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Expansive dynamics on locally compact groups

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Abstract. Let $\mathcal{G}$ be a second countable, Hausdorff topological group. If $\mathcal{G}$ is locally compact, totally disconnected and T is an expansive automorphism then it is shown that the dynamical system $(\mathcal{G}, T)$ is topologically conjugate to the product of a symbolic full-shift on a finite number of symbols, a totally wandering, countable-state Markov shift and a permutation of a countable coset space of $\mathcal{G}$ that fixes the defining subgroup. In particular if the automorphism is transitive then $\mathcal{G}$ is compact and $(\mathcal{G}, T)$ is topologically conjugate to a full-shift on a finite number of symbols.

Key words: automorphisms, locally compact groups, symbolic dynamics
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1. *Background and motivation*

In [**K1**] it was shown that an expansive automorphism of a compact, totally disconnected group is topologically conjugate to a full-shift on a finite number of symbols cross an automorphism of a finite group. The method of proof is to use the fact that a compact, totally disconnected group has arbitrarily small compact-open normal subgroups to code the system, $(\mathcal{G}, T)$, to a finite-state Markov shift. The Markov shift is topologically conjugate to $(\mathcal{G}, T)$, the alphabet is a finite group and the Markov shift has a group structure where the operation is defined using a symbol by symbol group operation. Then using the group structure of the Markov shift and symbolic dynamics techniques the Markov shift is, through a sequence of reductions, reduced to a topologically conjugate symbolic system which is a full-shift cross an automorphism of a finite group.

Since the time of the above-mentioned result, countable-state Markov shifts have been increasingly used in dynamics. This is the particularly true in the study of maps that exhibit some (but not uniform) hyperbolicity, maps of the interval and in symbolic dynamics itself. Likewise, the structure of locally compact, totally disconnected groups and their automorphisms has received increased attention. See, for example, [**CM**, **CH**].

In the present paper it is shown that an expansive automorphism of a locally compact, second countable, Hausdorff group is topologically conjugate to the product of a symbolic full-shift on a finite number of symbols, a totally wandering, countable-state Markov shift (in some cases) and a permutation of a countable coset space that fixes the defining subgroup. The strategy used in the case where the group is compact is followed but with several modifications. The first problem is that a locally compact, totally disconnected group has arbitrarily small compact-open subgroups but they need not be normal. This means that if $\mathcal{G}$ is locally compact, totally disconnected and T is an expansive automorphism then it is possible to code $(\mathcal{G}, T)$ to a countable-state Markov shift. The Markov shift is topologically conjugate to $(\mathcal{G}, T)$, the alphabet is a coset space of $\mathcal{G}$ but there is no easily described group structure on the countable-state Markov shift. Consequently, the constructions leading to the reductions used must be done in the group $\mathcal{G}$ itself rather than on the symbols of the Markov shift. Another problem is that there is wandering behavior that can occur in non-compact groups that cannot occur in compact groups. This leads to the introduction of totally wandering countable-state Markov shifts.

Following the main result there are remarks on the possible entropies of the transformations and several examples are provided.

The paper is organized as follows. Section 2 contains some notation from symbolic dynamics. Section 3 contains all the results and examples. First it is shown how to code an expansive automorphism of a locally compact, second countable, Hausdorff group to a countable-state Markov shift where each state has a finite number of predecessors and successors. Using the countable-state Markov shift presentation several important subsystems can be identified. Next the two fundamental constructions are explained. Then the constructions are applied to the countable-state Markov shift and the subsystems are used to obtain the desired result. The proof consists of a series of step-by-step reductions applied to countable-state Markov shifts. The reductions can best be understood by applying the same type of reductions to the examples supplied at the end of the paper.

2. *Symbolic dynamics*

Let $\mathcal{A}$ denote a finite or countably infinite set with the discrete topology and $\mathcal{A}^{\mathbb{Z}}$ denote the set of all two-sided infinite sequences on $\mathcal{A}$ with the product topology. Let σ be the *shift homeomorphism* of $\mathcal{A}^{\mathbb{Z}}$ to itself defined by $\sigma(x)_i = x_{i+1}$. A closed, shift-invariant subset, X, of $\mathcal{A}^{\mathbb{Z}}$ is a *subshift*. If each of the elements of $\mathcal{A}$ occurs in some point of X, we say that $\mathcal{A}$ is the *alphabet of* X. For a subshift X let $\mathcal{W}(X, n)$ denote the words of length n that can occur in X, that is, $w \in \mathcal{W}(X, n)$ if and only if $w = [x_0, \ldots, x_{n-1}]$ for some $x \in X$. Then let $\mathcal{W}(X)$ denote the union over all n of the $\mathcal{W}(X, n)$. For a word $w \in \mathcal{W}(X)$ define the *follower set of w, f(w)*, to be the symbols $a \in \mathcal{A}$ such that $wa \in \mathcal{W}(X)$. Define the *predecessor set of w, p(w)*, in a similar manner. The proofs that follow depend on understanding the sets $f(w)$ and $p(w)$. If $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a subshift the standard n-block presentation of X has alphabet $\mathcal{W}(X, n)$ and transitions by overlapping. That is, $[x_1, \ldots, x_n] \in f([x_0, \ldots, x_{n-1}]$ if and only if $[x_0, \ldots, x_n] \in \mathcal{W}(X, n+1)$. The shift transformation is defined as before. We will be dealing with finite or countably infinite state Markov shifts in the discussions that follow and the reader is referred to [**K2**] for further background.

