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SADDLE TOWERS AND MINIMAL k-NOIDS IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract Given $k \geqslant 2$, we construct a $(2k-2)$-parameter family of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation T, called *saddle towers*, which have total intrinsic curvature $4\pi(1-k)$, genus zero and $2k$ vertical Scherk-type ends in the quotient by T. Each of those examples is obtained from the conjugate graph of a Jenkins–Serrin graph over a convex polygonal domain with $2k$ edges of the same (finite) length. As limits of saddle towers, we obtain properly embedded minimal surfaces, called *minimal k-noids*, which are symmetric with respect to a horizontal slice (in fact they are vertical bi-graphs) and have total intrinsic curvature $4\pi(1-k)$, genus zero and k vertical planar ends.

Keywords: saddle tower; k-noid; conjugation; Jenkins–Serrin problem

AMS 2010 *Mathematics subject classification:* Primary 53A10; 49Q05; 58E12

1. Introduction

Scherk [**16**] found a singly periodic minimal surface in $\mathbb{R}^3$ invariant by a vertical translation, which can be seen as the desingularization of two orthogonal vertical planes. This is the conjugate surface of the doubly periodic minimal surface obtained from the graph surface of

$$u(x,y) = \log\left(\frac{\cos x}{\cos y}\right), \quad |x| < \tfrac{1}{2}\pi,\ |y| < \tfrac{1}{2}\pi,$$

rotating it by an angle π about the straight vertical lines in its boundary. Such singly periodic minimal surface can be seen in a one-parameter family of singly periodic minimal surfaces invariant by a vertical translation, by changing the angle between the vertical planes. They are called *singly periodic Scherk minimal examples*.

In general, consider a convex polygonal domain $\Omega \subset \mathbb{R}^2$ with $2k$ edges of length 1, with $k \geqslant 2$. Mark its edges alternately by $+\infty$ and $-\infty$. Jenkins and Serrin [**5**] gave necessary and sufficient conditions for the existence of a function u defined on Ω which goes to $\pm\infty$ on the edges, as indicated by the marking and whose graph surface is minimal. To

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satisfy such conditions, Ω is assumed to be different from a parallelogram bounded by two sides of length 1 and two sides of length $k-1$, for $k \geqslant 3$, which consist of the union of $k-1$ edges of Ω whose interior angles equal π (see, for example, [**9**, Proposition 1.3]).

The graph surface Σ_u of u is bounded by $2k$ vertical straight lines above the vertices of Ω. The conjugate minimal surface of Σ_u is then bounded by $2k$ horizontal geodesic curvature lines, lying in two horizontal planes at distance 1 from each other. By reflecting in one of the two symmetry planes, we obtain a fundamental domain for a properly embedded singly periodic minimal surface M of period $T = (0, 0, 2)$. In the quotient by T, M has genus zero and $2k$ ends asymptotic to flat vertical annuli (quotients of vertical half-planes by T). This kind of ends are classically called Scherk-type ends. We remark that by changing the length ℓ of the edges of Ω we get nothing but M rescaled by ℓ. This is why we can fix $\ell = 1$.

This procedure provides for $k = 2$ the one-parameter family of Scherk examples; and for any $k \geqslant 3$, a $(2k-3)$-parameter family of examples, which were constructed by Karcher [**6**, **7**] and called *saddle towers*. These examples have recently been classified by Pérez and Traizet [**13**] as the only complete embedded singly periodic minimal surfaces in $\mathbb{R}^3$ with genus zero and finitely many Scherk-type ends in the quotient.

In this paper we follow the same strategy in $\mathbb{H}^2 \times \mathbb{R}$ to construct properly embedded singly periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation T, with genus zero and $2k$ vertical Scherk-type ends in the quotient by T. We say that an end is a vertical Scherk-type end when it is asymptotic to the quotient by T of half a vertical geodesic plane.

Theorem 1.1. *Given an integer number $k \geqslant 2$ and a vertical translation T, there exists a $(2k-3)$-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with total (intrinsic) curvature $4\pi(1-k)$, genus zero and $2k$ vertical Scherk-type ends in the quotient by T. Moreover, they are symmetric with respect to a horizontal slice (in fact they are vertical bi-graphs). We call them* saddle towers.

Independently, Pyo [**14**] has recently constructed symmetric saddle towers following two different approaches: the conjugation method explained above and a barrier method (see Remark 3.2).

We observe that we do not have homotheties in $\mathbb{H}^2 \times \mathbb{R}$, so the length of T gives us another parameter of the family. Then, we obtain a $(2k-2)$-parameter family of saddle towers in $\mathbb{H}^2 \times \mathbb{R}$. The following theorem gives possible limits of saddle towers when the length of T goes to $+\infty$.

Theorem 1.2. *Given $k \geqslant 2$, there exists a $(2k-3)$-parameter family of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with total (intrinsic) curvature $4\pi(1-k)$, genus zero and k ends, each end asymptotic to a vertical geodesic plane. Those surfaces are invariant by the reflection in a horizontal slice (in fact they are vertical bi-graphs). We call them* minimal k-noids.

Theorem 1.2 includes the one-parameter family that Pyo [**15**] has independently constructed very recently. These are the examples described in Remark 4.3.

Meeks *et al*. proved in [**11**] that the only properly embedded minimal surfaces with genus zero and finite topology in the Euclidean space $\mathbb{R}^3$ are the plane, the helicoid and the catenoid. In particular, there are no examples with genus zero and $k \geqslant 3$ ends in $\mathbb{R}^3$. Theorem 1.2 says this is not the case in $\mathbb{H}^2 \times \mathbb{R}$.

Hauswirth and Rosenberg proved in [**3**] that, when it is finite, the total curvature of a complete embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is a multiple of 2π. The only examples they were allowed to give were minimal graphs over polygonal domains with $2(m+1)$ edges, whose vertices are located at the infinite boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ (usually called ideal polygonal domains), with boundary values $\pm\infty$ alternately. These graphs, which can also be obtained by taking limits of saddle towers, have total curvature $-2\pi m$. Hauswirth and Rosenberg suggested that it would be interesting to construct non-simply connected examples of finite total curvature; for example, an annulus of total curvature -4π. Theorem 1.2 includes such examples.

2. Preliminaries

Throughout the paper, all surfaces are supposed to be connected and orientable.

We consider the Poincaré disc model of $\mathbb{H}^2$,

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\},$$

with the hyperbolic metric

$$g_{-1} = \frac{4}{(1 - x^2 - y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2),$$

and denote by t the coordinate in $\mathbb{R}$. Consider in $\mathbb{H}^2 \times \mathbb{R}$ the usual product metric

$$\mathrm{d}s^2 = \frac{4}{(1 - x^2 - y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2.$$

2.1. Minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$

Let $\Omega \subset \mathbb{H}^2$ be an open domain and let $u \colon \Omega \to \mathbb{R}$ be a smooth function. The (vertical) graph of u is minimal in $\mathbb{H}^2 \times \mathbb{R}$ if, and only if,

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \tag{2.1}$$

where all terms are calculated in the metric of $\mathbb{H}^2$.

