Can. J. Math. Vol. **46 (1)** 1994, pp. 3–54.

HOMOMORPHISMS OF DISTRIBUTIVE LATTICES AS RESTRICTIONS OF CONGRUENCES. II. PLANARITY AND AUTOMORPHISMS

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ABSTRACT. We prove that any $\{0,1\}$-preserving homomorphism of finite distributive lattices can be realized as the restriction of the congruence relations of a *finite planar lattice with no nontrivial automorphisms* to an ideal of that lattice, where this ideal also has no nontrivial automorphisms. We also prove that any $\{0,1\}$-preserving homomorphism of finite distributive lattices with more than one element and any homomorphism of groups can be realized, simultaneously, as the restriction of the congruence relations and, respectively, the restriction of the automorphisms of a lattice L to those of an ideal of L; if the groups are both finite, then so is the lattice L.

1. **Introduction.** Let L be a lattice. It was proved in N. Funayama and T. Nakayama [6] that the congruence lattice of L is distributive. For a finite lattice L, the converse of this result was proved by R. P. Dilworth: Every finite distributive lattice D can be represented as the lattice of congruence relations of a suitable finite lattice L. The first published proof of this result is in G. Grätzer and E. T. Schmidt [12]. Another proof of this result, in the same spirit, by the present authors, is given in [7], pp. 81–84.

Based on the concept of *coloring*, which originated in S.-K. Teo [16], an entirely new proof of the above result was given by the present authors in [9]. It was furthermore proved that the finite lattice L can always chosen to be *planar* and to have only the identity mapping as an automorphism. This was applied in [9] to give a new proof of the following result due to V. A. Baranskiĭ [2], [3] and A. Urquhart [17]: for any finite distributive lattice D with more than one element and any group G, there is a lattice L whose congruence lattice Con L is isomorphic to D and whose automorphism group Aut L is isomorphic to G. If G is finite, then L can be chosen to be finite.

Given a lattice L and a convex sublattice L', it is well known that the *restriction* map of Con L to Con L', sending each congruence relation of L to its restriction to L', is a lattice homomorphism preserving 0 and 1. Based on the proof of Dilworth's representation theorem given in [7], it was shown by the present authors in [8] that, conversely, any $\{0,1\}$-preserving homomorphism of finite distributive lattices can be realized by restricting the congruence lattice of some finite lattice L to the congruence lattice of an ideal L' of L. See E. T. Schmidt [15] for an alternate proof of this result.

This research was supported by the NSERC of Canada.
Received by the editors February 5, 1992; revised January 27, 1993.
AMS subject classification: 06B10, 06D99; Secondary 08A30, 08A35.
Key words and phrases: Lattice, congruence lattice, automorphism group.

In this paper we apply the methods of [9] to prove the following theorem, which improves the result of [8] by showing that we can enforce planarity:

THEOREM 1. *Let D and D' be finite distributive lattices, and let $\psi\colon D \longrightarrow D'$ be a $\{0,1\}$-preserving lattice homomorphism. Then there exist a finite* planar *lattice L, an ideal L' of L, and lattice isomorphisms*

$$\varrho\colon D \longrightarrow \mathrm{Con}\, L, \qquad \varrho'\colon D' \longrightarrow \mathrm{Con}\, L'$$

such that $\psi\varrho'$ is the composition of ϱ with the restriction of Con L *to* Con L'. *Moreover, the lattices L and L' have no nontrivial automorphisms.*

By a *nontrivial* automorphism we mean one that is distinct from the identity mapping. In general, automorphisms of a lattice do not restrict to automorphisms of its ideals. However, we can construct lattices where this does happen:

THEOREM 2. *Let D and D' be finite distributive lattices with more than one element, and let $\psi\colon D \longrightarrow D'$ be a $\{0,1\}$-preserving lattice homomorphism. Let G and G' be groups, and let $\eta\colon G \longrightarrow G'$ be a group homomorphism. Then there exist a lattice L, an ideal L' in L, lattice isomorphisms*

$$\varrho\colon D \longrightarrow \mathrm{Con}\, L, \qquad \varrho'\colon D' \longrightarrow \mathrm{Con}\, L',$$

and group isomorphisms

$$\tau\colon G \longrightarrow \mathrm{Aut}\, L, \qquad \tau'\colon G' \longrightarrow \mathrm{Aut}\, L'$$

such that, for each $x \in D$, the congruence relation $x\psi\varrho'$ on L' is the restriction to L' of the congruence relation $x\varrho$ on L, and, for each $g \in G$, the automorphism $g\eta\tau'$ of L' is the restriction of the automorphism $g\tau$ of L.

If G and G' are finite, then the lattice L can be chosen to be finite.

By identifying D with Con L, D' with Con L', G with Aut L, and G' with Aut L', Theorem 2 can be paraphrased as follows: any pair ψ, a $\{0,1\}$-homomorphism of finite distributive lattices, and ϱ, a homomorphism of groups, can be simultaneously realized as the respective restrictions Con $L \longrightarrow$ Con L' and Aut $L \longrightarrow$ Aut L' for some lattice L and some ideal L' in L.

Note that we do not claim that the lattice L in Theorem 2 can be chosen to be planar—we do not know whether every (finite) group can be represented as the group of automorphisms of a planar lattice. If this were so, we could easily modify the constructions in Sections 6, 7 to get the stronger result.

The basic notation is explained in Section 2. In Section 3, we present the construction used to prove these theorems. It is based on the idea of coloring a chain, which originated in S.-K. Teo [16]. We discuss in Section 4 a generalization of this construction introduced in G. Grätzer and H. Lakser [11]. This is then applied in Section 5 to prove Theorem 1. In Section 6, we modify a construction of R. Frucht [4] and [5] to prove the special case of Theorem 2 where D and D' are the two-element chain. Finally, in Section 7, we prove Theorem 2.

2. **Notation.** For any finite distributive lattice D, we denote by $J(D)$ the partially ordered set of (nonzero) join-irreducible elements of D. Let $\mathfrak{M}_3$ denote the five-element modular nondistributive lattice, and let $\mathfrak{N}_5$ denote the five-element nonmodular lattice.

For a lattice A, let Con A denote the lattice of congruence relations on A, and let Ip A denote the the set of *prime intervals* in A, that is, the set of all intervals $\mathfrak{p} = [u, v]$, where $u \prec v$ (u is *covered by* v). We shall usually denote prime intervals in lower-case Fraktur font; $\mathfrak{p}$, $\mathfrak{q}$, and so on. If $I = [u, v]$ is an interval of A, then for any lattice B and $b \in B$, we use the notation $I \times \{b\}$ for the interval $[\langle u, b\rangle, \langle v, b\rangle]$ of $A \times B$. Note that if $\mathfrak{p}$ is prime, then $\mathfrak{p} \times \{b\} \in \mathrm{Ip}(A \times B)$.

For a (prime) interval $I = [u, v]$ in the lattice A, we shall denote by $\Theta_A(I)$ or $\Theta_A(u, v)$ the congruence relation generated by the interval I. If A is understood, we use the notation $\Theta(I)$ or $\Theta(u, v)$. Note that $u \equiv v \pmod{\Theta}$ is equivalent to $\Theta(I) \leq \Theta$.

For any structure A, let Aut A denote the group of automorphisms of A.

Condition (1) of Theorem 5 will be referred to as Condition (5.1), and so on.

We refer the reader to G. Grätzer [7] for the standard notation in lattice theory.

3. **The basic construction.** In this section we review the ideas introduced in [9] and show how the construction presented there can be modified to prove Theorem 1.

The starting point is the classical duality between finite distributive lattices D and the posets of their join-irreducible elements $J(D)$—see [7], p. 62. The lattice D is naturally isomorphic to the lattice of hereditary subsets of $J(D)$ (that is, those subsets H satisfying $x \in H$ and $y \leq x$ imply that $y \in H$); a hereditary subset H of $J(D)$ corresponds to the element $\bigvee(x \mid x \in H)$ of D. Let D and D' be finite distributive lattices and let $\psi\colon D \longrightarrow D'$ be a $\{0,1\}$-preserving homomorphism. The homomorphism ψ then determines an isotone map $\psi^*\colon J(D') \longrightarrow J(D)$ determined by setting

$$a\psi^* = \bigwedge\nolimits_D([a)_{D'}\psi^{-1}). \tag{3-1}$$

We can recover ψ from ψ^* by noting that

$$x\psi = \bigvee\nolimits_{D'}\left(\{a \in J(D) \mid a \leq x\}(\psi^*)^{-1}\right).$$

Finally, any isotone $\alpha\colon J(D') \longrightarrow J(D)$ is ψ^*, where ψ is given by

$$x\psi = \bigvee\nolimits_{D'}\left(\{a \in J(D) \mid a \leq x\}\alpha^{-1}\right).$$

In a finite lattice L, a join-irreducible congruence is a principle congruence $\Theta(a, b)$ where $[a, b]$ is a prime interval in L, that is, $a \prec b$. We first show how to construct a finite lattice whose poset of join-irreducible congruences is isomorphic to $J(D)$; then Con L is isomorphic to D. If the finite distributive lattice D has n join-irreducible elements, then, as the first step in our construction, we take a chain C of length n. The chain C has exactly n join-irreducible congruences; however, $J(\mathrm{Con}\ C)$ is a discrete poset, that is, any pair of elements is incomparable.

Our first task is to force *comparability* between certain pairs of join-irreducible congruences. This can be accomplished by what we call the $\mathfrak{N}_5$-*construction*—see

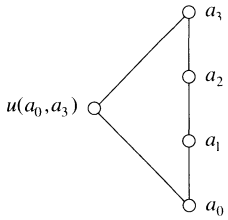

FIGURE 1

Figure 1. The lattice depicted there is the nonmodular lattice $\mathfrak{N}_5$, and it has exactly three join-irreducible congruences, $\Theta(a_0, a_1)$, $\Theta(a_1, a_2)$, $\Theta(a_2, a_3)$, and

$$\Theta(a_1, a_2) \prec \Theta(a_0, a_1)$$
$$\Theta(a_1, a_2) \prec \Theta(a_2, a_3).$$

We think of $\mathfrak{N}_5$ as a modification of the chain $\{a_0, a_1, a_2, a_3\}$ obtained by adding the atom $u(a_0, a_3)$, as indicated by how we draw the lattice. Now, using this idea alone, we can only get a poset of join-irreducible congruences of length 1 in which each minimal element is covered by either 0 or 2 elements, and each maximal element covers at most one element.

To get more general posets we must provide a construction that *identifies* various join-irreducible congruences. For example, to get a three-element chain, we take the lattice L depicted in Figure 2, a modification of the chain

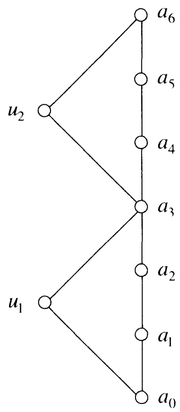

FIGURE 2

$$\{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$$

by the $\mathfrak{N}_5$-construction, and extend it to a lattice L' where the congruences

$$\Theta(a_0, a_1), \Theta(a_2, a_3), \Theta(a_4, a_5)$$

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are identified and the congruences

$$\Theta(a_3, a_4), \Theta(a_5, a_6)$$

are identified. If we can accomplish this, the poset $J(\text{Con } L')$ is then the chain

$$\Theta(a_1, a_2) < \Theta(a_4, a_5) < \Theta(a_3, a_4).$$

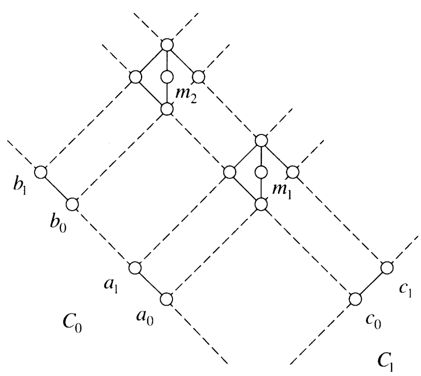
Given a chain C_0, various join-irreducible congruences can be identified by a construction we call the $\mathfrak{M}_3$-*construction*. We take a second chain C_1 and consider the product lattice $C_0 \times C_1$. Then

$$\text{Con } (C_0 \times C_1) \cong \text{Con } C_0 \times \text{Con } C_1,$$

and so we have a natural isomorphism

$$J\big(\text{Con } (C_0 \times C_1)\big) \cong J(\text{Con } C_0) \dot{\cup} J(\text{Con } C_1). \tag{3-2}$$

Given two prime intervals $[a_0, a_1]$, $[b_0, b_1]$ in C_0, we choose a prime interval $[c_0, c_1]$ in C_1 and add in the lattice $C_0 \times C_1$ a new atom m_1 to the interval $[\langle a_0, c_0\rangle, \langle a_1, c_1\rangle]$ and a new atom m_2 to the interval $[\langle b_0, c_0\rangle, \langle b_1, c_1\rangle]$, thereby getting a lattice L in which the intervals $[\langle a_0, c_0\rangle, \langle a_1, c_1\rangle]$ and $[\langle b_0, c_0\rangle, \langle b_1, c_1\rangle]$ are isomorphic to $\mathfrak{M}_3$—see Figure 3. Then, in Con L, $\Theta(a_0, a_1)$, $\Theta(b_0, b_1)$, $\Theta(c_0, c_1)$, the three join-irreducible congruences



L

FIGURE 3

under the isomorphism (3-2), are identified. Consequently, we can identify arbitrary join-irreducible congruences in a chain C_0 by taking a chain C_1 with the right number of prime intervals and applying the $\mathfrak{M}_3$-construction sufficiently often. We call C_0 the *working axis* and C_1 the *control axis*. Of course, we can also use C_0 as a control axis to identify distinct join-irreducible congruences of C_1. Indeed, since, in the notation of Theorem 1, distinct join-irreducible congruences of L' may need to be identified in L, identifying distinct join-irreducible congruences in C_1 may be necessary.

The $\mathfrak{M}_3$-construction can be formalized by the concept of *coloring*, an idea which originates in S.-K. Teo [16]. Let J be a set, which in practice will be the poset $J(D)$ of a finite distributive lattice D. A *coloring* of a chain C is a mapping

$$\varphi\colon \text{ Ip } C \longrightarrow J.$$

Following S.-K. Teo [16], for the chains C_0 and C_1 and colorings

$$\varphi_0\colon \text{ Ip } C_0 \longrightarrow J$$

and

$$\varphi_1\colon \text{ Ip } C_1 \longrightarrow J,$$

we define the lattice K, as follows: the lattice K is $C_0 \times C_1$ augmented with the elements $m(\mathfrak{p}_0, \mathfrak{p}_1)$, whenever $\mathfrak{p}_0 = [u_0, v_0] \in \text{Ip } C_0$, $\mathfrak{p}_1 = [u_1, v_1] \in \text{Ip } C_1$, and $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$; we require that the elements

(3-3) $$\langle u_0, u_1\rangle,\ \langle v_0, u_1\rangle,\ \langle u_0, v_1\rangle,\ m(\mathfrak{p}_0, \mathfrak{p}_1),\ \langle v_0, v_1\rangle$$

form a sublattice of K isomorphic to $\mathfrak{M}_3$, as illustrated by Figure 4. Then the poset of

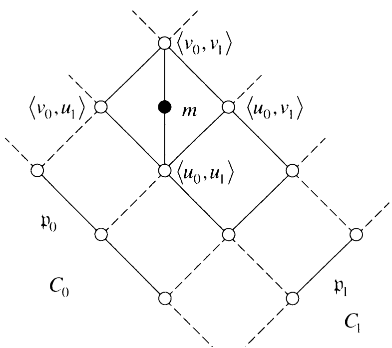


FIGURE 4

join-irreducible congruences of K is isomorphic to the discrete poset

$$(\text{Ip } C_0)\varphi_0 \cup (\text{Ip } C_1)\varphi_1.$$

In Teo's paper, $C_0 = C_1$ and $\varphi_0 = \varphi_1$, but the idea is the same.

Actually, in order to accomplish our purposes it is not necessary to apply the $\mathfrak{M}_3$-construction to *all* intervals of $C_0 \times C_1$ of the form $\mathfrak{p}_0 \times \mathfrak{p}_1$ with $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$. Indeed, in order to prove Theorem 1, it will be essential to use only a proper subset of that set of intervals. The following requirement is sufficient:

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Λ is a set of intervals in $C_0 \times C_1$ of the form $\mathfrak{p}_0 \times \mathfrak{p}_1$ satisfying

(3-4) If $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$, then $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$.

(3-5) For each $x \in (\mathrm{Ip}\ C_0)\varphi_0 \cap (\mathrm{Ip}\ C_1)\varphi_1$, there is a $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$ with $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1 = x$.

(3-6) For each $i = 0, 1$, if $\mathfrak{p}$, $\mathfrak{p}' \in \mathrm{Ip}\ C_i$ are distinct and $\mathfrak{p}\varphi_i = \mathfrak{p}'\varphi_i$, then there are $\bar{\mathfrak{p}} \in \mathrm{Ip}\ C_i$, $\mathfrak{q}$, $\mathfrak{q}' \in \mathrm{Ip}\ C_{1-i}$ with

$$\mathfrak{p} \times \mathfrak{q},\ \bar{\mathfrak{p}} \times \mathfrak{q},\ \mathfrak{p}' \times \mathfrak{q}',\ \bar{\mathfrak{p}} \times \mathfrak{q}' \in \Lambda, \qquad \text{if } i = 0,$$
$$\mathfrak{q} \times \mathfrak{p},\ \mathfrak{q} \times \bar{\mathfrak{p}},\ \mathfrak{q}' \times \mathfrak{p}',\ \mathfrak{q}' \times \bar{\mathfrak{p}} \in \Lambda, \qquad \text{if } i = 1.$$

Then, if K is the lattice we get by adding to $C_0 \times C_1$ the elements $m(\mathfrak{p}_0, \mathfrak{p}_1)$ for all intervals $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$, we again find that $J(\mathrm{Con}\ K)$ is isomorphic to the discrete poset $(\mathrm{Ip}\ C_0)\varphi_0 \cup (\mathrm{Ip}\ C_1)\varphi_1$—condition (3-6) guarantees that distinct prime intervals of C_i with the same color determine the same congruence.