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3. *The groups*

The groups will be Hausdorff, second countable, locally compact and totally disconnected (equivalently zero-dimensional in this setting). Let $\mathcal{G}$ be such a group and T be an automorphism of $\mathcal{G}$. T is expansive if there exists an open set $\mathcal{U}$ containing the identity element e and for each pair $g, h \in \mathcal{G}$, $g \neq h$, there is an $n \in \mathbb{Z}$ so that $T^n(g) \notin T^n(h)\mathcal{U}$. We say that such a set $\mathcal{U}$ *separates points*. We use van Dantzig's theorem, which states that in this setting any open set containing the identity element contains a compact-open subgroup $\mathcal{H}$. If the group is compact or meets various other conditions the subgroup can be taken to be normal. If the subgroup is normal the proofs of the following results are simplified and are somewhat more intuitive.

We will sometimes be working in the group $\mathcal{G}$ where we will use script letters and sometimes in a symbolic space where we will use Roman letters to denote the spaces, subsets, subgroups, etc. We need to be able to move back and forth between the two. This is illustrated in the first two observations.

OBSERVATION 3.1. *Let $\mathcal{G}$ be a second countable, Hausdorff topological group. If $\mathcal{G}$ is locally compact, totally disconnected and T is an expansive automorphism then $(\mathcal{G}, T)$ is topologically conjugate to a countable-state Markov shift $(\Sigma_{G/H}, \sigma)$. The symbols of the Markov shift correspond to cosets of a compact-open subgroup $\mathcal{H}$ of $\mathcal{G}$.*

Proof. T is expansive if we can find an open $\mathcal{U}$ that separates points as stated in the definition of expansive. Then by van Dantzig's theorem we can find a compact-open subgroup, $\mathcal{H}$, contained in $\mathcal{U}$ that also separates points. The partition of $\mathcal{G}$ into cosets of $\mathcal{H}$ is countable because $\mathcal{G}$ is second countable. In the standard way we code $(\mathcal{G}, T)$ into $((G/H)^{\mathbb{Z}}, \sigma)$. We will use gH as a symbol in the alphabet of $(G/H)^{\mathbb{Z}}$ corresponding to the coset $g\mathcal{H}$ in $\mathcal{G}$. This means that $g \in \mathcal{G}$ is coded to $x \in (G/H)^{\mathbb{Z}}$ by saying that $x_i = g_i H$ when $T^i(g) \in g_i\mathcal{H}$. The coding is one-to-one because $\mathcal{H}$ separates points. Let $X_{G/H}$ denote the image of $\mathcal{G}$ in $(G/H)^{\mathbb{Z}}$ and $\pi : \mathcal{G} \to X_{G/H}$ denote the coding map. So in this notation $\pi(g\mathcal{H}) = gH$. The set $X_{G/H}$ is a subshift of $(G/H)^{\mathbb{Z}}$ by construction.

Next we show that we can recode $X_{G/H}$ to obtain a countable-state Markov shift. Using the notation introduced previously we consider the sets of words $\mathcal{W}(X_{G/H}, n)$ and $\mathcal{W}(X_{G/H})$ and focus on the follower sets, $f(w)$, and predecessor sets, $p(w)$. Observe that $f(H) = \{gH : T(\mathcal{H}) \cap g\mathcal{H} \neq \phi\}$ while $p(H) = \{gH : g\mathcal{H} \cap T^{-1}(\mathcal{H}) \neq \phi\}$. This allows us to think of $f(H)$ defining a partition of $\mathcal{H}$ into cosets of $\mathcal{H} \cap T^{-1}(\mathcal{H})$ and $p(H)$ defining a partition of $\mathcal{H}$ into cosets of $T(\mathcal{H}) \cap \mathcal{H}$. Next observe that this picture carries over to all cosets of $\mathcal{H}$. We can think of $f(gH)$ partitioning $g\mathcal{H}$ into cosets of $\mathcal{H} \cap T^{-1}(\mathcal{H})$ and $p(gH)$ partitioning $g\mathcal{H}$ into cosets of $T(\mathcal{H}) \cap \mathcal{H}$. This means that the cardinalities of all $f(gH)$ are the same and the cardinalities of all the $p(gH)$ are the same.

Let H^n denote the word of all H's in $\mathcal{W}(X_{G/H}, n)$ corresponding to the subgroup $\mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-n+1}(\mathcal{H})$ in $\mathcal{G}$. Then we can think of $f(H^n)$ partitioning $\mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-n+1}(\mathcal{H})$ into cosets of $\mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-n}(\mathcal{H})$ and $p(H^n)$ partitioning $\mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-n+1}(\mathcal{H})$ into cosets of $T(\mathcal{H}) \cap \mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-n}(\mathcal{H})$. As for $\mathcal{H}$ and $g\mathcal{H}$ this partitioning translates to a partitioning of each $g_0\mathcal{H} \cap T^{-1}(g_1\mathcal{H}) \cap \cdots \cap T^{-n+1}(g_{n-1}\mathcal{H})$ by cosets of $\mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-n}(\mathcal{H})$

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indexed by $f(w)$, where w is the word corresponding to $g_0\mathcal{H} \cap T^{-1}(g_1\mathcal{H}) \cap \cdots \cap T^{-n+1}(g_{n-1}\mathcal{H})$. The same picture carries over to $p(H^n)$. So the cardinalities of the $f(w)$ is the same for all $w \in \mathcal{W}(X_{G/H}, n)$ and the same holds for the cardinalities of the $p(w)$. Since $\mathcal{G}$ is locally compact there is an N so that this cardinality is finite for all $n \geq N$.

