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ON THE INTEGRAL POINTS ON THE COMPLEMENT OF RAMIFICATION-DIVISORS

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Abstract Following a recent paper by Faltings, we study the integral points on $\boldsymbol{P}_2 \setminus \mathcal{D}$, where $\mathcal{D}$ is the branch locus of a projection from a surface $\mathcal{X}$; a crucial point in the analysis is that the pull-back of $\mathcal{D}$ in the Galois closure of the projection often splits into several components. As in the paper by Faltings, under certain assumptions we obtain finiteness of the integral points (Theorem 3.1); for instance, we shall find that it suffices if the projection is sufficiently general and if $\mathcal{X}$ has Kodaira number $\geqslant 0$ (Corollary 4.1). We have borrowed freely from Faltings's paper, for the whole geometrical setting. As to the arithmetic, our method is in part different, relying on the recent paper by Corvaja and Zannier and leading to apparently new conditions. We shall also use a more elementary approach to study a similar situation in arbitrary dimension, where the projection is taken from a hypersurface (Theorem 2.1).

In concrete terms, these results deal with certain diophantine equations $F(x_0, \dots, x_n) = c$.

Keywords: diophantine equations in many variables; varieties over global fields; diophantine geometry

AMS 2000 *Mathematics subject classification:* Primary 11D72; 11G35; 11G99

1. Introduction

Let $V \subset \boldsymbol{P}_n$ be an affine variety and let $\mathcal{D}$ be its divisor at infinity, i.e. the sum of the components of $\tilde{V} \setminus V$, where $\tilde{V}$ is the closure of V in $\boldsymbol{P}_n$. Many theorems on integral points apply to V only if $\mathcal{D}$ splits into several components; we may recall for instance Thue's theorem on curves defined by equations like $X^d + aY^d = c$, $d \geqslant 3$, $ac \neq 0$ (now there are $d \geqslant 3$ points at infinity), theorems by Schmidt on norm-form equations $N^k_{\boldsymbol{Q}}(L(\boldsymbol{X})) = c$ (now the conjugates of the linear form $L \in k[\boldsymbol{X}]$ define $\mathcal{D}$), theorems of Laurent on the integral points on subvarieties of $\boldsymbol{G}_m^n$ (observe that $\boldsymbol{G}_m^n$ has $2n$ components at infinity in its embedding into $\boldsymbol{P}_1^n$) and certain theorems of Vojta which require that $|\mathcal{D}|$ has $\geqslant r + \dim(V) + 1$ components, where r is the rank of the Néron–Severi group of $\tilde{V}$ (see, for example, [**16**], [**17**] and [**19**, Chapter III] for precise statements and for references); still other examples occur in the recent papers [**5**, **6**].

To our knowledge, only few results on integral points are known without such type of hypotheses. An example is provided by the deep theorem of Faltings on abelian varieties [**9**, Corollary 6.2 to Theorem 2]:

If $\mathcal{D}$ is an ample divisor in an abelian variety A, then $A\backslash\mathcal{D}$ has only finitely many integral points.

(Vojta [**18**] has then extended this to semiabelian varieties.) Other instances may occur when there is a finite cover $\tilde{W}$ of the variety $\tilde{V}$, unramified except possibly above points in $\mathcal{D}$, such that the pull-back $\mathcal{D}^*$ of $\mathcal{D}$ has more components than $\mathcal{D}$ itself. In fact, by a known principle going back to Hermite and Chevalley–Weil (see, for example, [**16**]) the hypothesis about ramification ensures that the integral points on V lift to (quasi-) integral points on $W := \tilde{W} \setminus \mathcal{D}^*$ (all defined over a fixed, though possibly larger, number field) so one may work replacing V with W, a variety which is more likely to satisfy some of the assumptions alluded to above. To show a known instance of this, let V be an affine curve; Siegel's celebrated theorem now asserts the finiteness of integral points if either $\operatorname{genus}(\tilde{V}) > 0$ or $\deg \mathcal{D} \geqslant 3$. Now, it suffices to prove the theorem with this last assumption, because if the genus is positive one may take an unramified cover of degree $\geqslant 3$ and apply to it the special case (see, for example, [**7**]).

When $\dim V > 1$ the required splitting of the pull-back of $\mathcal{D}$ does not generally occur. Exceptions may be constructed starting from a finite unramified cover $\tilde{W} \to \tilde{V}$ of complete varieties and defining $\mathcal{D}$ as the image of a divisor on $\tilde{W}$ (see [**5**, Example 1.4] for an example on abelian varieties); this is however artificial in a way (and of course does not work if $\tilde{V}$ is simply connected). Recently, Faltings [**10**] has shown new instances, when $V = \boldsymbol{P}_2 \setminus \mathcal{D}$ is the complement in $\boldsymbol{P}_2$ of an irreducible divisor $\mathcal{D}$ which is the branch locus of a suitable projection from a smooth surface $\mathcal{X}$. Under certain assumptions he proves the finiteness of S-integral points on V by working on the Galois closure of the cover $\mathcal{X} \to \boldsymbol{P}_2$; if the cover has degree n, then $\mathcal{D}$ splits into $n(n-1)/2$ components in the Galois closure.

The present paper just offers alternative approaches to these principles. First (see § 2) we shall study a simple situation where the projection is taken from a hypersurface $\mathcal{X}$ (in any dimension). Now the branch divisor $\mathcal{D}$ will be defined by a discriminant form; the familiar factorization of the discriminant over the Galois closure (as a product of root differences) makes then evident the alluded splitting and also allows a method which is different and more elementary than Faltings's one. In fact, we shall apply known results on $\boldsymbol{G}_m^N$ to confine the integral points of V on subvarieties of small dimension. Moreover, this method leads to effective conclusions via Baker's theory.