It is tempting to apply the $\mathfrak{N}_5$-construction on top of the $\mathfrak{M}_3$-construction to get an arbitrary poset of join-irreducible congruences. Let us introduce some more terminology. If $\varphi\colon C \longrightarrow J$ is a coloring of a chain C and if $x, y \in J$, then by an *(x, y)-interval of C with respect to φ* we mean an interval $\{a, b, c, d\}$ in C with

$$a \prec b \prec c \prec d$$

and with

$$[a, b]\varphi = [c, d]\varphi = y,$$
$$[b, c]\varphi = x.$$

Now, for example, to get a poset of join-irreducible congruences isomorphic to $J = \{x, y, u, v\}$, the (unordered) union of two two-element chains, with

$$x \prec y \text{ and } u \prec v,$$

we set the working axis to be the chain

$$C_0 : a_0 < a_1 < a_2 < a_3 < a_4 < a_5 < a_6$$

with the coloring $\varphi_0\colon \mathrm{Ip}\ C_0 \longrightarrow J$ such that $[a_0, a_3]$ is an (x, y)-interval and $[a_3, a_6]$ is a (u, v)-interval, that is, with

$$[a_0, a_1]\varphi_0 = [a_2, a_3]\varphi_0 = y$$
$$[a_3, a_4]\varphi_0 = [a_5, a_6]\varphi_0 = v$$
$$[a_1, a_2]\varphi_0 = x$$
$$[a_4, a_5]\varphi_0 = u,$$

and let C_1 be the chain

$$b < c < \cdots$$

with a coloring $\varphi_1\colon$ Ip $C_1 \longrightarrow J$ and with at least one prime interval $[c_0, c_1]$ with

$$[c_0, c_1]\varphi_1 = v,$$

and at least one prime interval $[c_2, c_3]$ with

$$[c_2, c_3]\varphi_1 = y.$$

We then apply the $\mathfrak{M}_3$-construction to $C_0 \times C_1$, getting a lattice K identifying (using somewhat loose notation, that is, a_0 for $\langle a_0, b\rangle$, *etc.*) the congruences

$$\Theta(a_0, a_1), \Theta(a_2, a_3)$$

and identifying the congruences

$$\Theta(a_3, a_4), \Theta(a_5, a_6).$$

See Figure 5, where the colors are indicated on the "inside" of the intervals.

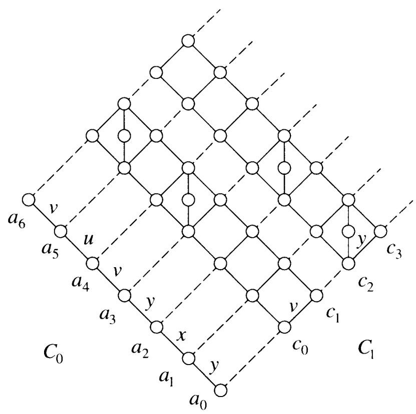

FIGURE 5

We then apply the $\mathfrak{N}_5$-construction to the chains $[a_0, a_3]$ and $[a_3, a_6]$ of K to get a lattice L where

$$\Theta(a_1, a_2) < \Theta(a_0, a_1) = \Theta(a_2, a_3) \text{ and } \Theta(a_4, a_5) < \Theta(a_3, a_4) = \Theta(a_5, a_6),$$

seemingly accomplishing our purpose. However, a close look at Figure 6, depicting the part of L involving only the bottom interval of C_1, yields a nasty surprise,

$$a_0 \equiv a_1 \quad (\Theta(b, c)) \quad \text{and} \quad a_3 \equiv a_4 \quad (\Theta(b, c)),$$

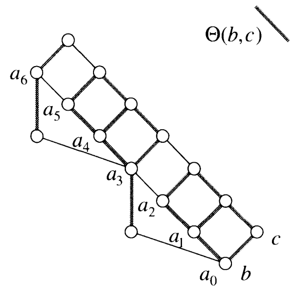

FIGURE 6

that is, $\Theta(b,c)$ is a join-irreducible congruence that includes both $\Theta(a_0, a_1)$ and $\Theta(a_3, a_4)$!

Indeed, proceeding in the above manner to attempt to get any arbitrary poset always results in getting a join-irreducible congruence that includes every nonisolated join-irreducible congruence. (An element of a poset is *isolated* if it is incomparable to every other element.) This problem is a familiar phenomenon in attempting to control congruences—it is usually easy to make congruences do at least what we desire, but much harder to make them do no more! It is precisely to avoid nasty surprises that we developed the rather technical theorems of Section 4.

The above difficulty is not hard to rectify. Rather than taking all of $C_0 \times C_1$, we take, for each (x, y)-interval $[a, b]$ of C_0, only the part $[a, b] \times [c)$ of $C_0 \times C_1$, where $[c, d]$ is a prime interval of C_1 colored by y. After applying the $\mathfrak{M}_3$-construction to this sublattice A, we then apply the $\mathfrak{N}_5$-construction to the intervals of the form $[\langle a, c\rangle, \langle b, c\rangle]$ by adding a new element $u(a, b)$ with

$$\langle a, c\rangle \prec u(a, b) \prec \langle b, c\rangle.$$

We note that, in order for A to be a sublattice of $C_0 \times C_1$, the colorings φ_0 and φ_1 must be coordinated—if $[a_1, b_1]$ is an (x_1, y_1)-interval in C_0 and $[a_2, b_2]$ is an (x_2, y_2)-interval and if $b_1 \leq a_2$, then there must be prime intervals $[c_1, d_1]$, $[c_2, d_2]$ in C_1, where $[c_1, d_1]\varphi_1 = y_1$, $[c_2, d_2]\varphi_1 = y_2$, with $d_1 \leq c_2$. For example, the coloring depicted in Figure 5 will not work—instead, we must color $[c_0, c_1]$ with y and $[c_2, c_3]$ with v.

Reverting to our example, given the poset $J = \{u, v, x, y\}$ with $x < y$ and $u < v$, we get the lattice K depicted in Figure 7 whose join-irreducible congruences represent J.

Note that we will always get a planar lattice since we apply the $\mathfrak{N}_5$-construction to the leftmost side of A.

As above, in order to apply the $\mathfrak{M}_3$-construction to the sublattice A of $C_0 \times C_1$ we have the requirement:

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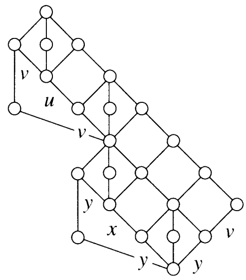

FIGURE 7

REQUIREMENT 1. Λ is a set of intervals in A of the form $\mathfrak{p}_0 \times \mathfrak{p}_1$ satisfying

(3-4) If $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$, then $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$.

(3-5) For each $x \in (\mathrm{Ip}\ C_0)\varphi_0 \cap (\mathrm{Ip}\ C_1)\varphi_1$, there is a $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$ with $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1 = x$.

(3-6) For each $i = 0, 1$, if $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ C_i$ are distinct and $\mathfrak{p}\varphi_i = \mathfrak{p}'\varphi_i$, then there are $\bar{\mathfrak{p}} \in \mathrm{Ip}\ C_i$, $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Ip}\ C_{1-i}$ with

$$\mathfrak{p} \times \mathfrak{q},\ \bar{\mathfrak{p}} \times \mathfrak{q},\ \mathfrak{p}' \times \mathfrak{q}',\ \bar{\mathfrak{p}} \times \mathfrak{q}' \in \Lambda, \qquad \text{if } i = 0,$$
$$\mathfrak{q} \times \mathfrak{p},\ \mathfrak{q} \times \bar{\mathfrak{p}},\ \mathfrak{q}' \times \mathfrak{p}',\ \mathfrak{q}' \times \bar{\mathfrak{p}} \in \Lambda, \qquad \text{if } i = 1.$$

Actually, in most cases, we shall have the following stronger form of (3-6):

(3-6′) For each $i = 0, 1$, if $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ C_i$ are distinct and $\mathfrak{p}\varphi_i = \mathfrak{p}'\varphi_i$, then there is a $\mathfrak{q} \in \mathrm{Ip}\ C_{1-i}$ with

$$\mathfrak{p} \times \mathfrak{q},\ \mathfrak{p}' \times \mathfrak{q} \in \Lambda \qquad \text{if } i = 0,$$
$$\mathfrak{q} \times \mathfrak{p},\ \mathfrak{q} \times \mathfrak{p}' \in \Lambda \qquad \text{if } i = 1.$$

We then have (3-6) with $\mathfrak{q}' = \mathfrak{q}$ and $\bar{\mathfrak{p}} = \mathfrak{p}$ (or, if we wish, $\bar{\mathfrak{p}} = \mathfrak{p}'$).

We also observe, for later use, that (3-5) and (3-6) together imply the following stronger form of (3-5):

(3-5′) For each $i = 0, 1$, if $\mathfrak{p} \in \mathrm{Ip}\ C_i$ and $\mathfrak{p}\varphi_i \in (\mathrm{Ip}\ C_{1-i})\varphi_{1-i}$, then there is a $\mathfrak{q} \in \mathrm{Ip}\ C_{1-i}$ with

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda, \qquad \text{if } i = 0,$$
$$\mathfrak{q} \times \mathfrak{p} \in \Lambda, \qquad \text{if } i = 1.$$

Indeed, and letting $i = 0$ without loss of generality,

$$x = \mathfrak{p}\varphi_0 \in (\mathrm{Ip}\ C_0)\varphi_0 \cap (\mathrm{Ip}\ C_1)\varphi_1,$$

and so, by (3-5), there are $\mathfrak{p}' \in \mathrm{Ip}\ C_0$ and $\mathfrak{q}_0 \in \mathrm{Ip}\ C_1$ with

$$\mathfrak{p}' \times \mathfrak{q}_0 \in \Lambda \text{ and } \mathfrak{p}'\varphi_0 = x = \mathfrak{p}\varphi_0.$$

If $\mathfrak{p}' = \mathfrak{p}$, set $\mathfrak{q} = \mathfrak{q}_0$ and we are done. Otherwise, the hypothesis of (3-6) holds, and so we are assured a $\mathfrak{q} \in \mathrm{Ip}\ C_1$ with

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda$$

and, again, we are done.

Except for ensuring that we have only the identity automorphism, this concludes the outline of the construction in [9].

We now apply these ideas to supply a construction for Theorem 1. Let D and D' be distributive lattices and let $\psi\colon D \longrightarrow D'$ be a $\{0,1\}$-preserving homomorphism. Note that if D' is the one-element lattice, then $J(D')$ is empty. If D is also the one-element lattice, then we set $L = L'$ to be the one-element lattice. So, in the sequel, we assume that $|D| > 1$ and thereby that $|J(D)| \neq \emptyset$. The homomorphism ψ yields the isotone mapping

$$\psi^*\colon J(D') \longrightarrow J(D)$$

of (3-1). Of course, if $|D'| = 1$, then ψ^* is the empty mapping. We must construct a lattice L with

$$J(\mathrm{Con}\ L) \cong J(D)$$

and with an ideal L' with

$$J(\mathrm{Con}\ L') \cong J(D')$$

such that each join-irreducible congruence Θ of L' extends to the congruence $\Theta\psi^*$ of L.

We take chains C_0, a working axis for L, and C_1, a control axis for L. The chain C_0 will have an ideal C_0', which will be a working axis for L', and the chain C_1 will have an ideal C_1', a control axis for L'. For each $i = 0, 1$, the chain C_i' will have a coloring

$$\varphi_i'\colon\ \mathrm{Ip}\ C_i' \longrightarrow J(D'),$$

and the chain C_i will have a coloring

$$\varphi_i\colon\ \mathrm{Ip}\ C_i \longrightarrow J(D).$$

Thus, each prime interval $\mathfrak{p}$ of C_i' will have *two* colors, $\mathfrak{p}\varphi_i'$ and $\mathfrak{p}\varphi_i$. The colorings φ_i will determine the lattice L, and the colorings φ_i' will determine its ideal L'.

The colorings on the working axis C_0 will have two tasks to perform. The first is to ensure that the partial orderings in $J(\mathrm{Con}\ L)$ and $J(\mathrm{Con}\ L')$ are correct, exactly as in the above outline of the construction in [9]. The second task, which was not required in [9], is to ensure that each $x \in J(D')$ is identified, in Con L, with $x\psi^*$.

We first describe the chain C_0' and its coloring φ_0'. Let s' be the number of isolated elements of $J(D')$, that is, the number of elements of $J(D')$ that are incomparable to all other elements of $J(D')$. Let

$$h_0', h_1', \ldots, h_{s'-1}'$$

be an arbitrary listing of the isolated elements. Then C_0' has an ideal

$$c_0 \prec c_1 \prec \cdots \prec c_{s'},$$

and, for each i, $0 \leq i < s'$, we set

$$[c_i, c_{i+1}]\varphi_0' = h_i'.$$

We list all the covering pairs in $J(D')$:

$$a_0' \prec b_0',\ a_1' \prec b_1', \ldots,\ a_{n'-1}' \prec b_{n'-1}'$$

(note: $a_i' \prec b_i'$ in $J(D')$, not in D'), so that

(3-7) $$\text{whenever } b_i' < b_j' \text{ in } D', \text{ then } i > j,$$

this latter to ensure that Requirement 1 be satisfied. We then set the rest of C_0' to be

$$c_{s'} \prec c_{s'+1} \prec \cdots \prec c_{s'+3n'}.$$

We extend the coloring φ_0' so that, for each i, the interval $[c_{s'+3i}, c_{s'+3i+3}]$ is an (a_i', b_i')-interval, that is, for each i, $0 \leq i < n'$, we set

$$[c_{s'+3i}, c_{s'+3i+1}]\varphi_0' = [c_{s'+3i+2}, c_{s'+3i+3}]\varphi_0' = b_i',$$
$$[c_{s'+3i+1}, c_{s'+3i+2}]\varphi_0' = a_i'.$$

We have thus defined the chain C_0',

$$c_0 \prec c_1 \prec \cdots \prec c_{s'+3n'}$$

and its coloring

$$\varphi_0' \colon \operatorname{Ip} C_0' \longrightarrow J(D'),$$

which, in contrast to the construction in [9], is always *surjective*.

We now describe the control axis C_1'. Let

$$e_0', e_1', \ldots, e_{r'-1}'$$

list all the nonisolated elements of $J(D')$ in such a manner that

(3-8) $$e_i' < e_j' \text{ implies } j < i,$$

in order to ensure, again, that Requirement 1 be satisfied. We set C_1' to be the chain

$$d_0 \prec d_1 \prec \cdots \prec d_{r'}$$

and define the coloring $\varphi_1' \colon \operatorname{Ip} C_1' \longrightarrow j(D')$ by setting

$$[d_i, d_{i+1}]\varphi_1' = e_i'$$

for each i, $0 \leq i < r'$. Note that, since each isolated color occurs exactly once on the working axis C_0', we do not need any isolated color on the control axis C_1'. Note also that, in contrast to the example of Figure 7, any nonisolated color may occur many times in C_0'—it may have many upper covers in $J(D')$—and so, for the sake of uniformity, the coloring of C_1' includes all nonisolated elements of $J(D')$.

We now extend C_0' to a chain C_0. Let

$$a_0 \prec b_0,\ a_1 \prec b_1, \dots,\ a_{n-1} \prec b_{n-1}$$

be a listing of all the covering pairs of $J(D)$, where, again,

$$b_i < b_j \text{ implies } j < i. \tag{3-9}$$

We extend C_0' by taking elements

$$c_{s'+3n'} \prec c_{s'+3n'+1} \prec \cdots \prec c_{s'+3n'+3n}$$

and defining the coloring $\varphi_0\colon\ \mathrm{Ip}\ C_0 \longrightarrow J(D)$ by setting

$$[c_{s'+3n'+3i}, c_{s'+3n'+3i+1}]\varphi_0 = [c_{s'+3n'+3i+2}, c_{s'+3n'+3i+3}]\varphi_0 = b_i,$$
$$[c_{s'+3n'+3i+1}, c_{s'+3n'+3i+2}]\varphi_0 = a_i,$$

for each i, $0 \leq i < n$, and setting

$$\mathfrak{p}\varphi_0 = \mathfrak{p}\varphi_0'\psi^*,$$

for each prime interval $\mathfrak{p} \in \mathrm{Ip}\ C_0'$.

Thus, for each $i < n$,

$$[c_{s'+3n'+3i}, c_{s'+3n'+3i+3}]$$

is the unique (a_i, b_i)-interval in

$$C_0 - [c_0, c_{s'+3n'-1}].$$

Finally, let

$$e_0, e_1, \dots, e_{r-1}$$

be a listing of *all* the elements of $J(D)$ with

$$e_i < e_j \text{ implying } j < i. \tag{3-10}$$

We extend C_1' to a chain C_1 by adding elements $d_{r'+1}, \dots, d_{r'+r}$ with

$$d_{r'} \prec d_{r'+1} \prec \cdots \prec d_{r'+r}.$$

We define the coloring $\varphi_1\colon\ \mathrm{Ip}\ C_1 \longrightarrow J(D)$ by setting

$$[d_{r'+i}, d_{r'+i+1}]\varphi_1 = e_i$$

for each i, $0 \leq i < r$, and setting

$$\mathfrak{p}\varphi_1 = \mathfrak{p}\varphi_1'\psi^*$$

for each prime interval $\mathfrak{p} \in \operatorname{Ip} C_1'$. Observe that φ_1 is surjective on $\operatorname{Ip}\,[d_{r'}, d_{r'+r}]$.

Note the asymmetry between the roles of $J(D')$ and $J(D)$ in the construction. All elements of $J(D')$ appear as colors in C_0', since we must control ψ^* as well as the partial order in C_0', and so no isolated element of $J(D')$ need appear as a color in C_1'. On the other hand, in the chain $[c_{s'+3n'}, c_{s'+3n'+3n}]$, we need only control the partial order, and thus no isolated colors appear. But, then, all elements of $J(D)$ must appear as colors in the chain $[d_{r'}, d_{r'+r}]$. This latter is also required to control ψ^*, as will be evident shortly.

We are now ready to describe the construction of the lattice L and its ideal L'. For each i, $0 \leq i < n'$, there is exactly one $\delta(s'+3i)$ with $d_{\delta(s'+3i)} < d_r'$, that is, with

$$[d_{\delta(s'+3i)}, d_{\delta(s'+3i)+1}] \in \operatorname{Ip} C_1',$$

satisfying

$$[d_{\delta(s'+3i)}, d_{\delta(s'+3i)+1}]\varphi_1' = [c_{s'+3i}, c_{s'+3i+1}]\varphi_0'.$$

Similarly, for each i, $0 \leq i < n$, there is exactly one $\delta(s'+3n'+3i)$ with $d_{r'} \leq d_{\delta(s'+3n'+3i)} < d_{r+r'}$, that is, with

$$[d_{\delta(s'+3n'+3i)}, d_{\delta(s'+3n'+3i)+1}] \in \operatorname{Ip}\,[d_{r'}, d_{r'+r}],$$

satisfying

$$[d_{\delta(s'+3n'+3i)}, d_{\delta(s'+3n'+3i)+1}]\varphi_1 = [c_{s'+3n'+3i}, c_{s'+3n'+3i+1}]\varphi_0.$$

Let the sublattice A of $C_0 \times C_1$ be defined by requiring that

$$\langle u, v\rangle \in A \iff \begin{cases} v \in C_1, & \text{in case } c_0 \leq u \leq c_{s'};\\ v \geq d_{\delta(s'+3i)}, & \text{in case } c_{s'+3i} < u \leq c_{s'+3i+3} \text{ for } 0 \leq i < n';\\ v \geq d_{\delta(s'+3n'+3i)}, & \text{in case } c_{s'+3n'+3i} < u \leq c_{s'+3n'+3i+3}\\ & \text{for } 0 \leq i < n. \end{cases}$$

That A is a sublattice of $C_0 \times C_1$ follows easily from (3-7), (3-8), (3-9), and (3-10). Note that A is the union of the sublattices

$$[c_0, c_{s'}] \times C_1,$$
$$[c_{s'+3i}, c_{s'+3i+3}] \times [d_{\delta(s'+3i)})_{C_1}$$

for all i, $0 \leq i < n'$, and

$$[c_{s'+3n'+3i}, c_{s'+3n'+3i+3}] \times [d_{\delta(s'+3n'+3i)})_{C_1}$$

for all i, $0 \leq i < n$.