Since $f(H^n)$ and $p(H^n)$ are finite for all $n \geq N$ and the cardinalities can only decrease as n increases, there is an M so that $f(H^{M+k}) = f(H^M)$ and $p(H^{M+k}) = p(H^M)$ for all $k \geq 0$. We have observed that the cardinalities of the $f(w)$'s and $p(w)$'s are the same for all $w \in \mathcal{W}(X_{g/H}, n)$, which means that $X_{G/H}$ is a Markov shift of memory at most M. Construct the standard M-block presentation of $X_{G/H}$, as previously described, to produce a countable-state Markov shift $(\Sigma_{G/H}, \sigma)$ topologically conjugate to $(\mathcal{G}, T)$. Passing to the standard M-block presentation of $X_{G/H}$ is just choosing the compact-open subgroup for the coding to be $\mathcal{H} \cap T^{-1}(\mathcal{H}) \cap \cdots \cap T^{-M+1}(\mathcal{H})$ instead of $\mathcal{H}$. So without loss of generality we may assume that $X_{G/H} = \Sigma_{G/H}$, where $\Sigma_{G/H}$ is a one-step countable-state Markov shift. □

The image of the coding map using the coset partition of $\mathcal{G}$ is the one-step Markov shift $\Sigma_{G/H}$. This means that the coset partition of $\mathcal{G}$ is a Markov partition of $\mathcal{G}$.

If $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$ then $\Sigma_{G/H}$ is a group with the group operation defined with a symbol by symbol operation and the arguments can be made by looking only at $\Sigma_{G/H}$ and ignoring $\mathcal{G}$. Moreover, if $\mathcal{G}$ is compact $\mathcal{H}$ can be chosen to be a normal subgroup, the coset partition will be finite and $\Sigma_{G/H}$ will be a subshift of finite type with the group operation defined symbol by symbol. This is the case examined in [**K2**].

If $\mathcal{H}$ is not a normal subgroup of $\mathcal{G}$ then $\Sigma_{G/H}$ is still a group but the group operation is complicated and is defined by pulling back to $\mathcal{G}$. The arguments that follow will depend on looking at $\mathcal{G}$ and $\mathcal{H}$.

In $\Sigma_{G/H}$ each symbol has the same number of followers, which is the cardinality of $f(H)$. Likewise, each symbol has the same number of predecessors, which is the cardinality of $p(H)$. This says that the cardinality of $f(H)$ is an eigenvalue for the transition matrix of $\Sigma_{G/H}$ with the column vector of all 1's an eigenvector. Likewise, the cardinality of $p(H)$ is an eigenvalue for the transition matrix with the row vector of all 1's an eigenvector. When $\mathcal{G}$ is compact the cardinalities of $f(H)$ and $p(H)$ must agree. If the transition matrix is countably infinite the cardinalities may be different as illustrated by Example 3.20.

OBSERVATION 3.2. *When $\Sigma_{G/H}$ is constructed as above there is a $\mathcal{G}$-action on $\Sigma_{G/H}$, where every $\mathcal{G}$-orbit is all of $\Sigma_{G/H}$.*

Proof. For each $g \in \mathcal{G}$ define the itinerary $i(g) \in \mathcal{G}^{\mathbb{Z}}$ by $i(g)_j = T^j(g)$ for each $j \in \mathbb{Z}$. Then $i(g)$ acts on a point in $\Sigma_{G/H}$ coordinate by coordinate using left multiplication on corresponding cosets in $\mathcal{G}$. It is clear that the $\mathcal{G}$-orbit of the point of all $H's$ is all of $\Sigma_{G/H}$. It follows that the $\mathcal{G}$-orbit of any point is all of $\Sigma_{G/H}$. □

Let $\Sigma_{G/H}$ be a countable-state Markov shift as constructed in Observation 3.1 with the $\mathcal{G}$-action as described in Observation 3.2. Using the transition graph of $\Sigma_{G/H}$ we describe two countable-state Markov shifts contained in $\Sigma_{G/H}$.

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Definition 3.3. We define the *irreducible component of the identity*. Define

$$\mathcal{J}_P = \{g \in \mathcal{G} : g \text{ is periodic and } T^n(g) \in \mathcal{H} \text{ for some } n \in \mathbb{Z}\}.$$

$\mathcal{J}_P$ is a T-invariant subset of $\mathcal{G}$. Let $\mathcal{J}_{IR}$ be the smallest closed subgroup of $\mathcal{G}$ containing $\mathcal{J}_P$. Then define the irreducible component of the identity, $\Sigma_{IR} \subseteq \Sigma_{G/H}$, to be $\pi(\mathcal{J}_{IR})$.

