Q_i(z) \\
&= v^{(-1)^{1-p}} u^{[(-1)^{n-i-p}-(-1)^{1-p}]/2} |(a_i/z)^{d_i} - (z/a_i)^{d_{i+1}}|^{(-1)^{n-i-p}} \cdot Q_i(z) \\
&\begin{cases} \le v^{(-1)^{1-p}} u^{[1-(-1)^{1-p}]/2} (|a_i/z|^{d_i} + |z/a_i|^{d_{i+1}}) Q_i(z) & \text{if } n-i-p \text{ is even}, \\ \ge v^{(-1)^{1-p}} u^{[-1-(-1)^{1-p}]/2} (|a_i/z|^{d_i} + |z/a_i|^{d_{i+1}})^{-1} Q_i(z) & \text{if } n-i-p \text{ is odd}, \end{cases}
\end{aligned} \tag{2.24}$$

where

$$Q_i(z) = \prod_{j=1}^{i-1} |1 - (a_j/z)^{D_j}|^{(-1)^{n-j-p}} \prod_{j=i+1}^{n-1} |1 - (z/a_j)^{D_j}|^{(-1)^{n-j-p}}. \tag{2.25}$$

For $1 \le i \le n-1$, consider $z \in \overline{A}_i$. If $1 \le j < i$, by (2.20), we have

$$|a_j/z|^{D_j} \le |a_i/z|^5 |a_j/a_i|^5 < 7K\varepsilon^{5(i-j)/2}/10 < K^{-9}. \tag{2.26}$$

If $i < j \le n-1$, then

$$|z/a_j|^{D_j} \le |z/a_i|^5 |a_i/a_j|^5 < 7K\varepsilon^{5(i-j)/2}/10 < K^{-9} \tag{2.27}$$

by (2.21). Since $e^x < 1+2x$ if $0 < x \le 1$ and $\varepsilon \le K^{-4}$, by (2.25)–(2.27) we have

$$Q_i(z) < \prod_{k=1}^{\infty}(1+7K\varepsilon^{5k/2}/5)^2 \le \exp\left(\frac{14K\varepsilon^{5/2}/5}{1-\varepsilon^{5/2}}\right) < 1+K^{-5} < 1.01 \tag{2.28}$$

and

$$Q_i(z) > \prod_{k=1}^{\infty}(1+7K\varepsilon^{5k/2}/5)^{-2} > 1/1.01 > 0.99. \tag{2.29}$$

For $p=1$, by Lemma 2.2(2) and (3a), for every $1 \le i \le n-1$, if $|z| \le s$, we have

$$|z^{D_i}/a_i^{D_i}| \le |s/a_1|^{D_i} |a_1/a_i|^{D_i} \le (sK^{-3}/2)^{5/K} u^{5(i-1)/K}. \tag{2.30}$$

If we notice Lemma 2.2(1), then

$$\sum_{i=1}^{n-1} |z^{D_i}/a_i^{D_i}| \le \frac{(sK^{-3}/2)^{5/K}}{1-u^{5/K}} \le \frac{K^{(10/K)-10}}{1-K^{-10}} < 1/200. \tag{2.31}$$

For $p=0$, by Lemma 2.2(2) and (4b), for every $1 \le i \le n-1$, if $|z| \le s$, we have

$$|z^{D_i}/a_i^{D_i}| \le |s/a_1|^{D_i} |a_1/a_i|^{D_i} \le (u^{1/2}/2)^{5/K} u^{5(i-1)/K}. \tag{2.32}$$

By Lemma 2.2(1), then

$$\sum_{i=1}^{n-1} |z^{D_i}/a_i^{D_i}| \le \frac{(u^{1/2}/2)^{5/K}}{1-u^{5/K}} \le \frac{K^{-5}}{1-K^{-10}} < 1/200. \tag{2.33}$$

Since $(1+2|a|)^{-1} \le |1+a|^{\pm 1} \le 1+2|a|$ if $0 \le |a| \le 1/2$, by (2.31) and (2.33) we know that

$$\prod_{i=1}^{n-1} |1 - z^{D_i}/a_i^{D_i}|^{(-1)^{n-i-p}} \le \prod_{i=1}^{n-1}(1+2|z/a_i|^{D_i}) < e^{1/100} < K. \tag{2.34}$$

Therefore,

$$\prod_{i=1}^{n-1} |1 - z^{D_i}/a_i^{D_i}|^{(-1)^{n-i-p}} \ge \prod_{i=1}^{n-1}(1+2|z/a_i|^{D_i})^{-1} > e^{-1/100} > 1/K. \tag{2.35}$$

(1) We first consider the case $p=1$. If $n-i$ is odd, by (2.22), (2.24) and (2.28), if $z \in \overline{A}_i$ we have

$$|f(z)| \le v \cdot (7K/5) \cdot 1.01 < 2Kv < s. \tag{2.36}$$

If $n-i$ is even, by (2.22), (2.24) and (2.29), for $z \in \overline{A}_i$ we have

$$|f(z)| \ge (v/u) \cdot (7K/5)^{-1} \cdot 0.99 > v/(2Ku) > K. \tag{2.37}$$

If n is odd, by Lemma 2.2(3a), (2.23) and (2.34), for every z such that $|z| \le s$, we have

$$|f(z)| = |z|^{d_1} \prod_{i=1}^{n-1} |a_i|^{D_i(-1)^{n-i-1}} \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i-1}} < |s/a_1|^{d_1} v u^{-1} \cdot 1.02 < s.$$

It follows that $f(\overline{\mathbb{D}}_s) \subset \mathbb{D}_s$ for odd n. If n is even and $|z| \le s$, by Lemma 2.2(3b), (2.23) and (2.35), we have

$$|f(z)| = |a_1/z|^{d_1} v \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i-1}} > |a_1/s|^{d_1} v/1.02 > K.$$

Therefore $f(\overline{\mathbb{D}}_s) \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_K$ for even n.

Note that f is very 'close' to $z \mapsto z^{d_n}$ on the outside of $\mathbb{D}_K$ since $|a_i|^{D_i}$ is extremely small, where $1 \le i \le n-1$. This means that f may exhibit some dynamics of $z \mapsto z^{d_n}$ if $|z| \ge K$. More specifically, by arguments completely similar to those for (2.34)–(2.35), if $|z| \ge K$, then

$$|f(z)| \ge |z|^{d_n} \prod_{i=1}^{n-1} \left(1 + 2\frac{|a_i|^{D_i}}{|z|^{D_i}}\right)^{-1} > K. \tag{2.38}$$

This means that $f(\overline{\mathbb{C}} \setminus \mathbb{D}_K) \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_K$. Then we have $f^{-1}(\overline{\mathbb{A}}_{s,K}) \subset \mathbb{A}_{s,K}$ for every $n \ge 2$ (see Figure 2).

(2) Now we consider the case $p = 0$. If $n - i$ is even, by (2.22), (2.24), (2.28) and Lemma 2.2(4a), if $z \in \overline{A_i}$ we have

$$|f(z)| \le v^{-1} u \cdot (7K/5) \cdot 1.01 < 2Ku/v < s. \tag{2.39}$$

If $n - i$ is odd, by (2.22), (2.24), (2.29) and Lemma 2.2(4a), for $z \in \overline{A_i}$ we have

$$|f(z)| \ge v^{-1} \cdot (7K/5)^{-1} \cdot 0.99 > 1/(2Kv) > M, \tag{2.40}$$

where $M = (2/s)^{1/d_n}$.

If n is even, by Lemma 2.2(4b), (2.23) and (2.34), for each z such that $|z| \le s$, we have

$$|f(z)| = |z|^{d_1} \prod_{i=1}^{n-1} |a_i|^{D_i(-1)^{n-i}} \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i}} < |s/a_1|^{d_1} v^{-1} \cdot e^{1/100} < s.$$

It follows that $f(\overline{\mathbb{D}}_s) \subset \mathbb{D}_s$ for even n. If n is odd and $|z| \le s$, by Lemma 2.2(4c), (2.23) and (2.35), we have

$$|f(z)| = |a_1/z|^{d_1} u v^{-1} \prod_{i=1}^{n-1} \left|1 - \frac{z^{D_i}}{a_i^{D_i}}\right|^{(-1)^{n-i}} \ge |a_1/s|^{d_1} u v^{-1} \cdot e^{-1/100} > M.$$

Therefore $f(\overline{\mathbb{D}}_s) \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_M$ for odd n.

If $|z| \ge M$, then

$$|f(z)| = |z|^{-d_n} \prod_{i=1}^{n-1} \left|1 - \frac{a_i^{D_i}}{z^{D_i}}\right|^{(-1)^{n-i}} \le M^{-d_n} \prod_{i=1}^{n-1} \left(1 + \frac{2|a_i|^{D_i}}{|z|^{D_i}}\right) < 2M^{-d_n} = s. \tag{2.41}$$

This means that $f(\overline{\mathbb{C}} \setminus \mathbb{D}_M) \subset \mathbb{D}_s$. Then we have $f^{-1}(\overline{\mathbb{A}}_{s,M}) \subset \mathbb{A}_{s,M}$ for every $n \ge 2$. □

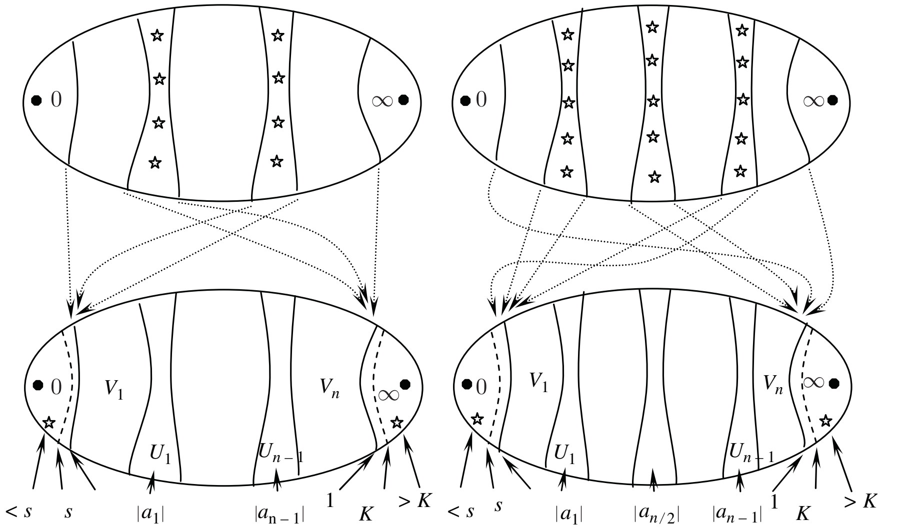

FIGURE 2. Sketch illustrating the mapping relation of $f_{1,d_1,\ldots,d_n}$, where n is odd and even respectively (from left to right). The small stars denote the critical points and critical values, and the numbers shown at the bottom of the figures denote the approximate coordinates.