We define a set Λ of intervals of the form $\mathfrak{p} \times \mathfrak{q}$, $\mathfrak{p} \in \operatorname{Ip} C_0$, $\mathfrak{q} \in \operatorname{Ip} C_1$, in A by requiring that if $\mathfrak{p} \times \mathfrak{q} \subseteq A$ with $\mathfrak{p} \in \operatorname{Ip} C_0'$, $\mathfrak{q} \in \operatorname{Ip} C_1'$, then

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda \iff \mathfrak{p}\varphi_0' = \mathfrak{q}\varphi_1',$$

if $\mathfrak{p} \times \mathfrak{q} \subseteq A$ with $\mathfrak{p} \in \operatorname{Ip} [c_{s'+3n'}, c_{s'+3n'+3n}]$, that is, with $\mathfrak{p} \notin \operatorname{Ip} C_0'$, and with $\mathfrak{q} \in \operatorname{Ip} C_1$, then

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda \iff \mathfrak{p}\varphi_0 = \mathfrak{q}\varphi_1,$$

if $\mathfrak{p} \times \mathfrak{q} \subseteq A$ with $\mathfrak{p} \in \operatorname{Ip} C_0$ and with $\mathfrak{q} \in \operatorname{Ip} [d_{r'}, d_{r'+r}]$, that is, with $\mathfrak{q} \notin \operatorname{Ip} C_1'$, then

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda \iff \mathfrak{p}\varphi_0 = \mathfrak{q}\varphi_1.$$

We set

$$A' = A \cap (C_0' \times C_1'),$$

that is, A' is the ideal $(\langle c_{s'+3n'}, d_{r'}\rangle]$ of A. We let Λ' be the set of those intervals in Λ that are subsets of A'. Thus

$$\Lambda' = \{\mathfrak{p} \times \mathfrak{q} \subseteq A \mid \mathfrak{p} \in \operatorname{Ip} C_0', \mathfrak{q} \in \operatorname{Ip} C_1', \mathfrak{p}\varphi_0' = \mathfrak{q}\varphi_1'\}.$$

LEMMA 3. *The set of intervals Λ' of A' satisfies Requirement 1 with respect to the colorings $\varphi_i'\colon \operatorname{Ip} C_i' \longrightarrow J(D')$. The set of intervals Λ of A satisfies Requirement 1 with respect to the colorings $\varphi_i\colon \operatorname{Ip} C_i \longrightarrow J(D)$.*

PROOF. We first establish Requirement 1 for Λ'. Condition (3-4) for Λ' and the colorings φ_i', and the implication

$$\text{if } \mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda', \text{ then } \mathfrak{p}_0\varphi_0' = \mathfrak{p}_1\varphi_1',$$

follow immediately from the definition of Λ' by virtue of the definition of Λ. It is also easy to see that condition (3-5),

for each $x \in (\operatorname{Ip} C_0')\varphi_0' \cap (\operatorname{Ip} C_1')\varphi_1'$, there is a $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda'$ with $\mathfrak{p}_0\varphi_0' = \mathfrak{p}_1\varphi_1' = x$,

holds. Indeed, since $(\operatorname{Ip} C_1')\varphi_1'$ consists only of nonisolated elements of $J(D')$, there is an i, $0 \leq i < n'$, with either

$$x = [c_{s'+3i}, c_{s'+3i+1}]\varphi_0'$$

or

$$x = [c_{s'+3i+1}, c_{s'+3i+2}]\varphi_0'.$$

In the first case, set

$$\mathfrak{p}_0 = [c_{s'+3i}, c_{s'+3i+1}], \qquad \mathfrak{p}_1 = [d_{\delta(s'+3i)}, d_{\delta(s'+3i)+1}].$$

Then, by definition of A', we have $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda'$.

In the second case, there is exactly one j, $0 \leq j < r'$ with

$$x = [d_j, d_{j+1}]\varphi_1'.$$

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Set

$$\mathfrak{p}_0 = [c_{s'+3i+1}, c_{s'+3i+2}], \quad \mathfrak{p}_1 = [d_j, d_{j+1}].$$

Since

$$x = [c_{s'+3i+1}, c_{s'+3i+2}]\varphi_0' < [c_{s'+3i}, c_{s'+3i+1}]\varphi_0' = [d_{\delta(s'+3i)}, d_{\delta(s'+3i)+1}]\varphi_1',$$

it follows by (3-8) that $d_j > d_{\delta(s'+3i)}$ in C_1', and so that

$$\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda',$$

establishing (3-5).

Finally, we establish (3-6),

for each $i = 0, 1$, if $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ C_i'$ are distinct and $\mathfrak{p}\varphi_i' = \mathfrak{p}'\varphi_i'$, then there are $\bar{\mathfrak{p}} \in \mathrm{Ip}\ C_i'$, $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Ip}\ C_{1-i}'$ with

$$\begin{aligned} &\mathfrak{p} \times \mathfrak{q},\ \bar{\mathfrak{p}} \times \mathfrak{q},\ \mathfrak{p}' \times \mathfrak{q}',\ \bar{\mathfrak{p}} \times \mathfrak{q}' \in \Lambda', && \text{if } i = 0,\\ &\mathfrak{q} \times \mathfrak{p},\ \mathfrak{q} \times \bar{\mathfrak{p}},\ \mathfrak{q}' \times \mathfrak{p}',\ \mathfrak{q}' \times \bar{\mathfrak{p}} \in \Lambda', && \text{if } i = 1,\end{aligned}$$

for Λ'. We, in fact, establish the stronger form (3-6′). Since φ_1' is injective, we need only take $i = 0$. Again, since φ_0' is injective on $\mathrm{Ip}\ [c_0, c_{s'}]$ and $(\mathrm{Ip}\ [c_0, c_{s'}])\varphi_0'$ consists of isolated elements of $J(D')$ while $(\mathrm{Ip}\ [c_{s'}, c_{s'+3n'}])\varphi_0'$ consists of nonisolated elements, there are j, k with $j \leq k$ and, without loss of generality, with

$$\mathfrak{p} \in \mathrm{Ip}\ [c_{s'+3j}, c_{s'+3j+3}]$$

and

$$\mathfrak{p}' \in \mathrm{Ip}\ [c_{s'+3k}, c_{s'+3k+3}].$$

Then, as in the proof of (3-5) above, there is a $t \geq \delta(s' + 3k)$ with

$$[d_t, d_{t+1}]\varphi_1' = \mathfrak{p}'\varphi_0'.$$

Setting $\mathfrak{q} = \mathfrak{q}' = [d_t, d_{t+1}]$ and $\bar{\mathfrak{p}} = \mathfrak{p}$ establishes (3-6) with $i = 0$, concluding the proof that Λ' satisfies Requirement 1 with respect to the colorings φ_i'.

We now establish Requirement 1 for Λ. Condition (3-4) for Λ and the colorings φ_i follows immediately from the definition of Λ—we need only observe that $\varphi_i = \varphi_i'\psi^*$ on $\mathrm{Ip}\ C_i'$.

To establish (3-5) and (3-6) for Λ, we observe that the full product is contained in A:

$$C_0' \times [d_{r'}, d_{r'+r}] \subseteq A. \tag{3-11}$$

We now establish condition (3-5),

for each $x \in (\mathrm{Ip}\ C_0)\varphi_0 \cap (\mathrm{Ip}\ C_1)\varphi_1$, there is a $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$ with $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1 = x$,

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for Λ. If $x \in \big(J(D')\big)\psi^*$, then, since

$$\varphi_0'\colon \operatorname{Ip} C_0' \longrightarrow J(D')$$

and

$$\varphi_1\colon \operatorname{Ip}\,[d_{r'}, d_{r'+r}] \longrightarrow J(D)$$

are surjective, and since $\varphi_0 = \varphi_0'\psi^*$, there are $\mathfrak{p}_0 \in \operatorname{Ip} C_0'$, $\mathfrak{p}_1 \in \operatorname{Ip}\,[d_{r'}, d_{r'+r}]$ with

$$x = \mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1.$$

Then, by (3-11) and the definition of Λ,

$$\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda.$$

Otherwise,

$$x \in (\operatorname{Ip}\,[c_{s'+3n'}, c_{s'+3n'+3n}])\varphi_0 \cap (\operatorname{Ip}\,[d_{r'}, d_{r'+r}])\varphi_1,$$

and we proceed in a manner similar to our proof of (3-5) for Λ' above. Indeed, there is an i, $0 \leq i < n$, with either

$$x = [c_{s'+3n'+3i}, c_{s'+3n'+3i+1}]\varphi_0$$

or

$$x = [c_{s'+3n'+3i+1}, c_{s'+3n'+3i+2}]\varphi_0.$$

In the first case, set

$$\mathfrak{p}_0 = [c_{s'+3n'+3i}, c_{s'+3n'+3i+1}], \qquad \mathfrak{p}_1 = [d_{\delta(s'+3n'+3i)}, d_{\delta(s'+3n'+3i)+1}].$$

Then, by definition of A, we have $\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda$.

In the second case, there is exactly one j, $r' \leq j < r' + r$ with

$$x = [d_j, d_{j+1}]\varphi_1.$$

Set

$$\mathfrak{p}_0 = [c_{s'+3n'+3i+1}, c_{s'+3n'+3i+2}], \qquad \mathfrak{p}_1 = [d_j, d_{j+1}].$$

Since

$$\begin{aligned} x = [c_{s'+3n'+3i+1}, c_{s'+3n'+3i+2}]\varphi_0 &< [c_{s'+3n'+3i}, c_{s'+3n'+3i+1}]\varphi_0 \\ &= [d_{\delta(s'+3n'+3i)}, d_{\delta(s'+3n'+3i)+1}]\varphi_1, \end{aligned}$$

it follows by (3-10) that $d_j > d_{\delta(s'+3n'+3i)}$ in $[d_{r'}, d_{r'+r}]$, and so that

$$\mathfrak{p}_0 \times \mathfrak{p}_1 \in \Lambda,$$

concluding our verification of (3-5).

Finally, we establish (3-6),

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for each $i = 0, 1$, if $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ C_i$ are distinct and $\mathfrak{p}\varphi_i = \mathfrak{p}'\varphi_i$, then there are $\bar{\mathfrak{p}} \in \mathrm{Ip}\ C_i$, $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Ip}\ C_{1-i}$ with

$$\mathfrak{p} \times \mathfrak{q},\ \bar{\mathfrak{p}} \times \mathfrak{q},\ \mathfrak{p}' \times \mathfrak{q}',\ \bar{\mathfrak{p}} \times \mathfrak{q}' \in \Lambda, \quad \text{if } i = 0,$$
$$\mathfrak{q} \times \mathfrak{p},\ \mathfrak{q} \times \bar{\mathfrak{p}},\ \mathfrak{q}' \times \mathfrak{p}',\ \mathfrak{q}' \times \bar{\mathfrak{p}} \in \Lambda, \quad \text{if } i = 1,$$

for Λ.

First, let $i = 0$, that is, let $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ C_0$ with $\mathfrak{p}\varphi_0 = \mathfrak{p}'\varphi_0$. We establish the stronger (3-6′).

If both $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ C_0'$, then, since φ_1 is surjective on $\mathrm{Ip}\ [d_{r'}, d_{r'+r}]$, let $\mathfrak{q}$ be a prime interval in $[d_{r'}, d_{r'+r}]$ with

$$\mathfrak{q}\varphi_1 = \mathfrak{p}\varphi_1 = \mathfrak{p}'\varphi_1.$$

By (3-11),

$$\mathfrak{p} \times \mathfrak{q},\ \mathfrak{p} \times \mathfrak{q}' \in \Lambda.$$

If both $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Ip}\ [c_{s'+3n'}, c_{s'+3n'+3n}]$, then we proceed as for Λ' above. There are j, k with $j \leq k$ and, without loss of generality, with

$$\mathfrak{p} \in \mathrm{Ip}\ [c_{s'+3n'+3j}, c_{s'+3n'+3j+3}]$$

and

$$\mathfrak{p}' \in \mathrm{Ip}\ [c_{s'+3n'+3k}, c_{s'+3n'+3k+3}].$$

Then, there is a $t \geq \delta(s' + 3n' + 3k)$ with

$$[d_t, d_{t+1}]\varphi_1 = \mathfrak{p}'\varphi_0.$$

Setting $\mathfrak{q} = [d_t, d_{t+1}]$ establishes (3-6′) in this case.

If $\mathfrak{p} \in \mathrm{Ip}\ C_0'$ and $\mathfrak{p}' \in \mathrm{Ip}\ [c_{s'+3n'}, c_{s'+3n'+3n}]$, then, as above, there is a prime interval $\mathfrak{q}$ in $[d_{r'}, d_{r'+r}]$ with

$$\mathfrak{p}' \times \mathfrak{q} \in \Lambda.$$

Then, by (3-11),

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda,$$

concluding the verification of (3-6) for the case $i = 0$.

Finally, we consider the case $i = 1$ of (3-6). Since φ_1 is injective on $\mathrm{Ip}\ [d_{r'}, d_{r'+r}]$, we may assume, without loss of generality, that

$$\mathfrak{p} \in \mathrm{Ip}\ C_1'.$$

Since φ_0' is surjective on $\mathrm{Ip}\ C_0'$, we conclude, by (3-5′) for Λ', that there is a $\mathfrak{q} \in \mathrm{Ip}\ C_0'$ with

$$\mathfrak{q} \times \mathfrak{p} \in \Lambda' \subseteq \Lambda.$$

If $\mathfrak{p}' \in \mathrm{Ip}\ [d_{r'}, d_{r'+r}]$, then, again by (3-11), we have

$$\mathfrak{q} \times \mathfrak{p}' \in \Lambda,$$

establishing the stronger version (3-6′) of (3-6) again. If, on the other hand, $\mathfrak{p}' \in \mathrm{Ip}\ C_1'$, then, as above, there is a $\mathfrak{q}' \in \mathrm{Ip}\ C_0'$ with

$$\mathfrak{q}' \times \mathfrak{p}' \in \Lambda' \subseteq \Lambda.$$

Observe that

$$\mathfrak{q}\varphi_0' = \mathfrak{p}\varphi_1'$$

and

$$\mathfrak{q}'\varphi_0' = \mathfrak{p}'\varphi_1'.$$

Then

$$\mathfrak{q}\varphi_0 = \mathfrak{q}\varphi_0'\psi^* = \mathfrak{p}\varphi_1'\psi^* = \mathfrak{p}\varphi_1 = \mathfrak{p}'\varphi_1 = \mathfrak{p}'\varphi_1'\psi^* = \mathfrak{q}'\varphi_0'\psi^* = \mathfrak{q}'\varphi_0.$$

Choose $\bar{\mathfrak{p}} \in \mathrm{Ip}\ [d_{r'}, d_{r'+r}]$ with $\bar{\mathfrak{p}}\varphi_1 = \mathfrak{q}\varphi_0$. Then, by (3-11),

$$\mathfrak{q} \times \bar{\mathfrak{p}},\ \mathfrak{q}' \times \bar{\mathfrak{p}} \in \Lambda,$$

establishing (3-6) in this final case. Note that it is only here that we can verify only (3-6), rather than the stronger (3-6′).

The proof of the lemma is hereby concluded. ∎

We now apply the $\mathfrak{M}_3$-construction to all of the intervals in Λ, thereby obtaining the lattice K and its ideal $K' = (\langle c_{s'+3n'}, d_{r'}\rangle]$, which is the lattice obtained from A' by applying the $\mathfrak{M}_3$-construction to all the intervals in Λ'. It should be fairly clear that $J(\mathrm{Con}\ K)$ is isomorphic to the *discrete* poset $J(D)$, that $J(\mathrm{Con}\ K')$ is isomorphic to the discrete poset $J(D')$, and that restriction of congruences from K to K' yields, under these isomorphisms, the *set mapping*

$$\psi^*\colon J(D') \longrightarrow J(D).$$

We then apply the $\mathfrak{N}_5$-construction to each of the intervals

$$[c_{s'+3i}, c_{s'+3i+3}] \times \{d_{\delta(s'+3i)}\}$$

for $0 \leq i < n'$, adding a new atom denoted $u(c_{s'+3i}, c_{s'+3i+3})$, and to each of the intervals

$$[c_{s'+3n'+3i}, c_{s'+3n'+3i+3}] \times \{d_{\delta(s'+3n'+3i)}\}$$

for $0 \leq i < n$, adding a new atom denoted $u(c_{s'+3n'+3i}, c_{s'+3n'+3i+3})$, thereby getting the lattice L and its ideal $L' = (\langle c_{s'+3n'}, d_{r'}\rangle]$. We then get isomorphisms of posets

$$J(\mathrm{Con}\ L) \cong J(D)$$

and

$$J(\mathrm{Con}\ L') \cong J(D')$$

such that the restriction of congruences from L to L' yields, under these isomorphisms, the isotone map

$$\psi^* : J(D') \longrightarrow J(D),$$

that is, we have lattice isomorphisms

$$\operatorname{Con} L \cong D$$

and

$$\operatorname{Con} L' \cong D'$$

such that, under these isomorphisms, the restriction of congruences from L to L' is the homomorphism ψ. A formal proof will be given in Section 5.

We now turn to the automorphisms of L and its ideal L', as constructed above. In all cases, L' will have only the identity automorphism, but, in certain special cases, L may have other automorphisms—in these cases our construction will have to be modified.

One such special case occurs when $J(D')$ is not empty and

$$[c_0, c_1]\varphi_0 = [c_0, c_1]\varphi_0'\psi^* = [d_{r'+r-1}, d_{r'+r}]\varphi_1, \tag{3-12}$$

that is, when the bottommost prime interval in C_0 has the same color under φ_0 as the topmost prime interval in C_1 under φ_1. Then the interval

$$[c_0, c_1] \times [d_{r'+r-1}, d_{r'+r}] \in \Lambda,$$

and so, in constructing L, we end up applying the $\mathfrak{M}_3$-construction to this interval by adding the new element

$$m = m([c_0, c_1], [d_{r'+r-1}, d_{r'+r}]).$$

Then, interchanging m and $\langle c_0, d_{r'+r}\rangle$ (and fixing all other elements) yields an automorphism of L. The relevant part of L is depicted in Figure 8. Usually, there is some freedom in coloring the intervals in C_0 and C_1, and so the difficulty (3-12) can be avoided. Sometimes, though, such avoidance is impossible. For example, if $J(D)$ has a unique minimal element x, and if $J(D')$ has isolated elements all of which map under ψ^* to x, then, under the construction presented above, (3-12) will always occur.