LEMMA 3.4. *The irreducible component of the identity, Σ_{IR}, is a countable-state Markov shift whose transition graph is the largest strongly connected subgraph (any vertex can be reached by a directed path beginning at any other vertex) of the transition graph for $\Sigma_{G/H}$ containing H.*

Proof. This is clear. □

In Corollary 3.16 there is an exact description of the dynamics of the possible Σ_{IR}.

Definition 3.5. We define the *accessible component of the identity*. Define

$$\mathcal{J}_H = \{g \in \mathcal{G} : T^n(g) \in \mathcal{H}, \text{ some } n \in \mathbb{Z}\}.$$

$\mathcal{J}_H$ is an open, T-invariant subset of $\mathcal{G}$. It is open because $\mathcal{J}_H = \bigcup_{n\in\mathbb{Z}} T^{-n}(\mathcal{H})$. It is clearly T-invariant. Define $\Sigma_H = \pi(\mathcal{J}_H)$. We know that Σ_H is a countable-state Markov shift because the transition graph can easily be written down. This is illustrated in Example 3.21. Let $\mathcal{J}_{AC}$ be the smallest closed subgroup of $\mathcal{G}$ containing $\mathcal{J}_H$. Then define the accessible component of the identity Σ_{AC} to be $\pi(\mathcal{J}_{AC})$. This is again a countable-state Markov shift because the transition graph can be written down. In Example 3.21 $\Sigma_{AC} = \mathbb{Q}_3 \oplus \mathbb{Q}_3$.

LEMMA 3.6. *$\mathcal{J}_{AC}$ is an open–closed, T-invariant subgroup of $\mathcal{G}$.*

Proof. By definition $\mathcal{J}_{AC}$ is a closed, T-invariant subgroup of $\mathcal{G}$. We need only prove that it is open.

Note that $\mathcal{H} \subseteq \mathcal{J}_{AC}$ and that both T and T^{-1} restricted to $\mathcal{J}_{AC}$ map $\mathcal{J}_{AC}$ onto $\mathcal{H}$. It follows that if $g \in \mathcal{J}_{AC}$ then $g\mathcal{H} \subseteq \mathcal{J}_{AC}$, and both T and T^{-1} restricted to $\mathcal{J}_{AC}$ map $\mathcal{J}_{AC}$ onto $g\mathcal{H}$. So $\mathcal{J}_{AC}$ is a union of of open–closed cylinder sets and is open. □

The irreducible component of the identity can be defined by taking its transition graph to be the largest strongly connected subgraph of the transition graph of $\Sigma_{G/H}$ containing H. The next lemma shows that the accessible component of the identity could have been defined in a similar manner.

LEMMA 3.7. *The transition graph for the accessible component of the identity is the largest (undirected) connected subgraph of the transition graph of $\Sigma_{G/H}$ containing H.*

Proof. The largest connected subgraph of the transition graph of $\Sigma_{G/H}$ containing H consists of all vertices and edges of the transition graph for $\Sigma_{G/H}$ that can be connected by a path to H, ignoring the way the edges are directed. Observe that Σ_{AC} being open in $\Sigma_{G/H}$ is equivalent to having no edge in the transition graph for $\Sigma_{G/H}$ connecting a vertex in the transition graph for Σ_{AC} to an outside vertex of the transition graph for $\Sigma_{G/H}$. □

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The transition graphs for the irreducible component of the identity and the accessible component of the identity can be quite different. In each of Examples 3.20 and 3.21 the transition graph for the irreducible component of the identity consists of H and the single edge from H to itself while the transition graph for the accessible component of the identity consists of the entire graph.

Definition 3.8. A countable-state Markov shift is said to be *totally wandering* if its cardinality is uncountably infinite, it has finitely many periodic points and every non-periodic point is wandering.

Remark 3.9.. The definition implies that if a countable-state Markov shift is totally wandering then its alphabet is infinite and the Markov shift is non-compact. Examples 3.20 and 3.21 are totally wandering Markov shifts.

LEMMA 3.10. *The structures of the inverse images of the follower and predecessor sets of H are*

$$\pi^{-1}(f(H)) = T(\mathcal{H})\mathcal{H} = \mathcal{H}(T(\mathcal{H})),$$
$$\pi^{-1}(p(H)) = T^{-1}(\mathcal{H})\mathcal{H} = \mathcal{H}(T^{-1}(\mathcal{H})).$$

Proof. Since the coset partition of $\mathcal{G}$ is a Markov partition we see that the inverse images of the follower and predecessor can be written as

$$\pi^{-1}(f(H)) = \bigcup_{\{g\mathcal{H}:T(\mathcal{H})\cap g\mathcal{H}\neq\phi\}} g\mathcal{H} = \bigcup_{\{g(T(\mathcal{H})):g(T(\mathcal{H}))\cap g\mathcal{H}\neq\phi\}} g(T(\mathcal{H})),$$
$$\pi^{-1}(p(H)) = \bigcup_{\{g\mathcal{H}:g\mathcal{H}\cap T^{-1}(\mathcal{H})\neq\phi\}} g\mathcal{H} = \bigcup_{\{g(T^{-1}(\mathcal{H})):g\mathcal{H}\cap g(T^{-1}(\mathcal{H}))\neq\phi\}} g(T^{-1}(\mathcal{H})).$$

Rewriting these equations in terms of subgroups $\mathcal{H}$, $T(\mathcal{H})$ and $T^{-1}(\mathcal{H})$ produces the desired result. □

