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Complete quenching phenomenon and instantaneous shrinking of support of solutions of degenerate parabolic equations with nonlinear singular absorption

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This paper deals with nonnegative solutions of the one-dimensional degenerate parabolic equations with zero homogeneous Dirichlet boundary condition. To obtain an existence result, we prove a sharp estimate for $|u_x|$. Besides, we investigate the qualitative behaviours of nonnegative solutions such as the quenching phenomenon, and the finite speed of propagation. Our results of the Dirichlet problem are also extended to the associated Cauchy problem on the whole domain $\mathbb{R}$. In addition, we also consider the instantaneous shrinking of compact support of nonnegative solutions.

Keywords: gradient estimates; quenching type of parabolic equations; irregular initial datum; free boundary; instantaneous shrinking of support

2010 *Mathematics subject classification:* Primary 35K55; 35K65; 35B99

1. Introduction

In this paper, we study the nonnegative solutions of the following equation:

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } I \times (0,\infty), \\ u(-l,t) = u(l,t) = 0 & t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{1.1}$$

where $\beta \in (0,1)$, $p > 2$; and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, that is,

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leqslant 0. \end{cases}$$

Note that the absorption term $\chi_{\{u>0\}}u^{-\beta}$ becomes singular when u is near to 0, and we impose $\chi_{\{u>0\}}u^{-\beta} = 0$ whenever $u = 0$. Through this paper, we assume that $f : [0,\infty) \to [0,\infty)$, is a nonnegative locally Lipschitz function, that is, for any $r > 0$, there is a real number $C(r) > 0$ such that

$$(H) \quad |f(s_1) - f(s_2)| \leqslant C(r)|s_1 - s_2|, \quad \forall s_1,\ s_2 \in [0,r]; \quad \text{and} \quad f(0) = 0.$$

If f is nondecreasing on $[0,\infty)$, we can then relax the locally Lipschitz property in (H), see lemma 2.3 below.

As already known, problem (1.1) in the semi-linear case ($p = 2$, and $f = 0$) can be considered as a limit of mathematical models arising in Chemical Engineering corresponding to catalyst kinetics of Langmuir-Hinshelwood type (see, e.g. [**1**, **30**] p. 68, [**27**] and reference therein). The semi-linear case was studied in many papers such as [**2**, **9**–**12**, **19**, **20**, **23**, **26**, **27**, **31**], and references therein. These papers focussed on studying the existence of solution, and the behaviour of solutions. The existence result of the semi-linear case was first proved by D. Phillips for the Cauchy problem (see theorem 1, [**27**]). The same result holds for the semi-linear equation with positive Dirichlet boundary condition (see theorem 1.2, [**27**]). Moreover, he proved a property of the finite speed of propagation of nonnegative solutions, that is, any solution with compact support initially has compact support at all later times $t > 0$. The finite speed of propagation was later studied for a more general formulation of the singular equation by means of some energy methods by J. I. Díaz, see [**15**].

The semi-linear problem of this type was also extended in many aspects. In [**12**], J. Dávila, and M. Montenegro proved the existence of solution with zero Dirichlet boundary condition with a source term $f(u)$ being sub-linear. Furthermore, they showed that the uniqueness result holds for a particular class of positive solutions, see theorem 1.10 in [**12**]. Recently, N. A. Dao, J. I. Díaz and P. Sauvy, [**11**] proved a uniqueness result for a class of solutions, which is different from the one of [**12**]. However, M. Winkler showed that the uniqueness result fails in general, see theorem 1.1, [**31**].

After that, the equations of this type was considered under more general forms. For example, Dao and Díaz [**9**] proved the existence of solution of equation (1.1) for the case $f = 0$. Furthermore, they also showed the behaviours of solutions such as the extinction phenomenon and the free boundary. We also mention here the porous medium of this type, which was studied by B. Kawohl and R. Kersner, [**24**].

Inspired by the above studies, we would like to investigate the existence of nonnegative solutions and the behaviours of solutions of equation (1.1). Before stating our main results, let us define the notion of a weak solution of equation (1.1).

Definition 1.1. Given $0 \leqslant u_0 \in L^1(I)$. A function $u \geqslant 0$ is called a weak solution of equation (1.1) if $f(u), u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0,\infty))$, and $u \in L^p_{loc}(0,\infty; W_0^{1,p}(I))$

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$\cap L^{\infty}_{loc}(\overline{I} \times (0,\infty)) \cap \mathcal{C}([0,\infty); L^1(I))$ satisfies equation (1.1) in the sense of distributions $\mathcal{D}'(I \times (0,\infty))$, that is:

$$\int_0^\infty \int_I \left(-u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\ \right) \mathrm{d}x\mathrm{d}t = 0,$$
$$\forall \phi \in \mathcal{C}^\infty_c(I \times (0,\infty)). \tag{1.2}$$

Then, we have the theorem on the existence of weak solutions.

THEOREM 1.2. *Let $0 \leqslant u_0 \in L^\infty(I)$, and f satisfy (H). Then, there exists a maximal weak bounded solution u of equation (1.1). Moreover, there exists a positive constants $C = C(\beta, p)$ such that*

$$|\partial_x u(x,\tau)|^p \leqslant Cu^{1-\beta}(x,\tau)\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f, \|u_0\|_\infty)\|u_0\|^\beta_\infty \right.$$
$$\left. + \mathit{Lip}(f,u_0)\|u_0\|_\infty^{1+\beta} + 1\right), \tag{1.3}$$

for a.e $(x,\tau) \in I \times (0,\infty)$, where $Lip(f,u_0)$ is the local Lipschitz constant of f on the closed interval $[0, 2\|u_0\|_\infty]$, and $\Theta(g,r) = \max_{0\leqslant s\leqslant r}\{|g(s)|\}$.

In addition, if $(u_0^{1/\gamma})_x \in L^\infty(I)$, with $\gamma = ((p/(p+\beta-1))$ then we have

$$|\partial_x u(x,\tau)|^p \leqslant Cu^{1-\beta}(x,\tau)\left(\|u_0\|_\infty^{1+\beta} + \Theta(f, \|u_0\|_\infty)\|u_0\|^\beta_\infty \right.$$
$$\left. + \mathit{Lip}(f,u_0)\|u_0\|_\infty^{1+\beta} + 1\right), \tag{1.4}$$

for a.e $(x,\tau) \in I \times (0,\infty)$, with $C = C(\beta, p, \|(u_0^{1/\gamma})_x\|_\infty) > 0$.

As a consequence of (1.3) (resp. (1.4)), we have

COROLLARY 1.3. *For any $\tau > 0$, there is a positive constant $C = C(\beta, p, \tau, |I|, \|u_0\|_\infty)$ such that*

$$|u(x,t) - u(y,s)| \leqslant C\left(|x-y| + |t-s|^{1/3}\right), \quad \forall x,\ y \in \overline{I}, \quad \forall t,\ s \geqslant \tau. \tag{1.5}$$

Furthermore, if $(u_0^{1/\gamma})_x \in L^\infty(I)$, then there is a constant $C = C(\beta, p, |I|, \|u_0\|_\infty, \|(u_0^{1/\gamma})_x\|_\infty)$ such that

$$|u(x,t) - u(y,s)| \leqslant C\left(|x-y| + |t-s|^{1/3}\right), \quad \forall x,\ y \in \overline{I}, \quad \forall t,\ s \geqslant 0. \tag{1.6}$$

REMARK 1.4. The above corollary implies that u is continuous up to the boundary. This result answers an open question stated in the Introduction of [**31**] for the semi-linear case.

REMARK 1.5. Estimate (1.6) says that u continues up to $t = 0$.

The second goal of this paper is to study the most striking phenomenon of equations of this type, the so-called quenching phenomenon that solution vanishes after a finite time. This property arises due to the presence of the singular term $u^{-\beta}\chi_{\{u>0\}}$.

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It occurs even starting with a positive unbounded initial data and there is a lack of uniqueness of solutions (see theorem 1.1, [**31**] again and see theorem 3 of Y. Belaud and J.I. Díaz, [**3**]). Then we have the following results:

THEOREM 1.6. *Assume as in theorem 1.2. Then, there is a finite time* $T_0 = T_0(\beta, p, \|u_0\|_\infty)$ *such that any solution of equation (1.1) vanishes after* T_0.

As a consequence of theorem 1.6, we show that the assumption $f(0) = 0$ is not only a necessary condition, but also a sufficient condition for the existence of solution.

COROLLARY 1.7. *The condition* $f(0) = 0$ *is a necessary and sufficient condition for the existence of a solution of equation (1.1).*

Beside of the consideration of the Dirichlet problem, we shall investigate also here the existence of solution of the Cauchy problem associated with equation (1.1).