Other sources of difficulty can occur when both $J(D)$ and $J(D')$ are discrete, that is, when both D and D' are Boolean. Then

$$n = n' = 0$$

and

$$r' = 0,$$

and so $C_0 = C_0'$ is a chain of length $s' = |J(D')|$, $C_1' = \{d_0\}$, and C_1 is a chain of length $r = |J(D)|$. The $\mathfrak{N}_5$-construction is not applied, and so $L = K$. One difficulty, similar to that of (3-12), that can then occur is that the colorings are chosen so that

$$[c_{s'-1}, c_{s'}]\varphi_0 = [c_{s'-1}, c_{s'}]\varphi_0'\psi^* = [d_0, d_1]\varphi_1, \tag{3-13}$$

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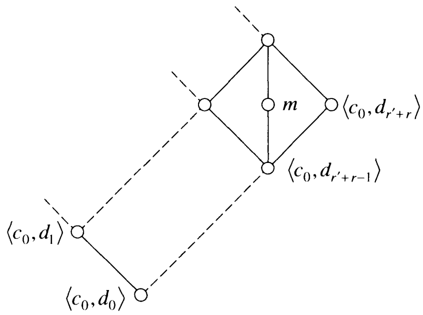

FIGURE 8

that is, when the topmost prime interval in C_0 has the same color under φ_0 as the bottommost prime interval in C_1 under φ_1. Then, as above, we get the new element $m([c_{s'-1}, c_{s'}], [d_0, d_1])$ which can be interchanged with $\langle c_{s'}, d_0\rangle$ to get a non-identity automorphism of L.

A second type of difficulty, for the case of both D and D' Boolean, can occur if

(3-14) $\psi\colon D \longrightarrow D'$ is an isomorphism.

Then $s' = r$ and so $C_0 \cong C_1$. If the colorings φ_0' and φ_1 are chosen so that

$$[c_i, c_{i+1}]\varphi_0 = [d_i, d_{i+1}]\varphi_1$$

for all i, then the automorphism of $A = C_0 \times C_1$

$$\langle c_i, d_j\rangle \longmapsto \langle c_j, d_i\rangle$$

determined by interchanging the axes extends to an automorphism of L. For example, let D' and D be the four-element Boolean lattice $\{0, a, b, 1\}$ with $0 < a < 1$ and $0 < b < 1$, and let ψ be the identity mapping. Then the chain C_0 is

$$c_0 \prec c_1 \prec c_2,$$

and C_1 is the chain

$$d_0 \prec d_1 \prec d_2.$$

If we chose the coloring φ_0' with $[c_0, c_1]\varphi_0' = a$ and $[c_1, c_2]\varphi_0' = b$ and the coloring φ_1 with $[d_0, d_1]\varphi_1 = a$ and $[d_1, d_2]\varphi_1 = b$, the resulting lattice L is depicted in Figure 9. (The ideal L' is the ideal whose maximal element is depicted as $\bullet$.) The interchange of the two axes is an automorphism.

We first present a formal proof that the difficulties (3-12), (3-13), and (3-14) are the only difficulties that can occur. Then, we discuss how these difficulties can be overcome.

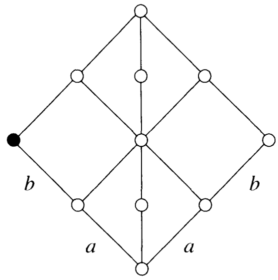

FIGURE 9

LEMMA 4. *The lattice L' has no automorphism other than the identity mapping. The lattice L has no automorphism other than the identity mapping* unless *one of the following three conditions holds:*

(1) $[c_0, c_1]\varphi_0 = [d_{r'+r-1}, d_{r'+r}]\varphi_1$.
(2) *D and D' are both Boolean and* $[c_{s'-1}, c_{s'}]\varphi_0 = [d_0, d_1]\varphi_1$.
(3) *D and D' are both Boolean and $\psi\colon D \longrightarrow D'$ is an isomorphism.*

PROOF. We first show that L' has no automorphism other than the identity mapping. Let $\alpha\colon L' \longrightarrow L'$ be an automorphism. If D' is Boolean, then C_1' is a singleton; so A is a chain isomorphic to C_0', and $L' = A$. Thus, α is an automorphism of a finite chain, and so is the identity mapping.

Otherwise, $n' > 0$. All the elements of $L' - A'$ are doubly irreducible in L', and none of them lies in an interval of L' which is a four-element Boolean lattice. The only elements of A' doubly irreducible in A' are the $\langle c_{s'+3i+3}, d_{\delta(s'+3i)}\rangle$ for $0 \leq i < n'$ and $\langle c_0, d_{r'}\rangle$. But, in L', we have the proper join

$$\langle c_{s'+3i+3}, d_{\delta(s'+3i)}\rangle = \langle c_{s'+3i+2}, d_{\delta(s'+3i)}\rangle \vee u(c_{s'+3i}, c_{s'+3i+3}).$$

Thus, the only element of A doubly-irreducible in L' is $\langle c_0, d_{r'}\rangle$. Now, observe that

$$[c_0, c_1]\varphi_0' \neq [d_{r'-1}, d_{r'}]\varphi_1'.$$

Indeed, if $s' > 0$ then $[c_0, c_1]\varphi_0'$ is an isolated element of $J(D')$, while, by definition of φ_1', no image of φ_1' is isolated. If $s' = 0$ then $[c_0, c_1]\varphi_0'$ is not minimal in $J(D')$, while, by (3-8), $[d_{r'-1}, d_{r'}]\varphi_1'$ is minimal. Consequently,

$$[c_0, c_1] \times [d_{r'-1}, d_{r'}] \notin \Lambda'$$

and so it remains an interval after we apply the $\mathfrak{M}_3$-construction to A'. Thus, $\langle c_0, d_{r'}\rangle$ is the only doubly-irreducible element of L' contained in an interval that is a four-element Boolean lattice. Thus,

$$\langle c_0, d_{r'}\rangle\alpha = \langle c_0, d_{r'}\rangle$$

and so

$$(L' - A')\alpha = L' - A',$$

that is, α restricts to an automorphism of A' that fixes $\langle c_0, d_{r'}\rangle$. But, then, the chain $\{c_0\} \times C_1'$ is fixed by α. It then follows easily that α is the identity mapping on A'. Since the elements of $L' - A'$ are determined uniquely by those of A', we conclude that α is the identity on L'. Thus, L' has only the identity automorphism.

Now, let none of (4.1), (4.2), (4.3) hold, and let α be an automorphism of L. We first consider the case when at least one of D, D' is not Boolean. Then, we proceed exactly as we did above for L' with D' not Boolean. Since (4.1) fails, we have

(3-15) $$[c_0, c_1] \times [d_{r'+r-1}, d_{r'+r}] \notin \Lambda,$$

and $\langle c_{s'+3n'+3n}, d_0\rangle$ is join-reducible in L. Thus,

$$A\alpha = A$$

and

$$\langle c_0, d_{r'+r}\rangle\alpha = \langle c_0, d_{r'+r}\rangle.$$

Consequently, as above, α is the identity mapping on A, and so is the identity mapping on L.

Next, we consider the case when both D and D' are Boolean. Then, since (4.2) fails, we also have

(3-16) $$[c_{s'-1}, c_{s'}] \times [d_0, d_1] \notin \Lambda.$$

Note that, in this case, $n' = n = 0$ and $c_{s'}$ is the maximum element of C_0, and $\langle c_{s'}, d_0\rangle$ is join-irreducible in $L = K$. Also, $r' = 0$. Then, by (3-15) and (3-16), $\langle c_{s'}, d_0\rangle$ and $\langle c_0, d_{r'+r}\rangle = \langle c_0, d_r\rangle$ are the only doubly-irreducible elements of L that lie in an interval that is a four-element Boolean lattice. Then, again,

$$A\alpha = A,$$

and, also,

$$\big(\{\langle c_{s'}, d_0\rangle, \langle c_0, d_{r'+r}\rangle\}\big)\alpha = \{\langle c_{s'}, d_0\rangle, \langle c_0, d_{r'+r}\rangle\}.$$

Since (4.3) fails, either

$$s' = |J(D')| \neq r = |J(D)|$$

or

$$\psi^* \colon J(D') \longrightarrow J(D)$$

is not surjective. Now, the height in A of $\langle c_{s'}, d_0\rangle$ is s' and that of $\langle c_0, d_{r'+r}\rangle$ is r. Thus, if $s' \neq r$, then both $\langle c_{s'}, d_0\rangle$ and $\langle c_0, d_{r'+r}\rangle$ are fixed by α. Consequently, α is the identity mapping on A, and therefore on L. If, on the other hand, $s' = r$, then either α is the identity mapping on A or

(3-17) $$\langle c_{s'}, d_0\rangle\alpha = \langle c_0, d_{r'+r}\rangle \text{ and } \langle c_0, d_{r'+r}\rangle\alpha = \langle c_{s'}, d_0\rangle.$$

Since ψ^* is not surjective, there is an i, $0 \leq i < r = s'$, with

$$\mathfrak{p} \times [d_i, d_{i+1}] \notin \Lambda$$

for all $\mathfrak{p} \in \operatorname{Ip} C_0$, that is, such that the interval

$$[\langle c_0, d_i \rangle, \langle c_{s'}, d_{i+1} \rangle]$$

of L is distributive. But, if (3-17) holds, then the interval

$$[\langle c_i, d_0 \rangle, \langle c_{i+1}, d_r \rangle] = [\langle c_0, d_i \rangle, \langle c_{s'}, d_{i+1} \rangle]\alpha$$

of L contains the non-distributive interval

$$[\langle c_i, d_j \rangle, \langle c_{i+1}, d_{j+1} \rangle]$$

isomorphic to $\mathfrak{M}_3$, where $[d_j, d_{j+1}]$ is an interval of C_1 with color $[c_i, c_{i+1}]\varphi_0$ under φ_1. Thus, (3-17) is impossible, and so in this case also α is the identity mapping on L, concluding the proof of the lemma. ■

We now describe how the exceptional cases described by (4.1), (4.2), (4.3) can be handled.

We first treat the case when at least one of D', D is not Boolean and (4.1) holds. There is some freedom in assigning the colorings φ_0 and φ_1, and we can often redefine these colorings so that (4.1) fails. There are certain cases, though, when that cannot be done. One such case is when $J(D')$ has isolated elements, $J(D)$ has a unique minimal element a, and ψ^* maps all the isolated elements of $J(D')$ to a. Then, no matter how we assign the colorings, $[c_0, c_1]\varphi_0'$ is isolated, and, because of (3-10),

$$[d_{r'+r-1}, d_{r'+r}]\varphi_1 = a.$$

Thus, perforce,

$$[c_0, c_1]\varphi_0 = [c_0, c_1]\varphi_0'\psi^* = [d_{r'+r-1}, d_{r'+r}]\varphi_1.$$

Note that the length of C_0 is greater than 1. We can get around the difficulty presented by (4.1) by modifying the chain C_1 by adding a new maximal element $d_{r'+r+1}$. We extend the coloring φ_1 by setting

$$[d_{r'+r}, d_{r'+r+1}]\varphi_1 = [c_1, c_2]\varphi_0, \tag{3-18}$$

extend A by adding $C_0 \times \{d_{r'+r+1}\}$, (that is, the definition of A reads as before, except that r is replaced by $r + 1$), and extend Λ by adding all intervals

$$\mathfrak{p} \times [d_{r'+r}, d_{r'+r+1}]$$

with

$$\mathfrak{p}\varphi_0 = [d_{r'+r}, d_{r'+r+1}]\varphi_1 \text{ and } \mathfrak{p} \neq [c_0, c_1],$$

thereby ensuring that

$$[c_0, c_1] \times [d_{r'+r}, d_{r'+r+1}] \notin \Lambda.$$

Note that C_0, C_0', C_1' and their colorings, and A', Λ' remain unchanged. Then, after applying the $\mathfrak{M}_3$- and $\mathfrak{N}_5$-constructions to this new A, the new lattice L has the property that $\langle c_0, d_{r'+r+1}\rangle$, its only doubly-irreducible element that lies in A, is the only doubly-irreducible element that lies in an interval that is a four-element Boolean lattice. As is evident by the relevant part of the proof of Lemma 4, L then has no automorphism other than the identity mapping. We must verify that Requirement 1 still holds. All that needs to be verified is (3-6) when $i = 1$ and when one of $\mathfrak{p}$, $\mathfrak{p}'$ equals $[d_{r'+r}, d_{r'+r+1}]$. So, let

$$\mathfrak{p} = [d_{r'+r}, d_{r'+r+1}].$$

Then $\mathfrak{p}'$ is in $[d_0, d_{r'+r}]$, the "old" C_1. Then, by (3-5′) for the "old" situation, there is a $\mathfrak{q}' \in \mathrm{Ip}\ C_0$ with

$$\mathfrak{q}' \times \mathfrak{p}' \in \Lambda.$$

If $\mathfrak{q}' \neq [c_0, c_1]$, then

$$\mathfrak{q}' \times \mathfrak{p} \in \Lambda$$

also, and we have (3-6′). Otherwise,

$$\mathfrak{q}' = [c_0, c_1].$$

Set

$$\mathfrak{q} = [c_1, c_2].$$

Then

$$\mathfrak{q} \times \mathfrak{p} \in \Lambda$$

and

$$\mathfrak{q}'\varphi_0 = \mathfrak{p}'\varphi_1 = \mathfrak{p}\varphi_1 = \mathfrak{q}\varphi_0,$$

the last equality by (3-18), the definition of $\mathfrak{p}\varphi_1$. Then, by definition of the "old" Λ, there is a unique

$$\bar{\mathfrak{p}} \in \mathrm{Ip}\ [d_{r'}, d_{r'+r}]$$

with

$$\mathfrak{q} \times \bar{\mathfrak{p}},\ \mathfrak{q}' \times \bar{\mathfrak{p}} \in \Lambda,$$

concluding the verification of (3-6).

Now, we consider the cases when both D and D' are Boolean. Then L' is a chain isomorphic to C_0.

If $\psi^* \colon J(D') \longrightarrow J(D)$ is an isomorphism, we can dispense entirely with the above construction. We let L be a chain of length $|J(D)|$, and set $L' = L$. It is then immediate that neither L nor L' has any automorphism other the identity mapping, that Con $L \cong D$, Con $L' \cong D'$, and that restriction of congruences is thereby the isomorphism ψ.

Thus, we may assume, henceforth, that (4.3) fails. Note that, since $J(D)$ and $J(D')$ are discrete, the colorings of C_0 and C_1 can be assigned in a completely arbitrary manner, subject only to the condition that they be injective. In most cases (4.1) and (4.2) can be avoided. Specifically, if there are $a, b \in J(D')$ with $a\psi^* \neq b\psi^*$, then we can define φ_0' and φ_1 so that

$$[c_0, c_1]\varphi_0' = a, \quad [d_0, d_1]\varphi_1 = a\psi^*$$

and

$$[c_{s'-1}, c_{s'}]\varphi_0' = b, \quad [d_{r-1}, d_r]\varphi_1 = b\psi^*.$$

See Figure 10 for a sketch of the resulting lattice L, where, again, the maximal element

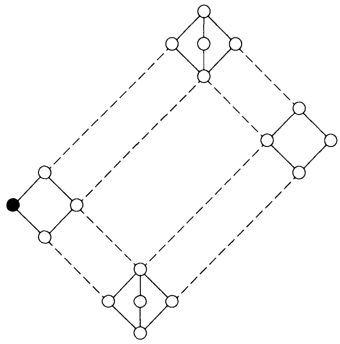

FIGURE 10

of the ideal L' is denoted by $\bullet$.

Similarly, if the image of ψ^* is a singleton $\{a\}$ and $J(D)$ has at least three elements, that is, if $r \geq 3$, then we can chose the coloring φ_1 so that the interval in C_1 colored by a is $[d_1, d_2]$ and, again, both (4.1) and (4.2) fail—see Figure 11 for an example.

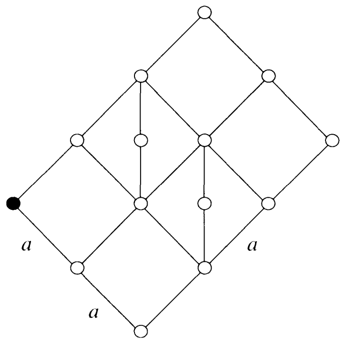

FIGURE 11

However, if the image of ψ^* is a singleton $\{a\}$ and $r \leq 2$, then, if we color $[d_0, d_1]$ by a, we have at least (4.1), and, if $r = 2$, if we color $[d_1, d_2]$ by a, we have (4.2). Thus, L as constructed will have an automorphism other than the identity map—see Figure 12a)

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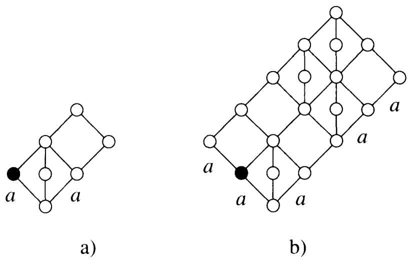


FIGURE 12

for an example. In this case, we must modify our construction of L. We extend the chain C_0 to a chain $\bar{C}_0$ by adding a new maximal element $c_{s'+1}$ and extend the coloring φ_0 by setting

$$[c_{s'}, c_{s'+1}]\varphi_0 = a,$$

and we extend the chain C_1 to a new chain $\bar{C}_1$ by adding at least two new elements $d_{r+1}, \ldots, d_{r+q}$ so that

$$d_r \prec d_{r+1} \prec \cdots \prec d_{r+q}$$

and so that $r + q \neq s' + 1$, that is, so that $\bar{C}_0$ and $\bar{C}_1$ have different lengths. We extend the coloring φ_1 to the new prime intervals by setting

$$[d_{r+i}, d_{r+i+1}]\varphi_1 = a$$

for all i, $0 \leq i < q$. We set

$$A = \bar{C}_0 \times \bar{C}_1$$

and

$$\Lambda = \{\mathfrak{p} \times \mathfrak{q} \mid \mathfrak{p}\varphi_0 = \mathfrak{q}\varphi_1\} - \{[c_{s'}, c_{s'+1}] \times [d_0, d_1],\ [c_0, c_1] \times [d_{r+q-1}, d_{r+q}]\}.$$

We then perform the $\mathfrak{M}_3$-construction on each interval in Λ to get our new L. Note that in this modification the ideal L' remains unchanged—it is still the chain $[\langle c_0, d_0\rangle, \langle c_{s'}, d_0\rangle]$. The modification of Figure 12 a) is depicted in Figure 12 b), where, again, the maximal element of L' is depicted by $\bullet$.

We observe that our new L has no non-identity automorphism. The elements $\langle c_{s'+1}, d_0\rangle$ and $\langle c_0, d_{r+q}\rangle$ are the only doubly-irreducible elements of L that lie in an interval that is a four-element Boolean lattice, and, since $s' + 1 \neq p + q$, they have different heights, and so are fixed by any automorphism. We then immediately conclude that there are no automorphisms other than the identity.