In [**12**], Nelli and Rosenberg proved a Jenkins–Serrin type theorem for simply connected bounded convex domains in $\mathbb{H}^2 \times \mathbb{R}$. Let $\Omega \subset \mathbb{H}^2$ be a simply connected bounded convex domain whose boundary consists of a finite number of geodesic arcs $A_1, \dots, A_n$, $B_1, \dots, B_m$ and a finite number of convex arcs $C_1, \dots, C_p$ (convex with respect to Ω), together with their endpoints, such that no two A_i edges and no two B_i edges have a common endpoint. They gave necessary and sufficient conditions (in terms of the lengths of the boundary arcs of Ω and of the perimeter of inscribed polygons in Ω whose vertices are among the vertices of Ω) for the existence and uniqueness (up to an additive constant, in the case the family of C_i arcs is empty) of a solution u for the minimal graph equation (2.1) such that

$$u_{|A_i} = +\infty, \quad u_{|B_i} = -\infty \quad \text{and} \quad u_{|C_i} = f_i,$$

for arbitrary continuous functions f_i.

Collin and Rosenberg [**1**] solved the Jenkins–Serrin problem for unbounded simply connected domains bounded by a finite number of complete geodesic arcs and a finite number of complete convex arcs, together with their endpoints at $\partial_\infty\mathbb{H}^2$, with the additional assumption that two consecutive boundary edges of Ω are asymptotic at their common endpoint at $\partial_\infty\mathbb{H}^2$. A general Jenkins–Serrin problem in $\mathbb{H}^2 \times \mathbb{R}$ was solved by the second author together with Mazet and Rosenberg in [**10**].

In this work we will consider the particular case where Ω is a convex polygonal domain with $2k$ geodesic edges $A_1, B_1, \ldots, A_k, B_k$ (cyclically ordered) of the same length $\ell \in (0, +\infty]$. When $\ell = +\infty$, we will assume Ω is a semi-ideal polygonal domain (see Definition 2.1 below). We will state the Jenkins–Serrin theorem for such a domain Ω. Before, we fix some notation.

Definition 2.1. Let Ω be a polygonal domain (i.e. a domain whose edges are geodesic arcs). The vertices of Ω that are at $\partial_\infty\mathbb{H}^2$ are called *ideal vertices*. We say that Ω is *semi-ideal* when it has an even number of vertices $p_1, \ldots, p_{2k}$ (cyclically ordered), such that the odd vertices p_{2i-1} are in $\mathbb{H}^2$ and the even vertices p_{2i} are at infinity $\partial_\infty\mathbb{H}^2$ (or vice versa).

For each ideal vertex p_i of Ω (if it exists), we consider a horocycle H_i at p_i. Assume H_i is small enough so that it only intersects $\partial\Omega$ at the boundary edges having p_i as an endpoint, and so that $H_i \cap H_j = \emptyset$, for every $i \neq j$. Given a polygonal domain $\mathcal{P}$ inscribed in Ω (i.e. a polygonal domain $\mathcal{P} \subset \Omega$ whose vertices are drawn from the set of endpoints of the A_i, B_i edges, possibly at infinity), we denote by $\Gamma(\mathcal{P})$ the part of $\partial\mathcal{P}$ outside the horocycles (observe that $\Gamma(\mathcal{P}) = \partial\mathcal{P}$ in the case Ω has no ideal vertices). Also let us call

$$\alpha(\mathcal{P}) = \sum_i |A_i \cap \Gamma(\mathcal{P})| \quad \text{and} \quad \beta(\mathcal{P}) = \sum_i |B_i \cap \Gamma(\mathcal{P})|,$$

where $|\cdot| = \text{length}_{\mathbb{H}^2}(\cdot)$.

Definition 2.2. Let Ω be a convex polygonal domain as above. We say that Ω is a *Jenkins–Serrin domain* when the following two additional conditions hold for some choice of horocycles at its ideal vertices:

(i) $\alpha(\Omega) = \beta(\Omega)$;

(ii) $2\alpha(\mathcal{P}) < |\Gamma(\mathcal{P})|$ and $2\beta(\mathcal{P}) < |\Gamma(\mathcal{P})|$ for every polygonal domain $\mathcal{P}$ inscribed in Ω, $\mathcal{P} \neq \Omega$.

Remark that condition (i) in the above definition does not depend on the choice of horocycles H_i; and if the inequalities of condition (ii) are satisfied for some choice of horocycles, then they continue to hold for 'smaller' horocycles.

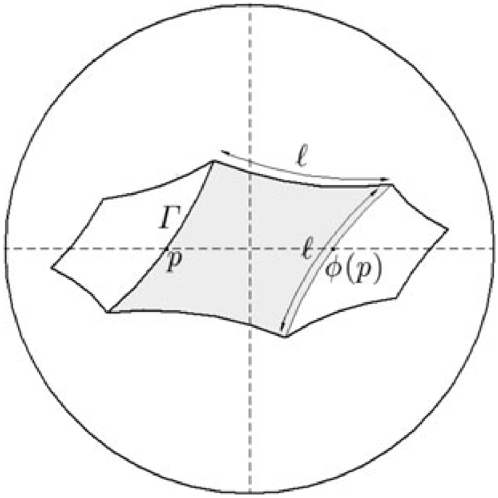

Figure 1. Consider the translation ϕ in the direction of the x-axis such that the edges of the geodesic square $\mathcal{D}$ determined by Γ and $\phi(\Gamma)$ (the shadowed region) have length ℓ. Translating twice $\mathcal{D}$ by ϕ we get a polygonal domain of eight edges of length ℓ which is not a Jenkins–Serrin domain.

Theorem 2.3 (Nelli and Rosenberg [12]; Collin and Rosenberg [1]; Mazet *et al.* [10]). *Let Ω be a convex polygonal domain with $2k$ edges $A_1, B_1, \dots, A_k, B_k$ (cyclically ordered) of the same length $\ell \in (0, +\infty]$. There exists a solution u for the minimal graph equation (2.1) such that*

$$u_{|A_i} = +\infty \quad \text{and} \quad u_{|B_i} = -\infty$$

if, and only if, Ω is a Jenkins–Serrin domain. Moreover, if it exists, then it is unique up to an additive constant.

Remark 2.4. In $\mathbb{R}^3$, the only convex polygonal domains with $2k$ edges of the same length $\ell \in (0, +\infty)$ which are not Jenkins–Serrin domains are parallelograms bounded by two sides of length ℓ and two sides of length $(k-1)\ell$, with $k \geqslant 3$, which consist of the union of $k-1$ edges of Ω whose interior angles equal π (see [**9**, Proposition 1.3]). In $\mathbb{H}^2 \times \mathbb{R}$, it is not so restrictive. For instance, we have the following.

- Let Γ be a geodesic arc of length ℓ, and let p be its middle point. Consider a geodesic γ passing through p, and a hyperbolic translation ϕ along γ such that the distance from the endpoints of Γ to the endpoints of $\phi(\Gamma)$ is ℓ. Call $\mathcal{D}$ the polygonal domain of four edges determined by Γ, $\phi(\Gamma)$. The convex polygonal domain obtained from $\mathcal{D}$ by translating it k times by ϕ (see Figure 1) is a polygonal domain of $4+2k$ edges of length ℓ which is not a Jenkins–Serrin domain.

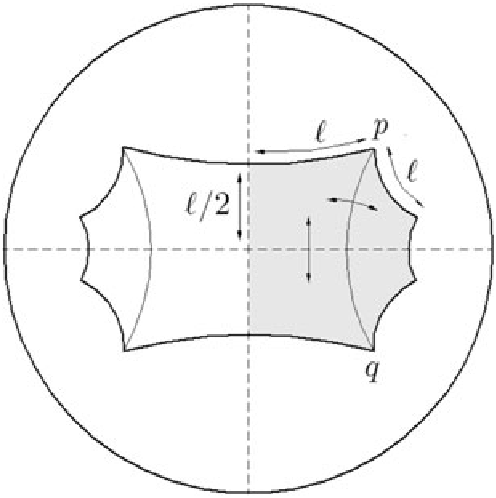

Figure 2. The shadowed region $\mathcal{D}$ is a geodesic hexagon whose edges have length ℓ and their interior angles are $\pi/2$ up to at two opposite vertices p, q where the interior angles are strictly smaller than $\pi/2$. By reflecting $\mathcal{D}$ in one of the edges who has not p nor q as an endpoint, we get a polygonal domain of ten edges of length ℓ which is not a Jenkins–Serrin domain.