THEOREM 2.5. *If* $|a_{n-1}| = (s_1 K^{-2})^{1/d_n}$ *and* $|a_i| = (s_1 K^{-5})^{1/d_{i+1}} |a_{i+1}|$ *for* $1 \le i \le n-2$*, where* $s_1 > 0$ *is small enough, then the Julia set of* $f_{1,d_1,\ldots,d_n}$ *is a Cantor set of circles. If* $|a_{n-1}| = (s_0^{(1/d_n)+(1-\xi)/3})^{1/d_n}$ *and* $|a_i| = (s_0^{1+(1/d_n)+2(1-\xi)/3})^{1/d_{i+1}} |a_{i+1}|$ *for* $1 \le i \le n-2$*, where* $s_0 > 0$ *is small enough, then the Julia set of* $f_{0,d_1,\ldots,d_n}$ *is a Cantor set of circles.*

Proof. We only focus on the case $p = 1$ since a similar proof can be used for the case $p = 0$ by using Lemma 2.4(2). We also use f to denote $f_{1,d_1,\ldots,d_n}$ for simplicity. Let U_i be the component of $f^{-1}(D)$ containing a_i, where $D = \mathbb{D}_s$ if $n - i$ is odd and $D = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_K$ if $n - i$ is even. By Lemma 2.4(1), it follows that the set of critical points $CP_i \subset U_i$ and U_i is a connected domain containing the annulus A_i. Moreover, $U_i \cap U_{i+1} = \emptyset$ since $f(U_i) \cap f(U_{i+1}) = \emptyset$ by Lemma 2.4(1), where $1 \le i < n - 2$. This means that $U_i \cap U_j = \emptyset$ for different i, j. Suppose that U_i has m_i boundary components. Since there are exactly D_i critical points in U_i and $f : U_i \to D$ is a branched covering with degree D_i, then the Riemann–Hurwitz formula tells us $\chi_{U_i} = 2 - m_i = D_i \chi_D - D_i = 0$, where χ denotes the Euler characteristic. This means that $m_i = 2$ and therefore U_i is an annulus surrounding the origin for every $1 \le i \le n - 1$.

For $1 \le i \le n - 2$, let V_{i+1} be the annulus domain between U_i and U_{i+1}. It is easy to see that $f : V_{i+1} \to \mathbb{A}_{s,K}$ is a covering map with degree d_{i+1}. Note that every component of $f^{-1}(\mathbb{A}_{s,K})$ is an annulus since $\mathbb{A}_{s,K}$ is doubly connected and contains no critical values. It follows that there exist two annuli V_1 and V_n, which lie between 0 and U_1, U_{n-1} and ∞ respectively, such that $f : V_1, V_n \to \mathbb{A}_{s,K}$ are covering maps with degree d_1 and d_n respectively. In fact, the restriction of f on ∂U_1 and ∂U_{n-1} has degrees d_1 and d_n, respectively, and there are no critical points in V_1 and V_n (see Figure 2).

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The Julia set of f is $J = \bigcap_{k\geq 0} f^{-k}(\mathbb{A}_{s,K})$. By construction, the components of J are compact sets nested between 0 and ∞ since each inverse branch $f^{-1} : \mathbb{A}_{s,K} \to V_j$ is conformal for every $0 \leq j \leq n$. Since the component of J cannot be a point and f is hyperbolic, every component of J is a Jordan curve (actually quasicircle) by Theorem 1.2 in [**PT**]. The dynamics on the set of Julia components of f is isomorphic to the one-sided shift on n symbols $\Sigma_n := \{0, 1, \ldots, n-1\}^{\mathbb{N}}$. In particular, J is homeomorphic to $\Sigma_n \times \mathbb{S}^1$, which is a Cantor set of circles as desired. This ends the proof of Theorem 2.5 and hence Theorem 1.1. □

Remark 2.6. Since f is hyperbolic, the Julia set of f is also a Cantor set of circles if we perturb some a_i gently, where $1 \leq i \leq n-1$. In the first version of our manuscript of this paper, only $d_i = n+1$ for every $1 \leq i \leq n$ was considered. In this case, it was shown that for every $n \geq 2$ and $1 \leq i \leq n-1$, if $|a_{n-i}| = [n/(n+1)]^{i-1} s^i$ for $0 < s \leq 1/10$, then the Julia set of $f_{1,n+1,\ldots,n+1}$ is a Cantor set of circles.

THEOREM 2.7. *Suppose that a_i is chosen as in Theorem 1.1 such that the Julia set of $f_{p,d_1,\ldots,d_n}$ is a Cantor set of circles for $n \geq 3$, then $f_{p,d_1,\ldots,d_n}$ is not topologically conjugate to any McMullen maps on their corresponding Julia sets.*

Proof. Since the dynamics on the set of Julia components of $f_{p,d_1,\ldots,d_n}$ is conjugate to the one-sided shift on n symbols $\Sigma_n := \{0, 1, \ldots, n-1\}^{\mathbb{N}}$ and, in particular, the set of Julia components of g_η is isomorphic to the one-sided shift on only two symbols $\Sigma_2 := \{0, 1\}^{\mathbb{N}}$, this means that $f_{p,d_1,\ldots,d_n}$ cannot be topologically conjugate to g_η on their corresponding Julia sets if $n \geq 3$. □

3. *Topological conjugacy between the Cantor circle Julia sets*

In this section we show that, for any given rational map whose Julia set is a Cantor set of circles, there exists a map $f_{p,d_1,\ldots,d_n}$ in (1.2) such that these two rational maps are topologically conjugate on their corresponding Julia sets.

LEMMA 3.1. *If f is a rational map whose Julia set is a Cantor set of circles, then there exist no critical points in $J(f)$.*

Proof. Suppose there exists a Julia component J_0 of f containing a critical point c_0 of f with multiplicity d. Then f is not one-to-one in any small neighborhood of c_0. It is known that $f(J_0)$ is a Julia component containing $f(c_0)$ [**Be**, Lemma 5.7.2]. Choose a small topological disk neighborhood U of $f(c_0)$ such that $U \cap f(J_0)$ is a simple curve. The component of $f^{-1}(U)$ containing c_0 is mapped onto U in the manner of $(d+1)$-to-one. Note that the component J' of $f^{-1}(U \cap f(J_0))$ containing c_0 is connected and contained in J_0. However, J' possesses star-like structure and hence is not a simple curve. This contradicts the assumption that J_0 is a Jordan closed curve since $J(f)$ is a Cantor set of circles. □

We say that a compact set $X \subset \overline{\mathbb{C}}$ *separates* 0 and ∞ if 0 and ∞ lie in the two different components of $\overline{\mathbb{C}} \backslash X$ respectively. Let X and Y be two disjoint compact sets that both separate 0 and ∞ respectively. We say $X \prec Y$ if X is contained in the component of $\overline{\mathbb{C}} \backslash Y$

which contains 0. Let A be an annulus whose closure separates 0 and ∞, we use $\partial_- A$ and $\partial_+ A$ to denote the two components of the boundary of A such that $\partial_- A \prec \partial_+ A$.

THEOREM 3.2. *Let f be a rational map whose Julia set is a Cantor set of circles. Then there exist $p \in \{0, 1\}$, positive integers $n \geq 2$ and $d_1, \ldots, d_n$ satisfying $\sum_{i=1}^{n}(1/d_i) < 1$ such that f is topologically conjugate to $f_{p,d_1,\ldots,d_n}$ on their corresponding Julia sets.*

Proof. Let $J(f)$ be the Julia set of f which is a Cantor set of circles, then every periodic Fatou component of f must be attracting or parabolic by Lemma 3.1. We only prove the attracting (hyperbolic) case in detail and explain the parabolic case by using the work of Cui [**Cui**].

In the following, we suppose that f is hyperbolic. There exist exactly two simply connected Fatou components of f and all other Fatou components are annuli. Let $\mathcal{D}$ and $\mathcal{A}$ be the collection of simply and doubly connected Fatou components of f respectively. We claim that $f(\mathcal{D}) \subset \mathcal{D}$ and there exists an integer $k \geq 1$ such that $f^{\circ k}(A) \in \mathcal{D}$ for every $A \in \mathcal{A}$. The assertion $f(\mathcal{D}) \subset \mathcal{D}$ is obvious since the image of a simply connected Fatou component under a rational map is again simply connected. If $f(A_1) = A_2$, where $A_1, A_2 \in \mathcal{A}$, then there exist no critical points in A_1 by Riemann–Hurwitz's formula. This means that each $A \in \mathcal{A}$ cannot be periodic since the cycle of every periodic attracting Fatou component must contain at least one critical point. On the other hand, by Sullivan's theorem, the Fatou components of a rational map cannot be wandering. This completes the proof of the claim.

Up to a Mobius transformation, we can assume that 0 and ∞, respectively, belong to the two simply connected Fatou components of f, which are denoted by D_0 and D_∞. Namely, $\mathcal{D} = \{D_0, D_\infty\}$. Since $f(\mathcal{D}) \subset \mathcal{D}$, we first suppose that $f(D_0) = D_0$ and $f(D_\infty) = D_\infty$. Let $f^{-1}(D_0) = D_0 \cup A_1 \cup \cdots \cup A_m$, where $A_1, \ldots, A_m$ are m annuli separating 0 and ∞ such that $A_i \prec A_{i+1}$ for every $1 \leq i \leq m-1$. It is easy to see $m \geq 1$. Otherwise, D_0 is completely invariant, then $J(f) = \partial D_0$ which contradicts the assumption that $J(f)$ is a Cantor set of circles.

Suppose that $\deg(f|_{D_0} : D_0 \to D_0) = d_1$ and $\deg(f|_{\partial_- A_i} : \partial_- A_i \to \partial D_0) = d_{2i}$ and $\deg(f|_{\partial_+ A_i} : \partial_+ A_i \to \partial D_0) = d_{2i+1}$ for $1 \leq i \leq m$. It follows that $\deg(f) = \sum_{j=1}^{2m+1} d_j$. Let W_1 be the annular domain between D_0 and A_1 and W_i be the annular domain between A_{i-1} and A_i, where $2 \leq i \leq m$. We have $f(W_i) = \overline{\mathbb{C}} \setminus \overline{D_0}$ and $\deg(f|_{W_i} : W_i \to \overline{\mathbb{C}} \setminus \overline{D_0}) = d_{2i-1} + d_{2i}$. This means that there exists at least one Fatou component $B_i \subsetneq W_i$ such that $f(B_i) = D_\infty$. If there exist $B_i' \neq B_i$ such that $B_i' \subsetneq W_i$ and $f(B_i') = D_\infty$, there must exist one component of $f^{-1}(D_0)$ in W_i, which contradicts the assumption that $A_1 \cup \cdots \cup A_m$ is the collection of all annular components of $f^{-1}(D_0)$. So there exists exactly one Fatou component $B_i \subsetneq W_i$ such that $f(B_i) = D_\infty$ and $\deg(f|_{B_i} : B_i \to D_\infty) = d_{2i-1} + d_{2i}$. A similar argument can be used to show that D_∞ is the only component of $f^{-1}(D_\infty)$ lying in the unbounded component of $\overline{\mathbb{C}} \setminus A_m$ which can be mapped onto D_∞. Therefore, $f^{-1}(D_\infty) = B_1 \cup \cdots \cup B_m \cup D_\infty$ and $\deg(f|_{D_\infty}) = d_{2m+1}$ since $\deg(f) = \sum_{j=1}^{2m+1} d_j$. Denote $\overline{\mathbb{C}} \setminus (D_0 \cup D_\infty)$ by E. The preimage $f^{-1}(E)$ consists of $2m+1$ annular components $E_1, \ldots, E_{2m+1}$ such that $E_i \prec E_{i+1}$ for $1 \leq i \leq 2m$. The map $f : E_i \to E$ is an unramified covering map with degree d_i, where $1 \leq i \leq 2m+1$ (see Figure 3).