We observe, also, that the new Λ still satisfies Requirement 1. Indeed, only (3-6) needs verification. First, let $i = 0$. Then, for any prime intervals $\mathfrak{p}$, $\mathfrak{p}'$ of $\bar{C}_0$ (they all have the same color a) set $\mathfrak{q} = [d_r, d_{r+1}]$, thereby verifying the stronger (3-6′). Finally, let $i = 1$, and let $\mathfrak{p}$, $\mathfrak{p}'$ be distinct prime intervals of $\bar{C}_1$ with $\mathfrak{p}\varphi_1 = \mathfrak{p}'\varphi_1$. Then

$$\mathfrak{p}\varphi_1 = \mathfrak{p}'\varphi_1 = a,$$

since φ_1 is injective on the prime intervals of the subchain C_1. If neither $\mathfrak{p}$ nor $\mathfrak{p}'$ is $[d_{r+q-1}, d_{r+q}]$, then we get (3-6′) with $\mathfrak{q} = [c_0, c_1]$. Otherwise, if, say, $\mathfrak{p} = [d_{r+q-1}, d_{r+q}]$, then we set

$$\mathfrak{q} = [c_1, c_2], \qquad \mathfrak{q}' = [c_0, c_1], \qquad \bar{\mathfrak{p}} = [d_r, d_{r+1}],$$

thereby getting (3-6).

Summarizing this section, we have:

SUMMARY. If D and D' are both boolean and $\varphi\colon D \longrightarrow D'$ is an isomorphism, we set $L = L' =$ a chain of length $|J(D)| = |J(D')|$.

Otherwise, we have chains C_i containing ideals C_i', $i = 0, 1$, and colorings φ_i: Ip $C_i \longrightarrow J(D)$, φ_i': Ip $C_i' \longrightarrow J(D')$ satisfying the following:

(a) $(\text{Ip } C_0)\varphi_0 \cup (\text{Ip } C_1)\varphi_1 = J(D)$.

(b) If $x \in J(D)$ is not isolated, then $x \in (\text{Ip } C_0)\varphi_0 \cap (\text{Ip } C_1)\varphi_1$.

(c) $(\text{Ip } C_0')\varphi_0' \cup (\text{Ip } C_1')\varphi_1' = J(D')$.

(d) If $x \in J(D')$ is not isolated, then $x \in (\text{Ip } C_0')\varphi_0' \cap (\text{Ip } C_1')\varphi_1'$.

(e) For each $i = 0, 1$, and each $\mathfrak{p} \in \text{Ip } C_i'$, we have $\mathfrak{p}\varphi_i = \mathfrak{p}\varphi_i'\psi^*$.

(f) For each covering pair $a \prec b$ in $J(D')$, there is an (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0', that is,

$$[c_j, c_{j+1}]\varphi_0' = [c_{j+2}, c_{j+3}]\varphi_0' = b$$

and

$$[c_{j+1}, c_{j+2}]\varphi_0' = a.$$

(g) For each covering pair $a \prec b$ in $J(D)$, there is an (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0 with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$, that is,

$$[c_j, c_{j+1}]\varphi_0 = [c_{j+2}, c_{j+3}]\varphi_0 = b$$

and

$$[c_{j+1}, c_{j+2}]\varphi_0 = a.$$

(h) For each covering pair $a \prec b$ in $J(D')$ and each (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0', there is a prime interval $[d_{\delta(j)}, d_{\delta(j)+1}]$ in C_1' with

$$[d_{\delta(j)}, d_{\delta(j)+1}]\varphi_1' = b.$$

(i) For each covering pair $a \prec b$ in $J(D)$ and each (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0 with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$, there is a prime interval $[d_{\delta(j)}, d_{\delta(j)+1}]$ in C_1 with $d_{\delta(j)+1} \notin C_1'$ and with

$$[d_{\delta(j)}, d_{\delta(j)+1}]\varphi_1 = b.$$

There is a sublattice A of $C_0 \times C_1$. For each covering pair $a \prec b$ in $J(D')$ and each (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0', if $x \in C_0$ with $x > c_j$, then

$$\langle x, y\rangle \in A \text{ if and only if } y \geq d_{\delta(j)}.$$

Similarly, for each covering pair $a \prec b$ in $J(D)$ and each (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0 with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$, if $x \in C_0$ with $x > c_j$, then

$$\langle x, y\rangle \in A \text{ if and only if } y \geq d_{\delta(j)}.$$

There is a set Λ of intervals $\mathfrak{p}_0 \times \mathfrak{p}_1$, $\mathfrak{p}_i \in \operatorname{Ip} C_i$ with

$$[c_j, c_{j+1}] \times [d_{\delta(j)}, d_{\delta(j)+1}], \quad [c_{j+2}, c_{j+3}] \times [d_{\delta(j)}, d_{\delta(j)+1}] \in \Lambda$$

for each (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0', for $a \prec b$ in $J(D')$, and each (a, b)-interval

$$\{c_j, c_{j+1}, c_{j+2}, c_{j+3}\}$$

in C_0 with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$, for $a \prec b$ in $J(D)$.

The set Λ of intervals of A satisfies Requirement 1 with respect to the colorings φ_i.

We have the ideal $A' = A \cap (C_0' \times C_1')$ of A. Let us denote its maximal element by $i_{A'}$, that is $A' = (i_{A'}]_A$. The set of intervals Λ', those elements of Λ that are intervals in A', satisfies Requirement 1 with respect to the colorings φ_i'.

We extend A first by applying the the $\mathfrak{M}_3$-construction to all intervals in Λ, and then, in the resulting lattice, applying the $\mathfrak{N}_5$-construction to all intervals

$$[\langle c_j, d_{\delta(j)}\rangle, \langle c_{j+3}, d_{\delta(j)}\rangle]$$

where $[c_j, c_{j+3}]$ is either an (a, b)-interval in C_0' for $a \prec b$ in $J(D')$ or $[c_j, c_{j+3}]$ is an (a, b)-interval in C_0, with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$, for $a \prec b$ in $J(D)$. The resulting lattice L is planar, and neither it nor its ideal $L' = (i_{A'}]$, determined by A', has any automorphism other than the identity mapping.

4. Generalized coloring. In [11], we presented a generalization of the $\mathfrak{M}_3$- and $\mathfrak{N}_5$-constructions of Section 3 as follows.

Let L be a lattice and let Λ be a set of *proper intervals* in L, that is, intervals with more than one element. We define a lattice $L[\Lambda]$ by adjoining the family of new pairwise distinct elements $\{m_I \mid I \in \Lambda\}$ to L, and requiring that $u \prec m_I \prec v$, for each $I = [u, v] \in \Lambda$.

We associate with $x \in L[\Lambda]$ the elements $\underline{x}$ and $\bar{x}$ of L: for $x \in L$, set $\underline{x} = \bar{x} = x$; for $I = [u, v] \in \Lambda$, set $\underline{m}_I = u$ and $\bar{m}_I = v$. We then, more formally, define the relation $\leq$ on the set $L[\Lambda]$ as follows:

$$x \leq y \text{ if and only if } x = y \text{ or } \bar{x} \leq_L \underline{y},$$

where $\leq_L$ denotes the partial ordering in L.

Then it follows easily that $\langle L[\Lambda], \leq \rangle$ is a lattice extending L. If X is a subset of $L[\Lambda]$, then $\bigvee X$ exists in $L[\Lambda]$ if and only if either there is an $x \in X$ such that, for all $y \in X$, we have $x \geq y$, in which case $\bigvee X = x$; or there is no such x and $\bigvee_L(\bar{x} \mid x \in X)$ exists, in which case

$$\bigvee X = \bigvee{}_L(\bar{x} \mid x \in X),$$

where $\bigvee_L$ is the complete join in L; and dually for $\bigwedge$.

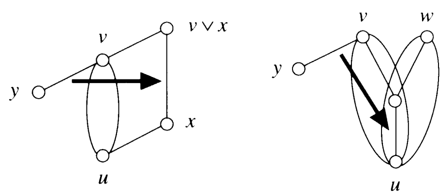
For the $\mathfrak{M}_3$-construction, Λ is a set of intervals in the lattice A that satisfy Requirement 1 of Section 3. For the $\mathfrak{N}_5$-construction, Λ is the set of intervals

$$[\langle c_i, d_{\delta(i)} \rangle, \langle c_{i+3}, d_{\delta(i)} \rangle]$$

in the lattice obtained by applying the $\mathfrak{M}_3$-construction to A, where the $[c_i, c_{i+3}]$ are (a, b)-intervals in the chain C_0.

The following result describes which congruences extend from L to $L[\Lambda]$:

THEOREM 5 (ONE POINT EXTENSION THEOREM [11]). *Let Λ be a set of nontrivial, nonprime intervals in the lattice L, and let Θ be a congruence relation on L. Then Θ has an extension $\Theta[\Lambda]$ to $L[\Lambda]$ if and only if Θ satisfies the following conditions and their duals (see Figure 13):*



Condition (3.1) Condition (3.2)

FIGURE 13

(1) For $[u, v] \in \Lambda$, $x, y \in L$ with $y < v$ and $u < x$,

$$y \equiv v \quad (\Theta) \quad \textit{implies that} \quad x \equiv v \vee x \quad (\Theta).$$

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(2) For $[u, v]$, $[u, w] \in \Lambda$, *with* $v \neq w$ *and* $y \in L$ *with* $y < v$,

$$y \equiv v \quad (\Theta) \quad \textit{implies that} \quad v \wedge w \equiv u \quad (\Theta).$$

The extension $\Theta[\Lambda]$ *of* Θ *to* $L[\Lambda]$ *is unique. It can be described as follows: For all* $a \in L[\Lambda]$, *set* $a \equiv a \ (\Theta[\Lambda])$. *For all* $a, b \in L[\Lambda]$, *with* $a \neq b$, *set*

$$a \equiv b \quad (\Theta[\Lambda])$$

if and only if the following three conditions hold:

(3) $\overline{a \wedge b} \equiv \underline{a \vee b}$ (Θ).

(4) $a \wedge b \in L$ *or* $a \wedge b \notin L$ *and there is an* $x_{a \wedge b} \in L$ *with*

$$x_{a \wedge b} > \underline{a \wedge b} \quad \textit{and} \quad x_{a \wedge b} \equiv \underline{a \wedge b} \quad (\Theta).$$

(5) $a \vee b \in L$ *or* $a \vee b \notin L$ *and there is a* $y_{a \vee b} \in L$ *with*

$$y_{a \vee b} < \overline{a \vee b} \quad \textit{and} \quad y_{a \vee b} \equiv \overline{a \vee b} \quad (\Theta).$$

An interesting special case can be developed by generalizing the concept of coloring from Section 3. Let P be a set of nontrivial intervals in a lattice L. A *(generalized) coloring* φ *of* L *by a set* X is a map $\varphi\colon P \longrightarrow X$. In this generalization, L need not be a chain, nor need the intervals in P be prime—they need only be nontrivial.

For each $i = 0, 1$, let A_i be a lattice with a coloring $\varphi_i\colon P_i \longrightarrow X$. Let A be a sublattice of $A_0 \times A_1$. We consider a set Λ of intervals in $A_0 \times A_1$ of the form $I_0 \times I_1$ with $I_0 \in P_0$ and $I_1 \in P_1$ such that each interval in Λ is a subset of A. In analogy to Requirement 1 of Section 3, we require of Λ:

REQUIREMENT 2. Λ is a set of intervals in A of the form $I_0 \times I_1$, $I_i \in P_i$, satisfying

(4-1) If $I_0 \times I_1 \in \Lambda$, then $I_0\varphi_0 = I_1\varphi_1$.

(4-2) For each $x \in P_0\varphi_0 \cap P_1\varphi_1$, there is a $I_0 \times I_1 \in \Lambda$ with $I_0\varphi_0 = I_1\varphi_1 = x$.

(4-3) For each $i = 0, 1$, if $I, I' \in P_i$ are distinct and $I\varphi_i = I'\varphi_i$, then there are $\bar{I} \in P_i$, $J, J' \in P_{1-i}$ with

$$I \times J,\ \bar{I} \times J,\ I' \times J',\ \bar{I} \times J' \in \Lambda, \qquad \text{if } i = 0,$$
$$J \times I,\ J \times \bar{I},\ J' \times I',\ J' \times \bar{I} \in \Lambda, \qquad \text{if } i = 1.$$

We form the lattice $A[\Lambda]$; let us denote the element $m_{I_0 \times I_1} \in A[\Lambda]$ by $m(I_0, I_1)$.

Recall that any congruence relation Θ on the lattice $A_0 \times A_1$ is of the form $\Theta_0 \times \Theta_1$, where, for $i = 0, 1$, Θ_i is a congruence relation on A_i. We consider congruence relations Θ on A which are restrictions of such congruence relations $\Theta_0 \times \Theta_1$ on $A_0 \times A_1$. The next result is an application of the One Point Extension Theorem to determine which such congruence relations on A extend to $A[\Lambda]$.

THEOREM 6 (EXTENDED COLORED PRODUCT EXTENSION THEOREM). *Let the set of intervals* Λ *satisfy Conditions* (4-1), (4-2), *and* (4-3). *The congruence relation* Θ *on* A *that is the restriction of* $\Theta_0 \times \Theta_1 \in \mathrm{Con}\,(A_0 \times A_1)$ *extends to* $A[\Lambda]$ *if and only if the following two conditions and the dual of the second condition*[1] *hold:*

[1] In dualizing, we only dualize for the lattices, not their *congruences*; thus, Condition (6.1) is self-dual.

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(1) For $I_0 \in P_0$, $I_1 \in P_1$, if $I_0\varphi_0 = I_1\varphi_1$, then

$$\Theta(I_0) \leq \Theta_0 \text{ is equivalent to } \Theta(I_1) \leq \Theta_1.$$

(2) For $i = 0, 1$, if $[u, v] \in P_i$ with $[u, v]\varphi_i \in P_0\varphi_0 \cap P_1\varphi_1$, and if there is a $y < v$ with $y \equiv v \ (\Theta_i)$, then $\Theta(u, v) \leq \Theta_i$.

In that event, the extension is unique.

PROOF. Since none of the intervals in Λ are prime, the One Point Extension Theorem applies. We note that in the present case Condition (5.1) is equivalent to the stronger condition:

(5.1$^+$) If $[u, v] \in \Lambda$ and there is a $y < v$ with $y \equiv v \ (\Theta)$, then $u \equiv v \ (\Theta)$.

Indeed, if $u = \langle u_0, u_1 \rangle$, $v = \langle v_0, v_1 \rangle$, $x = \langle u_0, v_1 \rangle$ then, by Condition (5.1),

$$v = \langle v_0, v_1 \rangle = \langle u_0, v_1 \rangle \vee \langle v_0, v_1 \rangle \equiv \langle u_0, v_1 \rangle \quad (\Theta),$$

and, similarly,

$$v = \langle v_0, v_1 \rangle \equiv \langle v_0, u_1 \rangle \quad (\Theta).$$

Consequently,

$$u = \langle u_0, u_1 \rangle = \langle u_0, v_1 \rangle \wedge \langle v_0, u_1 \rangle \equiv v \quad (\Theta),$$

establishing Condition (5.1$^+$).

Note also that Condition (5.2) follows immediately from Condition (5.1$^+$). Thus, in view of the One Point Extension Theorem and the principle of duality, we need only show that the conjunction of Conditions (6.1) and (6.2) is equivalent to Condition (5.1$^+$).

Let Conditions (6.1) and (6.2) hold. Let $[\langle u_0, u_1 \rangle, \langle v_0, v_1 \rangle] \in \Lambda$, and let $\langle y_0, y_1 \rangle < \langle v_0, v_1 \rangle$ with $\langle y_0, y_1 \rangle \equiv \langle v_0, v_1 \rangle \ (\Theta)$. Without loss of generality, we may assume that $y_0 < v_0$. Since $y_0 \equiv v_0 \ (\Theta_0)$, it follows from Conditions (4-1) and (6.2) that $u_0 \equiv v_0 \ (\Theta_0)$. Again, since $[u_0, v_0]\varphi_0 = [u_1, v_1]\varphi_1$, it follows from Condition (6.1) that $u_1 \equiv v_1 \ (\Theta_1)$, establishing Condition (5.1$^+$).

Now let Condition (5.1$^+$) hold. Then the following condition is an immediate consequence:

(4-4) If $[\langle u_0, u_1 \rangle, \langle v_0, v_1 \rangle] \in \Lambda$, then $u_0 \equiv v_0 \ (\Theta_0)$ if and only if $u_1 \equiv v_1 \ (\Theta_1)$.

Indeed, if $u_0 \equiv v_0 \ (\Theta_0)$, then set $y = \langle u_0, v_1 \rangle$ in Condition (5.1$^+$), and conclude that $\langle u_0, v_0 \rangle \equiv \langle u_1, v_1 \rangle \ (\Theta)$, that is, that $u_1 \equiv v_1 \ (\Theta_1)$. Similarly, if $u_1 \equiv v_1 \ (\Theta_1)$, then setting $y = \langle u_1, v_0 \rangle$ yields $u_0 \equiv v_0 \ (\Theta_0)$.

We first establish Condition (6.1). Let $I_0 \in P_0$, $I_1 \in P_1$ with $I_0\varphi_0 = I_1\varphi_1 = x$. Then, by Condition (4-2), there is $I \times J \in \Lambda$ with $I\varphi_0 = J\varphi_1 = x$. By Condition (4-4),

(4-5) $$\Theta(I) \leq \Theta_0 \iff \Theta(J) \leq \Theta_1.$$

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If $I = I_0$, then, obviously

$$\Theta(I_0) \leq \Theta_0 \iff \Theta(I) \leq \Theta_0. \tag{4-6}$$

If, on the other hand, $I \neq I_0$, then, by Condition (4-3) with $i = 0$ and $I' = I_0$, there are $\bar{I} \in P_0, J, J' \in P_1$ with

$$I \times J, \quad \bar{I} \times J, \quad I_0 \times J', \quad \bar{I} \times J' \in \Lambda.$$

Then, by Condition (4-4),

$$\Theta(I) \leq \Theta_0 \iff \Theta(J) \leq \Theta_1 \iff \Theta(\bar{I}) \leq \Theta_0 \iff \Theta(J') \leq \Theta_1 \iff \Theta(I_0) \leq \Theta_0.$$

That is, if $I \neq I_0$, we again have (4-6).

Similarly,

$$\Theta(I_1) \leq \Theta_1 \iff \Theta(J) \leq \Theta_1. \tag{4-7}$$

Combining (4-5), (4-6), and (4-7) we get

$$\Theta(I_0) \leq \Theta_0 \iff \Theta(I_1) \leq \Theta_1,$$

establishing Condition (6.1).

Next, we establish Condition (6.2).