However, this is certainly not sufficient to recover the full Faltings's results. In fact, as we shall point out in Remark 3.1, even if $\mathcal{X}$ admits some embeddings as a hypersurface the assumption that the mentioned projection factors through some such embedding is quite strong. Faltings himself points out that his results cannot always be directly obtained by embedding the relevant varieties into $\boldsymbol{G}_m^N$ or even semiabelian varieties; examples in this sense arise, for example, when $\mathcal{X} = \boldsymbol{P}_1 \times \boldsymbol{P}_1$ and the projection is obtained from three dimensional spaces of sections of a sheaf $\mathcal{O}(a, b)$ with a, b coprime integers $\geqslant 3$ (see [**10**, p. 246]).

We shall work then in the general cases studied by Faltings by applying the Main Theorem of the paper [**5**] (see § 3 below). We shall obtain sufficient conditions for finiteness of the integral points on $\boldsymbol{P}_2 \setminus \mathcal{D}$ *a priori* different from Faltings's ones; we do not know the precise implications about the two types of assumptions. Certainly our application enables us to prove finiteness when $\mathcal{X}$ has non-negative Kodaira number and to recover the results of [**10**] in the case $\mathcal{X} = \boldsymbol{P}_1 \times \boldsymbol{P}_1$ (see § 4 for these examples). This last example, because of the just mentioned observation of Faltings, also shows implicitly that the Main Theorem of [**5**] is not a direct consequence of the known ones about semiabelian varieties.

To conclude, we note that (by Proposition 2.3 below) these results in concrete terms assert the finiteness of the integer solutions to equations $F(x_0, x_1, x_2) = c$, for certain homogeneous irreducible forms F (i.e. those which represent the divisors $\mathcal{D}$ which may occur).

In the sequel, k will denote a number field and S a finite set of places of k, including the infinite ones. We adopt the usual notations $\mathcal{O}_S$ (respectively, $\mathcal{O}_S^*$) for the S-integers (respectively, S-units) of k. For the notion of a set of (quasi-) S-integral points, we refer to the definitions in [**16**], [**17**] or [**19**].

2. The case of projection from hypersurfaces

As announced above, in this section we shall study a simple case of Faltings's principle, on considering projections from hypersurfaces. This special setting will require just the theory of S-unit equations in three (homogeneous) variables; also, it will allow us to work in any dimension and moreover with effectiveness. A somewhat similar method has been sometimes used in special cases since long ago, for instance to deal with diophantine equations $4X^3 + 27Y^2 = c$, in which the left side is a discriminant (see, for example, [**4**]).

Let $\mathcal{X}$ be a hypersurface of degree $d > 1$ in $\boldsymbol{P}_{n+1}$, defined over k; also, let $Q \in \boldsymbol{P}_{n+1}(k)$, Q not in $\mathcal{X}$, and let H be a hyperplane defined over k, not containing Q; we consider the projection of $\mathcal{X}$ from Q to $H \cong \boldsymbol{P}_n$.

We define $\mathcal{D} \subset \boldsymbol{P}_n$ to be the branch locus of this projection and $T \subset \mathcal{D}$ as the set of points of $\mathcal{D}$ which are totally ramified in $\mathcal{X}$; that is, $P \in \mathcal{D}$ (respectively, $P \in T$) if and only if there are $\leqslant d - 1$ (respectively, $\leqslant 1$) distinct points of $\mathcal{X}$ on the line joining P and Q.

We have the following theorem.

Theorem 2.1. *Notation being as above, the Zariski closure of any set of quasi-S-integral points for $\boldsymbol{P}_n \setminus \mathcal{D}$ has dimension $\leqslant \dim T + 1$.*

Remark 2.2. We shall point out during the proof that, once Q, H are given, for a 'general' $\mathcal{X}$, we have $\dim T \leqslant \max(-1, n + 1 - d)$ (where $\dim(\emptyset) = -1$); by 'general' we mean that $\mathcal{X}$ is defined by an equation of degree d whose coefficient-vector lies outside a certain algebraic set. We note that the present notion of 'general' depends only on Q, H; on the contrary the exceptional variety containing Σ may depend on k, S and on an affine model for $\boldsymbol{P}_n \setminus \mathcal{D}$ (see for this Proposition 2.3).

Hence for general $\mathcal{X}$ the result confines the integral points on $\boldsymbol{P}_n \setminus \mathcal{D}$ on a subvariety of dimension $\leqslant \max(0, n + 2 - d)$. For $d = 2$ the result gives nothing, but on the other hand it is easily seen that now the problem amounts to quadratic diophantine equations $\Delta(x_0, \ldots, x_n) = c$, which may well have a Zariski dense set of integer solutions.

We also remark that it may be proved (e.g. by taking special equations for $\mathcal{X}$) that for a 'general' $\mathcal{X}$ the divisor $\mathcal{D}$, which has degree $d(d-1)$, will be irreducible. In view of the above introductory comments, this shows that the alluded 'method of covers' is now efficient.

We start with a simple fact, which describes the S-integral points on $\boldsymbol{P}_n \setminus \mathcal{D}$ by explicit diophantine equations. In practice this amounts to producing some affine embedding for $\boldsymbol{P}_2 \setminus \mathcal{D}$.

Proposition 2.3. *Let $\mathcal{D}$ be an effective divisor on $\boldsymbol{P}_n$, defined by a form $\Delta \in k[X_0, \dots, X_n]$. Let Σ be a set of quasi-S-integral points for the affine variety $\boldsymbol{P}_n \setminus \mathcal{D}$. Then there exists a finite set of places $S' \supset S$ of k such that each point of Σ has projective coordinates $(x_0 : \dots : x_n)$ with $x_i \in \mathcal{O}_{S'}$ and $\Delta(x_0, \dots, x_n) \in \mathcal{O}_{S'}^*$.*

Remark 2.4. Naturally the conditions $x_i \in \mathcal{O}_S$, $\Delta(x_0, \dots, x_n) \in \mathcal{O}_S^*$, may be in turn translated into a finite number of equations $\Delta(y_0, \dots, y_n) = c$, $c \in \mathcal{O}_S^*$, with unknowns $y_i \in \mathcal{O}_S$. (One may let c run through representatives for $\mathcal{O}_S^*$ modulo $\deg \Delta$-powers, which works because Δ is homogeneous and $\mathcal{O}_S^*$ is finitely generated.)