LEMMA 3.11. *The inverse image of the follower set of H, $\pi^{-1}(f(H))$, and the inverse image of the predecessor set of H, $\pi^{-1}(p(H))$, are compact-open subgroups of $\mathcal{G}$.*

Proof. Let $g_1, g_2 \in \pi^{-1}(f(H))$. There exist $h_1, h_2 \in \mathcal{H}$ such that $g_1h_1, g_2h_2 \in T(\mathcal{H})$ as seen in the proof of Lemma 3.10. Then $(g_1h_1)^{-1}g_2h_2 = h_1^{-1}g_1^{-1}g_2h_2 \in T(\mathcal{H})$. By Lemma 3.10 multiplying this on the left and right by elements of $\mathcal{H}$ produces an element of $\pi^{-1}(f(H))$. We conclude that $g_1^{-1}g_2 \in \pi^{-1}(f(H))$ and that $\pi^{-1}(f(H))$ is a subgroup. It is compact-open because it is a finite union of compact-open cosets.

A similar argument shows that $\pi^{-1}(p(H))$ is also a compact-open subgroup. □

Definition 3.12. Denote by $\mathcal{K}$ the non-empty compact-open subgroup of $\mathcal{G}$ defined by $\mathcal{K} = \pi^{-1}(p(H)) \cap \pi^{-1}(f(H))$.

Note that $\mathcal{H}$ is a subgroup of $\mathcal{K}$, the cosets of $\mathcal{H}$ partition $\mathcal{K}$ and all symbols in $K = \pi(\mathcal{K})$ have the same follower and the same predecessor sets. This is also true for each coset of $\mathcal{K}$.

Next we describe two constructions that will be used in the proof of Theorem 3.15.

Construction 3.13. The case when the cardinality of K is greater than one.

For notational purposes, let $\mathcal{H}_0 = \mathcal{H}$, $H_0 = H$, $\mathcal{H}_1 = \mathcal{K}$ and $H_1 = K$.

Consider two new systems which are Σ_{G/H_1} and $(H_1/H_0)^{\mathbb{Z}}$. The space Σ_{G/H_1} (when not countable) has all the properties of Σ_{G/H_0} including a $\mathcal{G}$-action defined on it and once again the orbit of any point under the action is the entire Markov shift. The space $(H_1/H_0)^{\mathbb{Z}}$ has a $H_1^{\mathbb{Z}}$-action on it when we use coordinate by coordinate left multiplication of the corresponding cosets of $\mathcal{H}_0$ in $\mathcal{H}_1$. The point is that the systems Σ_{G/H_0} and $\Sigma_{G/H_1} \times (H_1/H_0)^{\mathbb{Z}}$ are topologically conjugate. The conjugacy is defined on the symbol level. It is clear because there is the algebraic map $\mathcal{G}/\mathcal{H}_0 \to \mathcal{G}/\mathcal{H}_1$ with 'kernel' $\mathcal{H}_1/\mathcal{H}_0$. If $\mathcal{H}_0$ and $\mathcal{H}_1$ are normal subgroups of $\mathcal{G}$ this is just saying that $\mathcal{G}/\mathcal{H}_0$ is an extension of $\mathcal{H}_1/\mathcal{H}_0$ by $\mathcal{G}/\mathcal{H}_1$, so every element $\mathcal{G}/\mathcal{H}_0$ can be written as a pair of elements with the first from $\mathcal{H}_1/\mathcal{H}_0$ and the second from $\mathcal{G}/\mathcal{H}_1$. We have reduced the Markov shift Σ_{G/H_0} to a product of a new Markov shift cross a full-shift on a finite number of symbols. In the new Markov shift the cardinalities of the predecessor and follower sets are strictly smaller than in the old one. The new cardinalities of the predecessor and follower sets are the old cardinalities (which were finite) divided by the cardinality of H_1. Likewise, the cardinality of the alphabet of Σ_{G/H_1} is the cardinality of the alphabet of Σ_{G/H_0} divided by the cardinality of H_1. If the alphabet of Σ_{G/H_0} was infinite the new alphabet is infinite.

Construction 3.14. The case when the cardinality of K is one.

By Lemma 3.11 $\pi^{-1}(p(H))$ and $\pi^{-1}(f(H))$ are subgroups of $\mathcal{G}$ containing $\mathcal{H}$. The important point is that when the cardinality of K is one every element of $f(H)$ has a distinct follower set and every element of $p(H)$ has a distinct predecessor set. To see this assume that two elements of $f(H)$ have the same follower set. Using the fact that $\pi^{-1}(f(H))$ is a subgroup we can assume that one of them is H and the other is gH. But then we have both H and gH in K. The same reasoning applies to the predecessor sets. In turn each element of any follower set, $f(gH)$, has a unique follower set and each element of any predecessor set, $p(gH)$, has a unique predecessor set.