- It can be also considered a convex polygonal domain $\mathcal{D}$ of $2n$ edges of length ℓ, for $n \geqslant 3$, such that the interior angles at their vertices are smaller than or equal to $\pi/2$. By reflecting $\mathcal{D}$ in an edge (respectively k times in two opposite edges), we obtain a convex polygonal domain with $4n-2$ edges (respectively $2n+2(n-1)k$ edges) of length ℓ which is not a Jenkins–Serrin domain (see Figure 2).

2.2. Conjugate surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this subsection we will recall how to obtain minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ by conjugation from other known minimal examples. For more details see [**2**] and [**4**].

Let Σ be a simply connected Riemann surface and J be the rotation of angle $\pi/2$ on $T\Sigma$. Denote by $\langle \cdot, \cdot \rangle$ the Riemannian metric on Σ. Given a conformal minimal immersion $X\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$, let us call

- S the symmetric operator on Σ induced by the shape operator of $X(\Sigma)$;
- T the vector field such that $\mathrm{d}X(T)$ is the projection of $\partial/\partial t$ onto $T(X(\Sigma))$;
- N the induced unit normal vector field on $X(\Sigma)$;
- $\nu = \langle N, \partial/\partial t \rangle$ the angle function (in particular, $\|T\|^2 + \nu^2 = 1$).

Theorem 2.5 (Daniel [2, Theorem 4.2]). *Let $X\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion. There exists a conformal minimal immersion $X^*\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ such that*

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- *the metrics induced on* Σ *by* X *and* X^* *are the same;*
- *the symmetric operator on* Σ *induced by the shape operator of* $X^*(\Sigma)$ *is* $S^* = JS$*;*
- $\partial/\partial t = \mathrm{d}X^*(T^*) + \nu N^*$, *where* $T^* = JT$ *and* N^* *is the unit normal vector to* $X^*(\Sigma)$.

Definition 2.6. The immersion X^* obtained in Theorem 2.5, which is well defined up to isometries of $\mathbb{H}^2 \times \mathbb{R}$, is called the *conjugate immersion* of X; and $X^*(\Sigma)$ is called *conjugate surface* of $X(\Sigma)$.

Let us denote by $X = (h, f)\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ a conformal minimal immersion, and by $X^* = (h^*, f^*)\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ its conjugate immersion. Then f^* is the harmonic conjugate of f (see [**2**, Proposition 4.6]). This implies that the Hopf differentials of h and h^* are opposite:

$$Q_h := \langle h_w, h_w \rangle \,\mathrm{d}w^2 = -(f_w)^2 \,\mathrm{d}w^2 = (f_w^*)^2 \,\mathrm{d}w^2 = -Q_{h^*}, \tag{2.2}$$

where w is a conformal parameter on Σ for X (and then also for X^*). In [**4**], the conjugate immersion is defined using identity (2.2). Given a conformal minimal immersion $X = (h, f)\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$, there exists a conformal minimal immersion $X^* = (h^*, f^*)\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$, called conjugate immersion of X, such that the metrics induced on Σ by X and X^* coincide and $Q_{h^*} = -Q_h$. The following theorem shows these two definitions of conjugate immersion are equivalent.

Theorem 2.7 (Hauswirth *et al.* [4, Theorem 6]). *Let* $X_1 = (h_1, f_1)$, $X_2 = (h_2, f_2)\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ *be two isometric conformal minimal immersions such that* $Q_{h_1} = Q_{h_2}$. *Then,* X_1, X_2 *coincide up to an isometry of* $\mathbb{H}^2 \times \mathbb{R}$.

Some geometric properties of conjugate surfaces in $\mathbb{H}^2 \times \mathbb{R}$ are discussed in [**2**]. As in $\mathbb{R}^3$ (see Karcher [**6**–**8**]), we get the following lemma.

Lemma 2.8. *The conjugation exchanges the following Schwarz reflections.*

- *The symmetry with respect to a vertical plane containing a geodesic curvature line becomes the rotation with respect to a horizontal geodesic of* $\mathbb{H}^2$, *and vice versa.*
- *The symmetry with respect to a horizontal plane containing a geodesic curvature line becomes the rotation with respect to a vertical straight line, and vice versa.*

We will use the above correspondence to study the conjugate surfaces of minimal graphs defined on convex polygonal domains of $\mathbb{H}^2$. The surfaces constructed in this way are minimal graphs (and consequently embedded), as ensured by the following generalized version of Krust's Theorem.

Theorem 2.9 (Hauswirth *et al.* [4, Theorem 14]). *Let* $X(\Sigma)$ *be a (vertical) minimal graph over a convex domain* $\Omega \subset \mathbb{H}^2$. *Then* $X^*(\Sigma)$ *is a (vertical) minimal graph.*

We finish this section by showing the following convergence result.

Proposition 2.10. *For every n, let $X_n\colon \Sigma_n \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion. Suppose $M_n = X_n(\Sigma_n)$ can be written as the graph surface of $u_n\colon \Omega_n \to \mathbb{R}$, where Ω_n is a convex domain. Assume there exists a domain $\mathcal{U} \subset \mathbb{H}^2$ contained in Ω_n, for every n; $\{\Omega_n\}$ converges to a convex domain Ω, $\mathcal{U} \subset \Omega$; and $\{u_n\}$ converges uniformly on compact subsets of Ω to a minimal graph $u\colon \Omega \to \mathbb{R}$ (i.e. $M_n \to M$, where M is the graph surface of u). Then, after passing to a subsequence, $M_n^* \to M^*$ (up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$), where the convergence is uniform on compact subsets.*

We first prove the following technical lemma.

Lemma 2.11. *For every n, let $X_n\colon \Sigma_n \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion. Suppose X_n converges to a conformal minimal immersion $X\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$, in the sense that $M_n = X_n(\Sigma_n)$ converges (uniformly on compact subsets) to $M = X(\Sigma)$. If the sequence of conjugate surfaces $M_n^* = X_n^*(\Sigma_n)$ converges, it must converge to $M^* = X^*(\Sigma)$, up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. Let p be a point in Σ, and D a small neighbourhood of p in Σ. We denote by z a conformal parameter on D for X. Let $p_n \in \Sigma_n$ be a sequence of points such that $X_n(p_n)$ converges to $X(p)$. We can identify a neighbourhood of p_n in Σ_n with D, adapted to have z as a conformal parameter. We work locally in D, i.e. we consider the restriction of $X_n = (h_n, f_n)$, $X = (h, f)$, $X_n^* = (h_n^*, f_n^*)$ and $X^* = (h^*, f^*)$ to D. Suppose $X_n^* = (h_n^*, f_n^*)$ converges to a conformal minimal immersion $Y = (\tilde{h}, \tilde{f})\colon D \to \mathbb{H}^2 \times \mathbb{R}$. Let us prove that $Y = X^*$ in D. Since p has been chosen arbitrarily, that finishes the proof.