We do not know any explicit reference to such a (undoubtedly folklore) proposition (see however [**3**, Conjecture 4.2]) so we give a short proof of it. Let $\delta := \deg \Delta$; we may plainly assume that Δ has coefficients in $\mathcal{O}_S$. Observe that the rational functions $Q_i := X_i^\delta / \Delta(X_0, \dots, X_n)$, $i = 0, \dots, n$, are regular on $\boldsymbol{P}_n \setminus \mathcal{D}$. Therefore, (by definition of *quasi-S-integral set*) there exists a non-zero $c \in \mathcal{O}_S$ such that the values $cQ_i(P)$ are in $\mathcal{O}_S$ for all $P \in \Sigma$. By finiteness of class-number we may enlarge S to a finite set S' such that $\mathcal{O}_{S'}$ is a unique factorization domain. Then we may write $P = (x_0 : \dots : x_n)$, where the (projective) coordinates $x_i = x_i(P)$ are coprime S'-integers. Since $\Delta(x_0, \dots, x_n)$ divides cx_i^δ in $\mathcal{O}_{S'}$ for $i = 0, \dots, n$ we then conclude that $\Delta(x_0, \dots, x_n)$ divides in fact c in $\mathcal{O}_{S'}$. Enlarging further S' to assume that $c \in \mathcal{O}_{S'}^*$ we get the conclusion.

Proof of Theorem 2.1. We may assume that $Q = (0 : \dots : 0 : 1)$ and that H is defined by $X_{n+1} = 0$. Then, if $f(X_0, \dots, X_n, X_{n+1}) = 0$ is an equation of degree d defining $\mathcal{X}$, we may assume that f is monic with respect to X_{n+1} and that the projection consists of taking the first $n+1$ coordinates. We may write

$$f(X_0, \dots, X_n, X_{n+1}) = X_{n+1}^d + f_1(X_0, \dots, X_n) X_{n+1}^{d-1} + \dots + f_d(X_0, \dots, X_n),$$

where f_b is a homogeneous polynomial of degree b. We may normalize this equation by writing $X_{n+1} - (1/d) f_1$ in place of X_{n+1}, which amounts to the assumption $f_1 = 0$; this leaves $\mathcal{D}$ and T unchanged, so does not affect the result. Enlarging S we may also assume that f has coefficients in $\mathcal{O}_S$.

Note that T is defined in $\boldsymbol{P}_n$ by $f_2 = \dots = f_d = 0$. By well-known results this variety has dimension $\leqslant \max(-1, n+1-d)$ unless the coefficients of the f_i, $i = 2, \dots, d$, satisfy certain non-trivial algebraic relations, which depend only on n and the degree d. (Of course, the number of independent such relations can be bounded below more explicitly.)

Let $\Delta = \Delta(X_0, \dots, X_n)$ be the discriminant of f with respect to X_{n+1}, a homogeneous polynomial of degree $d(d-1)$. Then the branch locus $\mathcal{D}$ in $\boldsymbol{P}_n$ is defined by $\Delta = 0$.

Let Σ be a set of quasi-S-integral points for $\boldsymbol{P}_n \setminus \mathcal{D}$ and let us enlarge S as in Proposition 2.3. For each point $P \in \Sigma$ let us choose accordingly projective coordinates $P = (x_0 : \cdots : x_n)$ for P so that we may assume that the x_i lie in $\mathcal{O}_S$ and that $\Delta(x_0, \ldots, x_n) \in \mathcal{O}_S^*$. Let $\alpha_1, \ldots, \alpha_d$ be the (distinct) roots of $f(x_0, \ldots, x_n, X) = 0$ in $\bar{\boldsymbol{Q}}$. They generate over k a number field, depending on P, which is unramified except at places above S. Since the degree is bounded, a well-known result by Hermite implies that there are only finitely many number fields with this property (see [**16**, p. 49]), so all the roots α_i lie in a certain number field k' which is independent of $P \in \Sigma$; enlarging k we may assume that $k' = k$. Note that since we are assuming that f has coefficients in $\mathcal{O}_S$, the α_i are integral over $\mathcal{O}_S$ and since they lie in k, they lie in fact in $\mathcal{O}_S$. Now, we have the familiar identity

$$\Delta(x_0, \ldots, x_n) = \prod_{1 \leqslant i < j \leqslant d} (\alpha_i - \alpha_j)^2.$$

Since the left side lies in $\mathcal{O}_S^*$, we see that each difference $\beta_{ij} := \alpha_i - \alpha_j$, $i \neq j$, lies in $\mathcal{O}_S^*$ as well. Note the identities

$$\beta_{ij} + \beta_{jl} + \beta_{li} = 0. \tag{2.1}$$

Now, by a well-known result going back to Siegel and Mahler (see [**16**, 8.3] and [**19**]) the equation $x+y+z = 0$ has only finitely many non-proportional solutions $(x, y, z) \in (\mathcal{O}_S^*)^3$. Applying this to (2.1) with $\{i, j, l\} = \{1, 2, l\}$ and then with $\{i, j, l\} = \{1, j, l\}$ we see that β_{1l}/β_{12} and β_{jl}/β_{1l} take values in a certain finite set independent of P. Putting $\gamma := \beta_{12}$ we easily deduce equations

$$\alpha_j - \alpha_l = c_{jl}\gamma, \quad c_{jl} = c_{jl}(P), \quad \gamma = \gamma(P), \quad 1 \leqslant j \neq l \leqslant d,$$

where the c_{jl} lie in a finite set independent of P and where γ (which may depend on P) lies in $\mathcal{O}_S^*$. (See Remark 2.5 (A) for effectiveness of this point.)