Consider the new Markov shifts $\Sigma_{G/f(H)}$ and $\Sigma_{G/p(H)}$. The alphabets are the cosets of $\pi^{-1}(f(H))$ and $\pi^{-1}(p(H))$, respectively. Each is a factor of $\Sigma_{G/H}$ by a map defined on symbols. Moreover, each factor map is invertible. The map onto $\Sigma_{G/f(H)}$ is invertible by a two-block map that looks at the present symbol and one symbol into the future. This is well defined because each element of a fixed follower set has a distinct follower set. Likewise, the map onto $\Sigma_{G/p(H)}$ is invertible by a two-block map that looks at the present symbol and one symbol into the past. Each of the new Markov shifts (if not countable) has all of the properties of $\Sigma_{G/H}$ including a $\mathcal{G}$-action defined on it and once again the orbit of any point under the action is the entire Markov shift. The number of distinct follower sets in $\Sigma_{G/f(H)}$ is the number of distinct follower sets in $\Sigma_{G/H}$ divided by the cardinality of $f(H)$ and the number of distinct predecessor sets in $\Sigma_{G/p(H)}$ is the number of distinct predecessor sets in $\Sigma_{G/H}$ divided by the cardinality of $p(H)$. The cardinality of the alphabet of $\Sigma_{G/f(H)}$ is the cardinality of the alphabet of $\Sigma_{G/H}$ divided by the cardinality of $f(H)$ and the cardinality of the alphabet of $\Sigma_{G/p(H)}$ is the cardinality of the alphabet of $\Sigma_{G/H}$ divided by the cardinality of $p(H)$. If the alphabet of $\Sigma_{G/H}$ was infinite the new alphabets are infinite. Also observe that whether we use follower or predecessor sets to make the reduction the cardinalities of the new predecessor and follower sets, in both cases, are unchanged. They neither increase nor decrease. For notational purposes we again set $\mathcal{H}_1 = \pi^{-1}(f(H))$ and $H_1 = f(H)$ or $\mathcal{H}_1 = \pi^{-1}(p(H))$ and $H_1 = p(H)$ depending on which was used for the reduction.

The two constructions can be best understood by applying them to Example 3.19. First apply Construction 3.13 and then Construction 3.14 to obtain $(\Sigma_2, \sigma) \times (\Sigma_2, \sigma)$, which is topologically conjugate to (Σ_4, σ).

THEOREM 3.15. *Let $\mathcal{G}$ be a second countable, Hausdorff topological group. If $\mathcal{G}$ is locally compact, totally disconnected and T is an expansive automorphism then $(\mathcal{G}, T)$ is topologically conjugate to either:*

(1) $(\Sigma_N, \sigma) \times (C, \tau)$*; or*

(2) $(\Sigma_N, \sigma) \times (\Sigma_W, \sigma) \times (C, \tau)$*,*

where:

- (Σ_N, σ) *is a full-shift on a finite alphabet;*
- (Σ_W, σ) *is a totally wandering countable-state Markov shift;*
- (C, τ) *with C a countable coset space of $\mathcal{G}$ and τ a permutation that fixes the 'identity' element.*

Proof. We begin with the dynamical system $(\mathcal{G}, T)$. We use Observations 3.1 and 3.2 to produce a countable-state Markov shift $(\Sigma_{G/H}, \sigma)$ with a $\mathcal{G}$-action topologically conjugate to $(\mathcal{G}, T)$. Using the notation from the construction we consider the subgroup $\mathcal{K}$ defined by Definition 3.12. If the cardinality of K is greater than one call it H_1 and apply Construction 3.13. This produces $\Sigma_{G/H_1} \times (H_1/H_0)^{\mathbb{Z}}$ topologically conjugate to Σ_{G/H_0}. The Markov shift Σ_{G/H_1} has a $\mathcal{G}$-action and $(H_1/H_0)^{\mathbb{Z}}$ has a H_1-action. We think of factoring out a full-shift on a finite number of symbols. We can ignore the full-shift and concentrate on Σ_{G/H_1}. The set K in Σ_{G/H_1} has cardinality one. Consider the cardinalities of $p(H_1)$ and $f(H_1)$ in Σ_{G/H_1}. If one of the cardinalities is larger than one we can apply Construction 3.14 using either $p(H_1)$ or $f(H_1)$ if both are larger than one. Then continue to apply Constructions 3.13 or 3.14 as appropriate. Notice that since the cardinalities of $p(H)$ and $f(H)$ were finite, Construction 3.13 can be used only a finite number of times.

There are two possible outcomes.

(a) Construction 3.13 has been applied and in the resulting Markov shift the cardinalities of $p(H_\ell)$ and $f(H_\ell)$ are one. When the original alphabet is finite this is the only possible outcome.

(b) Construction 3.13 has been applied and in the resulting Markov shift the cardinalities of $p(H_\ell)$ and $f(H_\ell)$ are not both one but the cardinality of K is one and no further applications of Construction 3.14 will produce a Markov shift where the cardinality of K is greater than one. This is the case for Examples 3.20 and 3.21.

If outcome (a) occurs the original countable-state Markov shift satisfies result (1) of the theorem.

If outcome (b) occurs consider the accessible component of the identity, Σ_{AC}, in Σ_{G/H_ℓ}. We will show that Σ_{AC} is totally wandering.

First observe that either the cardinality of $p(H_\ell)$ or of $f(H_\ell)$ is greater than one and so the cardinality of Σ_{AC} is uncountably infinite.