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{on } \mathbb{R}. \end{cases} \tag{1.7}$$

In [**9**], Dao and Díaz studied equation (1.7) for the case $f = 0$. They proved the existence of solution. Moreover, they also studied the behaviours of solutions of equation (1.7) such as the quenching phenomenon, and the finite speed of propagation (see theorems 3.2 and 3.3, [**9**]). Of course, these properties still hold for any solution of problem (1.7) because the appearance of nonlinear absorption term $f(u)$ does not influence to these properties. In this paper, we will study the instantaneous shrinking of compact support (in short ISS), namely, if u_0 only goes to 0 uniformly as $|x| \to \infty$, then the support of solution is bounded for any $t > 0$. This property was first proved in the literature in the study of variational inequalities by H. Brezis and A. Friedman, see [**8**]. After that this phenomenon has been considered for quasilinear parabolic equations, see [**7**, **18**, **21**], and references therein for more details. Our main results for the Cauchy problem are as follows:

THEOREM 1.8. *Let* $0 \leqslant u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *and let* f *satisfy* (H). *Then, there exists a weak bounded solution* $u \in \mathcal{C}([0, \infty); L^1(\mathbb{R})) \cap L^p(0, T; W^{1,p}(\mathbb{R}))$, *satisfying equation (1.7) in* $\mathcal{D}'(\mathbb{R} \times (0, \infty))$. *Furthermore,* u *satisfies estimate (1.3) in* $\mathbb{R} \times (0, \infty)$.

Besides, if $(u_0^{1/\gamma})_x \in L^\infty(\mathbb{R})$, *then* u *satisfies estimate (1.4) in* $\mathbb{R} \times (0, \infty)$.

THEOREM 1.9. *Let* $0 \leqslant u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Suppose that* $u_0(x)$ *tends to* 0 *uniformly as* $|x| \to \infty$. *Then, any nonnegative solution of equation (1.7) has the ISS property.*

The paper is organized as follows: § 2 is devoted to prove gradient estimates, which are the main key of proving the existence of solution. In § 3, we shall give the proof of theorems 1.2, 1.6, and corollary 1.7. Finally, we give the proof of the existence of solution of problem (1.7) and theorem 1.9 in § 4.

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Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(p, \beta, \tau)$ means that C only depends on p, β, τ. We also denote by $I_r(x) = (x - r, x + r)$ to the open ball with centre at x and radius $r > 0$ in $\mathbb{R}$. If $x = 0$, we denote $I_r(0) = I_r$. Next $\partial_x u$ (resp. $\partial_t u$) means the partial derivative with respect to x (resp. t). We also write $\partial_x u = u_x$. Finally, the L^∞-norm of u is denoted by $\|u\|_\infty$.

2. Gradient estimate for the approximate solution

In this part, we shall modify Bernstein's technique to obtain a variety of estimates on $|u_x|$ depending on $f(u)$. Roughly speaking, the gradient estimates that we shall prove are of the type

$$|u_x(x,t)|^p \leqslant C u^{1-\beta}(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty). \tag{2.8}$$

It is known that such a gradient estimate of (2.8) plays a crucial role in proving the existence of solution (see e.g. [**12**, **27**, **31**] for the semi-linear case; and [**9**, **24**], for the case of quasilinear problems of this type). By the appearance of the nonlinear diffusion, p-Laplacian, we shall establish the gradient estimates for the solutions of the regularizing problem.

For any $\varepsilon > 0$, let us set

$$g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s), \qquad \text{with } \psi_\varepsilon(s) = \psi\left(\frac{s}{\varepsilon}\right),$$

and $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leqslant \psi \leqslant 1$ is a non-decreasing function such that $\psi(s) = \begin{cases} 0, & \text{if } s \leqslant 1, \\ 1, & \text{if } s \geqslant 2. \end{cases}$

Now fix $\varepsilon > 0$, we consider the following problem:

$$\begin{cases} \partial_t u - (a(u_x)u_x)_x + g_\varepsilon(u) + f(u)\psi_\varepsilon(u) = 0, & \text{in } I \times (0,\infty), \\ u(-l,t) = u(l,t) = \eta, & t \in (0,\infty), \\ u(x,0) = u_0(x) + \eta, & x \in I, \end{cases} \tag{2.9}$$

with $a(s) = b(s)^{((p-2)/2)}$, $b(s) = |s|^2 + \eta^2$; and $\eta \to 0^+$. Note that $a(u_x)$ is a regularization of $|u_x|^{p-2}$. Then, problem $(P_{\varepsilon,\eta})$ can be understood as a regularization of equation (1.1). The gradient estimates, presented in this framework are as follows:

LEMMA 2.1. *Let $0 \leqslant u_0 \in \mathcal{C}_c^\infty(I)$, $u_0 \neq 0$. Suppose that $f \in \mathcal{C}^1([0,\infty))$. Then, for any $0 < \eta < \varepsilon < \|u_0\|_\infty$, there exists a unique classical solution $u_{\varepsilon,\eta}$ of equation (2.9).*

(i) *Moreover, there is a positive constant $C = C(\beta, p)$ such that*

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \leqslant C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f, \|u_0\|_\infty)\|u_0\|_\infty^\beta + \Theta(f', \|u_0\|_\infty)\|u_0\|_\infty^{1+\beta} + 1\big), \tag{2.10}$$

for $(x,\tau) \in I \times (0,\infty)$. Recall that $\Theta(h,r) = \max_{0\leqslant s\leqslant r}\{|h(s)|\}$.

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(ii) *If we assume more that* $(u_0^{1/\gamma})_x \in L^\infty(I)$, *then there exists a positive constant* $C = C(\beta, p, \|(u_0^{1/\gamma})_x\|_\infty)$ *such that*

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \leqslant C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\|u_0\|_\infty^{1+\beta} + \Theta(f, \|u_0\|_\infty)\|u_0\|_\infty^\beta + \Theta(f', \|u_0\|_\infty)\|u_0\|_\infty^{1+\beta} + 1\big), \tag{2.11}$$

for any $(x,\tau) \in I \times (0,\infty)$.

Proof. (i) Obviously, equation (2.9) is not degenerated. Thus, the existence and uniqueness of solution $u_{\varepsilon,\eta} \in \mathcal{C}^\infty(\overline{I} \times [0,\infty))$ is well-known (see, e.g. [**21**, **25**, **32**, **33**]). In short, we denote $u = u_{\varepsilon,\eta}$. Then, we observe that η (resp. $\|u_0\|_\infty$) is a sub-solution (resp. super-solution) of equation (2.9). Thus, the strong comparison principle yields

$$\eta \leqslant u \leqslant \|u_0\|_\infty, \quad \text{in } I \times (0,\infty). \tag{2.12}$$

For any $0 < \tau < T < \infty$, let us consider a test function $\xi(t) \in \mathcal{C}_c^\infty(0,\infty)$, $0 \leqslant \xi(t) \leqslant 1$ such that

$$\xi(t) = \begin{cases} 1, & \text{on } [\tau, T], \\ 0, & \text{outside } \left(\frac{\tau}{2}, T + \frac{\tau}{2}\right). \end{cases}, \quad \text{and } |\xi_t| \leqslant \frac{c_0}{\tau},$$

and put

$$u = \varphi(v) = v^\gamma, \quad w(x,t) = \xi(t) v_x^2(x,t).$$

We write briefly $a(u_x) = a$, $(a(u_x))_x = a_x$, and $(a(u_x))_{xx} = a_{xx}$.

Then, we have

$$w_t - a w_{xx} = \xi_t v_x^2 + 2\xi v_x (v_t - a v_{xx})_x - 2\xi a v_{xx}^2 + 2\xi a_x v_x v_{xx}. \tag{2.13}$$

From the equation satisfied by u, we get

$$v_t - a v_{xx} = a_x v_x + a v_x^2 \frac{\varphi''}{\varphi'} - \frac{g_\varepsilon(\varphi)}{\varphi'} - \frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}. \tag{2.14}$$

Combining (2.13) and (2.14) provides us

$$w_t - a w_{xx} = \xi_t v_x^2 + 2\xi v_x \left(a_x v_x + a v_x^2 \frac{\varphi''}{\varphi'} - \frac{g_\varepsilon(\varphi)}{\varphi'} - \frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x - 2\xi a v_{xx}^2 + 2\xi a_x v_x v_{xx}. \tag{2.15}$$

Now, we define

$$L = \max_{\overline{I} \times [0,\infty)} \{w(x,t)\}.$$

If $L = 0$, then the conclusion (2.10) is trivial, and $|z_x(x,\tau)| = 0$, in $I \times (0,\infty)$. If not we have $L > 0$. This implies that $v_x(x_0,t_0) \neq 0$, and the function w must