Let $i = 0$. Let $[u, v] \in P_0$ with $[u, v]\varphi_0 \in P_1\varphi_1$, and let $y < v$ with $y \equiv v \ (\Theta_0)$. By Conditions (4-2) and (4-3), there is an interval $[u_1, v_1] \in P_1$ with $[\langle u, u_1\rangle, \langle v, v_1\rangle] \in \Lambda$. Now $\langle y, v_1\rangle < \langle v, v_1\rangle$ and $\langle y, v_1\rangle \equiv \langle v, v_1\rangle \ (\Theta)$. Then, by Condition (5.1^+), $\langle u, u_1\rangle \equiv \langle v, v_1\rangle \ (\Theta)$, and so $u \equiv v \ (\Theta_0)$, establishing Condition (6.2) for $i = 0$. A similar argument establishes Condition (6.2) for the case $i = 1$.

Consequently, we conclude that the conjunction of Conditions (6.1) and (6.2) is equivalent to Condition (5.1^+). Theorem 6 then follows by the One Point Extension Theorem. ■

We denote the extension of Θ to $A[\Lambda]$ by $\Theta_0 \times_{A.\Lambda} \Theta_1$.

Theorem 6 is a generalization of the Colored Product Extension Theorem, introduced in [11], and also used in [9].

5. **The proof of Theorem 1.** We refer the reader to the summary at the end of Section 3. If D and D' are both Boolean and ψ is an isomorphism, then, as observed there, taking $L = L'$ a chain of length $|J(D)|$ establishes the theorem. Otherwise we proceed via the chains C_0 and C_1 and their colorings.

The lattice L and its ideal L' are planar and admit only the identity automorphism, as demonstrated in Section 3. We need only formally establish isomorphisms

$$\varrho \colon D \longrightarrow \mathrm{Con}\, L$$

and

$$\varrho' : D' \longrightarrow \mathrm{Con}\, L'$$

such that ϱ followed by restriction to L' is $\psi\varrho'$.

The result of applying the $\mathfrak{M}_3$-construction to the lattice A for all the intervals in Λ is, in the notation of Section 4, the lattice $A[\Lambda]$, and its ideal $(i_{A'}]$ is $A'[\Lambda']$. In Section 3 these lattices were denoted K and K' respectively. We first determine the congruence relations on these lattices.

LEMMA 7. *The congruence relations of the lattice* $A[\Lambda]$ *are precisely those of the form* $\Theta_0 \times_{A,\Lambda} \Theta_1$ *where, for* $i = 0, 1$, Θ_i *is a congruence relation on* C_i *satisfying:*
(1) For $\mathfrak{p}_0 \in \mathrm{Ip}\, C_0$, $\mathfrak{p}_1 \in \mathrm{Ip}\, C_1$, *if* $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$, *then*

$$\Theta(\mathfrak{p}_0) \leq \Theta_0 \textit{ is equivalent to } \Theta(\mathfrak{p}_1) \leq \Theta_1.$$

PROOF. Denote the largest element of C_1 by d_t—usually, $t = r + r'$, but it may be larger, as in the discussion following the proof of Lemma 4 in Section 3. The smallest element of C_0 is denoted c_0. Since

$$C_0 \times \{d_t\} \subseteq A$$

and

$$\{c_0\} \times C_1 \subseteq A,$$

it follows easily that any congruence relation on A is the restriction of a congruence relation $\Theta_0 \times \Theta_1$, where Θ_i is a congruence relation on C_i for $i = 0, 1$.

We apply Theorem 6, the Extended Colored Product Extension Theorem. For each $i = 0, 1$, we set $P_i = \mathrm{Ip}\, C_i$. Since Λ satisfies Requirement 1 of Section 3, Conditions (4-1), (4-2), and (4-3) hold. Furthermore, Condition (7.1) is equivalent to Condition (6.1). Since all the intervals in P_0 and P_1 are prime, and since C_0 and C_1 are chains, Condition (6.2) and its dual hold trivially. Thus the lemma follows from Theorem 6. ■

Recall that Λ', the set of those elements of Λ that are intervals in A', satisfies Requirement 1 with respect to the colorings φ_i'. Thus, exactly as above, we have:

LEMMA 8. *The congruence relations of the lattice* $A'[\Lambda']$ *are precisely those of the form* $\Theta_0 \times_{A',\Lambda'} \Theta_1$ *where, for* $i = 0, 1$, Θ_i *is a congruence relation on* C_i' *satisfying:*
(1) For $\mathfrak{p}_0 \in \mathrm{Ip}\, C_0'$, $\mathfrak{p}_1 \in \mathrm{Ip}\, C_1'$, *if* $\mathfrak{p}_0\varphi_0' = \mathfrak{p}_1\varphi_1'$, *then*

$$\Theta(\mathfrak{p}_0) \leq \Theta_0 \textit{ is equivalent to } \Theta(\mathfrak{p}_1) \leq \Theta_1.$$

We now formally describe the application of the $\mathfrak{N}_5$-construction to the lattice $A[\Lambda]$ referred to in the summary at the end of Section 3. We let Γ be the set of those intervals of the form

$$[c_j, c_{j+3}] \times \{d_{\delta(j)}\}$$

where $[c_j, c_{j+3}]$ is either an (a, b)-interval in C_0' for some $a \prec b$ in $J(D')$, or is an (a, b)-interval in C_0 for some $a \prec b$ in $J(D)$ with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$. Then

$$L = A[\Lambda][\Gamma].$$

If Γ' is the set of those elements of Γ that are intervals in $A'[\Lambda']$, then it is the set of those intervals of the form

$$[c_j, c_{j+3}] \times \{d_{\delta(j)}\}$$

where $[c_j, c_{j+3}]$ is an (a, b)-interval in C_0' for some $a \prec b$ in $J(D')$. Then

$$L' = A'[\Lambda'][\Gamma'].$$

We describe the congruence relations of L:

LEMMA 9. *The congruence relations of the lattice L are precisely those of the form $(\Theta_0 \times_{A,\Lambda} \Theta_1)[\Gamma]$, where Θ_0 is a congruence relation on C_0 and Θ_1 is a congruence relation on C_1 satisfying:*

(1) For each $i, k \in \{0, 1\}$, and each $\mathfrak{p} \in \operatorname{Ip} C_i$, $\mathfrak{q} \in \operatorname{Ip} C_k$ with $\mathfrak{q}\varphi_k \le \mathfrak{p}\varphi_i$,

$$\text{if } \Theta(\mathfrak{p}) \le \Theta_i, \text{ then } \Theta(\mathfrak{q}) \le \Theta_k.$$

PROOF. By Lemma 7, the congruence relations of $A[\Lambda]$ are precisely those of the form $\Psi = (\Theta_0 \times_{A,\Lambda} \Theta_1)$ such that Condition (7.1) is satisfied. Any congruence relation on $L = A[\Lambda][\Gamma]$ is the extension of a congruence relation on $A[\Lambda]$. Note that Condition (7.1) is just the special case of Condition (9.1) with $i \neq k$ and $\mathfrak{q}\varphi_k = \mathfrak{p}\varphi_i$. Thus, we need only show that a congruence relation on $A[\Lambda]$ of the form

$$\Psi = \Theta_0 \times_{A,\Lambda} \Theta_1$$

extends to $A[\Lambda][\Gamma]$ if and only the full Condition (9.1) holds.

We apply Theorem 5. Note that no interval in Γ is prime. Condition (5.2) and its dual hold vacuously. Consequently, Ψ extends to L if and only if Condition (5.1) and its dual hold for Ψ and Γ, and then the extension is unique.

We show that in the present situation Condition (5.1) and its dual are equivalent to Condition (9.1).

Let Condition (9.1) hold.

First, we show that Condition (5.1) holds: If $[u, v] \in \Gamma$, then there are a, b in $J(D)$ or $J(D')$ with $a \prec b$ and $u = \langle c_j, d_{\delta(j)} \rangle$, $v = \langle c_{j+3}, d_{\delta(j)} \rangle$ where $[c_j, c_{j+3}]$ is an (a, b)-interval in C_0—see Figure 14. Now, $y < v$ in $A[\Lambda]$ implies that $y \le \langle c_{j+2}, d_{\delta(j)} \rangle$, by the definition of A. Thus, $y \equiv v \;\; (\Psi)$ implies that

$$c_{j+2} \equiv c_{j+3} \quad (\Theta_0).$$

Since the intervals $[\langle c_j, d_{\delta(j)} \rangle, \langle c_{j+1}, d_{\delta(j)} \rangle]$ and $[\langle c_{j+2}, d_{\delta(j)} \rangle, \langle c_{j+3}, d_{\delta(j)} \rangle]$ are projective in $A[\Lambda]$, we conclude that

$$c_j \equiv c_{j+1} \quad (\Theta_0).$$

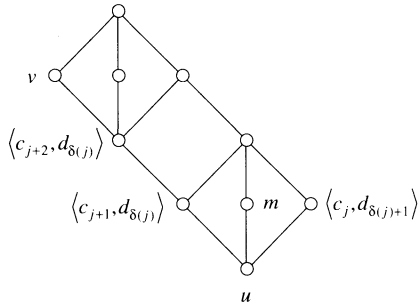

FIGURE 14

Furthermore, and using the fact that ψ^* is isotone in the case of $a, b \in J(D')$, we observe that

$$[c_{j+1}, c_{j+2}]\varphi_0 \leq [c_{j+2}, c_{j+3}]\varphi_0;$$

consequently, by Condition (9.1) with $i = k = 0$,

$$c_{j+1} \equiv c_{j+2} \quad (\Theta_0).$$

Thus

$$c_j \equiv c_{j+3} \quad (\Theta_0),$$

and so $u \equiv v \ \ (\Psi)$, and Condition (5.1) follows immediately.

Next, we verify the dual of Condition (5.1). Set

$$m = m([c_j, c_{j+1}], [d_{\delta(j)}, d_{\delta(j)+1}]),$$

the element that was added to $[c_j, c_{j+1}] \times [d_{\delta(j)}, d_{\delta(j)+1}]$ in going from A to $A[\Lambda]$. Let $y > u = \langle c_j, d_{\delta(j)}\rangle$. Then

$$y \geq \langle c_{j+1}, d_{\delta(j)}\rangle,$$

or

$$y \geq \langle c_j, d_{\delta(j)+1}\rangle,$$

or

$$y \geq m$$

—see Figure 14. If $y \equiv u \ \ (\Psi)$ we conclude, in each case, that

$$c_j \equiv c_{j+1} \quad (\Theta_0),$$

and so, exactly as above, that $u \equiv v \ \ (\Psi)$, establishing the dual of Condition (5.1).

Finally, let Condition (5.1) and its dual hold—all we really need is Condition (5.1). We, of course, have Condition (7.1). We next establish Condition (9.1).

We first consider the case where $\mathfrak{q}\varphi_k = \mathfrak{p}\varphi_i$ in the hypothesis of Condition (9.1). If $i \neq k$, then the conclusion follows by Condition (7.1). If $i = k$ and $\mathfrak{q} = \mathfrak{p}$, the conclusion is

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immediate. If $i = k$ and $\mathfrak{q} \neq \mathfrak{p}$, then, since Λ satisfies Condition (3-6) of Requirement 1, we get

$$\text{(5-1)} \qquad \text{if } \Theta(\mathfrak{p}) \leq \Theta_i, \text{ then } \Theta(\mathfrak{q}) \leq \Theta_i$$

by Condition (7.1).

We are then left with the case $\mathfrak{q}\varphi_k < \mathfrak{p}\varphi_i$. We can restrict ourselves to the case $i = k = 0$. Indeed, by (b) of the summary at the end of Section 3, there are prime intervals $\mathfrak{q}'$ and $\mathfrak{p}'$ in C_0 with

$$\mathfrak{q}'\varphi_0 = \mathfrak{q}\varphi_k$$

and

$$\mathfrak{p}'\varphi_0 = \mathfrak{p}\varphi_i.$$

(If $k = 0$, then $\mathfrak{q}' = \mathfrak{q}$, and if $i = 0$, then $\mathfrak{p}' = \mathfrak{p}$.) By Condition (7.1), we then need only show

$$\text{if } \Theta(\mathfrak{p}') \leq \Theta_0, \text{ then } \Theta(\mathfrak{q}') \leq \Theta_0.$$

Consequently, we may assume that $i = 0$ and that

$$\mathfrak{q}\varphi_0 < \mathfrak{p}\varphi_0$$

with

$$\Theta(\mathfrak{p}) \leq \Theta_0.$$

Since (Ip $C_0)\varphi_0$ contains all non-isolated elements of $J(D)$, we may further assume that

$$\mathfrak{q}\varphi_0 \prec \mathfrak{p}\varphi_0$$

(in $J(D)$). Set $a = \mathfrak{q}\varphi_0$, $b = \mathfrak{p}\varphi_0$, and let $[c_j, c_{j+3}]$ be an (a, b)-interval in C_0 with $c_{j+1}, c_{j+2}, c_{j+3} \notin C_0'$. By (5-1),

$$c_{j+2} \equiv c_{j+3} \quad (\Theta_0).$$

We consider the interval

$$[u, v] = [\langle c_j, d_{\delta(j)} \rangle, \langle c_{j+3}, d_{\delta(j)} \rangle] \in \Gamma$$

—see Figure 14. In Condition (5.1) set

$$y = \langle c_{j+2}, d_{\delta(j)} \rangle$$

and

$$x = \langle c_j, d_{\delta(j)+1} \rangle.$$

Then

$$y \equiv v \quad (\Psi)$$

and so, by Condition (5.1),

$$\langle c_{j+3}, d_{\delta(j)+1} \rangle = v \vee x \equiv x = \langle c_j, d_{\delta(j)+1} \rangle \quad (\Psi).$$

Taking the meet with $\langle c_{j+3}, d_{\delta(j)}\rangle$, we get

$$\langle c_{j+3}, d_{\delta(j)}\rangle \equiv \langle c_j, d_{\delta(j)}\rangle \quad (\Psi),$$

and so

$$c_{j+1} \equiv c_{j+2} \quad (\Theta_0).$$

Since

$$[c_{j+1}, c_{j+2}]\varphi_0 = a = \mathfrak{q}\varphi_0,$$

we conclude, by (5-1), that $\Theta(\mathfrak{q}) \leq \Theta_0$, establishing Condition (9.1), and thereby concluding the proof of Lemma 9. ∎

Similarly, for L', we have:

LEMMA 10. *The congruence relations of the lattice L' are precisely those of the form $(\Theta_0 \times_{A'.\Lambda'} \Theta_1)[\Gamma']$, where Θ_0 is a congruence relation on C_0' and Θ_1 is a congruence relation on C_1' satisfying:*

(1) For each $i, k \in \{0, 1\}$ and each $\mathfrak{p} \in C_i'$, $\mathfrak{q} \in \operatorname{Ip} C_k'$ with $\mathfrak{q}\varphi_k' \leq \mathfrak{p}\varphi_i'$,

$$\text{if } \Theta(\mathfrak{p}) \leq \Theta_i, \text{ then } \Theta(\mathfrak{q}) \leq \Theta_k.$$

We now proceed to establish the isomorphisms

$$\varrho \colon D \longrightarrow \operatorname{Con} L$$

and

$$\varrho' \colon D' \longrightarrow \operatorname{Con} L'.$$

We first define ϱ. Let $x \in D$. For each $i = 0, 1$, we define a congruence relation Θ_i^x on C_i; for $u, v \in C_i$ with $u < v$ set

$$u \equiv v \quad (\Theta_i^x) \iff \mathfrak{p}\varphi_i \leq x \text{ for each } \mathfrak{p} \in \operatorname{Ip}\,[u, v].$$

Then the congruences Θ_0^x, Θ_1^x satisfy Condition (9.1). Consequently we have the congruence $\Theta^x \in \operatorname{Con} L$ defined by

$$\Theta^x = (\Theta_0^x \times_{A.\Lambda} \Theta_1^x)[\Gamma].$$

Set $x\varrho = \Theta^x$. We show that ϱ is an isomorphism $D \longrightarrow \operatorname{Con} L$.

We first show that ϱ is surjective. Let Ψ be a congruence relation on L. By Lemma 9, there are congruence relations Ψ_0 on C_0 and Ψ_1 on C_1 satisfying Condition (9.1) such that

$$\Psi = (\Psi_0 \times_{A.\Lambda} \Psi_1)[\Gamma].$$

Set

$$x = \bigvee\nolimits_D \bigl(\mathfrak{p}\varphi_i \mid \mathfrak{p} \in \operatorname{Ip} C_i \text{ with } \Theta(\mathfrak{p}) \leq \Psi_i, i = 0, 1\bigr).$$

We need only show that, for each $i = 0, 1$, $\Psi_i = \Theta_i^x$.

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Let $u < v$ in C_i with $u \equiv v \ (\Psi_i)$ and let $\mathfrak{p} \in \mathrm{Ip}\ C_i$. Then

$$\Theta(\mathfrak{p}) \leq \Theta(u, v) \leq \Psi_i.$$

Then, by definition of x, $\mathfrak{p}\varphi_i \leq x$. But, by definition of Θ_i^x, we conclude that $u \equiv v \ (\Theta_i^x)$, thereby establishing

$$\Psi_i \leq \Theta_i^x.$$

On the other hand, let $u < v$ in C_i with $u \equiv v \ (\Theta_i^x)$. Then, by definition of Θ_i^x, for all $\mathfrak{p} \in \mathrm{Ip}\ [u, v]$, $\mathfrak{p}\varphi_i \leq x$. By the definition of x and since $\mathfrak{p}\varphi_i$ is join-irreducible, there is a $k = 0, 1$ and $\mathfrak{q} \in C_k$ with $\Theta(\mathfrak{q}) \leq \Psi_k$ and $\mathfrak{p}\varphi_i \leq \mathfrak{q}\varphi_k$. By Condition (9.1), we conclude that $\Theta(\mathfrak{p}) \leq \Psi_i$. Consequently, $u \equiv v \ (\Psi_i)$, and so

$$\Theta_i^x \leq \Psi_i$$

also, thereby showing that

$$\Psi_i = \Theta_i^x,$$

establishing the surjectivity of ϱ.

Next, we show that

(5-2) $$x\varrho \leq y\varrho \iff x \leq y.$$

If $x \leq y$, then, for each i, $\Theta_i^x \leq \Theta_i^y$, and, so, $x\varrho \leq y\varrho$. If, on the other hand, $x\varrho \leq y\varrho$, then $\Theta_i^x \leq \Theta_i^y$ for each $i \in \{0, 1\}$. Let $a \in J(D)$ with $a \leq x$. Then, by observation (a) in the summary at the end of Section 3, there is an $i \in \{0, 1\}$ and a $\mathfrak{p} \in \mathrm{Ip}\ C_i$ with

$$\mathfrak{p}\varphi_i = a \leq x.$$

Consequently, by the definition of Θ_i^x,

$$\Theta(\mathfrak{p}) \leq \Theta_i^x \leq \Theta_i^y$$

which, by definition of Θ_i^y, implies that

$$a = \mathfrak{p}\varphi_i \leq y.$$

Thus, if $x\varrho \leq y\varrho$, then $x \leq y$, establishing (5-2).

The surjectivity of ϱ and (5-2) establish that ϱ is an isomorphism.