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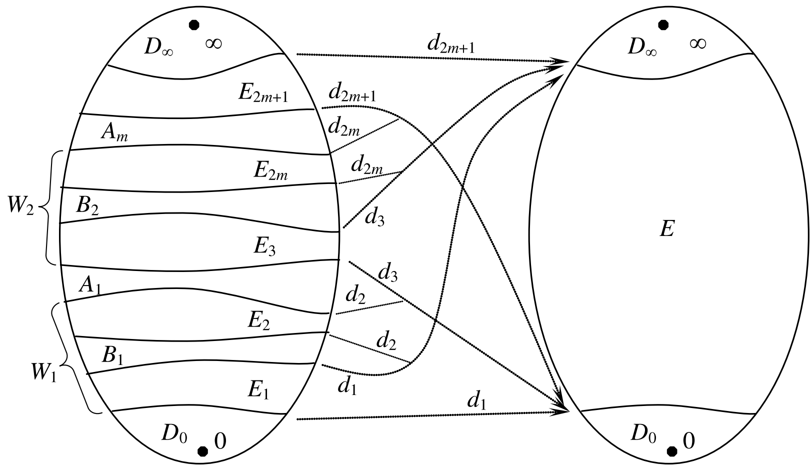

FIGURE 3. Sketch illustrating the mapping relation of f, where d_i, $1 \le i \le 2m+1$, denote the degrees of the restriction of f on the boundaries of Fatou components.

Let $n = 2m+1$ and $p = 1$. The assertion $\sum_{i=1}^{n} 1/d_i < 1$ follows from Grötzsch's modulus inequality since each E_i is essentially contained in E and $\text{mod}(E_i) = \text{mod}(E)/d_i$. In the following, we will construct a quasiconformal map $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which conjugates the dynamics on the Julia set of f to that of $f_{1,d_1,\ldots,d_n}$.

For simplicity, we denote $f_{1,d_1,\ldots,d_n}$ by F. Note that $F(0) = 0$ and $F(\infty) = \infty$. There exist two simply connected Fatou components D_0' and D_∞', both are invariant under F such that $0 \in D_0'$ and $\infty \in D_\infty'$. From the proof of Theorem 1.1, we know that $F^{-1}(D_0') = D_0' \cup A_1' \cup \cdots \cup A_m'$, where $A_1', \ldots, A_m'$ are m annuli separating 0 and ∞ such that $A_i' \prec A_{i+1}'$ for every $1 \le i \le m-1$. Moreover, $\deg(F|_{D_0'} : D_0' \to D_0') = d_1$, $\deg(F|_{\partial_- A_i'} : \partial_- A_i' \to \partial D_0') = d_{2i}$ and $\deg(F|_{\partial_+ A_i'} : \partial_+ A_i' \to \partial D_0') = d_{2i+1}$ for $1 \le i \le m$. Let W_1' be the annular domain between D_0' and A_1' and W_i' be the annular domain between A_{i-1}' and A_i', where $2 \le i \le m$. There exists exactly one Fatou component $B_i' \subsetneq W_i'$ such that $F(B_i') = D_\infty'$ and $\deg(F|_{B_i'} : B_i' \to D_\infty') = d_{2i-1} + d_{2i}$. We have $F^{-1}(D_\infty') = B_1' \cup \cdots \cup B_m' \cup D_\infty'$ and $\deg(F|_{D_\infty'}) = d_{2m+1}$. Similarly, let $E' := \overline{\mathbb{C}} \setminus (D_0' \cup D_\infty')$. There exist $2m+1$ annular components $E_1', \ldots, E_{2m+1}'$ of $F^{-1}(E')$ such that $E_i' \prec E_{i+1}'$ for $1 \le i \le 2m$. The map $F : E_i' \to E'$ is a covering with degree d_i, where $1 \le i \le 2m+1$.

By quasiconformal surgery, it can be seen that ∂D_0, ∂D_∞, $\partial D_0'$, $\partial D_\infty'$ and their preimages are all quasicircles and the dilatation is bounded by a fixed constant. There exists a quasiconformal mapping $\phi_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\phi_0(D_0) = D_0'$ and $\phi_0(D_\infty) = D_\infty'$, hence $\phi_0(\partial D_0) = \partial D_0'$ and $\phi_0(\partial D_\infty) = \partial D_\infty'$. Moreover, ϕ_0 can be chosen such that $\phi_0 \circ f = F \circ \phi_0$ on $\partial D_0 \cup \partial D_\infty$.

Now we construct a lift $\phi_{E_1} : E_1 \to E_1'$ of $\phi_0 : E \to E'$ as follows. For every $z \in E_1 \setminus \partial_- E_1$, we choose a simple curve $\gamma : [0, 1] \to E$ such that $\gamma(1) = f(z)$ and $\gamma(0) = w \in \partial_- E$. Since $f : E_1 \to E$ is a covering map, there exists a unique lift $\widetilde{\gamma} : [0, 1] \to E_1$ of γ such that $\widetilde{\gamma}(1) = z$ and $\widetilde{w} := \widetilde{\gamma}(0) \in \partial_- E_1$. Similarly, since $F : E_1' \to E'$ is a covering map, there exists a unique lift $\alpha : [0, 1] \to E_1'$ of $\phi_0(\gamma) : [0, 1] \to E'$ such that $\alpha(0) = \phi_0(\widetilde{w})$ since $\phi_0 \circ f = F \circ \phi_0$ on $\partial D_0 = \partial_- E_1$. Define $\phi_{E_1}(z) := \alpha(1)$. We know that $\phi_0 \circ f = F \circ \phi_{E_1}$ on E_1 and $\phi_{E_1} : E_1 \to E_1'$ is quasiconformal since f, F are both holomorphic covering maps with degree d_1 and $\phi_0 : E \to E'$ is quasiconformal. Now some parts of $\phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are defined as follows: $\phi_1|_{\overline{D_0}} = \phi_0|_{\overline{D_0}}$, $\phi_1|_{\overline{D_\infty}} = \phi_0|_{\overline{D_\infty}}$ and $\phi_1|_{E_1} = \phi_{E_1}$. Then $\phi_1 \circ f = F \circ \phi_1$ on ∂E_1. Similarly, there exists a unique quasiconformal mapping $\phi_{E_{2m+1}} : E_{2m+1} \to E'_{2m+1}$, which is the lift of $\phi_0 : E \to E'$, such that $\phi_0 \circ f = F \circ \phi_{E_{2m+1}}$ on E_{2m+1}. Define $\phi_1|_{E_{2m+1}} = \phi_{E_{2m+1}}$. Then, $\phi_1 \circ f = F \circ \phi_1$ on ∂E_{2m+1}.

Unlike the cases of E_1 and E_{2m+1}, the lift $\phi_{E_i} : E_i \to E_i'$ of $\phi_0 : E \to E'$ exists but is not unique for $2 \le i \le 2m$. We first show the existence of ϕ_{E_i}. Without loss of generality, suppose that i is even. Since $f : \partial_- E_i \to \partial D_\infty$ and $F : \partial_- E_i' \to \partial D'_\infty$ are both covering mappings with degree d_i, there exists a lift (not unique) $\phi_{E_i} : \partial_- E_i \to \partial_- E_i'$ of $\phi_0 : \partial D_\infty \to \partial D'_\infty$ such that $\phi_0 \circ f = F \circ \phi_{E_i}$ on $\partial_- E_i$. By using the same method of defining ϕ_{E_1}, there exists a unique lift of $\phi_0 : E \to E'$ defined from E_i to E_i', which we denote also by ϕ_{E_i} such that $\phi_0 \circ f = F \circ \phi_{E_i}$ on E_i. Note that $\phi_{E_i} : E_i \to E_i'$ is quasiconformal. Define $\phi_1|_{E_i} = \phi_{E_i}$. Then, $\phi_0 \circ f = F \circ \phi_1$ on $\bigcup_{i=1}^{2m+1} E_i$ and $\phi_1 \circ f = F \circ \phi_1$ on $\bigcup_{i=1}^{2m+1} \partial E_i$.

In order to unify the notation, let $D_{2i-1} := B_i$ and $D_{2i} := A_i$ for $1 \le i \le m$. Then we have $D_i \prec D_j$ for $1 \le i < j \le 2m$. We need to define ϕ_1 on $\bigcup_{i=1}^{2m} D_i$. For every D_i, where $1 \le i \le 2m$, its two boundary components $\partial_+ E_i$ and $\partial_- E_{i+1}$ are both quasicircles. Since ϕ_{E_i} and $\phi_{E_{i+1}}$ are both quasiconformal mappings, the map $\phi_1|_{\partial_+ E_i \cup \partial_- E_{i+1}}$ has a quasiconformal extension $\phi_{D_i} : \overline{D_i} \to \overline{D_i'}$ such that $\phi_{D_i}(D_i) = D_i'$. Now we obtain a quasiconformal mapping $\phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined as $\phi_1|_{E_i} := \phi_{E_i}$, $\phi_1|_{D_j} = \phi_{D_j}$ and $\phi_1|_{D_0 \cup D_\infty} = \phi_0$, where $1 \le i \le 2m+1$ and $1 \le j \le 2m$.

Next, we define ϕ_2. First, let $\phi_2|_{D_j} = \phi_1$ for $j \in \{0, 1, \ldots, 2m, \infty\}$. Then we lift $\phi_1 : E \to E'$ in an appropriate way to obtain $\phi_2 : E_i \to E_i'$ for $1 \le i \le 2m+1$. Finally, we check the continuity of the resulting map $\phi_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Now let us make this precise. In order to guarantee the continuity of ϕ_2 on $D_0 \cup E_1$, we need to have $\phi_2|_{\partial_- E_1} = \phi_1$. Then there exists only one way to lift $\phi_1 : E \to E'$ to obtain $\phi_2 : E_1 \to E_1'$. In order to guarantee the continuity of the lift ϕ_2, we need to check the continuity of ϕ_2 on the boundary $\partial_+ E_1$ first. In fact, $\phi_0|_E$ and $\phi_1|_E$ are homotopic to each other and $\phi_1|_{\partial E} = \phi_0|_{\partial E}$, so it follows that $\phi_2|_{\partial_+ E_1} = \phi_1|_{\partial_+ E_1}$ since $\phi_2|_{\partial_- E_1} = \phi_1|_{\partial_- E_1}$. This means that ϕ_2 is continuous on $\partial_+ E_1$. Similarly, we can lift $\phi_1 : E \to E'$ to obtain $\phi_2 : E_i \to E_i'$ for $2 \le i \le 2m+1$ and guarantee the continuity of ϕ_2. Above all, the map $\phi_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfies: (1) ϕ_2 is quasiconformal and the dilatation $K(\phi_2) = K(\phi_1)$; (2) $\phi_2|_{f^{-1}(D_0 \cup D_\infty)} = \phi_1$; (3) $\phi_1 \circ f = F \circ \phi_2$ on $\bigcup_{i=1}^{2m+1} E_i$ and hence $\phi_2 \circ f = F \circ \phi_2$ on $f^{-2}(\partial D_0 \cup \partial D_\infty)$.