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attain its maximum at a point $(x_0, t_0) \in I \times ((\tau/2), T + (\tau/2))$ since $w(x, t) = 0$ on $\partial I \times (0, \infty)$, and $w(., t) = 0$ outside $((\tau/2), T + (\tau/2))$. Therefore, we get

$$\begin{cases} w_t(x_0, t_0) = w_x(x_0, t_0) = 0, \quad \text{and} \\ 0 \geqslant w_{xx}(x_0, t_0) = 2\xi(t_0) v_{xx}^2(x_0, t_0) + 2\xi(t_0) v_x(x_0, t_0) v_{xxx}(x_0, t_0), \end{cases}$$

which implies

$$v_{xx}(x_0, t_0) = 0, \tag{2.16}$$

and

$$v_x(x_0, t_0) v_{xxx}(x_0, t_0) \leqslant 0, \tag{2.17}$$

At the moment, our argument focuses on the functions v, and w at the point (x_0, t_0). Note that by (2.16), inequality (2.15) reduces to

$$0 \leqslant w_t - a w_{xx} = \xi_t v_x^2 + 2\xi v_x \left(a_{xx} v_x + a_x v_x^2 \frac{\varphi''}{\varphi'} + a v_x^2 \left(\frac{\varphi''}{\varphi'} \right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'} \right)_x - \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'} \right)_x \right).$$

$$0 \leqslant \xi_t \xi^{-1} v_x^2 + 2 v_x \left(a_{xx} v_x + a_x v_x^2 \frac{\varphi''}{\varphi'} + a v_x^2 \left(\frac{\varphi''}{\varphi'} \right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'} \right)_x - \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'} \right)_x \right).$$

Or

$$-a v_x^3 \left(\frac{\varphi''}{\varphi'} \right)_x \leqslant \frac{1}{2} \xi_t \xi^{-1} v_x^2 + a_{xx} v_x^2 + a_x v_x^3 \frac{\varphi''}{\varphi'} - v_x \left(\frac{g_\varepsilon(\varphi)}{\varphi'} \right)_x - v_x \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'} \right)_x . \tag{2.18}$$

By using the fact $v_{xx}(x_0, t_0) = 0$ again, we get

$$\begin{cases} (a(u_x))_x = (p-2) b^{((p-4)/2)}(u_x) \varphi' \varphi'' v_x^3, \\ \left(\dfrac{\varphi''}{\varphi'} \right)_x = \left(\dfrac{\varphi''' \varphi' - \varphi''^2}{\varphi'^2} \right) v_x = -(\gamma - 1) v^{-2} v_x. \end{cases} \tag{2.19}$$

Next, we compute

$$\begin{aligned} (a(u_x))_{xx} &= (p-2)(p-4) b^{((p-6)/2)}(u_x) (\varphi' \varphi'')^2 v_x^6 \\ &+ (p-2) b^{((p-4)/2)}(u_x) (\varphi''^2 + \varphi' \varphi''') v_x^4 \\ &+ (p-2) b^{((p-4)/2)}(u_x) \varphi'^2 v_x v_{xxx}. \end{aligned}$$

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Thanks to (2.17), we obtain

$$(a(u_x))_{xx} \leqslant (p-2)(p-4)b^{((p-6)/2)}(u_x)(\varphi'\varphi'')^2 v_x^6 + \\ (p-2)b^{((p-4)/2)}(u_x)(\varphi''^2 + \varphi'\varphi''')v_x^4. \tag{2.20}$$

After that, we have

$$\begin{cases} v_x \left(\dfrac{g_\varepsilon(\varphi)}{\varphi'}\right)_x = \left(g'_\varepsilon - g_\varepsilon \dfrac{\varphi''}{\varphi'^2}\right) v_x^2 \\ = \left(\psi'_\varepsilon(\varphi)v^{-\beta} - \left(\beta + \frac{\gamma-1}{\gamma}\right)\psi_\varepsilon(\varphi)v^{-(1+\beta)\gamma}\right) v_x^2, \\ v_x \left(\dfrac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x = \left((f\psi_\varepsilon)' - (f\psi_\varepsilon)\dfrac{\varphi''}{\varphi'^2}\right) v_x^2 \\ = (f\psi_\varepsilon)' v_x^2 - f(\varphi(v))\psi_\varepsilon(\varphi(v))\left(\frac{\gamma-1}{\gamma}\right) v^{-\gamma} v_x^2. \end{cases}$$

Since $f, \psi_\varepsilon, \psi'_\varepsilon \geqslant 0$, and $0 \leqslant \psi_\varepsilon \leqslant 1$, we get

$$\begin{cases} v_x \left(\dfrac{g_\varepsilon(\varphi)}{\varphi'}\right)_x \geqslant -\left(\beta + \dfrac{\gamma-1}{\gamma}\right) v^{-(1+\beta)\gamma} v_x^2, \\ v_x \left(\dfrac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x \geqslant f'(\varphi(v))\psi_\varepsilon(\varphi(v))v_x^2 - \left(\dfrac{\gamma-1}{\gamma}\right) f(\varphi(v))v^{-\gamma} v_x^2. \end{cases} \tag{2.21}$$

Inserting (2.19)– (2.21) into (2.18) yields

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + \underbrace{\begin{matrix}(p-2)(p-4)b^{((p-6)/2)}(u_x)(\varphi'\varphi'')^2 v_x^8 \\ +(p-2)b^{((p-4)/2)}(u_x)(2\varphi''^2 + \varphi'\varphi''')v_x^6\end{matrix}}_{\mathcal{B}}$$

$$+ \left(\beta + \frac{\gamma-1}{\gamma}\right) v^{-(1+\beta)\gamma} v_x^2 + \left(\frac{\gamma-1}{\gamma}\right) f(\varphi(v))v^{-\gamma}v_x^2 \\ - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v_x^2 \geqslant (\gamma-1)v^{-2}a(z_x)v_x^4. \tag{2.22}$$

Now, we handle the term $\mathcal{B}$

$$\begin{aligned} \mathcal{B} &= (p-2)b^{((p-6)/2)}(u_x)v_x^6\left((p-4)(\varphi'\varphi'')^2 v_x^2 + (2\varphi''^2 + \varphi'\varphi''')b(u_x)\right) \\ &= (p-2)\varphi'^2 b^{((p-6)/2)}(u_x)v_x^8\left((p-2)\varphi''^2 + \varphi'\varphi'''\right) + \eta^2(p-2) \\ &\quad (2\varphi''^2 + \varphi'\varphi''')b^{((p-6)/2)}(u_x)v_x^6 \\ &\underbrace{= (p-2)(p(\gamma-1)-\gamma)\gamma^2(\gamma-1)v^{2(\gamma-2)}\varphi'^2 b^{((p-6)/2)}(u_x)v_x^8}_{\mathcal{B}_1} \\ &\quad + \underbrace{\eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)v^{2(\gamma-2)}b^{((p-6)/2)}(u_x)v_x^6}_{\mathcal{B}_2}. \end{aligned}$$

It is clear that $\mathcal{B}_1 \leqslant 0$, since $p(\gamma-1)-\gamma < 0$, thereby proves

$$\mathcal{B} \leqslant \mathcal{B}_2. \tag{2.23}$$

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From (2.22) and (2.23), we get

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2+\left(\beta+\frac{\gamma-1}{\gamma}\right)v^{-(1+\beta)\gamma}v_x^2 \\ +\left(\frac{\gamma-1}{\gamma}\right)f(\varphi(v))v^{-\gamma}v_x^2-f'(\varphi(v))\psi_\varepsilon(\varphi(v))v_x^2+\mathcal{B}_2 \\ \geqslant(\gamma-1)v^{-2}a(u_x)v_x^4.$$

Since $p>2$, $b^{((p-2)/2)}(.)$ is an increasing function, thereby proves

$$a(u_x)=b^{((p-2)/2)}(u_x)\geqslant|u_x|^{p-2}=\gamma^{p-2}v^{(\gamma-1)(p-2)}|v_x|^{p-2}.$$

It follows then from the last two inequalities

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2+\left(\beta+\frac{\gamma-1}{\gamma}\right)v^{-(1+\beta)\gamma}v_x^2 \\ +\left(\frac{\gamma-1}{\gamma}\right)f(\varphi(v))v^{-\gamma}v_x^2-f'(\varphi(v))\psi_\varepsilon(\varphi(v))v_x^2+\mathcal{B}_2 \\ \geqslant(\gamma-1)\gamma^{p-2}v^{(\gamma-1)(p-2)-2}|v_x|^{p+2}.$$

By noting that $\quad 2-(\gamma-1)(p-2)=(1+\beta)\gamma$, we get

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2+\left(\beta+\frac{\gamma-1}{\gamma}\right)v^{-(1+\beta)\gamma}v_x^2 \\ +\left(\frac{\gamma-1}{\gamma}\right)f(\varphi(v))v^{-\gamma}v_x^2-f'(\varphi(v))\psi_\varepsilon(\varphi(v))v_x^2+\mathcal{B}_2 \\ \geqslant(\gamma-1)\gamma^{p-2}v^{-(1+\beta)\gamma}|v_x|^{p+2}.$$

Multiplying both sides of the above inequality by $v^{(1+\beta)\gamma}$ yields

$$\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2+\left(\beta+\frac{\gamma-1}{\gamma}\right)v_x^2+\left(\frac{\gamma-1}{\gamma}\right)f(\varphi(v))v^{\beta\gamma}v_x^2 \\ -f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma}v_x^2 \\ +v^{(1+\beta)\gamma}\mathcal{B}_2\geqslant(\gamma-1)\gamma^{p-2}|v_x|^{p+2}. \tag{2.24}$$