By hypothesis, the real harmonic maps f_n converge to f, together with their derivatives. Then

$$Q_{h_n} = -(f_n)_z^2\,\mathrm{d}z^2 \to -f_z^2\,\mathrm{d}z^2 = Q_h. \tag{2.3}$$

(Observe that another consequence is that the real harmonic conjugate f_n^* of f_n converges to the real harmonic conjugate f^* of f, and then $\tilde{f} = f^*$.) Since $Q_{h_n^*} = -Q_{h_n}$, then we also have $Q_{h_n^*} \to -Q_h = Q_{h^*}$. On the other hand, we can prove similarly as in (2.3) that $Q_{h_n^*} \to Q_{\tilde{h}}$. Thus $Q_{\tilde{h}} = Q_{h^*}$.

Since $X_n \to X$ (respectively $X_n^* \to Y$), we know that the angle function ν_n of X_n (respectively ν_n^* of X_n^*) converges to the angle function ν of X (respectively ν_Y of Y). As $\nu_n = \nu_n^*$, we deduce that $\nu = \nu_Y$. We conclude that the metrics induced in D by X^* and Y coincide (see Equation (14) in [**4**]):

$$\mathrm{d}s_{X^*}^2 = 4\cosh^2\omega|Q_{h^*}| = 4\cosh^2\omega|Q_{\tilde{h}}| = \mathrm{d}s_Y^2,$$

where $\nu = \tanh\omega$. We finish the proof of Lemma 2.11 by Theorem 2.7. □

Proof of Proposition 2.10. By Lemma 2.11, it suffices to prove that the sequence $\{M_n^*\}$ converges.

Theorem 2.9 says that each M_n^* can be written as a graph u_n^* over a (not necessarily convex) domain $\Omega_n^* \subset \mathbb{H}^2$.

Let x be a point in $\mathcal{U}$, which is contained by hypothesis in every Ω_n. Denote by p_n the point in M_n which projects vertically on x, and let p_n^* be the corresponding point in M_n^*

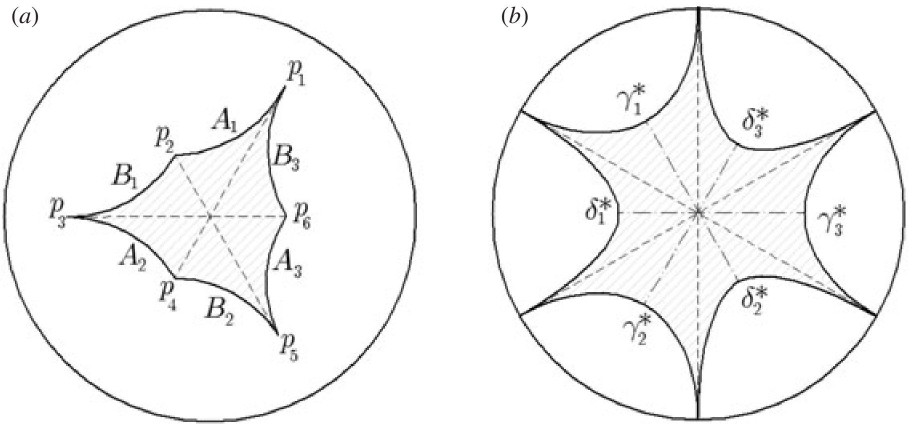

Figure 3. (a) An example of a (symmetric) bounded Jenkins–Serrin domain Ω with six edges of the same length. (b) The vertical projection over $\mathbb{H}^2$ of the conjugate surface (which is a graph) of the graph over Ω with boundary values $+\infty$ over $A_1 \cup A_2 \cup A_3$ and $-\infty$ over $B_1 \cup B_2 \cup B_3$.

by conjugation. Up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$, we can assume that p_n^* projects vertically on x, and $u_n^*(x) = 0$.

We call $M_{1,n}$ the part of M_n which projects over $\mathcal{U}$. Since u_n converges on $\mathcal{U}$, the sequence of absolute values of gradients $\{|\nabla u_n|\}$ is uniformly bounded on $\mathcal{U}$, after passing to a subsequence. That implies that the angle function of $M_{1,n}$ is uniformly bounded away from zero. Hence the same thing happens to its conjugate $M_{1,n}^*$. In particular, there exists a neighbourhood $\mathcal{V}$ of x in $\mathbb{H}^2$ such that $\mathcal{V} \subset \Omega_n^*$, for any n. Then, by passing to a subsequence, we can assume that there exists a domain Ω^* in the set of limit points of Ω_n^*.

Let y^* be a point in Ω^*. We have that $y^* \in \Omega_n^*$ for n big enough, and $\operatorname{dist}(y^*, \partial\Omega_n^*) \geqslant \delta$, for some small $\delta > 0$. Let $y_n \in \Omega_n$ be the point corresponding by conjugation to y^*. There exists a big compact set $K \subset \Omega$ containing y_n, for every n. By convergence of $\{u_n\}$, we have that $\{|\nabla u_n|\}$ is uniformly bounded in K. Arguing as above, we get that $\{|\nabla u_n^*(y^*)|\}$ is bounded. Since this holds for arbitrary y^*, we get by [**10**, Proposition 4.4] that u_n^* converges to a minimal graph u^* over Ω^*. This finishes Proposition 2.10. □

3. Saddle Towers in $\mathbb{H}^2 \times \mathbb{R}$

This section deals with the construction of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation T, which have total curvature $4\pi(1-k)$, genus zero and $2k$ vertical Scherk-type ends in the quotient by T (Theorem 1.1). The construction is similar to the one of Karcher's saddle towers in $\mathbb{R}^3$. We also call these new examples *saddle towers.*

Consider a convex Jenkins–Serrin domain Ω whose edges $A_1, B_1, A_2, B_2, \ldots, A_k, B_k$ (cyclically ordered) have length $\ell \in (0, +\infty)$. Denote by $p_1, \ldots, p_{2k}$ the vertices of Ω, such that p_{2i-1}, p_{2i} are the endpoints of A_i and p_{2i}, p_{2i+1} are the endpoints of B_i, for $i = 1, \ldots, k$ (as usual, we consider the cyclic notation $p_{2k+1} \equiv p_1$; see Figure 3).

By Theorem 2.3, there exists a solution $u\colon \Omega \to \mathbb{R}$ to the minimal graph equation (2.1) on Ω satisfying $u_{|A_i} = +\infty$ and $u_{|B_i} = -\infty$, for any $i = 1, \dots, k$.

The geometry of the graph surface Σ of u near $\partial\Omega$ is explained in [**12**]. When we approach a point in A_i (respectively B_i) within Ω, the tangent plane to Σ becomes vertical, asymptotic to $A_i \times \mathbb{R}$ (respectively $B_i \times \mathbb{R}$), i.e. the angle function ν goes to zero as we approach A_i, B_i. Moreover, Σ is bounded by $2k$ vertical straight lines passing through the vertices of Ω. Since Σ is asymptotic to $A_i \times \mathbb{R}$ (respectively $B_i \times \mathbb{R}$) over A_i (respectively B_i), the intrinsic distance on Σ from $\{p_{2i-1}\} \times \mathbb{R}$ to $\{p_{2i}\} \times \mathbb{R}$ (respectively from $\{p_{2i}\} \times \mathbb{R}$ to $\{p_{2i+1}\} \times \mathbb{R}$) is ℓ, which is never attained (ℓ is the asymptotic intrinsic distance at infinity).