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Next we see that the fixed point of all H_ℓ's is the only periodic point in Σ_{AC}. Let $f^1(H_\ell) = f(H_\ell)$, $f^2(H_\ell) = f(f(H_\ell))$ and so forth for all $f^k(H_\ell)$. Then $f^k(H_\ell) \subseteq f^{k+1}(H_\ell)$ for all k. Use the same notation for $p^k(H_\ell)$. Thinking of $f^k(H_\ell)$ as a collection of time-0 cylinder sets in Σ_{G/H_ℓ} we see that $\pi^{-1}(f^k(H_\ell))$ and $\pi^{-1}(p^k(H_\ell))$ are open–closed subgroups of $\mathcal{G}$ and each is the union of cosets of $\mathcal{H}_\ell$. Then we observe that $\mathcal{G}_{AC}$ is equal to the union of

$$\bigcup_{k\geq 1} \pi^{-1}(f^k(H_\ell)) \quad \text{and} \quad \bigcup_{k\geq 1} \pi^{-1}(p^k(H_\ell)).$$

When Construction 3.14 is applied using followers Σ_{G/H_ℓ} changes to $\Sigma_{G/f(H_\ell)}$ and the alphabet changes from 'cosets' of H_ℓ to 'cosets' of $f(H_\ell)$. The new coding map remains a topological conjugacy. The same follows if Construction 3.14 is applied using predecessors. If there were a periodic point other than the fixed point in Σ_{AC} repeated iterations of Construction 3.14 would eventually make the periodic orbit coded the same way as the fixed point. That is a contradiction.

Finally, we see that every point except the fixed point is wandering. Observe that if $x \in \Sigma_{AC}$ with $x_0 = H_\ell$ and $x_1 \neq H_\ell$ then $x_i \neq H_\ell$ for all $i \geq 1$. This follows because there are no periodic points except the fixed point. The same is true for $i \leq -1$ if $x_0 = H_\ell$ and $x_{-1} \neq H_\ell$. Suppose that $x \in \Sigma_{AC}$ with $x_0 = H_\ell$ and $x_1 \neq H_\ell$ the cylinder set of length two with H_ℓ at time 0 and x_1 at time 1 is an open set containing x that is wandering and so x is a wandering point. The same would show that x is a wandering point if $x_0 = H_\ell$ and $x_{-1} \neq H_\ell$. Expanding the reasoning shows that any point with H_ℓ at time 0 (except the fixed point) is wandering. Examining any other time-0 cylinder set, $gH \neq H$, and using the fact that there are no periodic points in gH lets us use the same reasoning to see that every point in gH is wandering.

When outcome (b) occurs we use the fact that $\mathcal{J}_{AC}$ is an open–closed subgroup of $\mathcal{G}/\mathcal{H}_\ell$ to divide out by it and see that the original countable-state Markov shift satisfies result (2) of the theorem. □

COROLLARY 3.16. *Let $\mathcal{G}$ be a second countable, Hausdorff topological group. If $\mathcal{G}$ is locally compact, totally disconnected and T is an expansive automorphism then the irreducible component of the identity in $(\mathcal{G}, T)$ is topologically conjugate to a full-shift on a finite alphabet.*

COROLLARY 3.17. *Let $\mathcal{G}$ be a second countable, Hausdorff topological group. If $\mathcal{G}$ is locally compact, totally disconnected and T is an expansive and transitive automorphism then $(\mathcal{G}, T)$ is topologically conjugate to a full-shift on a finite alphabet.*

Remark 3.18.. It is natural to try to formulate the notion of topological entropy for an expansive automorphism, T, of a second countable, locally compact, totally disconnected, Hausdorff topological group $\mathcal{G}$. In view of Theorem 3.15 we can consider the three building blocks of such an automorphism. The topological entropy of a full-shift on N symbols is $\log N$. The topological entropy of a permutation on a countable set can safely be said to be zero. The problem is with the totally wandering countable-state Markov shift. In general, there are many reasonable definitions for the topological entropy of a homeomorphism

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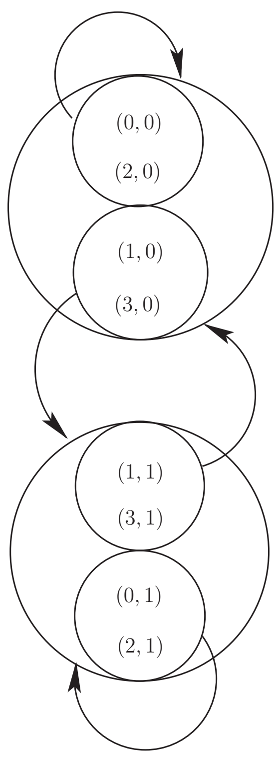
of a non-compact space. They all have difficulties and almost all can be shown to differ in specific cases. A discussion of some of the definitions and examples can be found in [**HKR**]. When the transition matrix for a countable-state Markov shift is irreducible things are somewhat better but there are still several problems. This is discussed in [**K2**]. If we consider the Markov shift defined in Example 3.21 there are several problems. On the one hand, there is only one invariant probability measure, which is the Dirac delta measure supported at the identity. It has zero entropy, so thinking in terms of the variational principle it is reasonable to declare the topological entropy of the Markov shift zero. On the other hand, the growth rate of the number of blocks occurring in the Markov shift that begin (or end) with any fixed symbol is 3^n, so thinking in terms of the (n, ϵ)-separated set definition of topological entropy it is reasonable to declare the topological entropy log 3.

Example 3.19. Define $\Sigma_A \subseteq (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ by the transition graph below.