Suppose we have obtained ϕ_k for some $k \ge 1$, then ϕ_{k+1} can be defined completely similarly to the process of the derivation of ϕ_2 from ϕ_1. Inductively, we can obtain a sequence of quasiconformal mappings $\{\phi_k\}_{k \ge 0}$ such that: (1) $K(\phi_k) = K(\phi_1) \ge K(\phi_0)$ for $k \ge 1$; (2) $\phi_{k+1}(z) = \phi_k(z)$ for $z \in f^{-k}(D_0 \cup D_\infty)$; (3) $\phi_k \circ f = F \circ \phi_k$ on $f^{-k}(\partial D_0 \cup \partial D_\infty)$. This means that $\{\phi_k\}_{k \ge 0}$ forms a normal family. Take a convergent subsequence of $\{\phi_k\}_{k \ge 0}$ whose limit we denote by ϕ_∞, then ϕ_∞ is a quasiconformal mapping satisfying $\phi_\infty \circ f = F \circ \phi_\infty$ on $\bigcup_{k \ge 0} f^{-k}(\partial D_0 \cup \partial D_\infty)$. Moreover, $K(\phi_\infty) \le K(\phi_1)$. Since ϕ_∞ is continuous, $\phi_\infty \circ f = F \circ \phi_\infty$ holds on the closure of $\bigcup_{k \ge 0} f^{-k}(\partial D_0 \cup \partial D_\infty)$, which is the Julia set of f. Therefore, $\phi = \phi_\infty$ is the quasiconformal mapping we want to find which conjugates f to F on their corresponding Julia sets. This ends the proof of the case $f(D_0) = D_0$ and $f(D_\infty) = D_\infty$.

The other three cases: (1) $f(D_0) = D_\infty$, $f(D_\infty) = D_\infty$; (2) $f(D_0) = D_\infty$, $f(D_\infty) = D_0$; and (3) $f(D_0) = D_0$, $f(D_\infty) = D_0$ can be proved completely similarly.

If one or both of the components D_0 and D_∞ are parabolic, there exists a perturbation f_ε of f such that f_ε is hyperbolic and the dynamics of f_ε are topologically conjugate to those of f on their corresponding Julia sets [**Cui**]. Then f has a 'model' in (1.2) since f_ε always does. This ends the proof of Theorem 3.2 and hence Theorem 1.2. □

From the proof of Theorem 3.2 in the hyperbolic case, we have the following immediate corollary.

COROLLARY 3.3. *If the parameters a_i are chosen as in Theorem 1.1, where $1 \le i \le n-1$, then each Julia component of $f_{p,d_1,\ldots,d_n}$ is a quasicircle.*

4. *Non-hyperbolic rational maps whose Julia sets are Cantor circles*

The rational maps

$$P_\lambda(z) = \frac{(1/n)[(1+z)^n - 1] + \lambda^{m+n} z^{m+n}}{1 - \lambda^{m+n} z^{m+n}}, \tag{4.1}$$

where $\lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ and $m, n \ge 2$ are both positive integers satisfying $1/m + 1/n < 1$, can be seen as a perturbation of the parabolic polynomial

$$\widetilde{P}(z) = \frac{(1+z)^n - 1}{n}. \tag{4.2}$$

Note that $\widetilde{P}$ has a parabolic fixed point at the origin with multiplier 1 and critical point -1 with multiplicity $n-1$. This means that there exists only one bounded and hence simply connected Fatou component of $\widetilde{P}$ in which all points are attracted to the origin. In particular, the Julia set of $\widetilde{P}$ is a Jordan curve with infinitely many cusps.

We hope that some of the properties of $\widetilde{P}$ stated above will also hold for P_λ when λ is small. However, there are obviously many differences between P_λ and $\widetilde{P}$. The degree of P_λ is $m+n$ and $P_\lambda(\infty) = -1$. There are $2(m+n) - 2$ critical points of P_λ: $m-1$ at ∞, $n-1$ very close to -1, and the remaining $m+n$ critical points lie near the circle $\mathbb{T}_{r_0/|\lambda|}$, where $r_0 = \sqrt[m+n]{n/m}$ (see Lemma 4.3). In fact, we will see that P_λ can be viewed as a 'parabolic' McMullen map at the end of this section since P_λ is conjugate to some g_η on their corresponding Julia sets.

Firstly, we show that the fixed parabolic Fatou component of $\widetilde{P}$ contains the Euclidean disk $\mathbb{D}(-3/4, 3/4)$ for every $n \ge 2$ and P_λ maps $\mathbb{D}(-3/4, 3/4)$ into itself if λ is small enough.

LEMMA 4.1.

(1) *For every $n \ge 2$, $\widetilde{P}(\overline{\mathbb{D}}(-3/4, 3/4)) \subset \mathbb{D}(-3/4, 3/4) \cup \{0\}$.*

(2) *If $0 < |\lambda| < 1/(3n)$, then $P_\lambda(\overline{\mathbb{D}}(-3/4, 3/4)) \subset \mathbb{D}(-3/4, 3/4) \cup \{0\}$. In particular, $\mathbb{D}(-3/4, 3/4)$ lies in the parabolic Fatou component of P_λ with parabolic fixed point 0.*

Proof. If $z \in \overline{\mathbb{D}}(-3/4, 3/4)$, then $|\widetilde{P}(z) + (1/n)| = |1+z|^n/n \le 1/n$. In particular, the inequality sign can be replaced by equality if and only if $z = 0$. This ends the proof of (1).

The proof of (2) will be divided into two cases: $|z|$ is small and not too small. For every $z = -(3/4) + (3/4)e^{i\theta} \in \partial\mathbb{D}(-3/4, 3/4)$, where $-\pi < \theta \le \pi$, we have $|1 + \widetilde{P}(z)| \le 5/2$ by (1) and $|\lambda z|^{m+n} < 1/2$ since $|\lambda| < 1/(3n)$. This means that

$$|P_\lambda(z) - \widetilde{P}(z)| = \left|\frac{\lambda^{m+n} z^{m+n}(1 + \widetilde{P}(z))}{1 - \lambda^{m+n} z^{m+n}}\right| \le 5|\lambda z|^{m+n}. \tag{4.3}$$

Since $|z| = (3/4)|1 - e^{i\theta}| = (3/4)|e^{-i\theta/2} - e^{i\theta/2}| = (3/4)|\sin\frac{\theta}{2}| \le (3/4)|\theta|$ and $|\lambda| < 1/(3n)$, we have

$$|P_\lambda(z) - \widetilde{P}(z)| \le 5(|\theta|/(4n))^{m+n}. \tag{4.4}$$

On the other hand, since $|\sin\theta| \ge (2/\pi)|\theta|$ if $|\theta| \le \pi/2$, we have

$$\begin{aligned} |\widetilde{P}(z) + (3/4)| &= \left|\frac{[(1/4) + (3/4)e^{i\theta}]^n - 1}{n} + \frac{3}{4}\right| \le \frac{|(1/4) + (3/4)e^{i\theta}|^n - 1}{n} + \frac{3}{4} \\ &= \frac{[1 - (3/4)\sin^2(\theta/2)]^{n/2} - 1}{n} + \frac{3}{4} \le \frac{[1 - (3\theta^2/4\pi^2)]^{n/2} - 1}{n} + \frac{3}{4}. \end{aligned} \tag{4.5}$$

If $|\theta| < 2\pi/n$, then $(3\theta^2/4\pi^2) < 2/n$. By Lemma 2.1(3), we have

$$|\widetilde{P}(z) + (3/4)| \le -\frac{(n/2)(3\theta^2/4\pi^2)}{3n} + \frac{3}{4} = \frac{3}{4} - \frac{\theta^2}{8\pi^2}. \tag{4.6}$$

Therefore, combining (4.4) and (4.6), it follows that if $|\theta| < 2\pi/n$, then

$$|P_\lambda(z) + (3/4)| \le |\widetilde{P}(z) + (3/4)| + |P_\lambda(z) - \widetilde{P}(z)| \le \frac{3}{4} - \frac{\theta^2}{8\pi^2} + 5\left(\frac{|\theta|}{4n}\right)^{m+n} \le 3/4. \tag{4.7}$$

If $(2\pi/n) \le |\theta| \le \pi$, from (4.5) and (4.6) we know that

$$|\widetilde{P}(z) + (3/4)| \le \frac{3}{4} - \frac{1}{2n^2}. \tag{4.8}$$

From (4.4) and (4.8), it follows that if $(2\pi/n) \le |\theta| \le \pi$, then

$$|P_\lambda(z) + (3/4)| \le \frac{3}{4} - \frac{1}{2n^2} + 5\left(\frac{|\theta|}{4n}\right)^{m+n} < 3/4. \tag{4.9}$$

Therefore, we have shown that $|P_\lambda(z) + (3/4)| \le 3/4$ for every $z \in \partial\mathbb{D}(-3/4, 3/4)$ and $|P_\lambda(z) + (3/4)| = 3/4$ if and only if $z = 0$. The proof is complete. □

As in the procedure in §2, we now locate the free critical points of P_λ. By a direct calculation, the bounded $m + 2n - 1$ critical points of P_λ are the solutions of

$$(1+z)^{n-1} + \lambda^{m+n} z^{m+n-1}\{[1 + (m/n)][(1+z)^n + n - 1] - z(1+z)^{n-1}\} = 0. \tag{4.10}$$

LEMMA 4.2. *If* $0 < |\lambda| < 1/(3n)$*, then there are* $n - 1$ *critical points of* P_λ *in*

$$\mathbb{D}(-1, |\lambda|) \subsetneq \mathbb{D}(-\tfrac{3}{4}, \tfrac{3}{4}).$$

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Proof. If $|z+1| \le |\lambda| < 1/(3n)$, then $|z| \cdot |1+z|^{n-1} \le (1+|\lambda|)|\lambda|^{n-1} < 1$ and

$$[1+(m/n)]|(1+z)^n+n-1| \le [1+(m/n)](|\lambda|^n+n-1) < m+n. \tag{4.11}$$

This means that if $|z+1| \le |\lambda|$, then

$$\begin{aligned}
&|\lambda^{m+n} z^{m+n-1}\{[1+(m/n)][(1+z)^n+n-1] - z(1+z)^{n-1}\}| \\
&\quad < |\lambda|^{n-1} \cdot |\lambda z|^{m-1} |\lambda|^2 |z|^n (m+n+1) < |\lambda|^{n-1} \cdot (2n)^{1-m} (9n^2)^{-1} e^{1/3} (m+n+1) \\
&\quad < |\lambda|^{n-1} \cdot (m+n-1)/(2n)^{m+1} < |\lambda|^{n-1}.
\end{aligned} \tag{4.12}$$

By Rouché's theorem, the proof is complete. □

Let $\widetilde{CP} := \{\widetilde{w}_j = (r_0/\lambda)\exp(\pi i(2j-1)/(m+n)) : 1 \le j \le m+n\}$ be the collection of the zeros of $m\lambda^{m+n} z^{m+n} + n = 0$, where $r_0 = \sqrt[m+n]{n/m}$. Since $h(x) = x^{1/x}$, $x > 0$, has maximal value $e^{1/e} < 3/2$ at $x = e$, we have

$$2/3 < 1/\sqrt[m]{m} < r_0 < \sqrt[n]{n} < 3/2. \tag{4.13}$$

The following lemma shows that the remaining $m+n$ critical points of P_λ are very 'close' to $\widetilde{CP}$.