Now, we divide the study of inequality (2.24) in two cases:

($\star$) **Case:** $3\gamma-4\leqslant 0$.
We have $\mathcal{B}_2\leqslant 0$. It follows then from (2.24) that

$$(\gamma-1)\gamma^{p-2}|v_x|^{p+2}\leqslant\left(\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}+\left(\beta+\frac{\gamma-1}{\gamma}\right)+\left(\frac{\gamma-1}{\gamma}\right)f(\varphi(v))v^{\beta\gamma}\right. \\ \left.-f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma}\right)v_x^2.$$

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Since $v_x(x_0,t_0)\neq 0$, we can simplify the term $|v_x|^2$ of both sides of the above inequality to obtain

$$(\gamma-1)\gamma^{p-2}|v_x|^p \leqslant \left(\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma} + \left(\beta+\frac{\gamma-1}{\gamma}\right) + \left(\frac{\gamma-1}{\gamma}\right) f(\varphi(v))v^{\beta\gamma} - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma}\right). \tag{2.25}$$

Since $\psi_\varepsilon(.)\leqslant 1$, there is a positive constant C depending on β, p such that

$$|v_x|^p \leqslant C\left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + 1 + f(\varphi(v))v^{\beta\gamma} + |f'(\varphi(v))|v^{(1+\beta)\gamma}\right). \tag{2.26}$$

Remind that $u=\varphi(v)=v^\gamma$. Thus, we infer from (2.12) and (2.26)

$$|v_x|^p \leqslant C\left(|\xi_t|\xi^{-1}(t_0)\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta}\Theta(f,\|u_0\|_\infty) + \|u_0\|_\infty^{1+\beta}\Theta(f',\|u_0\|_\infty)+1\right). \tag{2.27}$$

By multiplying both sides of (2.27) with $\xi(t_0)^{p/2}$, we get

$$\left(\xi(t_0)|v_x(x_0,t_0)|^2\right)^{p/2} \leqslant C\left(|\xi_t|\xi(t_0)^{(p/2)-1}\|u_0\|_\infty^{1+\beta} + \xi(t_0)^{p/2}\|u_0\|_\infty^{\beta}\Theta(f,\|u_0\|_\infty) + \xi(t_0)^{p/2}\|u_0\|_\infty^{1+\beta}\Theta(f',\|u_0\|_\infty) + \xi(t_0)^{p/2}\right).$$

Since $\xi(t)\leqslant 1$, and $|\xi_t(t)|\leqslant c_0\tau^{-1}$, there is a positive constant (still denoted by C) such that

$$w(x_0,t_0)^{p/2} \leqslant C\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta}\Theta(f,\|u_0\|_\infty) + \|u_0\|_\infty^{1+\beta}\Theta(f',\|u_0\|_\infty)+1\right).$$

Remind that $w(x_0,t_0)=\max_{(x,t)\in I\times[0,\infty)}\{w(x,t)\}$. The last estimate induces

$$w(x,t)^{p/2} \leqslant C\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta}\Theta(f,\|u_0\|_\infty) + \|u_0\|_\infty^{1+\beta}\Theta(f',\|u_0\|_\infty)+1\right),$$

for any $(x,t)\in I\times(0,\infty)$. By noting that $\xi(\tau)=1$, we obtain

$$|v_x(x,\tau)|^p = w(x,\tau)^p \leqslant C\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta}\Theta(f,\|u_0\|_\infty) + \|u_0\|_\infty^{1+\beta}\Theta(f',\|u_0\|_\infty)+1\right),$$

which implies

$$|u_x(x,\tau)|^p \leqslant Cu^{1-\beta}(x,\tau)\left(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \|u_0\|_\infty^{\beta}\Theta(f,\|u_0\|_\infty) + \|u_0\|_\infty^{1+\beta}\Theta(f',\|u_0\|_\infty)+1\right).$$

This inequality holds for any $\tau>0$, so we get (2.10).

($\star\star$) Case: $3\gamma-4>0 \Longleftrightarrow p<4(1-\beta)$.

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Now $b^{((p-6)/2)}(.)$ is a decreasing function, so we have

$$b^{((p-6)/2)}(z_x) \leqslant |z_x|^{((p-6)/2)} = (v_x^2\varphi'^2)^{((p-6)/2)},$$

which implies

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \leqslant \eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{2(\gamma-2)+(1+\beta)\gamma+(\gamma-1)(p-6)}|v_x|^p.$$

Note that $2(\gamma-2)+(1+\beta)\gamma+(\gamma-1)(p-6) = -2(\gamma-1)$. Then, we obtain

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \leqslant \eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{-2(\gamma-1)}|v_x|^p. \tag{2.28}$$

A combination of (2.28) and (2.24) gives us

$$\begin{aligned}\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2 + \left(\beta+\frac{\gamma-1}{\gamma}\right)v_x^2 + \left(\frac{\gamma-1}{\gamma}\right)f(\varphi(v))v^{\beta\gamma}v_x^2 \\ - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma}v_x^2 \\ + \eta^2(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{-2(\gamma-1)}|v_x|^p \\ \geqslant (\gamma-1)\gamma^{p-2}|v_x|^{p+2}.\end{aligned}$$

The fact $v = u^{1/\gamma} \geqslant \eta^{1/\gamma}$ implies $v^{-2(\gamma-1)} \leqslant \eta^{-((2(\gamma-1))/\gamma)}$. Therefore, there is constant $C = C(\beta,p) > 0$ such that

$$\begin{aligned}|v_x(x_0,t_0)|^{p+2} \leqslant C\Big(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + 1 + f(\varphi(v))v^{\beta\gamma} \\ - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma}\Big)v_x^2(x_0,t_0) \\ + C\eta^{2-((2(\gamma-1))/\gamma)}|v_x(x_0,t_0)|^p.\end{aligned}$$

Now, if $|v_x(x_0,t_0)| < 1$, then we have $w(x_0,t_0) = \xi(t_0)|v_x(x_0,t_0)|^2 < 1$, thereby proves $w(x,t) \leqslant 1$, in $I \times (0,\infty)$. Thus, estimate (2.10) follows immediately. If not, we have $|v_x(x_0,t_0)|^p \leqslant |v_x(x_0,t_0)|^{p+2}$, then it follows from the last inequality

$$\begin{aligned}|v_x(x_0,t_0)|^{p+2} \leqslant C\Big(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + f(\varphi(v))v^{\beta\gamma} \\ - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma} + 1\Big)v_x^2(x_0,t_0) \\ + C\eta^{\frac{2}{\gamma}}|v_x(x_0,t_0)|^{p+2},\end{aligned}$$

or

$$\begin{aligned}\left(1 - C\eta^{2/\gamma}\right)|v_x(x_0,t_0)|^p \leqslant C\Big(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + f(\varphi(v))v^{\beta\gamma} \\ - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma} + 1\Big).\end{aligned}$$

Because η is small enough, there exists a positive constant $C_1 = C_1(\beta,p)$ such that

$$|v_x(x_0,t_0)|^p \leqslant C_1\Big(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + f(\varphi(v))v^{\beta\gamma} - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma} + 1\Big).$$

This inequality is just a version of (2.25). By the same analysis as in ($\star$), we also obtain estimate (2.10).

(ii) Now, we prove estimate (2.11). For any $T \geqslant 1$ large enough, let us consider the cut-off function $\overline{\xi}(t) \in \mathcal{C}^\infty(\mathbb{R})$ instead of $\xi(t)$ above, $0 \leqslant \overline{\xi}(t) \leqslant 1$ such that

$$\overline{\xi}(t) = \begin{cases} 1, & \text{if } t < T, \\ 0, & \text{if } t > 2T, \end{cases} \qquad \text{and } |\overline{\xi}_t| \leqslant \frac{c_0}{T}.$$

The proof of estimate (2.11) is most likely to the one of estimate (2.10). In fact, we observe that

i) either $w(x,t)$ attains its maximum at the initial data, that is:

$$\max_{(x,t)\in I\times[0,\infty)} \{w(x,t)\} = w(x_0,0) = v_x^2(x_0,0) = |(u_0^{1/\gamma})_x|^2$$
$$\leqslant \|(u_0^{1/\gamma})_x\|_\infty^2, \quad \text{for some } x_0 \in I,$$

thereby proves

$$|u_x(x,t)|^p \leqslant \gamma^p \|(u_0^{1/\gamma})_x\|_\infty^p u^{1-\beta}(x,t), \quad \text{for any } (x,t) \in I \times (0,T), \tag{2.29}$$

or, ii) there is a point $(x_0,t_0) \in I \times (0,2T)$ such that $\max_{(x,t)\in I\times[0,\infty)} \{w(x,t)\} = w(x_0,t_0)$.