We may now split Σ into finitely many subsets so that the c_{jl} are fixed for P in a fixed subset; arguing separately with each subset we may then assume that the c_{jl} are independent of P.

From the last displayed equations we obtain, on setting $\alpha = \alpha_1$, $c_j := c_{1j}$ for $j \geqslant 2$ and $c_1 = 0$,

$$\alpha_j = \alpha + c_j\gamma, \quad j = 1, \ldots, d.$$

Since we are assuming $f_1 = 0$ we get

$$d\alpha + \left(\sum_{j=1}^{d} c_j\right)\gamma = 0,$$

hence $\alpha = c\gamma$ for a certain fixed c (that is, c depends only on the subset we are working with, not on the individual points P). Recalling that $f_b(x_0, \ldots, x_n)$ is the bth symmetric function of the α_j, we have

$$f_b(x_0, \ldots, x_n) = l_b\gamma^b, \quad b = 2, \ldots, d,$$

where $l_2, \ldots, l_d$ do not depend on P.*

* Similar conclusions naturally hold for 'discriminantal' equations $\Delta = c$, even if the f_i are not homogeneous.

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Now, consider the variety W defined in $\boldsymbol{P}_{n+1}$ by the equations $f_b(X_0,\dots,X_n)=l_bY^b$, $b=2,\dots,d$. Note that $\dim W\leqslant\dim T+1$, for otherwise intersecting W with the hyperplane $Y=0$ would produce a component of T of dimension $>\dim T$.

Note finally that by the last displayed equations Σ lies in the projection of W to $\boldsymbol{P}_n$, whose dimension is also $\leqslant\dim W\leqslant\dim T+1$. This completes the proof. □

Remark 2.5.

(A) As mentioned earlier, Baker's theory gives an effective procedure for actually finding a variety as in the statement, containing Σ, once $\mathcal{X}$ is given. In fact, a finite full set of non-proportional representatives for solutions of the S-unit equation (2.1) may be effectively found, as outlined, for example, in [**1**, Chapter 4]. Naturally this set depends on k and S, as well as on an explicit notion of 'quasi-S-integral'; note in fact that S has to be enlarged in the application of Proposition 2.3. Equivalently, one can work with 'fully' integral points, after specifying an affine model for $\boldsymbol{P}_2\setminus\mathcal{D}$.

(B) When $n=d+1$ and $\mathcal{X}$ is 'general', the result confines Σ in a curve; then Siegel's theorem may be possibly applied to show that Σ is actually finite.

(C) A similar method works even if the hypersurface $\mathcal{X}$ has a non-monic homogeneous equation

$$f:=f_0Y^d+f_1Y^{d-1}+\cdots+f_d=0,\quad f_i\in k[X_0,\dots,X_n],$$

and we again project on the first $n+1$ coordinates. We briefly describe the main differences with the above proof.

Again we may look at the integral points for $\boldsymbol{P}_n\setminus\mathcal{D}$ where $\mathcal{D}$ is now defined by the discriminant

$$\Delta:=f_0^{2d-2}\prod_{1\leqslant i<j\leqslant d}(\alpha_i-\alpha_j)^2,$$

the α_i being the roots, as above. As in the above proof, we may assume $\alpha_i\in k$ for the integral points in question, and write $\alpha_i=\mu_i/\delta_i$, with μ_i, δ_i coprime elements in $\mathcal{O}_S^*$. Then $f(\boldsymbol{x},Y)$ is divisible in $\mathcal{O}_S[Y]$ by $\prod(\delta_iY-\mu_i)$, whence $\delta_1\cdots\delta_d$ divides $f_0(\boldsymbol{x})$ in $\mathcal{O}_S$. We readily deduce that $\Delta(\boldsymbol{x})$ is divisible in $\mathcal{O}_S$ by $\prod_{i\neq j}(\delta_j\mu_i-\delta_i\mu_j)$, whence all the factors are in $\mathcal{O}_S^*$ (for the integral points $\boldsymbol{x}$ in question).

Put now $x_{ij}:=\delta_j\mu_i-\delta_i\mu_j$; these numbers all lie in $\mathcal{O}_S^*$ and they satisfy the identities

$$x_{1i}x_{2j}-x_{1j}x_{2i}+x_{12}x_{ij}=0,$$

to which we may apply the S-unit theorem. As a corollary the ratios $x_{1i}x_{2j}/x_{1j}x_{2i}$ have finitely many possibilities. On the other hand these ratios equal certain cross-ratios among the roots:

$$\frac{x_{1i}x_{2j}}{x_{1j}x_{2i}}=\frac{(\alpha_1-\alpha_i)(\alpha_2-\alpha_j)}{(\alpha_2-\alpha_i)(\alpha_1-\alpha_j)}.$$

Thus, again we find non-trivial fixed algebraic relations among the roots; as before, they confine the integral points in question in a suitable subvariety.

3. Projection from more general surfaces

In this section we shall closely follow Faltings's paper [**10**] for the geometric part. For the arithmetic, we shall use the Main Theorem of [**5**] in place of Faltings's method, which follows the approach of Faltings and Wüstholz [**11**]. Certainly there are analogies underlying the methods; for instance, [**5**] applies the Schmidt subspace theorem in a way which reminds us of the paper by Evertse and Ferretti [**8**], which in turn quantifies [**11**]. However, we do not know the precise extent of this link; for instance, the results below require assumptions which are *a priori* in part different from Faltings's.

For the reader's convenience we start by briefly recalling the context and the points of the paper [**10**] which we shall mainly need.

As in [**10**], we let $\mathcal{X}$ be a projective smooth geometrically irreducible algebraic surface, defined over $\bar{\boldsymbol{Q}}$, denoting by K its canonical class and by c_2 its Euler characteristic. We also let $\mathcal{L}$ be a very ample line bundle on $\mathcal{X}$, satisfying the ampleness conditions of [**10**, p. 234]. (In practice, it is required that the global sections separate points up to order 3 included, pairs of points up to order 2 and triples of points up to order 1.) We also assume that $K \otimes \mathcal{L}^{\otimes 3}$ is ample.