LEMMA 4.3. *If* $0 < |\lambda| < 1/(2^m n^2)$*, then (4.10) has a solution* w_j *such that* $|w_j - \widetilde{w}_j| < 2(m+n)/m$*, where* $1 \le j \le m+n$*. Moreover,* $w_i = w_j$ *if and only if* $i = j$*.*

Proof. Dividing by $(1+z)^{n-1}$ on both sides of (4.10), we have

$$1 + \lambda^{m+n} z^{m+n-1}\left(\frac{m}{n}z + \frac{m+n}{n}\left[1 + \frac{n-1}{(1+z)^{n-1}}\right]\right) = 0. \tag{4.14}$$

Or, in more useful form,

$$\frac{n}{m\lambda^{m+n}} + z^{m+n} + \frac{(m+n)z^{m+n-1}}{m}\left(1 + \frac{n-1}{(1+z)^{n-1}}\right) = 0. \tag{4.15}$$

Let $\Omega = \{z : |z^{m+n} + (n/m)\lambda^{-(m+n)}| \le \beta|\lambda| \cdot (n/m)|\lambda|^{-(m+n)}\}$, where $\beta = 2(m+n)/mr_0 < 3(m+n)/m$. If $z \in \Omega$, then $|\lambda^{m+n} z^{m+n} + (n/m)| < \beta|\lambda| \cdot n/m$ and $|z - \widetilde{w}_j| < \beta r_0$ for some $1 \le j \le 2n$ by Lemma 2.1(2). If $z \in \Omega$ and $0 < |\lambda| < 1/(2^m n^2)$, we have

$$\frac{n-1}{|1+z|^{n-1}} < \frac{n-1}{[(|\lambda|^{-1} - \beta)r_0 - 1]^{n-1}} < \frac{n-1}{[(2^{m+1}n^2/3) - 3 - (2n/m)]^{n-1}} < \frac{1}{15} \tag{4.16}$$

and

$$\beta|\lambda| \le \frac{2(m+n)}{2^m n^2 \cdot mr_0} < \frac{3}{2^m n}\left(\frac{1}{m} + \frac{1}{n}\right) < \frac{1}{4} \quad \text{therefore} \quad \frac{1+\beta|\lambda|}{2(1-\beta|\lambda|)} < \frac{5}{6}. \tag{4.17}$$

Therefore, if $z \in \Omega$ and $0 < |\lambda| < 1/(2^m n^2)$, from (4.16) and (4.17) we have

$$\begin{aligned}
&\left|\frac{(m+n)z^{m+n-1}}{m}\left(1 + \frac{n-1}{(1+z)^{n-1}}\right)\right| = \frac{m+n}{m|\lambda|^{m+n}}\left|\frac{\lambda^{m+n}z^{m+n}}{z}\left(1 + \frac{n-1}{(1+z)^{n-1}}\right)\right| \\
&\quad < \frac{m+n}{m|\lambda|^{m+n}} \frac{(\beta|\lambda|+1)n/m}{r_0(1/|\lambda| - \beta)} \cdot \frac{16}{15} = \frac{n\beta|\lambda|}{m|\lambda|^{m+n}} \frac{1+\beta|\lambda|}{2(1-\beta|\lambda|)} \cdot \frac{16}{15} < \frac{n\beta|\lambda|}{m|\lambda|^{m+n}}.
\end{aligned} \tag{4.18}$$

Applying Rouché's theorem to (4.15) and then using Lemma 2.1(2), the proof of the first assertion is complete. By means of the same argument as for (2.15), if $0 < |\lambda| < 1/(2^m n^2)$ we have

$$\frac{(r_0/|\lambda|)\cdot \sin(\pi/(m+n))}{2(m+n)/m} \geq \frac{mr_0}{(m+n)^2|\lambda|} > \frac{2^{m+1}m}{3[(m/n)+1]^2} > 1. \tag{4.19}$$

This means that $w_i = w_j$ if and only if $i = j$. The proof is complete. □

Let $CP := \{w_j : 1 \leq j \leq m+n\}$ be the $m+n$ critical points of P_λ lying near the circle $\mathbb{T}_{r_0/|\lambda|}$ and $CV := \{P_\lambda(w_j) : 1 \leq j \leq m+n\}$. Let CP_{-1} be the collection of $n-1$ critical points of P_λ near -1 (see Lemma 4.2) and $CV_{-1} = \{P_\lambda(z) : z \in CP_{-1}\}$.

Let T_0 be the Fatou component of P_λ containing the attracting petal at the origin and $U := \mathbb{D}(-3/4, 3/4)$. By Lemmas 4.1(2) and 4.2, we know that $CP_{-1} \cup CV_{-1} \subset U \subset T_0$. Since $P_\lambda(\infty) = -1$, it follows that there exists a neighborhood of ∞ such that P_λ maps it to a neighborhood of -1. Let T_∞ be the Fatou component such that $\infty \in T_\infty$ and U_0, U_∞ be the components of $P_\lambda^{-1}(U)$ such that $0 \in \overline{U}_0$ and $\infty \in U_\infty$. Obviously, we have $U \subset U_0 \subset T_0$ and $U_\infty \subset T_\infty$.

LEMMA 4.4. *If $0 < |\lambda| \leq 1/(2^{10m}n^3)$, there exists an annular neighborhood A_1 of CP containing $\mathbb{T}_{1/|\lambda|} \cup CP$ such that $P_\lambda(A_1) \subset \overline{U'}_\infty \subset U_\infty$, where U'_∞ is a neighborhood of ∞.*

Proof. It is known from Lemma 4.3 that CP 'almost' lies uniformly on the circle $\mathbb{T}_{r_0/|\lambda|}$ and all the finite poles of P_λ lie on the circle $\mathbb{T}_{1/|\lambda|}$. Define the annulus

$$A_1 = \{z : 1/(2|\lambda|) < |z| < 2/|\lambda|\}. \tag{4.20}$$

Note that

$$\frac{r_0}{|\lambda|} + \frac{2(m+n)}{m} < \frac{3}{2|\lambda|} + 2 + \frac{2n}{m} < \frac{2}{|\lambda|} \tag{4.21}$$

and

$$\frac{r_0}{|\lambda|} - \frac{2(m+n)}{m} > \frac{2}{3|\lambda|} - 2 - \frac{2n}{m} > \frac{1}{2|\lambda|}. \tag{4.22}$$

We have $\mathbb{T}_{1/|\lambda|} \cup CP \subset A_1$ by Lemma 4.3. If $z \in A_1$ and $|\lambda| \leq 1/(2^{10m}n^3)$, then

$$\begin{aligned}|P_\lambda(z)+1| &\geq \frac{(|z|-1)^n}{n(|\lambda z|^{m+n}+1)} \geq \frac{((1/2|\lambda|)-1)^n}{n(2^{m+n}+1)} = \frac{(1-2|\lambda|)^n}{2^n n|\lambda|^n(2^{m+n}+1)}\\ &> \frac{2}{|\lambda|^{1+(n/m)}} + 1.\end{aligned} \tag{4.23}$$

In fact,

$$\frac{(1-2|\lambda|)^n}{2^{m+n}+1} > \frac{[1-(2/2^{10m}n^3)]^n}{2^{m+n}+1} > \frac{0.9}{2^{m+n}+1} > \frac{1}{2^{m+n+1}} + 2^n n|\lambda|^n. \tag{4.24}$$

This means that (4.23) follows by

$$2^{m+2n+2}n|\lambda|^n \leq |\lambda|^{1+(n/m)}. \tag{4.25}$$

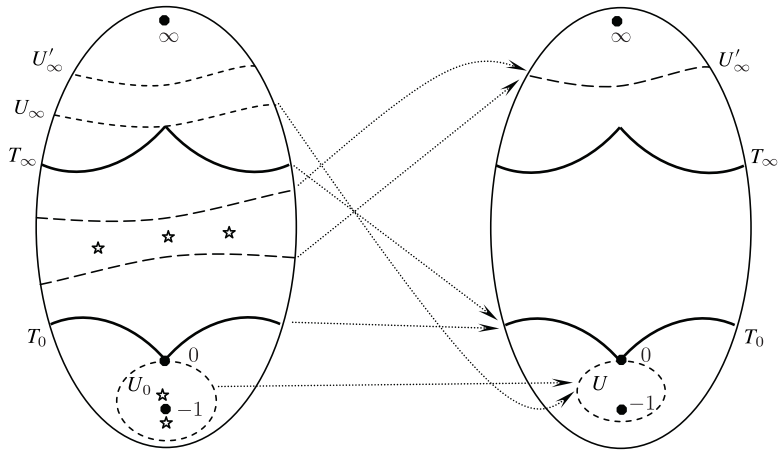

FIGURE 4. Sketch illustrating the mapping relation of P_λ. The small pentagons denote the critical points.

This is true because $|\lambda| \leq 1/(2^{10m}n^3)$. Now we have proved that if $z \in A_1$ and $|\lambda| \leq 1/(2^{10m}n^3)$, then $|P_\lambda(z)| > 2/|\lambda|^{1+(n/m)}$.

On the other hand, if $|z| \geq 2/|\lambda|^{1+(n/m)}$, then

$$|P_\lambda(z) + 1| \leq \frac{(|z| + 1)^n + 1}{|\lambda z|^{m+n} - 1} \leq \frac{(1 + |z|^{-1})^n + |z|^{-n}}{2^m - |z|^{-n}} < \frac{1}{2}. \tag{4.26}$$

This means that $P_\lambda(z) \in \mathbb{D}(-1, 1/2) \subset U$. Let U'_∞ be the component of $P_\lambda^{-1}(\mathbb{D}(-1, 1/2))$ containing $\{z : |z| \geq 2/|\lambda|^{1+(n/m)}\}$, then it follows that $P_\lambda(A_1) \subset \overline{U'}_\infty \subset U_\infty$ (see Figure 4). □

Proof of Theorem 1.3. For every λ such that $0 < |\lambda| \leq 1/(2^{10m}n^3)$, let $A := \overline{\mathbb{C}} \setminus (U \cup U'_\infty)$. Since $P_\lambda : U'_\infty \to \mathbb{D}(-1, 1/2)$ is proper with degree m, it follows that U'_∞ is simply connected and A is an annulus. Note that $P_\lambda^{-1}(U'_\infty)$ is an annulus since there are $m + n$ critical points in $P_\lambda^{-1}(U'_\infty)$ on which the degree of P_λ is $m + n$. This means that $P_\lambda^{-1}(A)$ consists of two disjoint annuli I_1 and I_2 and $I_1 \cup I_2 \subset A$. The degree of the restriction of P_λ on I_1 and I_2 are m and n respectively.

The following argument is very similar to that of Theorem 1.1. The Julia set of P_λ is $J_\lambda = \bigcap_{k \geq 0} P_\lambda^{-k}(A)$. By construction, the components of J_n are compact sets nested between -1 and ∞ since $P_\lambda^{-1} : A \to I_j$ is conformal for $j = 1$ or 2. Since the component of J_n cannot be a point and the proof of Theorem 1.2 in [**PT**] can also be applied to geometrically finite rational maps (see [**PT**, §9] and [**TY**]), we know that every component of J_n is a Jordan curve. The dynamics of P_λ on the set of Julia components is isomorphic to the one-sided shift on two symbols $\Sigma_2 := \{0, 1\}^{\mathbb{N}}$. In particular, J_λ is homeomorphic to $\Sigma_2 \times \mathbb{S}^1$, which is a Cantor set of circles, as claimed. □

Remark 4.5. From the proof of Theorems 1.3 and 3.2, we know that the dynamics on the Julia set of P_λ is conjugate to that of some g_η with the form (1.1). Therefore, we can view P_λ as a 'parabolic' McMullen map since the only difference is that the super-attracting basin and its preimages of g_η have been replaced by a fixed parabolic basin and its preimages of P_λ (see Figure 5).