Then, we repeat the proof in *i*) to get for $(x,t) \in I \times (0,\infty)$

$$|u_x(x,t)|^p \leqslant C(\beta,p) u^{1-\beta}(x,t) \big(T^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f,\|u_0\|_\infty)\|u_0\|_\infty^\beta + \Theta(f',\|u_0\|_\infty)\|u_0\|_\infty^{1+\beta} + 1\big).$$

Since $T \geqslant 1$, we obtain from the above inequality

$$|u_x(x,t)|^p \leqslant C u^{1-\beta}(x,t) \big(\|u_0\|_\infty^{1+\beta} + \Theta(f,\|u_0\|_\infty)\|u_0\|_\infty^\beta + \Theta(f',\|u_0\|_\infty)\|u_0\|_\infty^{1+\beta} + 1\big). \tag{2.30}$$

A combination of (2.29) and (2.30) implies that there is a positive constant (still denoted by C) depending only on $\beta, p, \|(u_0^{1/\gamma})_x\|_\infty$ such that

$$|u_x(x,t)|^p \leqslant C u^{1-\beta}(x,t) \big(\|u_0\|_\infty^{1+\beta} + \Theta(f,\|u_0\|_\infty)\|u_0\|_\infty^\beta + \Theta(f',\|u_0\|_\infty)\|u_0\|_\infty^{1+\beta} + 1\big), \tag{2.31}$$

for $(x,t) \in I \times (0,\infty)$.

This puts an end to the proof of lemma 2.1. □

If f is only a local Lipschitz function on $[0,\infty)$, we have then

LEMMA 2.2. *Assume as in lemma 2.1. Suppose that f is only a locally Lipschitz function on $[0,\infty)$. Then estimate (2.10) becomes*

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \leqslant C u_{\varepsilon,\eta}^{1-\beta}(x,\tau) \big(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f,\|u_0\|_\infty)\|u_0\|_\infty^\beta + Lip(f,u_0)\|u_0\|_\infty^{1+\beta} + 1\big), \tag{2.32}$$

for $(x,\tau) \in I \times (0,\infty)$, where $Lip(f,u_0)$ is the local Lipschitz constant of f on the closed interval $[0, 2\|u_0\|_\infty]$.

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Moreover, if $(u_0^{1/\gamma})_x \in L^\infty(I)$*, then estimate (2.11) becomes*

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \leqslant C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\|u_0\|_\infty^{1+\beta} + \Theta(f,\|u_0\|_\infty)\|u_0\|_\infty^{\beta} + Lip(f,u_0)\|u_0\|_\infty^{1+\beta} + 1\big), \tag{2.33}$$

with $C = C(\beta, p, \|(u_0^{1/\gamma})_x\|_\infty) > 0$*.*

Proof. At the beginning, we regularize f on $[0,\infty)$. To do it, we extend f by 0 in $(-\infty,0)$ (still denoted by f). Let f_n be the standard regularization of f on $\mathbb{R}$. Then, we consider the following problem:

$$\begin{cases} \partial_t u - (a(u_x)u_x)_x + g_\varepsilon(u) + f_n(u)\psi_\varepsilon(u) = 0, & \text{in } I \times (0,\infty), \\ u(-l,t) = u(l,t) = \eta, & t \in (0,\infty), \\ u(x,0) = u_0(x) + \eta, & x \in I. \end{cases} \tag{2.34}$$

Let ε, η be in lemma 2.1. Then, equation (2.34) possesses a unique classical solution u_n. Thanks to lemma 2.1, we have

$$|\partial_x u_n(x,t)|^p \leqslant C u_n^{1-\beta}(x,\tau)\big(\tau^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f_n,\|u_0\|_\infty)\|u_0\|_\infty^{\beta} + \Theta(f_n',\|u_0\|_\infty)\|u_0\|_\infty^{1+\beta} + 1\big), \tag{2.35}$$

for any $(x,t) \in I \times (0,\infty)$. One hand, we observe that for any $n \geqslant 1$

$$\Theta(f_n,\|u_0\|_\infty) \leqslant \Theta\left(f,\|u_0\|_\infty + \frac{1}{n}\right) \leqslant \Theta(f, 2\|u_0\|_\infty). \tag{2.36}$$

Other hand, Rademacher's theorem (see, e.g., [**17**]) ensures that

$$\Theta(f_n',\|u_0\|_\infty) \leqslant Lip\left(f,\|u_0\|_\infty + \frac{1}{n}\right) \leqslant Lip(f, 2\|u_0\|_\infty). \tag{2.37}$$

From (2.35), (2.36), and (2.37), we observe that $|\partial_x u_n(x,t)|$ is bounded by a constant not depending on n. Then, the classical argument allows us to pass to the limit as $n \to \infty$ in (2.35) in order to get the gradient estimate (2.32).

Similarly, we also obtain estimate (2.33). □

If f in lemma 2.2 is nondecreasing on $[0,\infty)$ then we can relax the term $Lip(f,u_0)$ in (2.32) and (2.33).

LEMMA 2.3. *Let f be a continuous and a nondecreasing function on $[0,\infty)$. Then, there exists a positive constant $C = C(\beta,p)$ such that*

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \leqslant C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\tau^{-1}\|u_0\|_\infty^{1+\beta} + f(\|u_0\|_\infty)\|u_0\|_\infty^{\beta} + 1\big), \tag{2.38}$$

for $(x,\tau) \in I \times (0,\infty)$*. Note that* $\Theta(f,\|u_0\|_\infty) = f(\|u_0\|_\infty)$ *in this case.*

Furthermore, if $(u_0^{1/\gamma})_x \in L^\infty(I)$*, then there is a positive constant* $C = C(\beta, p, \|(u_0^{1/\gamma})_x\|_\infty)$ *such that*

$$|\partial_x u_{\varepsilon,\eta}(x,\tau)|^p \leqslant C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\big(\|u_0\|_\infty^{1+\beta} + f(\|u_0\|_\infty)\|u_0\|_\infty^{\beta} + 1\big). \tag{2.39}$$

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Proof. We can assume without loss of generality that $f \in \mathcal{C}^1([0,\infty))$. If not, we work on the standard regularization of f, that is, f_n above. Note that f_n is also a nondecreasing function.

The proof of this lemma is most likely to the one of lemma 2.1. In fact, we just make a slight change in (2.25) in order to remove the term containing f'. Let us recall inequality (2.25) here for a convenience.

$$(\gamma-1)\gamma^{p-2}|v_x|^p \leqslant \left(\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma} + \left(\beta+\frac{\gamma-1}{\gamma}\right) + \left(\frac{\gamma-1}{\gamma}\right) f(\varphi(v))v^{\beta\gamma} \right.$$
$$\left. - f'(\varphi(v))\psi_\varepsilon(\varphi(v))v^{(1+\beta)\gamma}\right).$$

Since $f' \geqslant 0$, we have

$$(\gamma-1)\gamma^{p-2}|v_x|^p \leqslant \left(\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma} + \left(\beta+\frac{\gamma-1}{\gamma}\right) + \left(\frac{\gamma-1}{\gamma}\right) f(\varphi(v))v^{\beta\gamma}\right).$$

Obviously, the term containing f' does not appear in the last inequality. Then, we just repeat the proof of lemma 2.1 to get estimate (2.38).

Finally, by the same argument as in the proof of *ii*) lemma 2.1, we get (2.39). □

Remark 2.4. Note that the solution in lemma 2.3 is unique because of the monotonicity of f.

Remark 2.5. Note that the estimates in the proof of lemma 2.1 (resp. lemmas 2.2, 2.3) are independent of η, ε. This observation allows us to pass to the limit as $\eta, \varepsilon \to 0$ in order to get gradient estimates for solution u of equation (1.1).

Next, we pass to the limit as $\eta \to 0$ to obtain a solution of the following problem.

$$(P_\varepsilon)\begin{cases} \partial_t u - \partial_x\left(|\partial_x u|^{p-2}\partial_x u\right) + g_\varepsilon(u) + f(u)\psi_\varepsilon(u) = 0 & \text{in } I\times(0,\infty), \\ u(-l,t) = u(l,t) = 0 & t\in(0,\infty), \\ u(x,0) = u_0(x) & \text{on } I. \end{cases}$$

Theorem 2.6. *Let $0 \leqslant u_0 \in L^\infty(I)$, and let f satisfy (H). Then, there exists a unique weak solution u_ε of equation (P_ε). Furthermore, u_ε is bounded by $\|u_0\|_\infty$, and it fulfills gradient estimate (2.32) for a.e $(x,t) \in I\times(0,\infty)$.*

Besides, if $(u_0^{1/\gamma})_x \in L^\infty(I)$, then u_ε satisfies estimate (2.33) for a.e $(x,t) \in I\times(0,\infty)$.

Proof. Note that we can regularize initial data u_0 if necessary. Then, the proof of this theorem is obtained by passing $\eta \to 0$ in equation (2.9). It is today a classical argument, see, for example, [**16**, **32**, **33**]. Thus, we leave the details to the reader. □

Remark 2.7. Up to now, we have not used the assumption $f(0) = 0$ yet. However, this assumption will be used in the step of passing $\varepsilon \to 0$.