Similarly, for each $x \in D'$ and $i \in \{0, 1\}$, we have the congruence relation $\Theta_i'^x$ on C_i' given by

$$u \equiv v \quad (\Theta_i'^x) \iff \mathfrak{p}\varphi_i' \leq x \text{ for each } \mathfrak{p} \in \mathrm{Ip}\ [u, v]$$

for $u < v$ in C_i'. Setting

$$x\varrho' = \Theta'^x = (\Theta_0'^x \times_{A', \Lambda'} \Theta_1'^x)[\Gamma']$$

yields an isomorphism $\varrho' \colon D' \longrightarrow \mathrm{Con}\ L'$.

We now determine ϱ followed by restriction to L'. Let $x \in D$. Since ϱ' is an isomorphism, the restriction to L' of $x\varrho = \Theta^x$ is of the form

$$y\varrho' = \Theta'^y = (\Theta'^y_0 \times_{A',\Lambda'} \Theta'^y_1)[\Gamma']$$

for some $y \in D'$, where, for each i, Θ'^y_i is the restriction to C'_i of Θ^x_i.

We claim that

$$y = x\psi.$$

We need only show for each $a \in J(D')$ that

$$a \leq y \iff a\psi^* \leq x.$$

By (c) of the summary at the end of Section 3, there is an $i \in \{0, 1\}$ and a $\mathfrak{p} \in \operatorname{Ip} C'_i$ with $a = \mathfrak{p}\varphi'_i$. Then

$$a \leq y \iff \mathfrak{p}\varphi'_i \leq y \iff \Theta(\mathfrak{p}) \leq \Theta'^y_i \text{ in } \operatorname{Con} C'_i$$

by definition of Θ'^y_i. Thus, since Θ'^y_i is the restriction to C'_i of Θ^x_i,

$$a \leq y \iff \Theta(\mathfrak{p}) \leq \Theta^x_i \text{ in } \operatorname{Con} C_i.$$

So, by definition of Θ^x_i,

$$a \leq y \iff \mathfrak{p}\varphi_i \leq x.$$

Now,

$$\mathfrak{p}\varphi_i = \mathfrak{p}\varphi'_i\psi^* = a\psi^*$$

and consequently

$$a \leq y \iff a\psi^* \leq x,$$

showing that ϱ followed by restriction to L' is $\psi\varrho'$, and thereby concluding the proof of Theorem 1.

6. **Representing groups.** In this section, we construct, for each group homomorphism $\eta\colon G \longrightarrow G'$, a simple lattice H with automorphism group isomorphic to G and containing an ideal H' with automorphism group isomorphic to G' such that each automorphism of H restricts to an automorphism of H' where the restriction is naturally equivalent to the mapping η. The results in this section are related to results in [13] and [1]. Although our construction is inspired by R. Frucht [4], the presentation is self-complete and does not use any of the results in these works.

We use the term *digraph* for a directed graph (the edges have a direction) without multiple edges—for each pair of vertices v, w, there is at most one edge from v to w. The digraphs we consider here will also have *no loops*, that is, the ends of any edge will be distinct. Thus a digraph $\mathbf{D}$ is a structure $\langle V, R\rangle$ where V is the set of vertices and R, the set of (directed) edges, is a subset of V^2 disjoint from the diagonal. We say that a digraph $\mathbf{D}' = \langle V', R'\rangle$ is a *full subgraph* of a digraph $\mathbf{D} = \langle V, R\rangle$ if $V' \subseteq V$ and $R' = R \cap V'^2$, that is, if for any v, $w \in V'$, there is an edge from v to w in $\mathbf{D}'$ iff there is an edge from v to w in $\mathbf{D}$.

The term *graph* denotes an undirected graph without multiple edges and without loops. Thus a graph $\mathbf{G}$ is a structure $\langle V, E\rangle$, where V is the set of vertices and E, the set of (undirected) edges of $\mathbf{G}$, is a set of two-element subsets of V. We say that a graph $\mathbf{G}'$ is a *full subgraph* of a graph $\mathbf{G}$ if all the vertices of $\mathbf{G}'$ are vertices of $\mathbf{G}$, and any pair of vertices of $\mathbf{G}'$ are joined by an edge of $\mathbf{G}'$ iff they are joined by an edge of $\mathbf{G}$.

By a *labeled digraph* $\mathbf{D} = \langle V, R, \lambda\rangle$ we mean a digraph with vertices V, a set R of directed edges (that is, $R \subseteq V^2$ and R is disjoint from the diagonal), and a surjective mapping $\lambda\colon R \longrightarrow X$, where X is a set of *labels*. An *automorphism* α of a labeled digraph $\mathbf{D}$ is a bijection $\alpha\colon V \longrightarrow V$ such that $R\alpha^2 = R$ and such that, for each edge $\langle x, y\rangle \in R$, we have that $\langle x, y\rangle\lambda = \langle x\alpha, y\alpha\rangle\lambda$.

We associate with each group G its *Cayley digraph* $\mathbf{D}(G) = \langle G, R_G, \lambda_G\rangle$, where

$$R_G = \{\langle g, h\rangle \in G^2 \mid g \neq h\}$$

and $\langle g, h\rangle\lambda_G = gh^{-1}$. The automorphism group of $\mathbf{D}(G)$ is isomorphic to G; we associate with the element $g \in G$ the automorphism α_g defined by $h\alpha_g = hg$ for each $h \in G$.

Given the groups G, G' and the homomorphism $\eta\colon G \longrightarrow G'$, we construct a labeled digraph $\mathbf{D}(\eta) = \langle V_\eta, R_\eta, \lambda_\eta\rangle$, where

$$V_\eta = G \mathbin{\dot\cup} G',$$
$$R_\eta = R_G \mathbin{\dot\cup} R_{G'} \mathbin{\dot\cup} \{\langle g, g\eta\rangle \mid g \in G\},$$

and the labeling λ_η is defined by setting

$$\langle g, h\rangle\lambda_\eta = \begin{cases} \langle g, h\rangle\lambda_G = gh^{-1}, & \text{if } \langle g, h\rangle \in R_G;\\ \langle g, h\rangle\lambda_{G'} = gh^{-1}, & \text{if } \langle g, h\rangle \in R_{G'};\\ \eta, & \text{if } g \in G \text{ and } h = g\eta. \end{cases}$$

Thus $\mathbf{D}(\eta)$ is the disjoint union of $\mathbf{D}(G)$ and $\mathbf{D}(G')$ with new directed edges $\langle g, g\eta\rangle$, all with label η, added—see Figure 15. We note that the Cayley digraph $\mathbf{D}(G')$ is a full subgraph of the digraph $\mathbf{D}(\eta)$.

With each $g \in G$, we associate a mapping $\alpha_g\colon V_\eta \longrightarrow V_\eta$ by setting

$$h\alpha_g = \begin{cases} hg, & \text{if } h \in G;\\ hg\eta, & \text{if } h \in G'. \end{cases}$$

We recall the isomorphism $G' \longrightarrow \operatorname{Aut} \mathbf{D}(G')$ which associates with each $g \in G'$ the automorphism α'_g of $\mathbf{D}(G')$ determined by setting $h\alpha'_g = hg$.

LEMMA 11. *For each $g \in G$, the mapping α_g is an automorphism of the labeled digraph $\mathbf{D}(\eta)$. The mapping $\bar\eta\colon G \longrightarrow \operatorname{Aut} \mathbf{D}(\eta)$, defined by setting*

$$g\bar\eta = \alpha_g,$$

is a group isomorphism. For each $g \in G$, α_g restricts to the automorphism $\alpha'_{g\eta}$ of $\mathbf{D}(G')$.

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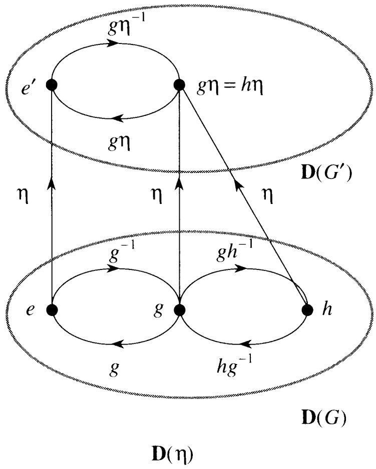

FIGURE 15

PROOF. Let $g \in G$. It is clear that α_g is a bijection on V_η.

If g_1, g_2 are distinct elements of G, then

$$\langle g_1, g_2\rangle\lambda_\eta = g_1 g_2^{-1} = (g_1 g)(g_2 g)^{-1} = \langle g_1\alpha_g, g_2\alpha_g\rangle\lambda_\eta.$$

If g_1, g_2 are distinct elements of G', then

$$\langle g_1, g_2\rangle\lambda_\eta = g_1 g_2^{-1} = (g_1 g\eta)(g_2 g\eta)^{-1} = \langle g_1\alpha_g, g_2\alpha_g\rangle\lambda_\eta.$$

If $g_1 \in G$, then

$$\langle g_1, g_1\eta\rangle\lambda_\eta = \eta$$

and

$$g_1\alpha_g\eta = (g_1 g)\eta = (g_1\eta)(g\eta) = g_1\eta\alpha_g.$$

Consequently,

$$\langle g_1\alpha_g, g_1\eta\alpha_g\rangle \in R_\eta$$

and, clearly,

$$\langle g_1\alpha_g, g_1\eta\alpha_g\rangle\lambda_\eta = \eta.$$

Thus, for each $g \in G$, the mapping α_g is an automorphism of the labeled digraph $\mathbf{D}(\eta)$.

We now show that each automorphism ϱ of the labeled digraph $\mathbf{D}(\eta)$ is of the form α_g for some $g \in G$. Let us denote the identity of G by e and the identity of G' by e'. Set

$$g = e\varrho.$$

Since the vertex e of $\mathbf{D}(\eta)$ has a directed edge labeled by η exiting from it, namely, the edge $\langle e, e'\rangle$, it follows that so does g; therefore,

$$g \in G.$$

We claim that $\varrho = \alpha_g$. Indeed, if $h \in G$, then

$$h = \langle h, e\rangle\lambda_\eta = \langle h\varrho, e\varrho\rangle\lambda_\eta = \langle h\varrho, g\rangle\lambda_\eta = h\varrho g^{-1},$$

and so

$$h\varrho = hg = h\alpha_g.$$

Now,

$$\eta = \langle e, e'\rangle\lambda_\eta = \langle e\varrho, e'\varrho\rangle\lambda_\eta = \langle g, e'\varrho\rangle\lambda_\eta,$$

and so

$$e'\varrho = g\eta.$$

Then, if $h \in G'$, we get as above, with e replaced by e',

$$h\varrho = hg\eta = h\alpha_g,$$

establishing our claim.

Thus $\bar{\eta}\colon G \longrightarrow \operatorname{Aut} \mathbf{D}(\eta)$ is surjective. Clearly, $\alpha_g = \alpha_h$ implies that $g = h$ and, equally clearly, $\alpha_{gh} = \alpha_g\alpha_h$—thus $\bar{\eta}$ is an isomorphism.

The final claim of the lemma is clear; the proof is complete. ∎

We now show how to associate a lattice with any labeled digraph in such a manner that the automorphism group of the digraph corresponds to the automorphism group of the lattice, and such that any full subgraph corresponds to a sublattice. This is done in several steps; with the labeled digraph we first associate a (unlabeled) digraph, with that digraph we then associate a graph, and, finally, with the resulting graph we associate a lattice.

We first show how to associate a digraph with any labeled digraph. Note that any ordinal γ can be considered to be a digraph; $\langle\beta_1, \beta_2\rangle$ is a directed edge iff $\beta_1 < \beta_2$; then the only digraph automorphism of γ is the identity, and distinct ordinals are nonisomorphic digraphs. Let $\mathbf{D} = \langle V, R, \lambda\rangle$ be a labeled digraph and let X denote the set of labels. For each $x \in X$, we choose an ordinal $\gamma_x > 3$ so that distinct ordinals are chosen for distinct x. For each directed edge $\langle v, w\rangle \in R$, we take a digraph $\mathbf{O}_{v,w} = \langle V_{v,w}, R_{v,w}\rangle$ corresponding to the ordinal $\gamma_{\langle v,w\rangle\lambda}$ in such a manner that the digraphs associated with distinct edges are disjoint. Let us denote the vertex of $\mathbf{O}_{v,w}$ corresponding to $0 \in \gamma_{\langle v,w\rangle\lambda}$ by $o_{v,w}$. We construct a digraph $\mathbf{D}_0 = \langle V_0, R_0\rangle$ from $\mathbf{D}$ by replacing each directed edge $\langle v, w\rangle$ of $\mathbf{D}$ by the subgraph depicted in Figure 16.

More formally,

$$V_0 = V \,\dot\cup\, \dot{\bigcup}\{V_{v,w} \mid \langle v, w\rangle \in R\}$$

and

$$R_0 = \{\langle v, o_{v,w}\rangle \mid \langle v, w\rangle \in R\} \cup \{\langle o_{v,w}, w\rangle \mid \langle v, w\rangle \in R\} \cup \bigcup\{R_{v,w} \mid \langle v, w\rangle \in R\}.$$

Each automorphism of $\mathbf{D}$ extends naturally to an automorphism of $\mathbf{D}_0$ in a unique manner; if $v \longmapsto v'$ and $w \longmapsto w'$, then $o_{v,w} \longmapsto o_{v',w'}$ and, since $\langle v, w\rangle\lambda = \langle v', w'\rangle\lambda$, there

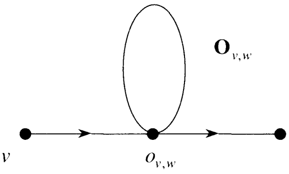

FIGURE 16

is a unique isomorphism $\mathbf{O}_{v,w} \longrightarrow \mathbf{O}_{v',w'}$ with $o_{v,w} \longmapsto o_{v',w'}$. It is also easy to see that each automorphism α of $\mathbf{D}_0$ is such an extension of an automorphism of $\mathbf{D}$. Indeed, the elements v of V satisfy the following two conditions in $\mathbf{D}_0$:

If $w_1 \neq w_2$ and $\langle w_1, v\rangle, \langle w_2, v\rangle \in R_0$, then $\langle w_1, w_2\rangle \notin R_0$.
If $w_1 \neq w_2$ and $\langle v, w_1\rangle, \langle v, w_2\rangle \in R_0$, then $\langle w_1, w_2\rangle \notin R_0$.

Each element of any $\mathbf{O}_{v,w}$ fails at least one of them, since $\mathbf{O}_{v,w}$ is an ordinal with at least four elements. Thus, $V\alpha = V$. If $v\alpha = v'$ and $w\alpha = w'$, then, since $o_{v,w}$ is the unique element y of V_0 with $\langle v, y\rangle$, $\langle y, w\rangle \in R_0$ and $o_{v',w'}$ is the unique element y of V_0 with $\langle v', y\rangle$, $\langle y, w'\rangle \in R_0$, we conclude that $o_{v,w}\alpha = o_{v',w'}$ and so it follows easily that $V_{v,w}\alpha = V_{v',w'}$, that is, that $\mathbf{O}_{v,w}$ is isomorphic to $\mathbf{O}_{v',w'}$, that is, that $\langle v, w\rangle\lambda = \langle v', w'\rangle\lambda$.

We next associate with the digraph $\mathbf{D}_0$ a graph $\mathbf{G} = \langle V_1, E\rangle$ where E is the set of (undirected) edges and $V_0 \subseteq V_1$. We do this, following R. Frucht [4], by replacing each directed edge $\langle v, w\rangle$ of $\mathbf{D}_0$ by the graph depicted in Figure 17. It then follows

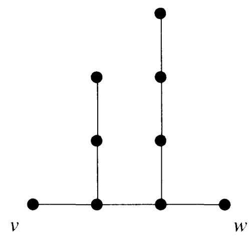

FIGURE 17

easily that any automorphism of $\mathbf{G}$ is the extension of a mapping $V_0 \longrightarrow V_0$ which is an automorphism of $\mathbf{D}_0$, and that any automorphism of $\mathbf{D}_0$ extends uniquely to an automorphism of $\mathbf{G}$.

Thus, summarizing, with each labeled digraph $\mathbf{D}$ with vertex set V we associate a graph $\mathbf{G} = \langle V_1, E\rangle$, with vertex set $V_1 \supseteq V$ and set of edges E, such that each automorphism $V \longrightarrow V$ of $\mathbf{D}$ extends uniquely to an automorphism $V_1 \longrightarrow V_1$ of $\mathbf{G}$, and such that each automorphism of $\mathbf{G}$ is so obtained. It is also clear that the above construction preserves the property of being a full subgraph.

Finally, as in R. Frucht [5], from $\mathbf{G}$ we form the lattice

$$H = V_1 \dot\cup E \dot\cup \{0, 1\},$$

where, for all $v \in V_1$ and $a \in E$, the relations $0 < v < 1$ and $0 < a < 1$ hold; let $v < a$ in H iff $v \in a$. Note that H is of length three. It is easy to prove that the lattice H is simple if the graph $\mathbf{G}$ has the following property:

(6-1) For $v \in V_1$, there are $a_0, a_1 \in E$ with $v \notin a_0, a_1$ and $a_0 \cap a_1 = \emptyset$.

The automorphisms of the lattice H correspond to automorphisms of the graph $\mathbf{G}$ since the vertices of $\mathbf{G}$ are the atoms of H. It is also clear that the lattice associated with any full subgraph of $\mathbf{G}$ is a $\{0,1\}$-sublattice of H.

Applying these results to the labeled digraph $\mathbf{D}(\eta)$ and its full subgraph $\mathbf{D}(G')$, we get a graph $\mathbf{G}$ and a full subgraph $\mathbf{G}'$ and so, a lattice H_0, a $\{0,1\}$-sublattice H_0', and isomorphisms $\tau\colon G \longrightarrow \operatorname{Aut} H_0$, $\tau'\colon G' \longrightarrow \operatorname{Aut} H_0'$ such that, for each $g \in G$, the automorphism $g\tau$ restricts to the automorphism $(g\eta)\tau'$ of H_0'. The graph $\mathbf{G}$ satisfies (6-1); thus the lattice H_0 is a simple lattice.

If the group G' is the trivial group $\{e'\}$, then H_0' is the three-element chain $\{0, e', 1\}$ and e' is fixed by each automorphism of H_0. We set $H = H_0$ and set $H' = \{0, e'\}$. Let us denote the element e' by i_H. Then H is a simple lattice of length 3 whose automorphism group is isomorphic to G. The lattice H contains a simple ideal $H' = (i_H]$ of length 1 such that each automorphism of H is an extension of the identity map on H'.

If the group G' is not trivial, then the graph $\mathbf{G}'$ satisfies (6-1), and so H_0' is a simple lattice. Let H' be a lattice isomorphic to H_0' and disjoint from H_0, and let $\mu\colon H' \longrightarrow H_0$ denote the embedding of H' as the sublattice H_0' of H_0. Let H_1 be the ordinal sum of H' and H_0 with the unit element of H' identified with the zero element of H_0—we will denote this element of H_1 by i_H. Let the set Λ of intervals of H_1 be defined by setting

$$\Lambda = \{[x, x\mu] \mid x \in H' - \{0, i_H\}\}$$

where 0 denotes the zero of H'. We apply the One Point Extension of Section 4 to get the lattice $H = H_1[\Lambda]$—see Figure 18. It is easy to see by Theorem 5 that H is a simple lattice. Now H' is the ideal $(i_H]$ of H. It is also easy to see that there is a one-to-one correspondence between the automorphisms of H and those of H_0 whereby the automorphisms of H_0' correspond to the automorphisms of H'; the doubly-irreducible element $m_{[x,x\mu]}$ ties together the element $x\mu$ of H_0 with the element x of H'. We observe that that the length of H is 6, and that of H' is 3.