This means that

$$f((0,0)) = f((2,0)) = f((1,1)) = f((3,1)) = \{(0,0), (2,0)(1,0), (3,0)\},$$
$$f((1,0)) = f((3,0)) = f((0,1)) = f((2,1)) = \{(0,1), (2,1)(1,1), (3,1)\}.$$

In this case the finite-state Markov shift is a compact group where the group operation is coordinate by coordinate addition (with no carry). The subgroup $\mathcal{K}$ is $\{(0,0), (2,0)\}$. The Markov shift Σ_A is topologically conjugate to the full-shift on four symbols.



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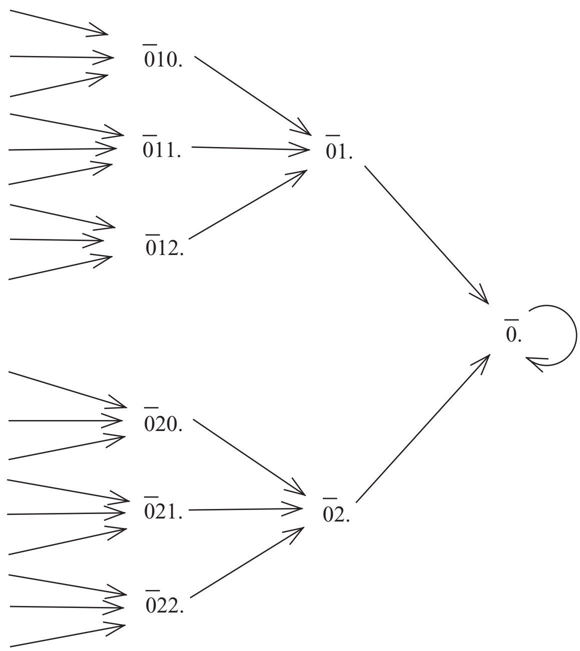
Example 3.20. Consider the 3-adic numbers, $\mathbb{Q}_3$, expressed as the subgroup of $(\mathbb{Z}/3\mathbb{Z})^{\mathbb{Z}}$ consisting of the sequences that are eventually all zeros to the left. The group operation is coordinate by coordinate addition in $\mathbb{Z}/3\mathbb{Z}$ with a carry to the right. The automorphism, σ^{-1}, is multiplication by 3, which is the inverse of the usual shift on the sequences. If we choose for $\mathcal{H}$ the subgroup of sequences that are zero for all coordinates to the left of the time-0 entry, we produce the countable-state Markov shift defined by the transition graph



Here $\bar{0}$ means zeros from that coordinate to the left. Note that the transition graph for the irreducible component of the identity consists of the single vertex $\bar{0}$. and the edge going from it to itself while the transition graph for the accessible component of the identity is the entire graph.

Example 3.21. Consider $\mathbb{Q}_3 \oplus \mathbb{Q}_3$, the direct sum of two copies of the 3-adic numbers. Express the entries in each copy of $\mathbb{Q}_3$ as in Example 3.20. The automorphism is multiplication by 1/3 in the first copy of $\mathbb{Q}_3$ and multiplication by 3 in the second copy. Take $\mathcal{H}$ to be the subgroup $\{(\bar{0}., \bar{0}.)\}$ in the previous notation. This gives the countable-state Markov shift defined by the transition graph

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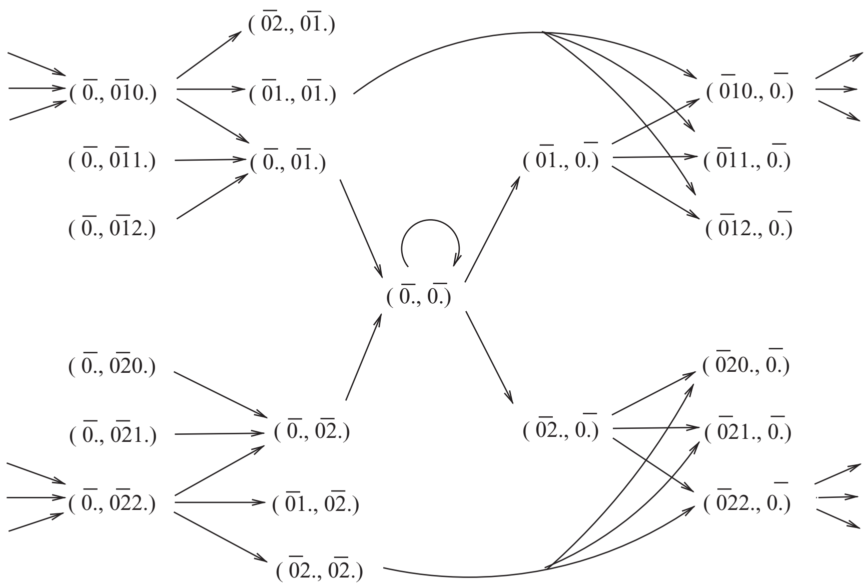

Once again the transition graph for the irreducible component of the identity consists of the single vertex $(\bar{0}., \bar{0}.)$ and the edge going from it to itself while the transition graph for the accessible component of the identity is the entire graph. This example illustrates the $\mathcal{J}_H$ used in the construction of $\mathcal{J}_{AC}$ for the accessible component of the identity.

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