We then consider three-dimensional subspaces $E \subset \Gamma(\mathcal{X}, \mathcal{L})$ of global sections, giving a map $f = f_E : \mathcal{X} \to \boldsymbol{P}(E) = \boldsymbol{P}_2$, which will be regular for E in an open subset of the corresponding Grassmannian; since $\mathcal{L}$ is very ample, f_E will correspond to a regular projection and will then be finite. We assume at once that

$$n := \mathcal{L} \cdot \mathcal{L} = \text{degree of } f_E > 8. \tag{3.1}$$

Further, we define $Z \subset \mathcal{X}$ as the discriminant locus of f_E and $\mathcal{D} := f_E(Z) \subset \boldsymbol{P}(E)$ as the branch locus in $\boldsymbol{P}_2$. It is observed [**10**, pp. 236, 240] that Z corresponds to a section of $K + 3\mathcal{L}$ (which is a kind of Hurwitz formula; see, for example, [**15**, Proposition 6.19]). Hence Z is ample. Similarly to § 2, we are interested in the integral points for $V := \boldsymbol{P}_2 \setminus \mathcal{D}$.

Remark 3.1. It is natural to ask whether such integral points may be investigated with the simple method of § 2. For this, we should realize $\mathcal{D}$ as the branch locus of a projection from a hypersurface in $\boldsymbol{P}_3$; thus one could try to send $\mathcal{X}$ birationally and regularly to $\boldsymbol{P}_3$ and to factor f_E through such map. Now, first of all $\mathcal{X}$ cannot be generally embedded in $\boldsymbol{P}_3$ as a smooth surface. In these cases sending $\mathcal{X}$ to $\boldsymbol{P}_3$ by a regular birational map would produce a singular locus (often a curve) on the image surface; this locus would appear in the ramification of any projection from $\boldsymbol{P}_3$ to $\boldsymbol{P}_2$, so in practice the resulting branch locus in $\boldsymbol{P}_2$ would bring an additional component besides $\mathcal{D}$. Of course the approach could still lead to non-trivial arithmetical conclusions, but weaker. Secondly, even if $\mathcal{X}$ may be embedded as a smooth hypersurface in $\boldsymbol{P}_3$, the degrees of such embeddings are bounded. (One can prove this, for example, by recalling that the self-intersection K^2 in a smooth hypersurface of degree d in $\boldsymbol{P}_3$ equals $d(d-4)^2$; see, for example, [**14**, Example A.2.7].) Thus for large n the present projections cannot be in any case recovered that way.

We continue to follow [**10**], defining $\mathcal{Y} \to \mathcal{X} \to \boldsymbol{P}_2$ as the associated Galois cover. The surface $\mathcal{Y}$ may be embedded in the nth power of $\mathcal{X}$ over $\boldsymbol{P}_2$; the fibre of $\mathcal{Y}$ over a point

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in $\boldsymbol{P}_2 \setminus \mathcal{D}$ classifies orderings of the n points in f_E^{-1} [**10**, p. 236]. We consider only those subspaces E as above, generating $\mathcal{L}$ and such that

(i) Z is smooth and irreducible,

(ii) f_E is birational onto $\mathcal{D}$,

(iii) $\mathcal{D}$ has only cusps and simple double points as singularities, and

(iv) $\mathcal{Y}$ is smooth with Galois group S_n.

In [**10**, Proposition 1(i),(ii)] it is proved that these conditions are verified for all E in a certain open dense subset of the Grassmannian, a condition which we assume, saying that E is 'general'.

We now state the main result of this section.

Theorem 3.2. *For a general E as above, suppose that $\mathcal{D}$ has at least one double point and that*

$$(n-9)K^2 + 6(n-7)K.\mathcal{L} + 3n(3n-19) + 4c_2 > 0. \tag{3.2}$$

Then every set of S-integral points for $\boldsymbol{P}_2 \setminus \mathcal{D}$ is finite.

In Faltings's paper, the same conclusion is obtained assuming, in place of (3.2), that $d\mathcal{L} - \alpha Z$ is ample on $\mathcal{X}$ for some $\alpha > 12$ (see [**10**, Theorem 1]; here $d := Z.\mathcal{L} = K.\mathcal{L} + 3n$). As remarked earlier, we do not know the exact implications between the two types of assumptions.

That $\mathcal{D}$ has at least one double point amounts to an inequality somewhat similar to (3.2). This is stated in § 4, together with some instance of the validity of it and of (3.2); see, for example, Corollary 4.1.

Proof. In Faltings's paper it is observed that the inverse image of Z in $\mathcal{Y}$ splits in the union of curves Z_{ij} (the set of fixed points of the transposition (i,j)), for subsets $\{i,j\}$ of $\{1,\dots,n\}$, $i \neq j$.* For a general E, the Z_{ij} are irreducible as soon as $\mathcal{D}$ contains some double point [**10**, Lemma 1]. Moreover, for distinct indices $i < j < l$, Z_{ij}, Z_{il}, Z_{jl} intersect with different tangents at points which give rise to cusps of $\mathcal{D}$; for disjoint pairs $i < j$ and $l < m$, Z_{ij} and Z_{lm} intersect transversally, giving rise to double points of $\mathcal{D}$; otherwise there are no intersections among the Z_{ij} (see [**10**, p. 239] for these conclusions).

As in [**10**] (but for a different purpose!), we consider the divisor $A := \sum_{j=2}^{n} Z_{1j}$ on $\mathcal{Y}$. It is the pull-back of Z under the first projection $\mathcal{Y} \to \mathcal{X}$. Since Z is ample on $\mathcal{X}$ by assumption and since f_E is finite (quasi-finite would suffice), A is ample on $\mathcal{Y}$ (e.g. by [**13**, Example 5.7(d), p. 232]).