Proposition 3.1. *The conjugate surface Σ^* of Σ is a (vertical) minimal graph, whose boundary is of the form $\partial\Sigma^* = \gamma_1^* \cup \delta_1^* \cup \cdots \cup \gamma_k^* \cup \delta_k^*$, where*

$$\gamma_1^*, \dots, \gamma_k^* \subset \{t = 0\} \quad \textit{and} \quad \delta_1^*, \dots, \delta_k^* \subset \{t = \ell\}$$

are geodesic curvature lines. Let us call Ω^*, $\tilde{\delta}_i^*$ the respective vertical projection of Σ^*, δ_i^* over $\{t = 0\}$. Then

- *the curves γ_i^*, $\tilde{\delta}_i^*$ are strictly concave (with respect to Ω^*);*
- *γ_i^* and $\tilde{\delta}_i^*$ (respectively $\tilde{\delta}_i^*$ and γ_{i+1}^*) are asymptotic at their common endpoint at $\partial_\infty \mathbb{H}^2$;*
- *$\partial\Omega^* = \gamma_1^* \cup \tilde{\delta}_1^* \cup \cdots \cup \gamma_k^* \cup \tilde{\delta}_k^*$ (cyclically ordered);*
- *$\Sigma^* - \partial\Sigma^* \subset \{0 < t < \ell\}$.*

Proof. Since Ω is convex, Theorem 2.9 says that the conjugate surface Σ^* of Σ is a minimal graph over a domain $\Omega^* \subset \mathbb{H}^2$. By Lemma 2.8, we know that the conjugation transforms vertical straight lines into horizontal geodesic curvature lines. Then $\partial\Sigma^*$ consists of $2k$ horizontal geodesic curvature lines $\gamma_1^*, \delta_1^*, \dots, \gamma_k^*, \delta_k^*$. Assume those boundary curves are ordered so that two consecutive ones correspond by conjugation to vertical straight lines in $\partial\Sigma$ through consecutive vertices of Ω. For every $i = 1, \dots, k$, let $\gamma_i, \delta_i \subset \partial\Sigma$ be the straight lines which correspond by conjugation to $\gamma_i^*, \delta_i^* \subset \partial\Sigma^*$; and $\tilde{\gamma}_i^*$, $\tilde{\delta}_i^*$ be the vertical projection of γ_i^*, δ_i^* over $\{t = 0\} \equiv \mathbb{H}^2$, respectively.

Consider the surface M obtained by extending Σ^* by symmetry with respect to the horizontal plane containing γ_1^*. If $\tilde{\gamma}_1^*$ is convex (with respect to Ω^*) at some point, then we will obtain by the maximum principle that M is contained in a vertical plane, a contradiction with the fact that Σ^* is a graph. Similarly, we deduce that none of vertical projections of the curves in $\partial\Sigma^*$ has a convexity point. This proves (1).

The asymptotic intrinsic distance at infinity between γ_i, δ_i (respectively δ_i, γ_{i+1}) is ℓ since the surface Σ is asymptotically vertical. By Theorem 2.5, Σ, Σ^* are isometric and have the same angle function. Thus the asymptotic intrinsic distance between γ_i^*, δ_i^* (respectively δ_i^*, γ_{i+1}^*) is ℓ, and the unit normal vector field N^* to Σ^* is asymptotically horizontal between γ_i^* and δ_i^* (respectively between δ_i^* and γ_{i+1}^*). In particular, $\tilde{\delta}_i^*$ shares an endpoint with $\tilde{\gamma}_i^*$ and the other with $\tilde{\gamma}_{i+1}^*$. Observe that this proves (3).

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To finish (2), it remains to prove that the endpoints of each $\tilde{\delta}_i^*$ (and then also of $\tilde{\gamma}_i^*$) are at $\partial_\infty \mathbb{H}^2$. (As $\tilde{\gamma}_i^*$, $\tilde{\delta}_i^*$ are strictly concave, they must arrive tangentially.) Fix a point $p^* \in \delta_i^*$, and let $\tilde{p}^* \in \tilde{\delta}_i^*$ be its vertical projection. The point p^* corresponds by conjugation to a point $p \in \delta_i$, which divides δ_i in two curves of infinite length. Since Σ and Σ^* are isometric, then each component of $\tilde{\delta}_i^* - \{\tilde{p}^*\}$ has infinite length as well, and finishes at a common endpoint with $\tilde{\gamma}_i^*$ or $\tilde{\gamma}_{i+1}^*$. Since all $\tilde{\gamma}_i^*$, $\tilde{\delta}_i^*$, $\tilde{\gamma}_{i+1}^*$ are strictly concave (with respect to Ω^*), we deduce that the endpoints of $\tilde{\delta}_i^*$ must be at $\partial_\infty \mathbb{H}^2$.

Recall that the asymptotic intrinsic distance between γ_i^*, δ_i^* (respectively δ_i^*, γ_{i+1}^*) is ℓ, for any $i = 1, \ldots, k$. We can assume that $\gamma_1^* \subset \{t = 0\}$ and $\delta_1^* \subset \{t = \ell\}$. We know that either $\gamma_2^* \subset \{t = 0\}$ or $\gamma_2^* \subset \{t = 2\ell\}$. Let us prove that the second case is impossible. We call $q_1^* \in \partial_\infty \mathbb{H}^2$ the common endpoint of $\tilde{\gamma}_1^*$, $\tilde{\delta}_1^*$. Since Σ^* is asymptotic to $\{q_1^*\} \times (0, \ell)$ when we approach q_1^* within Ω^*, then Σ^* is locally below δ_1^* near the asymptotic point (q_1^*, ℓ) at infinity. Since N^* is horizontal along δ_1^*, then Σ^* is locally below $\{t = \ell\}$ in a small neighbourhood of δ_1^*. In particular, Σ^* is locally below δ_1^* near (q_2^*, ℓ), where q_2^* is the common endpoint of $\tilde{\delta}_1^*$, $\tilde{\gamma}_2^*$. Thus Σ^* cannot be asymptotic to $\{q_2^*\} \times (\ell, 2\ell)$, and then $\gamma_2^* \subset \{t = 0\}$. Arguing similarly we prove

$$\gamma_1^*, \ldots, \gamma_k^* \subset \{t = 0\} \quad \text{and} \quad \delta_1^*, \ldots, \delta_k^* \subset \{t = \ell\}.$$

Finally, we obtain (4) by the maximum principle using horizontal slices. □

Since Σ^* is a graph, it is in particular embedded. By reflecting Σ^* in the horizontal plane $\{t = \ell\}$ we get a surface M whose boundary consists of horizontal geodesic curvature lines at heights 0 and 2ℓ, which differ by the translation by $T = (0, 0, 2\ell)$. Moreover, M is embedded, as $\Sigma^* - \partial\Sigma^* \subset \{0 < t < \ell\}$.

Extending Σ^* by symmetry with respect to the horizontal planes at heights multiple of ℓ, we obtain an embedded singly periodic minimal surface $\mathcal{M}$ with period $T = (0, 0, 2\ell)$. Furthermore, $\mathcal{M}$ is proper, by item (2) in Proposition 3.1. It is easy to see that the quotient of $\mathcal{M}$ by T has genus 0 and $2k$ ends asymptotic to flat vertical annuli (named vertical Scherk-type ends). $\mathcal{M}$ is called a *saddle tower*.

Moreover, Nelli and Rosenberg [**12**], using the Gauss–Bonnet Theorem, proved that Σ has total curvature $2\pi(1 - k)$. Thus the same holds for Σ^*, and the fundamental domain of $\mathcal{M}$ has total curvature $4\pi(1 - k)$.