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3. Existence of a maximal solution

Proof of theorem 1.2: Thanks to theorem 2.6, problem (P_ε) has a unique (bounded) weak solution u_ε. Furthermore, u_ε satisfies gradient estimate (2.32).

Now, we claim that $\{u_\varepsilon\}_{\varepsilon>0}$ is a nondecreasing sequence. Indeed, we observe that $\psi_{\varepsilon_1}(s) \geqslant \psi_{\varepsilon_2}(s)$ for any $\varepsilon_1 < \varepsilon_2$, thereby proves

$$g_{\varepsilon_1}(u_{\varepsilon_1}) \geqslant g_{\varepsilon_2}(u_{\varepsilon_1}), \text{ and } f(u_{\varepsilon_1})\psi_{\varepsilon_1}(u_{\varepsilon_1}) \geqslant f(u_{\varepsilon_1})\psi_{\varepsilon_2}(u_{\varepsilon_1}).$$

These facts imply that u_{ε_1} is a sub-solution of equation satisfied by u_{ε_2}. By the comparison principle, we obtain

$$0 \leqslant u_{\varepsilon_1} \leqslant u_{\varepsilon_2}, \quad \text{in } I \times (0, \infty).$$

Thus, there is a nonnegative function u such that as $\varepsilon \to 0$

$$u_\varepsilon(x,t) \downarrow u(x,t), \quad \text{for } (x,t) \in I \times (0,\infty).$$

Now, we claim that

$$u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0,\infty)). \tag{3.40}$$

By one hand, it follows from the energy estimate of the regularized equation that

$$\|u_\varepsilon(t)\|_{L^1(I)}, \|g_\varepsilon(u_\varepsilon)\|_{L^1(I\times(0,\infty))}, \text{ and } \|f(u_\varepsilon)\|_{L^1(I\times(0,\infty))} \leqslant \|u_0\|_{L^1(I)}, \tag{3.41}$$

for any $\varepsilon > 0$. On the other hand, the monotonicity of $\{u_\varepsilon\}_\varepsilon$ yields

$$g_\varepsilon(u_\varepsilon) \geqslant g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}.$$

Thanks to (3.41) and Fatou's Lemma, there exists a nonnegative function $\Phi \in L^1(I \times (0,\infty))$ such that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = \Phi \geqslant u^{-\beta}\chi_{\{u>0\}}, \tag{3.42}$$

and $\|\Phi\|_{L^1(I\times(0,\infty))}$, $\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(I\times(0,\infty))}$ are also bounded by $\|u_0\|_{L^1(I)}$. Then we get claim (3.40). We will prove now that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } \ L^1(I \times (0,\infty)). \tag{3.43}$$

Next, (3.41) allows us to apply a result of L. Boccardo et al., the so-called almost everywhere convergence of the gradients (see [**5**, **6**]) in order to obtain

$$\partial_x u_\varepsilon(x,t) \to \partial_x u(x,t), \qquad \text{for a.e } \ (x,t) \in I \times (0,\infty),$$

up to a subsequence. Therefore, u also satisfies estimate (2.32) for a.e $(x,t) \in I \times (0,\infty)$. In addition, we have

$$\partial_x u_\varepsilon \to \partial_x u, \quad \text{in } \ L^r(I \times (\tau, T)), \text{ for any } \ 0 < \tau < T < \infty, \text{ and for } \ r \geqslant 1. \tag{3.44}$$

Let us show that u must satisfy equation (1.1) in the sense of distribution.

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For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, $\phi \in \mathcal{C}_c^\infty(I \times (0,\infty))$, in the equation satisfied by u_ε. Then, using integration by parts yields

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u_\varepsilon)\phi_t + \frac{1}{\eta}|\partial_x u_\varepsilon|^p \psi'\left(\frac{u_\varepsilon}{\eta}\right)\phi + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon \phi_x \psi_\eta(u_\varepsilon) \\ + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi + f(u_\varepsilon)\psi_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi \ \Big)\, \mathrm{d}x\mathrm{d}s = 0,$$

with $\Psi_\eta(u) = \int_0^u \psi_\eta(s)\mathrm{d}s$. Note that the test function $\psi_\eta(.)$ plays a role in isolating the singularity when u_ε is near to 0. Thus, there is no problem of going to the limit as $\varepsilon \to 0$ in the above identity to get

$$\int_{Supp(\phi)} \Big(-\Psi_\eta(u)\phi_t + \frac{1}{\eta}|u_x|^p \psi'\left(\frac{u}{\eta}\right)\phi + |u_x|^{p-2}u_x \phi_x \psi_\eta(u) \\ + u^{-\beta}\psi_\eta(u)\phi + f(u)\psi_\eta(u)\phi\Big)\, \mathrm{d}x\mathrm{d}s = 0. \tag{3.45}$$

Next, we will go to the limit as $\eta \to 0$ in equation (3.45).

We first note that $u^{-\beta}\psi_\eta(u)(x,t) \uparrow u^{-\beta}\chi_{\{u>0\}}(x,t)$, for any $(x,t) \in I \times (0,\infty)$. By (3.40), the Monotone Convergence Theorem implies that $u^{-\beta}\psi_\eta(u) \uparrow u^{-\beta}\chi_{\{u>0\}}$ in $L^1(I \times (0,\infty))$, thereby proves

$$u^{-\beta}\psi_\eta(u)\phi \to u^{-\beta}\chi_{\{u>0\}}\phi, \quad \text{in} \ \ L^1(I \times (0,\infty)). \tag{3.46}$$

Since $f(0) = 0$, it follows from the Dominated Convergence Theorem that

$$\lim_{\eta\to 0} \int f(u)\psi_\eta(u)\phi \ \mathrm{d}x\mathrm{d}s = \int f(u)\phi \ \mathrm{d}x\mathrm{d}s. \tag{3.47}$$

On the other hand, we have

$$\lim_{\eta\to 0} \int_{Supp(\phi)} \frac{1}{\eta}|\partial_x u|^p \psi'\left(\frac{u}{\eta}\right)\phi \ \mathrm{d}x\mathrm{d}s = 0. \tag{3.48}$$

In fact, we have

$$\frac{1}{\eta}\int_{Supp(\phi)} |\partial_x u|^p \psi'\left(\frac{u}{\eta}\right)\phi \ \mathrm{d}x\mathrm{d}s = \frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} |\partial_x u|^p \psi'\left(\frac{u}{\eta}\right)\phi \ \mathrm{d}x\mathrm{d}s.$$

Since u satisfies estimate (2.32), we have

$$\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} |\partial_x u|^p \left|\psi'\left(\frac{u}{\eta}\right)\right||\phi| \ \mathrm{d}x\mathrm{d}s \leqslant C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{1-\beta}\mathrm{d}x\mathrm{d}s \\ \leqslant 2C\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}\mathrm{d}x\mathrm{d}s,$$

where the constant $C > 0$ is independent of η. Moreover, $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $I \times (0,\infty)$ by (3.40). Thus, we get

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}\mathrm{d}x\mathrm{d}s = 0,$$

thereby proves the conclusion (3.48).

A combination of (3.46)–(3.48) deduces

$$\int_{Supp(\phi)} \left(-u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right) \mathrm{d}x\mathrm{d}s = 0. \tag{3.49}$$

In other words, u satisfies equation (1.1) in $\mathcal{D}'(I \times (0,\infty))$.

As mentioned above, we prove (3.43) now. The fact that u_ε is a weak solution of (P_ε) leads to

$$\int_{Supp(\phi)} \left(-u_\varepsilon\phi_t + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon \partial_x\phi + g_\varepsilon(u_\varepsilon)\phi + f(u_\varepsilon)\psi_\varepsilon(u_\varepsilon)\phi\right) \mathrm{d}x\mathrm{d}s = 0,$$

for $\phi \in \mathcal{C}_c^\infty(I \times (0,\infty))$, $\phi \geqslant 0$. Letting $\varepsilon \to 0$ in the last equation deduces

$$\int_{Supp(\phi)} \left(-u\phi_t + |u_x|^{p-2}u_x\phi_x\right) \mathrm{d}x\mathrm{d}s + \lim_{\varepsilon\to 0}\int_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi \; \mathrm{d}x\mathrm{d}s \\ + \int_{Supp(\phi)} f(u)\phi \; \mathrm{d}x\mathrm{d}s = 0. \tag{3.50}$$

By (3.49) and (3.50), we get

$$\lim_{\varepsilon\to 0}\int_0^\infty \int_I g_\varepsilon(u_\varepsilon)\phi \; \mathrm{d}x\mathrm{d}s = \int_0^\infty \int_I u^{-\beta}\chi_{\{u>0\}}\phi \; \mathrm{d}x\mathrm{d}s. \tag{3.51}$$

According to (3.51) and Fatou's Lemma, we obtain

$$\int_0^\infty \int_I \Phi\phi \; \mathrm{d}x\mathrm{d}s = \int_0^\infty \int_I \liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)\phi \; \mathrm{d}x\mathrm{d}s \leqslant \int_0^\infty \int_I u^{-\beta}\chi_{\{u>0\}}\phi \; \mathrm{d}x\mathrm{d}s,$$

$\forall\phi \in \mathcal{C}_c^\infty(I \times (0,\infty))$, $\phi \geqslant 0$. The last inequality and (3.42) yield

$$u^{-\beta}\chi_{\{u>0\}} = \Phi, \quad \text{a.e in } \; I \times (0,\infty).$$

Then, we get (3.43).