Observe now (as in [**10**]) that by the usual principle based on the Chevalley–Weil theorem and the theorem of Hermite mentioned in § 2, any set Σ of integral points on $\boldsymbol{P}_2 \setminus \mathcal{D}$ lifts to a set Σ' of integral points on $\mathcal{Y} \setminus \bigcup_{i<j} Z_{ij}$ [**16**, p. 50]. Therefore, it will suffice to prove that Σ' is finite.

* In the case when $\mathcal{X}$ is a hypersurface, these $n(n-1)/2$ curves correspond to the factorization of the discriminant, which has been used in § 2.

In turn, Σ' is *a fortiori* a set of S-integral points for $\mathcal{Y} \setminus A$, and it is with this last variety that we shall mainly work. We start by proving the following.

Theorem 3.3. *Under the assumptions for Theorem 3.2, any set Σ^* of (quasi-) S-integral points for $\mathcal{Y} \setminus A$ is contained in a curve.*

Proof. This result seems not to appear in [**10**]. The proof will follow from the Main Theorem of [**5**], which we adapt for the reader's convenience to the present context.

Theorem CZ. *Let $\mathcal{Y}$, $A = \sum_{i=2}^n Z_{1i}$, Σ^* be as above. Assume that the Z_{1i} are distinct irreducible divisors with the following properties:*

(i) *no three of the Z_{1i} share a common point;*

(ii) *defining ξ_i as the minimal positive solution of the equation $Z_{1i}^2 T^2 - 2(A.Z_{1i})T + A^2 = 0$ (which exists), for $i = 2, \dots, n$, we have the inequality*

$$2A^2\xi_i > (A.Z_{1i})\xi_i^2 + 3A^2.$$

Then Σ^* is contained in a curve.

In order to apply this statement, we need just to check (ii); in fact recall that by the above observations extracted from [**10**] the Z_{1i} are distinct, irreducible and no three of them intersect. For the rest of (ii) we are going to use again [**10**], where some intersection indices are computed.

On p. 243 of [**10**] we find the formula

$$Z_{1j}.Z_{1l} = (n-3)!\gamma, \quad j \neq l, \quad j, l \neq 1, \tag{3.3}$$

where γ is the number of cusps on $\mathcal{D}$, which is computed (p. 244) as

$$\gamma = 2K^2 - c_2 + 9K.\mathcal{L} + 12n. \tag{3.4}$$

There is also a formula for $Z_{ij}.Z_{lm}$ where i, j, l, m are all distinct, in terms of the number of double points on $\mathcal{D}$, but we shall not need it. We shall need the self-intersections

$$Z_{1j}^2 = (n-2)!(\rho - \gamma), \qquad \rho := Z^2 = (K + 3\mathcal{L})^2 = K^2 + 6K.\mathcal{L} + 9n. \tag{3.5}$$

We shall use these formulae to check (ii) of Theorem CZ. First we have

$$A.Z_{1i} = Z_{1i}^2 + \sum_{j>1,\ j\neq i} Z_{1j}.Z_{1i} = (n-2)!(\rho - \gamma) + (n-2)!\gamma = (n-2)!\rho. \tag{3.6}$$

This yields

$$A^2 = \sum_{i=2}^{n} A.Z_{1i} = (n-1)!\rho. \tag{3.7}$$

Let us then evaluate the numbers ξ_i defined in the statement of Theorem CZ. Using (3.5), (3.6) and (3.7) we see that ξ_i is the minimal positive solution of the equation

$$(\rho - \gamma)T^2 - 2\rho T + (n-1)\rho = 0. \tag{3.8}$$

The discriminant of this equation is $4\Delta := 4(\rho^2 - (n-1)\rho(\rho-\gamma))$; as remarked in [**5**] it is never negative (see [**13**, Example 1.9(a), p. 368]). From this it follows that (recall $\rho = Z^2 > 0$ as Z is ample)

$$(n-1)\gamma \geqslant (n-2)\rho. \tag{3.9}$$

The roots in question are, for $\rho \neq \gamma$,

$$\xi_{\pm} = \frac{\rho \pm \sqrt{\Delta}}{(\rho-\gamma)}. \tag{3.10}$$

We now split the proof according to the sign of $\rho - \gamma$.

Case A ($\boldsymbol{\rho < \gamma}$). Now $\Delta = \rho^2 + (n-1)\rho(\gamma-\rho) > \rho^2$; the unique positive one of the roots (3.10) is

$$\xi := \xi_- = \frac{\sqrt{\Delta} - \rho}{\gamma - \rho}.$$

In view of formulae (3.5)–(3.7), (ii) of Theorem CZ amounts to $2(n-1)\xi > \xi^2 + 3(n-1)$.

Now, the roots of the polynomial in U given by $P(U) := U^2 - 2(n-1)U + 3(n-1)$ are $(n-1) \pm \sqrt{(n-1)^2 - 3(n-1)}$ and we have to check whether ξ is contained in the interval $\mathcal{I}$ between them. Observing the inequality $\sqrt{(n-1)^2 - 3(n-1)} \geqslant n-3$ for $n \geqslant 5$, we see that $\mathcal{I}$ contains the interval $[2, 2n-4]$, whence it suffices to verify that $2 < \xi < 2n-4$, i.e. that

$$\rho + 2(\gamma - \rho) < \sqrt{\Delta} < \rho + (2n-4)(\gamma - \rho).$$

The three involved terms are all positive, so we may square to find the equivalent inequalities

$$4\rho(\gamma-\rho) + 4(\gamma-\rho)^2 < (n-1)\rho(\gamma-\rho) < 2(2n-4)\rho(\gamma-\rho) + (2n-4)^2(\gamma-\rho)^2.$$

Now, we are assuming $n > 5$ and $\gamma - \rho > 0$, while $\rho = Z^2 > 0$ follows from the ampleness of Z. Hence the right inequality is certainly true. The left one amounts to $4\gamma < (n-1)\rho$. In view of the formula (3.4) and of $\rho := Z^2 = (K + 3\mathcal{L})^2 = K^2 + 6K.\mathcal{L} + 9n$ this is equivalent to

$$(n-9)K^2 + 6(n-7)K.\mathcal{L} + 3n(3n-19) + 4c_2 > 0,$$

which is just (3.2). This completes the verification in the Case A.