To finish Theorem 1.1, it remains to prove that, given $k \geqslant 2$ and $\ell \in (0, +\infty)$, there exist $2k - 3$ possible Jenkins–Serrin domains Ω with $2k$ edges $A_1, B_1, A_2, B_2, \ldots, A_k, B_k$ of length ℓ, after identifying them by isometries of $\mathbb{H}^2$. Up to an isometry of $\mathbb{H}^2$ we can assume that A_1 is fixed, i.e. the vertices p_1, p_2 are fixed. Observe that once we have chosen the vertices $p_3, \ldots, p_{2k-1}$, then the vertex p_{2k} is determined by p_{2k-1} and p_1, as Ω is a Jenkins–Serrin domain. Each vertex p_i, $i = 3, \ldots, 2k-1$, is at distance ℓ from p_{i-1}, hence p_i is determined by the interior angle θ_{i-1} at p_{i-1} (i.e. the interior angle at p_{i-1} between the edges in $\partial\Omega$ which have p_{i-1} as a common endpoint). Since Ω is convex, $0 \leqslant \theta_{i-1} \leqslant \pi$. Additional constraints for θ_{i-1} come from the facts that $\partial\Omega$ is closed, and that Ω is a Jenkins–Serrin domain. We have obtained that the space of Jenkins–Serrin domains Ω with $2k$ edges of length ℓ, one of them fixed, has $2k - 3$ freedom parameters $\theta_2, \ldots, \theta_{2k-2}$. This proves Theorem 1.1.

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Remark 3.2 (symmetric case).

(1) In the case the vertices p_{2i-1} of Ω are at distance λ from a point of $\mathbb{H}^2$, say the origin $\mathbf{0}$, and the vertices p_{2i} of Ω are at distance μ from $\mathbf{0}$ (see Figure 3(a)), then the graph Σ over Ω can be obtained (up to a vertical translation) by reflection from the minimal graph Σ_T over a triangle T with vertices $p_1, p_2, \mathbf{0}$ with boundary values $+\infty$ along A_1 and 0 along $\partial T - A_1$. Then Σ contains k geodesic arcs at height 0 meeting at $\mathbf{0} \in \Sigma$ equiangularly (as usual, we are identifying $\mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$):

- k geodesic arcs of length $\lambda + \mu$, if k is odd;
- $k/2$ geodesic arcs of length 2λ and $k/2$ geodesic arcs of length 2μ, when k is even.

Those horizontal geodesics give us by conjugation k vertical geodesic curvature lines in Σ^* (of the same length as in Σ) meeting with angle π/k. Then Σ^* can be obtained from Σ_T^* by symmetries.

(2) By uniqueness of the Jenkins–Serrin graphs, when $\lambda = \mu$ we have that Σ_T is symmetric with respect to the vertical plane which bisects T at its vertex $\mathbf{0}$. That symmetry says that Σ_T^* contains half a geodesic, and then we can obtain Σ^* from half a Σ_T^* bounded by a horizontal geodesic curvature line γ at height 0, a vertical geodesic curvature line α of length 2λ and half a horizontal geodesic curve L at height $\ell/2$; α, L meeting at an angle $\pi/(2k)$. These symmetric examples are the ones Pyo has constructed in [**14**].

4. Properly embedded minimal surfaces of genus zero in $\mathbb{H}^2 \times \mathbb{R}$

In this section we obtain as a limit of saddle towers with $2k$ vertical Scherk-type ends and period vector $(0, 0, 2\ell)$, with $\ell \to +\infty$, a properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with total curvature $4\pi(1-k)$, genus zero and k ends asymptotic to vertical geodesic planes (Theorem 1.2). It will be the conjugate surface of a Jenkins–Serrin graph over a convex semi-ideal polygonal domain.

Consider a convex semi-ideal Jenkins–Serrin domain Ω with $2k$ vertices $p_1, \dots, p_{2k}$ cyclically ordered so that the vertices p_{2i-1} are in the interior of $\mathbb{H}^2$, and the vertices p_{2i} are at $\partial_\infty \mathbb{H}^2$, for $i = 1, \dots, k$. As in the previous section, call A_i the edge of Ω whose endpoints are p_{2i-1}, p_{2i}, and B_i the edge of Ω whose endpoints are p_{2i}, p_{2i+1}. We also require that Ω satisfies the following additional condition.

(★) For each $p_{2i} \in \partial_\infty \mathbb{H}^2$, there exists a sufficiently small horocycle H_{2i} such that it only intersects $\partial\Omega$ along A_i, B_i, and

$$\operatorname{dist}_{\mathbb{H}^2}(p_{2i-1}, H_{2i}) = \operatorname{dist}_{\mathbb{H}^2}(p_{2i+1}, H_{2i}).$$

Observe we can choose the horocycles H_{2i} so that, for any $i = 1, \dots, k$,

$$\operatorname{dist}_{\mathbb{H}^2}(p_{2i-1}, H_{2i}) = \operatorname{dist}_{\mathbb{H}^2}(p_{2i+1}, H_{2i}) = \ell_0,$$

for some $\ell_0 > 0$, independently of i. Also we can choose them small enough so that

$$\mathrm{dist}_{\mathbb{H}^2}(p_{2i-1}, p_{2i+1}) < 2\ell_0, \quad \text{for any } i = 1, \ldots, k. \tag{4.1}$$

Consider the nested sequence of horocycles $H_{2i}(n)$ at p_{2i}, $n \geqslant 0$, converging to p_{2i} as $n \to +\infty$, such that $H_{2i}(0) = H_{2i}$ and $\mathrm{dist}_{\mathbb{H}^2}(H_{2i}(n+1), H_{2i}(n)) = 1$. We set

$$\ell_n = \ell_0 + n.$$

We are going to obtain Ω as limit of convex Jenkins–Serrin domains Ω_n as $n \to +\infty$, each Ω_n with $2k$ edges of length ℓ_n.

Firstly, we remark the following fact. Condition ($\star$) ensures the existence of a horocycle C_{2i} at p_{2i} passing through p_{2i-1}, p_{2i+1}. Call D_{2i} the component of $\mathbb{H}^2 - C_{2i}$ whose only point of $\partial_\infty \mathbb{H}^2$ at its infinite boundary is p_{2i} (i.e. D_{2i} is the horodisk at p_{2i} bounded by the horocycle C_{2i}), and $\bar{D}_{2i} = D_{2i} \cup C_{2i}$. We get the following lemma, since Ω is a Jenkins–Serrin domain.

Lemma 4.1. *Every interior vertex* p_{2j-1} *of* Ω, *for* $j \notin \{i, i+1\}$, *is contained in* $\mathbb{H}^2 - \bar{D}_{2i}$.

Proof. Suppose there exists some $p_{2j-1} \in \bar{D}_{2i}$, with $j \notin \{i, i+1\}$. Then for n large we have

$$\mathrm{dist}_{\mathbb{H}^2}(p_{2j-1}, H_{2i}(n)) \leqslant \ell_n = \mathrm{dist}_{\mathbb{H}^2}(p_{2i-1}, H_{2i}(n)) = \mathrm{dist}_{\mathbb{H}^2}(p_{2i+1}, H_{2i}(n)).$$

Let γ be the geodesic from p_{2j-1} to p_{2i}, and $\mathcal{P}$ be the component of $\Omega - \gamma$ containing A_i on its boundary. Clearly, $\mathcal{P}$ is a polygonal domain inscribed in Ω. It holds $\beta(\mathcal{P}) = \alpha(\mathcal{P}) - \ell_n$ for this choice of horocycles, and then

$$|\Gamma(\mathcal{P})| = \mathrm{dist}_{\mathbb{H}^2}(p_{2j-1}, H_{2i}(n)) + \alpha(\mathcal{P}) + \beta(\mathcal{P}) \leqslant 2\alpha(\mathcal{P}).$$

And this holds for every n large enough, a contradiction as Ω is a Jenkins–Serrin domain. □

Now let us construct the Jenkins–Serrin domains Ω_n. All the vertices $p_{2i-1} \in \mathbb{H}^2$ of Ω will be vertices of each Ω_n as well. Let us obtain the vertices $p_{2i}(n)$ of Ω_n such that, for each $i = 1, \ldots, k$:

(a) $\mathrm{dist}_{\mathbb{H}^2}(p_{2i}(n), p_{2i-1}) = \mathrm{dist}_{\mathbb{H}^2}(p_{2i}(n), p_{2i+1}) = \ell_n$;

(b) $p_{2i}(n) \in \Omega$ and $p_{2i}(n) \to p_{2i}$ as $n \to +\infty$.