It remains to show that u is the maximal solution of problem (1.1).

PROPOSITION 3.1. *Let v be a weak solution of problem (1.1). Then, we have*

$$v(x,t) \leqslant u(x,t), \quad \textit{for a.e } \; (x,t) \in I \times (0,\infty).$$

Proof. For any $\varepsilon > 0$, we observe that $g_\varepsilon(v) \leqslant v^{-\beta}\chi_{\{v>0\}}$, and $f(v)\psi_\varepsilon(v) \leqslant f(v)$. Thus, we get

$$\partial_t v - \left(|v_x|^{p-2}v_x\right)_x + g_\varepsilon(v) + f(v)\psi_\varepsilon(v) \leqslant 0.$$

This means that v is a sub-solution of equation (P_ε), so the comparison theorem yields

$$v(x,t) \leqslant u_\varepsilon(x,t), \quad \text{for a.e } \; (x,t) \in I \times (0,\infty).$$

The conclusion follows by letting $\varepsilon \to 0$ in the last inequality.

This puts an end to the proof of theorem 1.2. □

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If f is a global Lipschitz function, or f is a nondecreasing function on $[0,\infty)$, then the existence result holds for L^1-initial data.

Theorem 3.2. *Let $0 \leqslant u_0 \in L^1(I)$. Suppose that f is a global Lipschitz function on $[0,\infty)$, and $f(0)=0$. Then there exists a maximal weak solution u of equation (1.1). Furthermore, we have*

$$\|u(t)\|_\infty \leqslant C(p,|I|)t^{-(1/\lambda)}\|u_0\|_{L^1(I)}^{p/\lambda}, \quad \forall t>0, \text{ with } \lambda = 2(p-1). \tag{3.52}$$

Besides, for any $\tau>0$, u satisfies the following gradient estimate

$$|u_x(x,t)|^p \leqslant C(\beta,p)u^{1-\beta}(x,t)\left(\tau^{-1}\|u(\tau)\|_\infty^{1+\beta} + \Theta(f,\|u(\tau)\|_\infty)\|u(\tau)\|_\infty^{\beta} + Lip(f)\|u(\tau)\|_\infty^{1+\beta}+1\right), \tag{3.53}$$

for a.e $(x,t)\in I\times(\tau,\infty)$, where $Lip(f)$ is the global Lipschitz constant of f.

Theorem 3.3. *Let $0 \leqslant u_0 \in L^1(I)$. Suppose that f is continuous and nondecreasing on $[0,\infty)$, and $f(0)=0$. Then, equation (1.1) possesses a maximal weak solution u satisfying the universal bound (3.52). Moreover, for any $\tau>0$, we have*

$$|u_x(x,t)|^p \leqslant C(\beta,p)u^{1-\beta}(x,t)\left(\tau^{-1}\|u(\tau)\|_\infty^{1+\beta} + f(2\|u(\tau)\|_\infty)\|u(\tau)\|_\infty^{\beta}+1\right), \tag{3.54}$$

for a.e $(x,t)\in I\times(\tau,\infty)$.

Proof. The proof of theorems 3.2 and 3.3 is just a combination of the one of theorem 1.2 and the L^1-framework argument in [**9**] (see also [**11**] for the semi-linear case). Then, we leave the details to the reader. □

Remark 3.4. We emphasize that our existence results also hold for a class of $\mathcal{C}^1$-functions $f(u,x,t)$ such that $f(0,x,t)=0$, $\forall(x,t)\in I\times(0,\infty)$, see [**10**].

Next, we give the proof of theorem 1.6 and corollary 1.7.

Proof of theorem 1.6. It is sufficient to show that the quenching result holds for the maximal solution u.

Indeed, let v be the maximal solution of the equation:

$$\begin{cases} \partial_t v - (|v_x|^{p-2}v_x)_x + v^{-\beta}\chi_{\{v>0\}} = 0 & \text{in } I\times(0,\infty),\\ v(-l,t)=v(l,t)=0 & t\in(0,\infty),\\ v(x,0)=u_0(x) & \text{in } I. \end{cases} \tag{3.55}$$

Thanks to the result of theorem 13, [**9**], there is a finite time $T_0 = T_0(\beta,p,\|u_0\|)$ such that

$$v(x,t)=0, \quad \forall (x,t)\in I\times(T_0,\infty).$$

It follows from the construction of u and v that

$$u\leqslant v, \quad \forall(x,t)\in I\times(0,\infty).$$

Thus, we get the conclusion. □

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Proof of corollary 1.7. If $f(0) = 0$, then the existence result follows from theorem 1.2 above.

Next, assume that problem (1.1) possesses a weak solution w. Thanks to theorem 1.6, there is a finite time T_0 such that

$$w(x,t) = 0, \quad \text{for } x \in I,\ t > T_0.$$

Thus, it follows from problem (1.1) that $f(0) = 0$. □

4. The Cauchy problem

4.1. Existence of a weak solution

We first give the proof of theorem 1.8.

Proof. Let u_r be the maximal solution of the following equation

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } I_r \times (0,\infty), \\ u(-r,t) = u(r,t) = 0, & \forall t \in (0,\infty), \\ u(x,0) = u_0(x), & \text{in } I_r, \end{cases} \tag{4.56}$$

see theorem 1.2. It is clear that $\{u_r\}_{r>0}$ is a nondecreasing sequence. Moreover, the strong comparison principle deduces

$$u_r(x,t) \leqslant \|u_0\|_{L^\infty(\mathbb{R})}, \quad \text{for } \ (x,t) \in I_r \times (0,\infty). \tag{4.57}$$

Thus, there exists a function u such that $u_r \uparrow u$ as $r \to \infty$. We will show that u is a solution of problem (1.7).

First, the energy estimate provides us

$$\begin{cases} \|u_r(.,t)\|_{L^1(I_r)} \leqslant \|u_0\|_{L^1(\mathbb{R})}, \quad \text{for any } \ t \in (0,\infty), \\ \\ \|f(u_r)\|_{L^1(I_r\times(0,\infty))},\ \|u_r^{-\beta}\chi_{\{u_r>0\}}\|_{L^1(I_r\times(0,\infty))} \leqslant \|u_0\|_{L^1(\mathbb{R})}. \end{cases} \tag{4.58}$$

It follows immediately from the Monotone Convergence Theorem that $u_r(t)$ converges to $u(t)$ in $L^1(\mathbb{R})$, and $f(u_r)$ converges to $f(u)$ in $L^1(\mathbb{R}\times(0,\infty))$ as $r \to \infty$, likewise

$$\begin{cases} \|u(.,t)\|_{L^1(\mathbb{R})} \leqslant \|u_0\|_{L^1(\mathbb{R})}, \quad \text{for any } \ t \in (0,\infty), \\ \\ \|f(u)\|_{L^1(\mathbb{R}\times(0,\infty))} \leqslant \|u_0\|_{L^1(\mathbb{R})}. \end{cases} \tag{4.59}$$

Furthermore, u_r satisfies the gradient estimate

$$\begin{aligned} |\partial_x u_r(x,t)|^p \leqslant C u_r^{1-\beta}(x,t) \big(t^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f, \|u_0\|_\infty)\|u_0\|_\infty^\beta \\ + Lip(f,u_0)\|u_0\|_\infty^{1+\beta} + 1\big), \end{aligned} \tag{4.60}$$

for a.e $(x,t) \in I_r \times (0,\infty)$. By the same argument as in the proof of theorem 1.2, there is a subsequence of $\{u_r\}_{r>0}$ (still denoted as $\{u_r\}_{r>0}$) such that $\partial_x u_r \overset{r\to\infty}{\longrightarrow} \partial_x u$, for a.e $(x,t) \in \mathbb{R}\times(0,\infty)$.