Case B ($\boldsymbol{\rho = \gamma}$). Now, in view of (3.8), we have $\xi_i = (n-1)/2$ and we have to check that $P((n-1)/2) < 0$; now, $P((n-1)/2) = (n-1)(15-3n)/4$, so what is needed follows from $n > 5$.

Case C ($\boldsymbol{\rho > \gamma}$). Now $\Delta < \rho^2$ and again the least positive one between the roots (3.10) is

$$\xi := \xi_- = \frac{\rho - \sqrt{\Delta}}{\rho - \gamma}.$$

As before it suffices to check whether $\xi \in (2, 2n-4)$, namely whether

$$2(\rho - \gamma) < \rho - \sqrt{\Delta} < (2n-4)(\rho - \gamma). \tag{3.11}$$

Now, the left inequality amounts to $\sqrt{\Delta} < \rho - 2(\rho - \gamma)$. The right side of this ($= 2\gamma - \rho$) is positive, by (3.9) and since $n > 5$. Therefore, we reduce to verify $\Delta < (\rho - 2(\rho - \gamma))^2$, i.e. $-(n-1)\rho(\rho - \gamma) < -4\rho(\rho - \gamma) + 4(\rho - \gamma)^2$, which is true because $n > 5$.

We remain with the right side of (3.11), which amounts to

$$\sqrt{\Delta} > \rho - (2n-4)(\rho - \gamma).$$

Now, this follows immediately from $n > 5$ and

$$\sqrt{\Delta} = \sqrt{\rho(\rho - (n-1)(\rho - \gamma))} \geqslant \sqrt{(\rho - (n-1)(\rho - \gamma))^2} = \rho - (n-1)(\rho - \gamma).$$

All of this completes the verification of the assumptions of Theorem CZ, so we may conclude that Σ^* lies on a certain curve, proving Theorem 3.3. □

A fortiori, the original set Σ' also lies on a curve and to prove the full Theorem 3.2, we now show that Σ' is in fact finite; for this we apply Siegel's theorem and use that Σ' is a set of S-integral points not merely for $\mathcal{Y} \setminus A$, but for $\mathcal{Y} \setminus \bigcup_{i<j} Z_{ij}$.

Now, $\bigcup_{i<j} Z_{ij}$ is equal to the union of the supports of the divisors $A_i := \sum_{j \neq i} Z_{ij}$. We have $A_1 = A$ and all the A_i are ample (A_i is the pull-back of Z under the ith projection from $\mathcal{Y}$ to $\mathcal{X}$ and on the other hand $S_n = \mathrm{Aut}(\mathcal{Y}/\mathcal{X})$ acts transitively on $\{A_1, \dots, A_n\}$).

Now, let C be an irreducible (affine) curve on $\mathcal{Y} \setminus \bigcup_{i<j} Z_{ij}$, having infinite intersection with Σ'. We forget about the genus of C an apply Siegel's theorem merely to conclude that the closure $\tilde{C}$ of C in $\mathcal{Y}$ cannot have more than two points at infinity. This means that $\tilde{C}$ cannot intersect $\bigcup_{i=1}^n |A_i|$ in more than two points.

On the other hand, $\tilde{C}$ must intersect non-trivially each divisor A_i (which is ample). We conclude that there exists a point $P \in \tilde{C}$ belonging to at least $n/2$ of the supports $|A_i|$. Further, if $P \in |A_i|$, then P lies in Z_{ij}, for some $j = j(i) \neq i$. A first case now occurs when among these Z_{ij} there appear Z_{ab} and Z_{cd} for some disjoint sets $\{a, b\}$ and $\{c, d\}$. Now the point P cannot lie on any other Z_{ij}, whence $P \in |A_i|$ only if $i \in \{a, b, c, d\}$, which leads to $n \leqslant 8$, a contradiction.

The second case is when a pair Z_{ab}, Z_{ac} occurs; now P cannot lie in any other Z_{ij} except Z_{bc}. We then conclude that $P \in |A_i|$ only if $i \in \{a, b, c\}$, leading to $n \leqslant 6$ and a contradiction.

This completes the proof of Theorem 3.2. □

Note that the final argument shows the finiteness of integral points on $\mathcal{Y} \setminus \bigcup_{i \in I} |A_i|$ for any subset I of $\{1, \dots, n\}$ with more than eight elements.

4. Further remarks and examples

We start by translating the condition that $\mathcal{D}$ has at least a double point; for this we use a formula on p. 244 of [**10**], stating that the number δ of double points on $\mathcal{D}$ is

$d^2/2 - 15d + 24n - 3K^2 + c_2$. Using $d = Z.\mathcal{L} = K.\mathcal{L} + 3n$, we see that our condition amounts to

$$2\delta = (K.\mathcal{L})^2 - 6K^2 + 6(n-5)K.\mathcal{L} + 3n(3n-14) + 2c_2 > 0. \tag{4.1}$$

Since $\mathcal{L}$ is ample, by [**13**, Example 1.9(a), p. 368] we have $(K.\mathcal{L})^2 \geqslant K^2\mathcal{L}^2 = nK^2$, whence (4.1) follows from

$$(n-6)K^2 + 6(n-5)K.\mathcal{L} + 3n(3n-14) + 2c_2 > 0. \tag{4.2}$$

If we merely want to prove the finiteness of integral points on $\mathcal{Y} \setminus \bigcup Z_{ij}$ and forget about Theorem 3.3, then (3.2) may be replaced by similar but slightly weaker inequalities. The proof is the same as above, but working with $\bigcup_{\{i,j\}\in J} Z_{ij}$ in place of A, where J is another suitable set of pairs, such that no three of the relevant divisors intersect (as is required in Theorem CZ). We leave the details to the interested reader.