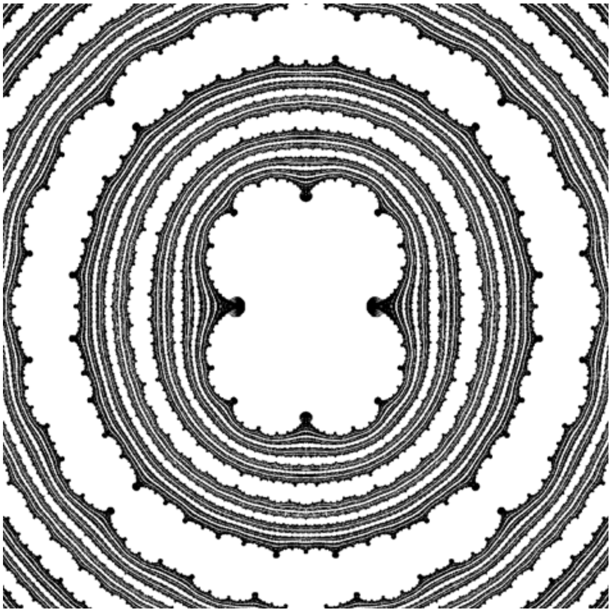

FIGURE 5. The Julia set of P_λ, where $m = 3$, $n = 2$ and λ is small enough such that J_λ is a Cantor set of circles. All the Fatou components of P_λ are iterated onto the fixed parabolic component (the 'cauliflower' in the center of this figure) with parabolic fixed point 1.

5. *More non-hyperbolic examples*

In this section we will construct more non-hyperbolic rational maps whose Julia sets are Cantor circles but which were not included in the previous section. Inspired by Theorem 1.1, for every $n \geq 2$, we define

$$P_n(z) = A_n \frac{(n+1)z^{(-1)^{n+1}(n+1)}}{nz^{n+1}+1} \prod_{i=1}^{n-1} (z^{2n+2} - b_i^{2n+2})^{(-1)^{i-1}} + B_n, \tag{5.1}$$

where $|b_i| = s^i$ for some $0 < s \leq 1/(25n^2)$ and

$$A_n = \frac{1}{1+(2n+2)C_n} \prod_{i=1}^{n-1} (1 - b_i^{2n+2})^{(-1)^i}, \quad B_n = \frac{(2n+2)C_n}{1+(2n+2)C_n}$$

$$\text{and} \quad C_n = \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2}}. \tag{5.2}$$

LEMMA 5.1.
(1) *$P_n(1) = 1$ and $P_n'(1) = 1$.*
(2) *$1 - s^{2n+1}/(n+1) < |A_n| < 1 + s^{2n+1}/(n+1)$ and $|B_n| < s^{2n+1}/(3n+3)$.*

Proof. It is easy to see $P_n(1) = 1$ by a straightforward calculation. Note that

$$F_n(z) := \frac{zP_n'(z)}{P_n(z) - B_n} = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}(2n+2)z^{2n+2}}{z^{2n+2} - b_i^{2n+2}} + (-1)^{n+1}(n+1) - \frac{n(n+1)z^{n+1}}{nz^{n+1}+1}. \tag{5.3}$$

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This means that

$$\frac{P_n'(1)}{P_n(1)-B_n} = (2n+2)\sum_{i=1}^{n-1}\frac{(-1)^{i-1}b_i^{2n+2}}{1-b_i^{2n+2}} + (2n+2) \times \sum_{i=1}^{n-1}(-1)^{i-1} + (-1)^{n+1}(n+1) - n$$
$$= (2n+2)\sum_{i=1}^{n-1}\frac{(-1)^{i-1}b_i^{2n+2}}{1-b_i^{2n+2}} + 1 := (2n+2)C_n + 1. \tag{5.4}$$

Therefore, we have

$$P_n'(1) = (1-B_n)[(2n+2)C_n+1] = 1. \tag{5.5}$$

It follows that 1 is a parabolic fixed point of P_n. This completes the proof of (1).

For (2), since $|1-b_i^{2n+2}|^{-1} \le 1+2|b_1|^{2n+2}$ for $1 \le i \le n-1$ and $0 < s \le 1/(25n^2) \le 1/100$, then

$$(2n+2)|C_n| < (2n+2)(1+2|b_1|^{2n+2})\sum_{i=1}^{n-1}|b_i|^{2n+2}$$
$$\le \frac{(2n+2)(1+2s^{2n+2})s^{2n+2}}{1-s^{2n+2}} < \frac{s^{2n+1}}{4n+4}. \tag{5.6}$$

We have

$$|B_n| = \left|\frac{(2n+2)C_n}{1+(2n+2)C_n}\right| < (2n+2)|C_n|[1+(4n+4)|C_n|] < \frac{s^{2n+1}}{3n+3} \tag{5.7}$$

and

$$|A_n| < [1+(4n+4)|C_n|]\prod_{i=1}^{n-1}(1+2|b_i|^{2n+2})$$
$$< \left(1+\frac{s^{2n+1}}{2n+2}\right)(1+5s^{2n+2}) < 1+\frac{s^{2n+1}}{n+1}. \tag{5.8}$$

Moreover, we have

$$|A_n| > [1-(2n+2)|C_n|]\prod_{i=1}^{n-1}(1-|b_i|^{2n+2})$$
$$> \left(1-\frac{s^{2n+1}}{4n+4}\right)\left(1-\frac{s^{2n+2}}{1-s^{2n+2}}\right) > 1-\frac{s^{2n+1}}{n+1}. \tag{5.9}$$

The proof is complete. □

Let us first explain some ideas behind the construction. For $n \ge 2$, define $\widetilde{Q}(z) = (z^{n+1}+n)/(n+1)$ and $\varphi(z) = 1/z$, then $Q(z) := \varphi \circ \widetilde{Q} \circ \varphi^{-1}(z) = (n+1)z^{n+1}/(nz^{n+1}+1)$ satisfies: ∞ is a critical point of Q with multiplicity n which is attracted to the parabolic fixed point 1. Since $\{b_i\}_{1\le i\le n-1}$ are very small, the rational map P_n can be viewed as a small perturbation of Q. The terms A_n and B_n here guarantee that 1 is always

TABLE 1. The proof of Lemma 5.3.

		$\sum_{1\leqslant j<i}(-1)^j$	$\sum_{i<j\leqslant n-1}(-1)^{j-1}$	$(1+(-1)^{n+1})/2$
odd n	odd i	0	-1	1
	even i	-1	0	1
even n	odd i	0	0	0
	even i	-1	1	0

a parabolic fixed point of P_n (see Lemma 5.1). It can be shown that P_n maps an annular neighborhood of $\mathbb{T}_{|b_i|}$ into T_0 or T_∞ according to whether i is odd or even, where T_0 and T_∞ denote the Fatou components containing 0 and ∞ respectively (see Lemma 5.5). The Fatou component T_∞ is always parabolic while T_0 is attracting or mapped to T_∞ according to whether n is odd or even. The proof of Theorem 1.4 will based on the mixed arguments in the previous two sections.

If $|z| \le 1$, then $|\widetilde{Q}(z)| \le 1$. This means that the fixed parabolic Fatou component of $\widetilde{Q}$ contains the unit disk for every $n \ge 2$. Therefore, the parabolic Fatou component of Q contains the exterior of the closed unit disk $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$. Although the polynomial Q has been perturbed into P_n, we still have following lemma.

LEMMA 5.2. $P_n(\overline{\mathbb{C}}\backslash\mathbb{D}) \subset (\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}) \cup \{1\}$. *In particular, the disk* $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$ *lies in the parabolic Fatou component of* P_n *with parabolic fixed point 1.*

The proof of Lemma 5.2 is very subtle, and will be delayed until the next section.

LEMMA 5.3. *Let* $n \ge 2$ *and* $1 \le i \le n-1$, *then*

$$\sum_{1\le j<i}(-1)^j + \sum_{i<j\le n-1}(-1)^{j-1} + \frac{1+(-1)^{n+1}}{2} = 0. \tag{5.10}$$

Proof. The argument is based on several cases shown in Table 1. □

As before, we first locate the critical points of P_n. Note that 0 and ∞ are both critical points of P_n with multiplicity n and the degree of P_n is n^2+n. The remaining $2(n^2-1)$ critical points of P_n are the solutions of $F_n(z) = 0$ (see equation (5.3)).

For $1 \le i \le n-1$, let $\widetilde{CP}_i := \{\widetilde{w}_{i,j} = b_i \exp(\pi i(2j-1)/(2n+2)) : 1 \le j \le 2n+2\}$ be the collection of $2n+2$ points lying on $\mathbb{T}_{|b_i|}$ uniformly. The following lemma is similar to Lemmas 2.3 and 4.3.

LEMMA 5.4. *For every* $\widetilde{w}_{i,j} \in \widetilde{CP}_i$, *where* $1 \le i \le n-1$ *and* $1 \le j \le 2n+2$, *there exists* $w_{i,j}$, *which is a solution of* $F_n(z) = 0$, *such that* $|w_{i,j} - \widetilde{w}_{i,j}| < s^{n+1/2}|b_i|$. *Moreover,* $w_{i_1,j_1} = w_{i_2,j_2}$ *if and only if* $(i_1, j_1) = (i_2, j_2)$.

Proof. Note that $F_n(z) = 0$ is equivalent to

$$\sum_{i=1}^{n-1}(-1)^{i-1}\frac{z^{2n+2}+b_i^{2n+2}}{z^{2n+2}-b_i^{2n+2}} + \frac{1+(-1)^{n+1}}{2} - \frac{nz^{n+1}}{nz^{n+1}+1} = 0. \tag{5.11}$$

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Multiplying both sides of (5.11) by $z^{2n+2} - b_i^{2n+2}$, where $1 \le i \le n-1$, we have

$$(-1)^{i-1}(z^{2n+2} + b_i^{2n+2}) + (z^{2n+2} - b_i^{2n+2})\, G_i(z) = 0, \tag{5.12}$$

where

$$G_i(z) = \sum_{1 \le j \le n-1, j \ne i} (-1)^{j-1} \frac{z^{2n+2} + b_j^{2n+2}}{z^{2n+2} - b_j^{2n+2}} + \frac{1 + (-1)^{n+1}}{2} - \frac{nz^{n+1}}{nz^{n+1}+1}. \tag{5.13}$$

Let $\Omega_i = \{z : |z^{2n+2} + b_i^{2n+2}| \le s^{n+(1/2)}|b_i|^{2n+2}\}$, where $1 \le i \le n-1$. If $z \in \Omega_i$, then $|z|^{n+1} \le (1 + s^{n+(1/2)})|b_i|^{n+1} \le (1 + s^{n+(1/2)})s^{n+1}$ by Lemma 2.1(2). So

$$\left|\frac{nz^{n+1}}{nz^{n+1}+1}\right| \le \frac{n(1+s^{n+(1/2)})s^{n+1}}{1 - n(1+s^{n+(1/2)})s^{n+1}} \le \frac{(1+100^{-5/2})s^{n+(1/2)}/5}{1-(1+100^{-5/2})100^{-5/2}/5} < 0.3s^{n+(1/2)}$$

since $s \le 1/(25n^2) \le 1/100$. For every $z \in \Omega_i$, if $1 \le j < i$, we have

$$|z/b_j|^{2n+2} = |z/b_i|^{2n+2}|b_i/b_j|^{2n+2} < (1 + s^{n+(1/2)})s^{(2n+2)(i-j)}. \tag{5.14}$$