By (4.1), $\mathrm{dist}_{\mathbb{H}^2}(p_{2i-1}, p_{2i+1}) < 2\ell_0 < 2\ell_n$. This guarantees that the circles of radius ℓ_n centred at p_{2i-1}, p_{2i+1} intersect at exactly two points, each one lying in a different component of $\mathbb{H}^2 - \gamma_{2i}$, where γ_{2i} is the complete geodesic passing through p_{2i-1}, p_{2i+1} (see Figure 4). We define $p_{2i}(n)$ as the intersection point of those circles which is contained in the component of $\mathbb{H}^2 - \gamma_{2i}$ having p_{2i} at its boundary at infinity. The point $p_{2i}(n)$ lies in the region of Ω bounded by A_i, $H_{2i}(n)$, B_i and γ_{2i}. By construction, $p_{2i}(n)$ verifies conditions (a) and (b) above, and the domain Ω_n is convex, has $2k$ edges of length ℓ_n and converges to Ω as $n \to +\infty$. Call $A_i(n)$ the edge of Ω_n whose endpoints are p_{2i-1}, $p_{2i}(n)$, and $B_i(n)$ the edge of Ω_n whose endpoints are $p_{2i}(n)$, p_{2i+1}.

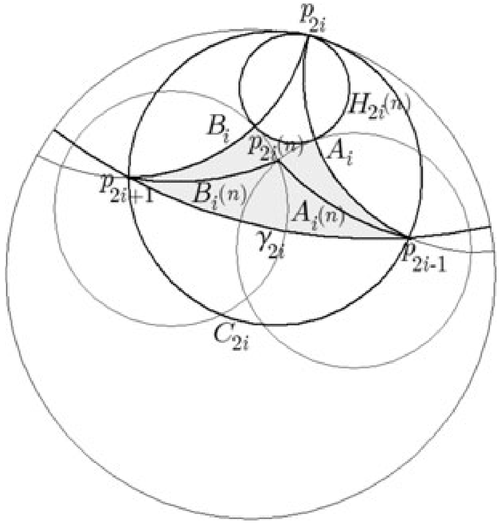

Figure 4. Construction of the vertex $p_{2i}(n)$ of Ω_n as the intersection point in the shadowed region of the circles of radius ℓ_n centred at p_{2i-1}, p_{2i+1}.

Lemma 4.2. *For n big enough, Ω_n is a Jenkins–Serrin domain.*

Proof. By construction, $\alpha(\Omega_n) = \beta(\Omega_n)$. Suppose there exists an inscribed polygonal domain $\mathcal{P}$ in Ω_n, $\mathcal{P} \neq \Omega_n$, such that $|\partial\mathcal{P}| \leqslant 2\alpha(\mathcal{P})$ (the case $|\partial\mathcal{P}| \leqslant 2\beta(\mathcal{P})$ follows similarly). Since $\mathcal{P} \neq \Omega_n$, there is at least an interior geodesic γ_1 in $\partial\mathcal{P}$ (i.e. $\gamma_1 \subset \partial\mathcal{P} \cap \Omega_n$). We can assume that there are no two consecutive interior geodesics γ_1, γ_2: we would replace $\mathcal{P}$ by another inscribed polygonal domain satisfying the same properties by replacing the geodesics γ_1, γ_2 by the geodesic γ_3 such that $\gamma_1 \cup \gamma_2 \cup \gamma_3$ bounds a geodesic triangle contained in Ω_n. In a similar way, we can assume that

$$\partial\mathcal{P} = A_{i_1}(n) \cup \gamma_1 \cup \cdots \cup A_{i_j}(n) \cup \gamma_j \cup A_{i_{j+1}}(n) \cup \cdots \cup A_{i_s}(n) \cup \gamma_s,$$

where each γ_j is either an interior geodesic or a $B_i(n)$ edge, and at least $\gamma_1 \subset \Omega_n$. In particular, each γ_j joins an even vertex $q_{2j}(n) = p_{2i_j}(n)$ to an odd vertex $q_{2j+1} = p_{2i_{j+1}-1}$. Remark that when γ_j is a $B_i(n)$ edge, then $\gamma_j = B_{i_j}(n)$ and $i_{j+1} = i_j + 1$.

As

$$\sum_{j=1}^{s} |\gamma_j| = |\partial\mathcal{P}| - \alpha(\mathcal{P}) \leqslant \alpha(\mathcal{P}) = s\ell_n,$$

there must be some interior geodesic $\gamma_j \subset \partial\mathcal{P}$ whose length is smaller than or equal to ℓ_n. Take the hyperbolic circle $S(n)$ of centre $q_{2j}(n)$ and radius ℓ_n, and let $D(n)$ be the hyperbolic disc bounded by $S(n)$ (see Figure 5). Then the vertex q_{2j+1} lies in $\overline{D(n)} = D(n) \cup S(n)$. Let us prove that this is not possible for n large.

The circles $S(n)$ converge to the horocycle C_{2i_j} as $n \to +\infty$. And by Lemma 4.1, q_{2j+1} cannot be contained in the closed horodisc $\bar{D}_{2i_j}$ bounded by C_{2i_j}. Then $q_{2j+1} \in \mathbb{H}^2 - \overline{D(n)}$ for n large enough. □

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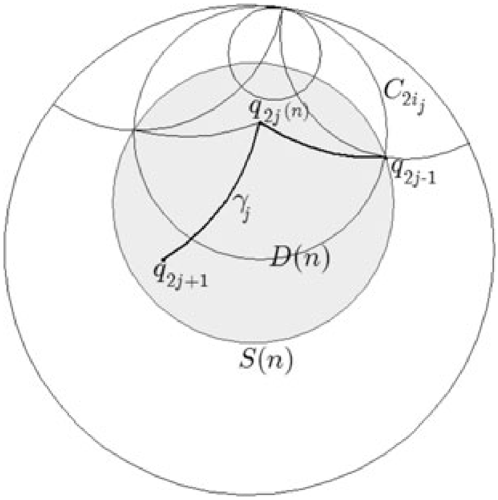

Figure 5. The circle $S(n)$ of radius ℓ_n centred at $q_{2j}(n)$ converges to the horocycle C_{2i_j} at p_{2i_j} as $n \to +\infty$.

Observe that the inscribed polygonal domain $\mathcal{P}_0$ whose vertices are the vertices p_{2i-1} of Ω, is contained in all the domains Ω_n. Fix a point $p_0 \in \mathcal{P}_0$.

By Theorem 2.3, there exists a solution u (respectively u_n, for any n) to the minimal graph equation defined over Ω (respectively Ω_n) with boundary values $+\infty$ over A_i (respectively $A_i(n)$) and $-\infty$ over B_i (respectively $B_i(n)$). Denote by Σ (respectively Σ_n) the graph surface of u (respectively u_n). Up to a vertical translation we can assume $u(p_0) = u_n(p_0) = 0$. (Observe that we could have exchanged the edges A_i, B_i, but the graph we would have obtained would be Σ up to a symmetry about $\mathbb{H}^2 \times \{0\}$.)