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By (4.60), we obtain

$$|u_x(x,t)|^p \leqslant Cu^{1-\beta}(x,t)\left(t^{-1}\|u_0\|_\infty^{1+\beta} + \Theta(f,\|u_0\|_\infty)\|u_0\|_\infty^\beta + Lip(f,u_0)\|u_0\|_\infty^{1+\beta} + 1\right), \tag{4.61}$$

for a.e $(x,t) \in \mathbb{R} \times (0,\infty)$, and

$$\partial_x u_r \overset{r\to\infty}{\longrightarrow} u_x, \quad \text{in } L^q_{loc}(\mathbb{R}\times(0,\infty)), \quad \forall q \geqslant 1. \tag{4.62}$$

Now, we show that u satisfies equation (1.7) in the sense of distribution. Indeed, using the test function $\psi_\eta(u_r)\phi$ for the equation satisfied by u_r gives us

$$\int_{Supp(\phi)} \left(-\Psi_\eta(u_r)\phi_t + |\partial_x u_r|^{p-2}\partial_x u_r \phi_x \psi_\eta(u_r) + \frac{1}{\eta}|\partial_x u_r|^{p-2}\partial_x u_r \psi'\left(\frac{u_r}{\eta}\right)\phi \right.$$
$$\left. + u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi + f(u_r)\psi_\eta(u_r)\phi\right)\, dsdx = 0, \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty)).$$

We first take care of the term $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi$ in passing $r \to \infty$ and $\eta \to 0$. It is not difficult to see that $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r) = u_r^{-\beta}\psi_\eta(u_r)$ is bounded by $\eta^{-\beta}$. Then for any $\eta > 0$, the Dominated Convergence Theorem yields $u_r^{-\beta}\psi_\eta(u_r) \overset{r\to\infty}{\longrightarrow} u^{-\beta}\psi_\eta(u)$ in $L^1_{loc}(\mathbb{R}\times(0,\infty))$.

By (4.58), we obtain

$$\|u^{-\beta}\psi_\eta(u)\|_{L^1(\mathbb{R}\times(0,\infty))} \leqslant \|u_0\|_{L^1(\mathbb{R})}.$$

Next, using the Monotone Convergence Theorem deduces $u^{-\beta}\psi_\eta(u) \uparrow u^{-\beta}\chi_{\{u>0\}}$ in $L^1(\mathbb{R}\times(0,\infty))$, as $\eta \to 0$, thereby proves

$$\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(\mathbb{R}\times(0,\infty))} \leqslant \|u_0\|_{L^1(\mathbb{R})}. \tag{4.63}$$

Thanks to (4.62), (4.58) and (4.57), there is no problem of passing to the limit as $r \to \infty$ in the indicated variational equation in order to get

$$\int_{Supp(\phi)} \left(-\Psi_\eta(u)\phi_t + |u_x|^{p-2}u_x\phi_x\psi_\eta(u) + \frac{1}{\eta}|u_x|^{p-2}u_x\psi'\left(\frac{u}{\eta}\right)\phi \right.$$
$$\left. + u^{-\beta}\psi_\eta(u)\phi + f(u)\psi_\eta(u)\phi\right)\, \mathrm{d}s\mathrm{d}x = 0, \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty)).$$

By (4.59), (4.61), and (4.63), we can proceed similarly as the proof of (3.45)–(3.48) to obtain after letting $\eta \to 0$

$$\int_{Supp(\phi)} \left(-u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right)\, \mathrm{d}x\mathrm{d}s = 0,$$
$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty)). \tag{4.64}$$

Or u satisfies equation (1.1) in the sense of distributions.

Then, it remains to prove that $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$. Let us first claim that

$$u \in \mathcal{C}([0,\infty); L^1_{loc}(\mathbb{R})). \tag{4.65}$$

In order to prove (4.65), we use a compactness result by Porretta, [**28**]. We present it here for the reader convenience.

Lemma 4.1 (Theorem 1.1, [**28**]). *Let $p>1$ and p' its conjugate exponent $(1/p+1/p'=1)$, $a,b \in \mathbb{R}$, and define the space*

$$V_1^p(a,b) = \{u : \Omega \times (a,b) \to \mathbb{R}; \quad u \in L^p(a,b; W_0^{1,p}(\Omega)),$$
$$u_t \in L^{p'}(a,b; W^{-1,p'}(\Omega)) + L^1(\Omega \times (a,b))\},$$

where Ω is a bounded set in $\mathbb{R}^N$. Then, we have

$$V_1^p(a,b) \subset \mathcal{C}([a,b]; L^1(\Omega)).$$

For any $r>0$, we extend u_r by 0 outside I_r, still denoted as u_r. Use u_r as a test function to the equation satisfied by u_r to get

$$\int_0^T \int_{\mathbb{R}} |\partial_x u_r|^p \mathrm{d}x\mathrm{d}s \leqslant \frac{1}{2}\int_{I_r} u_0^2(x)\mathrm{d}x \leqslant \frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}\|u_0\|_{L^\infty(\mathbb{R})}, \quad \text{for } T>0,$$

which implies $\|u_x\|^p_{L^p(\mathbb{R}\times(0,T))}$ is also bounded by $1/2\|u_0\|_{L^1(\mathbb{R})}\|u_0\|_{L^\infty(\mathbb{R})}$, or $u_x \in L^p(\mathbb{R}\times(0,T))$. By (4.59) and the boundedness of u, it follows from the Interpolation Theorem that
$u \in L^p(\mathbb{R}\times(0,T))$, for any $T>0$. Thus, we have

$$u \in L^p(0,T; W^{1,p}(\mathbb{R})). \tag{4.66}$$

According to (4.66), (4.59) and (4.63), we get from the equation satisfied by u

$$u_t \in L^{p'}(a,b; W^{-1,p'}(\mathbb{R})) + L^1(\mathbb{R}\times(0,T)). \tag{4.67}$$

Then, a local application of lemma 4.1 yields the claim (4.65).

Note that the last conclusion does not imply $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$ since the proof of theorem 1.1, [**28**] depends on the boundedness of Ω.

To prove $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$, it suffices to show that $u(t)$ is continuous at $t=0$ in $L^1(\mathbb{R})$, that is: $\lim_{t\to 0}\|u(t)-u_0\|_{L^1(\mathbb{R})} \to 0$.

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In fact, we have that, for any $m \geqslant 1$

$$\begin{aligned}
\int_{\mathbb{R}} |u(x,t)-u_0(x)|\mathrm{d}x &\leqslant \int_{I_m} |u(x,t)-u_0(x)|\mathrm{d}x + \int_{\mathbb{R}\setminus I_m} |u(x,t)-u_0(x)|\mathrm{d}x \\
&\leqslant \int_{I_m} |u(x,t)-u_0(x)|\mathrm{d}x + \int_{\mathbb{R}\setminus I_m} u(x,t)\mathrm{d}x + \int_{\mathbb{R}\setminus I_m} u_0(x)\mathrm{d}x \\
&= \int_{I_m} |u(x,t)-u_0(x)|\mathrm{d}x - \left(\int_{I_m} (u(x,t)-u_0(x))\mathrm{d}x\right) \\
&\quad + \int_{\mathbb{R}} u(x,t)\mathrm{d}x - \int_{I_m} u_0(x)\mathrm{d}x + \int_{\mathbb{R}\setminus I_m} u_0(x)\mathrm{d}x \\
&\leqslant 2\int_{I_m} |u(x,t)-u_0(x)|\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\mathrm{d}x \\
&\quad - \int_{I_m} u_0(x)\mathrm{d}x + \int_{\mathbb{R}\setminus I_m} u_0(x)\mathrm{d}x \\
&= 2\int_{I_m} |u(x,t)-u_0(x)|\mathrm{d}x + 2\int_{\mathbb{R}\setminus I_m} u_0(x)\mathrm{d}x.
\end{aligned}$$

Taking $\limsup\limits_{t\to 0}$ in both sides of the above inequality we get

$$\limsup_{t\to 0}\int_{\mathbb{R}} |u(x,t)-u_0(x)|\mathrm{d}x \leqslant 2\limsup_{t\to 0}\int_{I_m} |u(x,t)-u_0(x)|\mathrm{d}x + 2\int_{\mathbb{R}\setminus I_m} u_0(x)\mathrm{d}x.$$

By $u \in \mathcal{C}([0,\infty); L^1_{loc}(\mathbb{R}))$, we obtain from the last inequality

$$\limsup_{t\to 0}\int_{\mathbb{R}} |u(x,t)-u_0(x)|\mathrm{d}x \leqslant 2\int_{\mathbb{R}\setminus I_m} u_0(x)\mathrm{d}x.$$

Then the result follows as $m \to \infty$. Then, we get the proof of theorem 1.8. □

Remark 4.2. The existence result also holds for f a nondecreasing function on $[0,\infty)$.

4.2. Instantaneous shrinking of support of solutions

Now, we prove theorem 1.9.

Proof. Let u be a solution of problem (1.7). Since $f(u) \geqslant 0$, we have for any $q \in (0,1)$

$$f(u) + u^{-\beta}\chi_{\{u>0\}} \geqslant c_0 u^q,$$

with $c_0 = ((1)/(\|u_0\|^{\beta+q}_{L^\infty(\mathbb{R})}))$. This implies that u is a sub-solution of the following problem:

$$\begin{cases} \partial_t w - (|w_x|^{p-2}w_x)_x + c_0 w^q = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ w(x,0) = u_0(x), & \text{in } \mathbb{R}, \end{cases} \tag{4.68}$$

Note that problem (4.68) has a unique solution w (see, e.g, [**21**, **33**]). Thus, the comparison principle implies

$$u(x,t) \leqslant w(x,t), \quad \text{in } \mathbb{R}\times(0,\infty).$$

Thanks to the result of Herrero [**21**], w has an instantaneous shrinking of compact support, so does u.

Thus, we obtain the conclusion. □

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