Next, let us briefly discuss inequality (3.2) and Faltings's condition, which we label (F), that, for some rational $\alpha > 12$, $d\mathcal{L} - \alpha Z$ is ample. This amounts to the ampleness of $(d - 3\alpha)\mathcal{L} - \alpha K = (K.\mathcal{L} + 3n - 3\alpha)\mathcal{L} - \alpha K$.

We note that (3.2) is satisfied for large enough n, once $\mathcal{X}$ is given.

Condition (F) certainly implies that $(d\mathcal{L} - \alpha Z).\mathcal{L} > 0$, whence, recalling $d = Z.\mathcal{L}$, we deduce $n > \alpha > 12$. We shall see below some instances when both (4.1) and (3.2) hold for $n \geqslant 9$. (However, the ampleness conditions on $\mathcal{L}$ may imply that n is larger.)

The main example detailed in [**10**] is when $\mathcal{X} = \boldsymbol{P}_1 \times \boldsymbol{P}_1$ and $\mathcal{L} = \mathcal{O}(a, b)$ for integers $a, b \geqslant 3$. Now the finiteness of integral points follows from [**10**] and also from Theorem 3.2 above. In fact, we have in this case $n = 2ab$, $K^2 = 8$, $c_2 = 4$, $K.\mathcal{L} = -2(a + b)$, whence (3.2) amounts to

$$8(2ab-9) - 12(a+b)(2ab-7) + 6ab(6ab-19) + 16 > 0,$$

which is easily found to be true for $a, b \geqslant 2$. Similarly for (4.1) (required also in [**10**]).

As another application, we have the following corollary.

Corollary 4.1. *If $\mathcal{X}$ has Kodaira number $\geqslant 0$, if $n \geqslant 9$ and if E is 'general' in the above sense, then every set of S-integral points for $\boldsymbol{P}_2 \setminus \mathcal{D}$ is finite.*

We do not know whether (F) is always true under such assumptions.

Proof. First we observe that $K.(K + \mathcal{L}) \geqslant 0$ on our assumptions. To show this recall that by [**13**], proof of Theorem 5.8, there is a sequence of smooth surfaces and maps

$$\mathcal{X} = \mathcal{X}_0 \to \mathcal{X}_1 \to \cdots \to \mathcal{X}_r =: \mathcal{Y}$$

such that $\mathcal{Y}$ is minimal and each $\mathcal{X}_i$ is obtained blowing up a point in $\mathcal{X}_{i+1}$. If ε is the composite map then we have that $K_{\mathcal{X}} = \varepsilon^* K_{\mathcal{Y}} + E_1 + \cdots + E_r$, where the E_i are effective, mutually orthogonal and satisfy $E_i^2 = -1$; also, $K_{\mathcal{X}}^2 = K_{\mathcal{Y}}^2 - r$. All of this follows at once from the computation of the intersection product form on a blow-up (see [**2**, II]). Then

$$K_{\mathcal{X}}.(K_{\mathcal{X}} + \mathcal{L}) = K_{\mathcal{Y}}^2 - r + \varepsilon^* K_{\mathcal{Y}}.\mathcal{L} + (E_1 + \cdots + E_r).\mathcal{L}.$$

Now, $\mathcal{Y}$ is minimal and with Kodaira number (a birational invariant of non-singular surfaces) equal to the one of $\mathcal{X}$, hence $\geqslant 0$. It follows (see [**13**, V.6]) that $K_{\mathcal{Y}}^2 \geqslant 0$.

Next, $mK_{\mathcal{Y}}$ contains effective divisors for some $m > 0$ (by definition of Kodaira number [**12**, p. 572]) whence the same is true of $\varepsilon^*(mK_{\mathcal{Y}})$. It follows that $\varepsilon^* K_{\mathcal{Y}}.\mathcal{L} \geqslant 0$. Finally, $(E_1 + \cdots + E_r).\mathcal{L} \geqslant r$ because $\mathcal{L}$ is ample and the E_i are effective; these inequalities prove the claim.

To complete the proof it suffices now to show that both (4.2) (and hence (4.1)) and (3.2) are satisfied for $n \geqslant 9$. To this end we first recall that $c_2 \geqslant 0$, for otherwise $\mathcal{X}$ would be birational to a ruled surface (see [**12**, pp. 554, 558] or [**2**, Theorem X.4]); but then the Kodaira number would be -1 (see [**12**, p. 575], [**13**, p. 422]). Now, $K.\mathcal{L} \geqslant 0$ (because some mK, $m > 0$, contains effective divisors), so the claim is immediate if $K^2 \geqslant 0$. If not, observe that, since $6(n-5) \geqslant n-6 > 0$, the left side of (4.2) is $\geqslant 6(n-5)(K^2 + K.\mathcal{L}) + 3n(3n-14) + 2c_2$ and is positive by the above. This same argument shows as well the positivity of the left side of (3.2) since $6(n-7) \geqslant n-9 \geqslant 0$, concluding the proof. (In place of the opening argument one could also use the assumption that $K+3\mathcal{L}$ is ample; this yields $K.(K+3\mathcal{L}) \geqslant 0$, which suffices. Sometimes the argument however can be used to prove the assumption.) $\square$

To go further, we recall that if $\mathcal{X}$ has negative Kodaira number (necessarily -1) and if $\mathcal{X}$ is minimal, then $\mathcal{X}$ is either rational or ruled [**12**, p. 575]; hence the only cases when Corollary 4.1 does not apply occur when $\mathcal{X}$ is birationally equivalent to a product $\boldsymbol{P}_1 \times C$ for some curve C. In case $\mathcal{X}$ is ruled, it is again easy to test inequalities (3.2) and (4.1) using, for example, the formulae for K and for intersection products given in [**13**, V.2]. Possibly the same could be done in the most general case.

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