If $i < j \le n-1$, by the first statement of Lemma 2.1(2), we have

$$|b_j/z|^{2n+2} = |b_i/z|^{2n+2}|b_j/b_i|^{2n+2} \le (1 + 2 \cdot s^{n+(1/2)})s^{(2n+2)(j-i)}. \tag{5.15}$$

From (5.14), (5.15) and Lemma 5.3, we have

$$\begin{aligned}
&\left|G_i(z) + \frac{nz^{n+1}}{nz^{n+1}+1}\right| \\
&\quad = \left|\sum_{1 \le j < i} (-1)^j \frac{1 + (z/b_j)^{2n+2}}{1 - (z/b_j)^{2n+2}} + \sum_{i<j\le n-1} (-1)^{j-1} \frac{1 + (b_j/z)^{2n+2}}{1 - (b_j/z)^{2n+2}} + \frac{1 + (-1)^{n+1}}{2}\right| \\
&\quad < 3 \cdot (1 + 2 \cdot s^{n+(1/2)})\left(\sum_{1 \le j < i} s^{(2n+2)(i-j)} + \sum_{i<j\le n-1} s^{(2n+2)(j-i)}\right) \\
&\quad < 6 \cdot (1 + 2 \cdot s^{n+(1/2)})^2 s^{2n+2}.
\end{aligned} \tag{5.16}$$

The first inequality in (5.16) follows from the inequality $2x/(1-x) \le 3x$ if $x < 1/3$ (here $x \le (1 + 2 \cdot s^{n+(1/2)})\, s^{2n+2} < 10^{-10}$). So we have

$$|G_i(z)| < 6 \cdot (1 + 2 \cdot s^{n+(1/2)})^2 s^{2n+2} + 0.3s^{n+(1/2)} < 0.4s^{n+(1/2)}. \tag{5.17}$$

Therefore, if $z \in \Omega_i$, then

$$|z^{2n+2} - b_i^{2n+2}| \cdot |G_i(z)| < (2 + s^{n+(1/2)})|b_i|^{2n+2} \cdot 0.4s^{n+(1/2)} < s^{n+(1/2)}|b_i|^{2n+2}. \tag{5.18}$$

From (5.12) and by Rouché's theorem, there exists a solution $w_{i,j}$ of $F_n(z) = 0$ such that $w_{i,j} \in \Omega_i$ for every $1 \le j \le 2n+2$. In particular, $|w_{i,j} - \widetilde{w}_{i,j}| < s^{n+(1/2)}|b_i|$ by the second statement of Lemma 2.1(2). The assertion $w_{i_1,j_1} = w_{i_2,j_2}$ if and only if $(i_1, j_1) = (i_2, j_2)$ can be verified similarly as for equations (2.14) and (2.15). The proof is complete. □

For $1 \le i \le n-1$, let $CP_i := \{w_{i,j} : 1 \le j \le 2n+2\}$ be the collection of critical points of P_n which lie close to the circle $\mathbb{T}_{|b_i|}$.

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LEMMA 5.5. *There exist* $n-1$ *annuli* $\{A_i\}_{i=1}^{n-1}$ *satisfying* $A_{n-1} \prec \cdots \prec A_1$ *and two simply connected domains* U_0 *and* U_∞ *which contain* 0 *and* ∞*, respectively, such that*

(1) $U_\infty \supset \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ *and* $P_n(\overline{U}_\infty) \subset U_\infty \cup \{1\}$;

(2) $A_i \supset \mathbb{T}_{|b_i|} \cup CP_i$, $P_n(\overline{A}_i) \subset U_0$ *for odd* i *and* $P_n(\overline{A}_i) \subset U_\infty$ *for even* i;

(3) $P_n(\overline{U}_0) \subset U_\infty$ *for even* n *and* $P_n(\overline{U}_0) \subset U_0$ *for odd* n.

Proof. Let $U_\infty := \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ be the exterior of the closed unit disk. Then (1) is obvious if we apply Lemma 5.2. Let $\varepsilon = s^{n+(1/2)}$ and $A_i = \mathbb{A}_{|b_i|(1-2\varepsilon),|b_i|(1+2\varepsilon)}$. From (5.1), we know that

$$|R_n(z)| := \left|\frac{P_n(z) - B_n}{A_n} \cdot \frac{nz^{n+1}+1}{n+1}\right| = |z|^{(-1)^{n+1}(n+1)} |z^{2n+2} - b_i^{2n+2}|^{(-1)^{i-1}} H_i(z), \tag{5.19}$$

where

$$H_i(z) = \prod_{j=1}^{i-1} |b_j|^{(2n+2)(-1)^{j-1}} \prod_{j=i+1}^{n-1} |z|^{(2n+2)(-1)^{j-1}} \cdot Q_i(z) \tag{5.20}$$

and

$$Q_i(z) = \prod_{j=1}^{i-1} |1 - (z/b_j)^{2n+2}|^{(-1)^{j-1}} \prod_{j=i+1}^{n-1} |1 - (b_j/z)^{2n+2}|^{(-1)^{j-1}}. \tag{5.21}$$

If $z \in \overline{A}_i$, where $1 \le i \le n-1$, we have

$$Q_i(z) < \prod_{j=1}^{i-1} \left(1 + 3|b_i/b_j|^{2n+2}\right) \prod_{j=i+1}^{n-1} \left(1 + 3|b_j/b_i|^{2n+2}\right) < (1 + 6s^{2n+2})^2 \tag{5.22}$$

and

$$Q_i(z) > \prod_{j=1}^{i-1} (1 + 3|b_i/b_j|^{2n+2})^{-1} \prod_{j=i+1}^{n-1} (1 + 3|b_j/b_i|^{2n+2})^{-1} > (1 + 6s^{2n+2})^{-2}. \tag{5.23}$$

Note that $\varepsilon = s^{n+(1/2)} \le (5n)^{-2n-1} \le 10^{-5}$. If n is even and $1 \le i \le n-1$ is odd, then for $z \in \overline{A}_i$ we have

$$\begin{aligned} |R_n(z)| &= \frac{|z^{2n+2} - b_i^{2n+2}|}{|z|^{n+1}} \frac{1}{s^{(i-1)(n+1)}} Q_i(z) < \frac{|b_i|^{n+1}[1 + (1+2\varepsilon)^{2n+2}]}{(1-2\varepsilon)^{n+1}} \frac{(1+6s^{2n+2})^2}{s^{(i-1)(n+1)}} \\ &= \frac{1 + (1+2\varepsilon)^{2n+2}}{(1-2\varepsilon)^{n+1}} (1+6s^{2n+2})^2 s^{n+1} < 2.1 \cdot s^{n+1}. \end{aligned}$$

If n and $1 \le i \le n-1$ are both even, then for $z \in \overline{A}_i$ we have

$$\begin{aligned} |R_n(z)| &= \frac{|b_{i-1}|^{2n+2}|z|^{2n+2}}{|z|^{n+1}|z^{2n+2} - b_i^{2n+2}|} \frac{1}{s^{(i-2)(n+1)}} Q_i(z) \\ &> \frac{(1-2\varepsilon)^{n+1}}{1 + (1+2\varepsilon)^{2n+2}} (1 - 6s^{2n+2})^2 > 0.49. \end{aligned}$$

This means that if n is even and $1 \le i \le n-1$ is odd, for $z \in \overline{A}_i$ we have

$$\begin{aligned} |P_n(z)| &< \left|\frac{2.1 \cdot s^{n+1} \cdot (n+1)A_n}{nz^{n+1}+1}\right| + |B_n| \\ &\le \frac{2.1(s^{n+(1/2)}/5) \cdot (1 + s^{2n+1}/(n+1))}{1 - n(1+2\varepsilon)s^{n+1}} + \frac{s^{2n+1}}{3n+3} < s^{n+(1/2)} \end{aligned}$$

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by Lemma 5.1(2). If n and $1 \le i \le n-1$ are both even, then for $z \in \overline{A}_i$ we have

$$|P_n(z)| > \left|\frac{0.49(n+1)A_n}{nz^{n+1}+1}\right| - |B_n| \ge \frac{0.49(n+1)[1-s^{2n+1}/(n+1)]}{1+n(1+2\varepsilon)s^{n+1}} - \frac{s^{2n+1}}{3n+3} > \frac{n+1}{3} \ge 1.$$

By completely similar arguments one can show that, if n is odd, for $z \in \overline{A}_i$ we have

$$|P_n(z)| < s^{n+(1/2)} \text{ for odd } i \text{ and } |P_n(z)| > 1 \text{ for even } i. \tag{5.24}$$

Let $U_0 = \mathbb{D}_r$, where $r = s^{n+(1/2)}$. This proves (2).

If n is odd, for every z such that $|z| \le s^{n+(1/2)}$ we have

$$\begin{aligned}|P_n(z)| &\le \left|\frac{(n+1)A_n}{nz^{n+1}+1}\right||z|^{n+1}\prod_{i=1}^{n-1}|b_i|^{(2n+2)(-1)^{i-1}}\prod_{i=1}^{n-1}\left|1-\frac{z^{2n+2}}{b_i^{2n+2}}\right|^{(-1)^{i-1}} + |B_n| \\ &\le \frac{(n+1)[1+s^{2n+1}/(n+1)]}{1-ns^{n^2+(n/2)}}s^{3(n+1)/2}\prod_{i=1}^{n-1}\left(1+2\frac{|z|^{2n+2}}{|b_i|^{2n+2}}\right) \\ &\quad + \frac{s^{2n+1}}{3n+3} < s^{n+(1/2)}.\end{aligned}$$

It follows that $P_n(\overline{\mathbb{D}}_r) \subset \mathbb{D}_r$ for odd n, where $r = s^{n+(1/2)}$.

If n is even, then P_n maps a neighborhood of 0 to that of ∞. For every z such that $|z| \le s^{n+(1/2)}$ we have

$$|P_n(z)| \ge \frac{(n+1)s^{-(n+1)/2}(1-s^{2n+1}/(n+1))}{1+ns^{n^2+(n/2)}}\prod_{i=1}^{n-1}\left(1-2\frac{|z|^{2n+2}}{|b_i|^{2n+2}}\right) - \frac{s^{2n+1}}{3n+3} > n > 1. \tag{5.25}$$

This ends the proof of (3). The proof is complete. □

Proof of Theorem 1.4. Let $A := \overline{\mathbb{C}}\setminus(U_0 \cup U_\infty)$. The Julia set of P_n is equal to $\bigcap_{k\ge 0} P_n^{-k}(A)$. Note that P_n is geometrically finite. The argument is completely similar to the proofs of Theorems 1.1 and 1.3. The set of Julia components of P_n is isomorphic to the one-sided shift on n symbols $\Sigma_n := \{0, 1, \ldots, n-1\}^{\mathbb{N}}$. In particular, the Julia set of P_n is homeomorphic to $\Sigma_n \times \mathbb{S}^1$, which is a Cantor set of circles, as desired (see Figure 6). We omit the details here. □

6. *Proof of Lemma 5.2*

This section will be devoted to proving Lemma 5.2, which is the key ingredient in the proof of Lemma 5.5 and hence in Theorem 1.4.