Note finally that if G and G' are finite, then so is the labeled digraph $\mathbf{D}(\eta)$. Since there are only finitely many labels, we can choose all the graphs $\mathbf{O}_{v,w}$ to be finite, and thus the resulting digraph $\mathbf{D}(\eta)_0$ can be chosen to be finite. Then the corresponding graph is finite, and so the lattice H is finite.

We thus have the following lemma:

LEMMA 12. *Let G, G' be groups and let $\eta\colon G \longrightarrow G'$ be a group homomorphism. Then there is a simple lattice H of length at most 6 with an ideal $H' = (i_H]$ of length at most 3 which is also simple. There are isomorphisms*

$$\tau_H\colon G \longrightarrow \operatorname{Aut} H$$

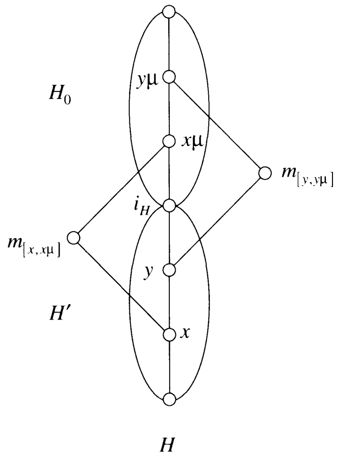

FIGURE 18

and

$$\tau'_{H'}\colon G' \longrightarrow \operatorname{Aut} H'$$

such that, for each $g \in G$, the automorphism $g\tau_H$ of H restricts to the automorphism $g\eta\tau'_{H'}$ of H'. Furthermore, if G and G' are finite, then so is the lattice H.

As a final comment, we should like to remark that, starting with the labeled graph $\mathbf{D}(G)$ rather than the more complicated $\mathbf{D}(\eta)$ if we wish, we have a proof that any group G can be represented as the group of automorphisms of a graph. This proof is an alternative to the proof in G. Sabidussi [14] and is more in the spirit of R. Frucht's [4] proof for the finite case.

7. **The proof of Theorem 2.** We show how to combine the results of Section 3 and Section 5 to prove Theorem 2.

Let us, henceforth, denote by 0_H the smallest element and by 1_H the largest element of the lattice H constructed in Section 6. We modify the chain C_1 by replacing $d_{r'}$, the largest element of C'_1, by the lattice H, thereby obtaining a lattice A_1 which is *never* a chain—see Figure 19. The ideal $(0_H]$ of A_1 will be regarded the same as the chain C'_1, and the dual ideal $[1_H)$ will be regarded the same as the chain $[d_{r'})$, that is, the same as $(C_1 - C'_1) \cup \{d_{r'}\}$.The former topmost prime interval $[d_{r'-1}, d_{r'}]$ of C'_1 now corresponds to $[d_{r'-1}, 0_H]$, and the former prime interval $[d_{r'}, d_{r'+1}]$, the bottommost prime interval of C_1 that is in C'_1, now corresponds to $[1_H, d_{r'+1}]$. We set A'_1 to be the ideal $(i_H]$ of A_1. In the construction in this section, A_1 will play the role of C_1 and A'_1 will play the role of C'_1.

For uniformity of notation, we henceforth set

$$A'_0 = C'_0$$

and

$$A_0 = C_0.$$

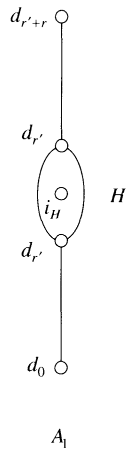

FIGURE 19

We extend the coloring φ_1' of Ip $(0_H]$ = Ip C_1' to A_1'. Since, by the hypothesis of Theorem 2, $J(D')$ is not empty, we choose any *minimal* $a \in J(D')$ and set $\mathfrak{p}\varphi_1' = a$ for all $\mathfrak{p} \in$ Ip $[0_H, i_H]$. We similarly extend the coloring φ_1 by setting $\mathfrak{p}\varphi_1 = a\psi^*$ for all $\mathfrak{p} \in$ Ip H. Then we still have

$$\mathfrak{p}\varphi_1 = \mathfrak{p}\varphi_1'\psi^* \text{for all } \mathfrak{p} \in \text{Ip } A_1'.$$

In Section 3 we did not have with the automorphisms of L' those difficulties associated with the automorphisms of L discussed just before Lemma 4, essentially because C_1' is trivial if D' is Boolean. However, here, these difficulties can arise for L'. If D' is Boolean, then $A_1' = [0_H, i_H]$; if $[c_{s'-1}, c_{s'}]\varphi_0' = a$, a as above, (recall, $[c_{s'-1}, c_{s'}]$ is the topmost prime interval in A_0'), then the doubly-irreducible $\langle c_{s'}, 0_H \rangle$ can be switched with some m introduced in applying the $\mathfrak{M}_3$-construction to an interval $[c_{s'-1}, c_{s'}] \times [0_h, u]$, where u is an atom of $H' = (i_H]$.

A similar difficulty can occur on the other side of L' when D' is Boolean and G' is trivial, that is, when i_H is join-irreducible in A_1'—then we may have $[c_0, c_1]\varphi_0' = [0_H, i_H]\varphi_1'$.

These difficulties can be alleviated exactly as in the discussion following the proof of Lemma 4 of Section 3—extend A_0' and C_1' by adding more prime intervals colored a. Finally, if D' is Boolean and G' is trivial, we can thereby arrange that A_0' and A_1', which is in this event a chain, are of different lengths, thereby preventing the interchange of axes in L'.

Note that for each (a, b)-interval $[c_j, c_{j+3}]$ in C_0, $[a, b]$ a prime interval in $J(D')$ or in $J(D)$, we still have $d_{\delta(j)}$ in the chain $(A_1 - H) \cup \{1_H\}$. Then, exactly as in Section 3, we consider the sublattice of $A_0 \times A_1$ determined by

$$\langle u, v\rangle \in A \iff \begin{cases} v \in A_1, & \text{in case } c_0 \leq u \leq c_{s'}; \\ v \geq d_{\delta(s'+3i)}, & \text{in case } c_{s'+3i} < u \leq c_{s'+3i+3} \text{ for } 0 \leq i < n'; \\ v \geq d_{\delta(s'+3n'+3i)}, & \text{in case } c_{s'+3n'+3i} < u \leq c_{s'+3n'+3i+3} \\ & \text{for } 0 \leq i < n. \end{cases}$$

Note that $A'_0 \times H \subseteq A$.

Again, as in Section 3, we define a set Λ of intervals of the form $\mathfrak{p} \times \mathfrak{q}$, $\mathfrak{p} \in \operatorname{Ip} A_0$, $\mathfrak{q} \in \operatorname{Ip} A_1$, in A by requiring that

if $\mathfrak{p} \times \mathfrak{q} \subseteq A$ with $\mathfrak{p} \in \operatorname{Ip} A'_0$, $\mathfrak{q} \in \operatorname{Ip} A'_1$, then

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda \iff \mathfrak{p}\varphi'_0 = \mathfrak{q}\varphi'_1,$$

except that if D' is Boolean, if $\mathfrak{p}$ is the topmost prime interval in A'_0, and if $\mathfrak{q}$ is a bottommost prime interval in A'_1, then

$$\mathfrak{p} \times \mathfrak{q} \notin \Lambda,$$

and if D' is Boolean and G' is trivial, then

$$[c_0, c_1] \times [0_H, i_H] \notin \Lambda;$$

if $\mathfrak{p} \times \mathfrak{q} \subseteq A$ with $\mathfrak{p} \notin \operatorname{Ip} A'_0$, and with $\mathfrak{q} \in \operatorname{Ip} A_1$, then

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda \iff \mathfrak{p}\varphi_0 = \mathfrak{q}\varphi_1,$$

except that, if D' and D are both Boolean, if $\mathfrak{p}$ is the topmost prime interval in A_0, and if $\mathfrak{q}$ is a bottommost prime interval in A_1, then

$$\mathfrak{p} \times \mathfrak{q} \notin \Lambda;$$

if $\mathfrak{p} \times \mathfrak{q} \subseteq A$ with $\mathfrak{p} \in \operatorname{Ip} A_0$ and with $\mathfrak{q} \notin \operatorname{Ip} A'_1$, then

$$\mathfrak{p} \times \mathfrak{q} \in \Lambda \iff \mathfrak{p}\varphi_0 = \mathfrak{q}\varphi_1,$$

except that, if D' and D are both Boolean and $\mathfrak{q}$ is the topmost prime interval in A_1, then

$$[c_0, c_1] \times \mathfrak{q} \notin \Lambda.$$

We set

$$A' = A \cap (A'_0 \times A'_1),$$

that is, A' is the ideal $(\langle c_{s'+3n'}, i_H\rangle]$ of A, where $c_{s'+3n'}$ is the largest element of the chain A'_0. We let Λ' be the set of those intervals in Λ that are subsets of A'.

Then, exactly as in Lemma 3 of Section 3, we have

LEMMA 13. *The set of intervals* Λ' *of* A' *satisfies Requirement 1 with respect to the colorings* $\varphi_i'\colon$ Ip $A_i' \longrightarrow J(D')$. *The set of intervals* Λ *of* A *satisfies Requirement 1 with respect to the colorings* $\varphi_i\colon$ Ip $A_i \longrightarrow J(D)$.

We then have:

LEMMA 14. *The congruence relations of the lattice* $A[\Lambda]$ *are precisely those of the form* $\Theta_0 \times_{A.\Lambda} \Theta_1$ *where, for* $i = 0, 1$, Θ_i *is a congruence relation on* A_i *satisfying:*
(1) For $\mathfrak{p}_0 \in$ Ip A_0, $\mathfrak{p}_1 \in$ Ip A_1, *if* $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$, *then*

$$\Theta(\mathfrak{p}_0) \leq \Theta_0 \textit{ is equivalent to } \Theta(\mathfrak{p}_1) \leq \Theta_1.$$

PROOF. The proof proceeds exactly as that of Lemma 7, except that, since A_1 is not a chain, we must verify Condition (6.2) and its dual. However, all the prime intervals in H, the part of A_1 that is not a chain, have the same color under φ_1 and this color lies in (Ip $A_0)\varphi_0$. Condition (6.2) and its dual thereby follow by Condition (14.1). ∎

Similarly, we have:

LEMMA 15. *The congruence relations of the lattice* $A'[\Lambda']$ *are precisely those of the form* $\Theta_0 \times_{A'.\Lambda'} \Theta_1$ *where, for* $i = 0, 1$, Θ_i *is a congruence relation on* A_i' *satisfying:*
(1) For $\mathfrak{p}_0 \in$ Ip A_0', $\mathfrak{p}_1 \in$ Ip A_1', *if* $\mathfrak{p}_0\varphi_0' = \mathfrak{p}_1\varphi_1'$, *then*

$$\Theta(\mathfrak{p}_0) \leq \Theta_0 \textit{ is equivalent to } \Theta(\mathfrak{p}_1) \leq \Theta_1.$$

We now proceed exactly as in Section 5, with the obvious exception that C_i' there is now A_i' and C_i there is now A_i. The definitions of the families of intervals Γ and Γ' are the same, and we again have lattices

$$L = A[\Lambda][\Gamma]$$

and

$$L' = A'[\Lambda'][\Gamma'].$$

The mappings

$$\varrho\colon D \longrightarrow \text{Con } L$$

and

$$\varrho'\colon D' \longrightarrow \text{Con } L'$$

are the same, and we have:

LEMMA 16. ϱ *and* ϱ' *are isomorphisms, and composing* ϱ *with restriction of congruence relations to* L' *yields* $\psi\varrho'$.

We now determine the automorphism groups of L and L'.

Let $g \in G$. We extend the automorphism $g\tau_H$ of H, determined in Section 6, trivially to the rest of A_1, thereby getting $\alpha_1 \in$ Aut A_1. We let $\alpha_0 \in$ Aut A_0 be the identity map. Since $A_0' \times H \subseteq A$ and α_1 maps H to H and acts trivially outside of H, it is immediate that $\alpha_0 \times \alpha_1$ restricts to an automorphism of A.

Clearly, α_0 preserves φ_0, and, since all prime intervals in H have the same color under φ_1, α_1 preserves φ_1. Furthermore, since automorphisms preserve height, the set of exceptional intervals $\mathfrak{p} \times \mathfrak{q}$ that are excluded from Λ is preserved by the automorphism $\alpha_0 \times \alpha_1$ on A. Consequently, the restriction of $\alpha_0 \times \alpha_1$ to A extends to an automorphism $\alpha[\Lambda]$ of $A[\Lambda]$. Since α_1 acts trivially outside of H, the automorphism $\alpha[\Lambda]$ acts trivially on the set of intervals Γ, and thereby extends to an automorphism $g\tau\colon L \longrightarrow L$.

Similarly, for each $g \in G'$, we extend $g\tau'_{H'} \in \operatorname{Aut} H'$ to $g\tau'\colon L' \longrightarrow L'$. We thus have injective group homomorphisms

$$\tau\colon G \longrightarrow \operatorname{Aut} L,$$
$$\tau'\colon G' \longrightarrow \operatorname{Aut} L',$$

where, for each $g \in G$, $g\eta\tau'$ is the restriction of $g\tau$ to L'. To complete the proof of Theorem 2 we need only show that τ and τ' are surjective.

We proceed very much as in the proof of Lemma 4. Recall that c_0 is the minimum element of the chain A_0. For the sake of notational convenience, let p henceforth be the length of A_0; then c_p is the maximum element of A_0. Similarly, we denote the maximum element of A_1 by d_t (it covers exactly one element, denoted d_{t-1}) and the minimum element by d_0—this could possibly be 0_H.

Let $\alpha\colon L \longrightarrow L$ be an automorphism. All the elements of $L - A$ are doubly-irreducible in L, and none of them lies in an interval of L that is a four-element Boolean lattice. The only elements of A doubly-irreducible in L are $\langle c_0, d_t\rangle$ and, in the event that both D' and D are Boolean, $\langle c_p, d_0\rangle$. Since

$$[c_0, c_1] \times [d_{t-1}, d_t] \notin \Lambda,$$

the interval $[\langle c_0, d_{t-1}\rangle, \langle c_1, d_t\rangle]$ in L is a four-element Boolean lattice.

Thus, if at least one of D' or D is not Boolean,

$$\langle c_0, d_t\rangle\alpha = \langle c_0, d_t\rangle,$$

that is, α maps $\{c_0\} \times A_1$ onto itself. It then follows easily that α restricted to A is of the form $\alpha_0 \times \alpha_1$ restricted to A, where α_0 is the trivial automorphism of A_0 and α_1 is the extension to A_1 by the trivial action of an automorphism $g\tau_H$ of H for some $g \in G$. Since the elements of $L\psi - A$ are determined uniquely by those of A, we conclude that

$$\alpha = g\tau.$$

If both D' and D are Boolean, then, since

$$[c_{p-1}, c_p] \times [d_0, u] \notin \Lambda$$

for any atom u of A_1, the elements $\langle c_p, d_0\rangle$ and $\langle c_0, d_t\rangle$ are the doubly-irreducible elements of L that lie in an interval that is a four-element Boolean lattice. Now, the ideal

$$[\langle c_0, d_0\rangle, \langle c_p, d_0\rangle] = A_0 \times \{d_0\}$$

is a chain. However, the ideal

$$[\langle c_0, d_0\rangle, \langle c_0, d_t\rangle] = \{c_0\} \times A_1$$

is not a chain, since H is not. So, again,

$$\langle c_0, d_t\rangle\alpha = \langle c_0, d_t\rangle,$$

and we conclude that

$$\alpha = g\tau$$

for some $g \in G$. Thus, τ is surjective.

The proof that τ' is surjective proceeds in an identical manner, except for the case when D' is Boolean and when G' is the trivial group—in this case both A'_0 and A'_1 are chains. However, by explicit construction, in this case A'_0 and A'_1 have different lengths, and we then conclude that L' has only the trivial automorphism, which is of course $e'\tau'$.

This concludes the proof of Theorem 2.

We remark that, in contrast to the proof of Theorem 1, we did not treat the case when both D' and D are Boolean and when both ψ and η are isomorphisms as a special case.

REFERENCES

1. M. Adams and J. Sichler, *Cover set lattices*, Can. J. Math. **32**(1980), 1177–1205.

2. V. A. Baranskiĭ, *On the independence of the automorphism group and the congruence lattice for lattices*, Abstracts of lectures of the 15th All-Soviet Algebraic Conference, Krasnojarsk, July 1979, **1**, p. 11.

3. ———, *On the independence of the automorphism group and the congruence lattice for lattices*, Izv. Vyssh. Uchebn. Zaved. Mat. **12**(1984), 12–17.

4. R. Frucht, *Herstellung von graphen mit vorgegebener abstrakter gruppe*, Compos. Math. **6**(1938), 239–250.

5. ———, *Lattices with a given group of automorphisms*, Can. J. Math. **2**(1950), 417–419.

6. N. Funayama and T. Nakayama, *On the distributivity of a lattice of lattice-congruences*, Proc. Imp. Acad. Tokyo **18**(1942), 553–554.

7. G. Grätzer, *General lattice theory*, Academic Press, New York, N. Y., Birkhäuser Verlag, Basel; Akademie Verlag, Berlin, 1978.

8. G. Grätzer and H. Lakser, *Homomorphisms of distributive lattices as restrictions of congruences*, Can. J. Math. **38**(1986), 1122–1134.

9. ———, *Congruence lattices of planar lattices*, Acta Math. Hungar. **60**(1992), 251–268.

10. ———, *On complete congruence lattices of complete lattices*, Trans. Amer. Math. Soc. **327**(1991), 385–405.

11. ———, *On congruence lattices of* $\mathfrak{m}$*-complete lattices*, J. Austral. Math. Soc. Ser. A **52**(1992), 57–87.

12. G. Grätzer and E. T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13**(1962), 179–185.

13. A. Pultr and V. Trnková, *Combinatorial, algebraic and topological representations of groups, semigroups and categories*, Academia, Prague, 1980.

14. G. Sabidussi, *Graphs with given infinite groups*, Monatsh. Math. **64**(1960), 64–67.

15. E. T. Schmidt, *Homomorphisms of distributive lattices as restrictions of congruences*, Acta Sci. Math. (Szeged) **51**(1987), 209–215.

16. S.-K. Teo, *Representing finite lattices as complete congruence lattices of complete lattices*, Abstracts of papers presented to the Amer. Math. Soc. 88T-06-207.

17. A. Urquhart, *A topological representation theory for lattices*, Algebra Universalis **8**(1978), 45–58.

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