The domains Ω_n converge to Ω. Since Ω is a Jenkins–Serrin domain and $u_n(p_0) = 0$ for any $n \in \mathbb{N}$, a subsequence of the u_n converges uniformly on compact sets of Ω to a solution u_∞ of the minimal graph equation with the same boundary values as u (it can be proved as the convergence of the sequence $\{u_m\}$ at the end of the proof of Theorem 4.9 in [**10**]). By uniqueness, $u_\infty = u$. Hence the minimal graphs Σ_n converge to Σ, after taking a subsequence.

Observe that the vertical straight lines $\Gamma_i = \{p_{2i-1}\} \times \mathbb{R}$ are contained in the boundary of all the Σ_n and also of Σ. In particular, none of the distances $\mathrm{dist}_{\Sigma_n}(\Gamma_i, \Gamma_j)$ can diverge, for any $i, j \in \{1, \dots, k\}$. After passing to a subsequence, we can assume that there exists a constant $C > 0$ such that, for any $i, j \in \{1, \dots, k\}$,

$$\mathrm{dist}_{\Sigma_n}(\Gamma_i, \Gamma_j) \leqslant C, \quad \text{for any } n \in \mathbb{N},$$

and $\mathrm{dist}_{\Sigma}(\Gamma_i, \Gamma_j) \leqslant C$.

Denote by Σ^* (respectively Σ_n^*) the conjugate surface of Σ (respectively Σ_n). By Theorem 2.9, Σ^* is a minimal graph, as Σ is a minimal graph over Ω, which is convex. Moreover, $\partial\Sigma = \Gamma_1 \cup \dots \cup \Gamma_k$, so the boundary of Σ^* is composed of k horizontal geodesic curvature lines Γ_i^*, by Lemma 2.8. Since Σ, Σ^* are isometric, then $\mathrm{dist}_{\Sigma^*}(\Gamma_i^*, \Gamma_j^*) \leqslant C$

for every $i, j \in \{1, \ldots, k\}$. We want to prove that all the curves Γ_i^* lie in the same horizontal plane, say $\{t = 0\}$, and Σ^* is contained in one of the half-spaces determined by $\{t = 0\}$.

For any n, Σ_n is a graph over the convex domain Ω_n and the boundary of Σ_n equals $\Gamma_1 \cup \eta_1(n) \cup \cdots \Gamma_k \cup \eta_k(n)$, where each Γ_i is defined as above and $\eta_i(n) = \{p_{2i}(n)\} \times \mathbb{R}$, for any $i = 1, \ldots, k$. By Proposition 3.1, Σ_n^* is a graph over a domain Ω_n^* and

$$\partial \Sigma_n^* = \Gamma_1^*(n) \cup \eta_1^*(n) \cup \cdots \cup \Gamma_k^*(n) \cup \eta_k^*(n),$$

where $\Gamma_1^*(n), \ldots, \Gamma_k^*(n)$ (respectively $\eta_1^*(n), \ldots, \eta_k^*(n)$) are horizontal geodesic curvature lines contained in the same horizontal plane, and both planes are at distance ℓ_n from each other.

Call $\tilde{\Gamma}_i^*(n)$ (respectively $\tilde{\eta}_i^*(n)$) the vertical projection of $\Gamma_i^*(n)$ (respectively $\eta_i^*(n)$) over $\{t = 0\}$. Then $\partial \Omega_n^* = \tilde{\Gamma}_1^*(n) \cup \tilde{\eta}_1^*(n) \cup \cdots \cup \tilde{\Gamma}_k^*(n) \cup \tilde{\eta}_k^*(n)$, and two consecutive curves in $\partial \Omega_n^*$ are asymptotic at $\partial_\infty \mathbb{H}^2$. Proposition 2.10 ensures that the graphs Σ_n^* converge to the graph Σ^*, up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. It could happen that the boundary values of the graphs over the boundary curves $\tilde{\Gamma}_i^*(n)$ would diverge to $-\infty$; but this is not possible as $\mathrm{dist}_{\Sigma_n^*}(\Gamma_i^*(n), \Gamma_j^*(n)) \leqslant C$, for any $i, j \in \{1, \ldots, k\}$ and any $n \in \mathbb{N}$. Therefore, up to a vertical translation we can assume that $\Gamma_1^*(n), \ldots, \Gamma_k^*(n) \subset \{t = 0\}$; the curves $\tilde{\Gamma}_i^*(n) = \Gamma_i^*(n)$ converge to the curves Γ_i^*. So $\partial \Sigma^* \subset \{t = 0\}$.

Suppose $\eta_1^*(n), \ldots, \eta_k^*(n) \subset \{t = \ell_n\}$ (if they are contained in $\{t = -\ell_n\}$ we argue similarly). The height of each curve $\eta_i^*(n)$ diverge to $+\infty$, hence its projection $\tilde{\eta}_i^*(n)$ converge to the geodesic $\tilde{\eta}_i^* \in \mathbb{H}^2$ joining the corresponding endpoints of Γ_i^*, Γ_{i+1}^* at $\partial_\infty \mathbb{H}^2$. Moreover, since the graph Σ_n^* is contained in $\{t \geqslant 0\}$ for any n, then the same holds for Σ^*. If we reflect Σ^* in the plane $\{t = 0\}$, we get a properly embedded minimal surface M of genus zero and k planar ends in $\mathbb{H}^2 \times \mathbb{R}$ (the ends of M are asymptotic to the vertical geodesic planes $\tilde{\eta}_i^* \times \mathbb{R}$).

Collin and Rosenberg [**1**] proved that Σ (and so Σ^*) has total curvature $2\pi(1 - k)$. Hence M has total curvature $4\pi(1 - k)$.

To complete the proof of Theorem 1.2, it remains to show that, given $k \geqslant 2$, there exist $2k - 3$ possible semi-ideal Jenkins–Serrin domains Ω satisfying condition $(\star)$, after identifying them by isometries of $\mathbb{H}^2$. Firstly we observe that, for $i = 1, \ldots, k$, the vertex $p_{2i} \in \partial_\infty \mathbb{H}^2$ is determined once we have chosen p_{2i-1} and p_{2i+1}, since Ω satisfies condition $(\star)$. Thus we have to compute the parameters which determine the vertices having odd subindex. We can assume that p_1 is fixed as well as the direction of the geodesic arc α_1 from p_1 to p_3. So the vertex p_3 is determined by the length of α_1. That gives the first parameter. For $i = 2, \ldots, k-1$, the vertex p_{2i+1} is determined by both the direction and the length of the geodesic arc α_i joining p_{2i-1}, p_{2i+1} (the direction of α_i is given by the interior angle at p_{2i-1} between α_{i-1} and α_i). So we have two additional parameters for the remaining $k - 2$ vertices of Ω, and the total number of freedom parameters equals $2k - 3$. This finishes Theorem 1.2.

Remark 4.3 (symmetric case). In the case the vertices p_{2i-1} of Ω are at distance λ from the origin $\mathbf{0}$ of $\mathbb{H}^2$, then we can get Σ^* as limit of symmetric surfaces as at item (1) of Remark 3.2 (when $\mu \to +\infty$). In particular, Σ^* contains k vertical geodesic curvature lines (of infinite length when k is odd; or $k/2$ of length 2λ and $k/2$ of infinite length when k is even). These are the examples constructed by Pyo in [**15**].

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