Proof. Let $\widetilde{R}(z) = 1/P_n(1/z)$, then Lemma 5.2 reduces to proving $\widetilde{R}(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$. Let $w = z^{n+1}$, by a straightforward calculation we have

$$R(w) := \widetilde{R}(z) = \frac{w+n}{n+1}\cdot\frac{1}{S(w)}, \tag{6.1}$$

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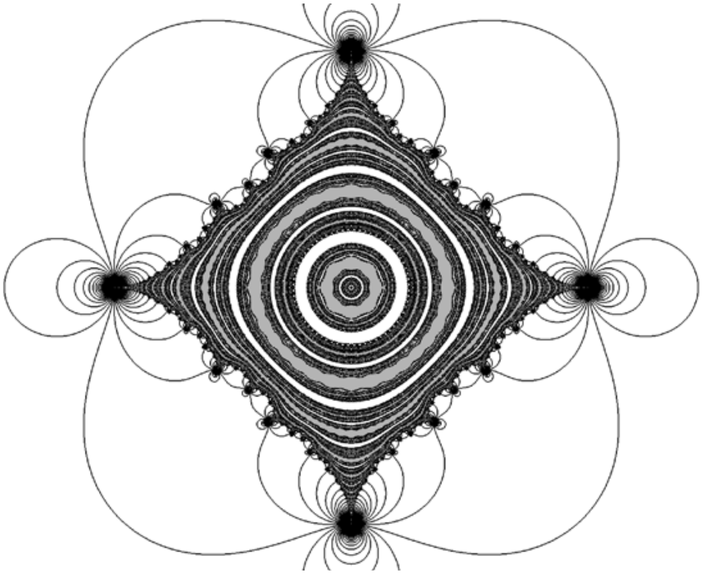

FIGURE 6. The Julia set of P_3, which is a Cantor set of circles. The parameter s is chosen small enough. The gray parts in the figure denote the Fatou components, which are iterated to the attracting Fatou component containing the origin, while the white parts denote the Fatou components iterated to the parabolic Fatou component whose boundary contains the parabolic fixed point one. Some equipotentials of the Fatou coordinate have been drawn in the parabolic Fatou component and its preimages. Figure range: $[-1.6, 1.6] \times [-1.2, 1.2]$.

where

$$S(w) = A_n \prod_{i=1}^{n-1} (1 - b_i^{2n+2} w^2)^{(-1)^{i-1}} + \frac{w+n}{n+1} B_n = 1 + \frac{w-1}{1+(2n+2)C_n} \left(\frac{H(w)-1}{w-1} + 2C_n \right) \tag{6.2}$$

and

$$H(w) = \prod_{i=1}^{n-1} (1 - b_i^{2n+2})^{(-1)^i} \prod_{i=1}^{n-1} (1 - b_i^{2n+2} w^2)^{(-1)^{i-1}}. \tag{6.3}$$

Since $H(1) = 1$, it follows that $H'(1)$ is a finite number. In fact,

$$I(w) := \frac{H'(w)}{H(w)} = -2w \sum_{i=1}^{n-1} \frac{(-1)^{i-1} b_i^{2n+2}}{1 - b_i^{2n+2} w^2}. \tag{6.4}$$

We know that $I(1) = H'(1) = -2C_n$. For every small enough $w - 1$, we can write $S(w)$ as

$$S(w) = 1 + \frac{(w-1)^2}{1+(2n+2)C_n} \cdot \frac{(H(w)-1)/(w-1) + 2C_n}{w-1} =: 1 + \frac{(w-1)^2}{1+(2n+2)C_n} \cdot \Phi(w), \tag{6.5}$$

where

$$\Phi(w) = \sum_{k\geq 2} \frac{H^{(k)}(1)}{k!}(w-1)^{k-2}. \tag{6.6}$$

The next step is to estimate $H^{(k)}(1)$ for every $k \geq 2$.

For every $k \geq 1$, let

$$Y_k(w) = \sum_{i=1}^{n-1}(-1)^{i-1}\bigg(\frac{b_i^{2n+2}}{1-b_i^{2n+2}w^2}\bigg)^k. \tag{6.7}$$

In particular, $Y_1(1) = C_n$ and

$$Y_k'(w) = 2kwY_{k+1}(w). \tag{6.8}$$

If $|w| = 1$, we have

$$|Y_k(w)| \leq \bigg|\frac{b_1^{2n+2}}{1-b_1^{2n+2}}\bigg|^k\bigg(1+\sum_{i=2}^{n-1}\bigg|\frac{b_i^{2n+2}(1-b_1^{2n+2})}{b_1^{2n+2}(1-b_i^{2n+2})}\bigg|^k\bigg) \leq \frac{11}{10}\bigg|\frac{b_1^{2n+2}}{1-b_1^{2n+2}}\bigg|^k. \tag{6.9}$$

Similarly, we have $|Y_k(w)| \geq (9/10)|b_1^{2n+2}/(1-b_1^{2n+2})|^k$. This means that

$$\bigg|\frac{Y_{k+1}(w)}{Y_k(w)}\bigg| \leq \frac{11}{9}\bigg|\frac{b_1^{2n+2}}{1-b_1^{2n+2}}\bigg| \leq 2s^{2n+2} < 1/2. \tag{6.10}$$

We first claim that $|I^{(k)}(1)| \leq 2^{k+1}k!|C_n|$ for every $k \geq 0$. Since $I^{(0)}(w) = -2wY_1(w)$ and $I^{(1)}(w) = -2Y_1(w) - 4w^2Y_2(w)$, it can be proved inductively that $I^{(k)}(w)$ can be written as

$$I^{(k)}(w) = \sum_{j=1}^{2^k} Q_{k,j}(w) = \sum_{j=1}^{2^k} P_{k,j}(w)Y_{k,j}(w), \tag{6.11}$$

where $P_{k,j}(w)$ is a polynomial with degree at most $k+1$ and $Y_{k,j} = Y_l$ for some $1 \leq l \leq k+1$. Note that some terms $Q_{k,j}$ may be equal to zero (the degree of the corresponding polynomial $P_{k,j}$ is regarded as $-\infty$) and the formula (6.11) can be simplified, but what we need is this 'long' expansion. In particular, without loss of generality, for $1 \leq j \leq 2^k$, we require further that

$$\begin{aligned} P_{k+1,2j-1}(w)Y_{k+1,2j-1}(w) &= P_{k,j}'(w)Y_{k,j}(w) \quad \text{and} \\ P_{k+1,2j}(w)Y_{k+1,2j}(w) &= P_{k,j}(w)Y_{k,j}'(w). \end{aligned} \tag{6.12}$$

Since $\deg(P_{k,j}) \leq k+1$ and $Y_{k,j} = Y_l$ for some $1 \leq l \leq k+1$, it follows that

$$\begin{aligned} |P_{k+1,2j-1}&(1)Y_{k+1,2j-1}(1)| + |P_{k+1,2j}(1)Y_{k+1,2j}(1)| \\ &= |P_{k,j}'(1)Y_l(1)| + |P_{k,j}(1)Y_l'(1)| \\ &\leq (k+1)|P_{k,j}(1)Y_l(1)| + 2(k+1)|P_{k,j}(1)Y_{l+1}(1)| \\ &\leq 2(k+1)|P_{k,j}(1)Y_{k,j}(1)| \end{aligned} \tag{6.13}$$

since $|Y_{l+1}(1)/Y_l(1)| \leq 1/2$ for every $l \geq 1$ by (6.10).

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Denoting $||I^{(k)}(1)|| := \sum_{j=1}^{2^k} |P_{k,j}(1)Y_{k,j}(1)|$, we have $||I^{(k)}(1)|| \leq 2k||I^{(k-1)}(1)||$. This means that

$$|I^{(k)}(1)| \leq ||I^{(k)}(1)|| \leq 2^k k!||I^{(0)}(1)|| = 2^{k+1}k!|C_n|. \tag{6.14}$$

This proves the claim $|I^{(k)}(1)| \leq 2^{k+1}k!|C_n|$ for every $k \geq 0$.

Secondly, we check by induction that $|H^{(k)}(1)| \leq 4^k k!|C_n|$ for $k \geq 1$. For $k = 1$, we have $|H'(1)| = 2|C_n| < 4|C_n|$. Assume that $|H^{(i)}(1)| \leq 4^i i!|C_n|$ for every $1 \leq i \leq k$. By (6.4), we have $H'(w) = H(w)I(w)$. So

$$\begin{aligned} |H^{(k+1)}(1)| &\leq |I^{(k)}(1)| + \sum_{i=1}^{k} \frac{k!}{i!(k-i)!}|H^{(i)}(1)| \cdot |I^{(k-i)}(1)| \\ &\leq 2^{k+1}k!|C_n|(1 + 2^{k+1}|C_n|) \leq 4^{k+1}(k+1)!|C_n| \end{aligned} \tag{6.15}$$

since $|I^{(k-i)}(1)| \leq 2^{k-i+1}(k-i)!|C_n|$ and $|H^{(i)}(1)| \leq 4^i i!|C_n|$ for every $1 \leq i \leq k$.

If $w = e^{i\theta}$ for $|\theta| \leq 1/20$, then $|w - 1| < |\theta| \leq 1/20$. By (6.6) and (6.15), we have

$$|\Phi(w)| \leq \sum_{k \geq 2} 4^k |C_n|(1/20)^{k-2} \leq 16|C_n| \sum_{k \geq 0} 5^{-k} = 20|C_n|. \tag{6.16}$$

By (6.5) and (6.16), it follows that

$$|S(w)| \geq 1 - \frac{\theta^2}{1 - (2n+2)|C_n|} 20|C_n| \geq 1 - \frac{s^{2n+1}}{n+1}\theta^2 \tag{6.17}$$

since $n \geq 2$ and $|C_n| < s^{2n+1}/[8(n+1)^2]$ by (5.6).

On the other hand, if $w = e^{i\theta}$ for $0 \leq |\theta| \leq \pi$, then

$$\left|\frac{w+n}{n+1}\right| = \left(1 - \frac{4n}{(n+1)^2}\sin^2\frac{\theta}{2}\right)^{1/2} \leq \left(1 - \frac{4n}{\pi^2(n+1)^2}\theta^2\right)^{1/2} \leq 1 - \frac{2n}{(n+1)^2\pi^2}\theta^2 \tag{6.18}$$

since $(1 - x)^{1/2} \leq 1 - x/2$ for $0 \leq x < 1$. This means that if $w = e^{i\theta}$ for $|\theta| \leq 1/20$, then

$$|R(w)| \leq \left(1 - \frac{2n}{(n+1)^2\pi^2}\theta^2\right)\left(1 - \frac{s^{2n+1}}{n+1}\theta^2\right)^{-1} \leq 1. \tag{6.19}$$

Moreover, $|R(w)| = 1$ if and only if $w = 1$.

If $w = e^{i\theta}$ for $|\theta| > 1/20$, by (6.2) and Lemma 5.1(2) we have

$$|S(w)| \geq \left(1 - \frac{s^{2n+1}}{n+1}\right)\prod_{i=1}^{n-1}(1 - |b_i|^{2n+2}) - \frac{s^{2n+1}}{3n+3} \geq 1 - \frac{3s^{2n+1}}{n+1}. \tag{6.20}$$

By (6.18) and (6.20), we have

$$|R(w)| \leq \left(1 - \frac{2}{20^2(n+1)\pi^2}\right)\left(1 - \frac{3s^{2n+1}}{n+1}\right)^{-1} < 1. \tag{6.21}$$

This means that $R(w)$ maps the boundary of the unit disk into the unit disk except at $w = 1$. Since $R(w) \neq \infty$ if $|w| \leq 1$, we know that $R(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$. Therefore, $\widetilde{R}(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}$ and $\widetilde{R}$ maps $\{z \in \mathbb{C} : z^{n+1} = 1\}$ onto 1. This ends the proof of Lemma 5.2. □

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