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A GENERALIZATION OF CENTRALIZER NEAR-RINGS

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1. Introduction

Let G be a group with identity 0 and let $\mathscr{A}$ be a group of automorphisms of G. The centralizer near-ring determined by G and $\mathscr{A}$ is the set $C(\mathscr{A};G)=\{f:G\to G\,|\,f\alpha=\alpha f$ for all $\alpha\in\mathscr{A}$ and $f(0)=0\}$, forming a near-ring under function addition and function composition. This class of near-rings has been extensively studied (for example see [1], [2], [5] and [6]) and it is known that every finite simple near-ring with identity which is not a ring is isomorphic to $C(\mathscr{A};G)$ for a suitable pair $(\mathscr{A},G)$ see [6] page 131, Corollary 4.59 and Theorem 4.60.

As illustrated in [1] a key to the study of the near-ring $C(\mathscr{A};G)$ is the orbit structure of G determined by $\mathscr{A}$. For each $v\in G$ the stabilizer of v is $\operatorname{stab}(v)=\{\alpha\in\mathscr{A}\,|\,\alpha v=v\}$, a subgroup of $\mathscr{A}$. If v, $w\in G$ belong to the same $\mathscr{A}$-orbit then there exists a $\beta\in\mathscr{A}$ such that $w=\beta v$ and we have $\operatorname{stab}(w)=\beta\operatorname{stab}(v)\beta^{-1}$. So two elements of G from the same orbit have conjugate stabilizers.

Definition 1. Let G be a group and $\mathscr{A}$ a group of automorphisms of G. We will call the pair $(\mathscr{A},G)$ *normal* if

(a) G has finitely many $\mathscr{A}$-orbits, and

(b) if $v,w\in G$ belong to the same $\mathscr{A}$-orbit and if $\operatorname{stab}(v)\subseteq\operatorname{stab}(w)$ then $\operatorname{stab}(v)=\operatorname{stab}(w)$.

We remark that if G is a finite group then $(\mathscr{A},G)$ is normal. Also we note that (b) is the finiteness condition used by Meldrum and Zeller in [5].

Although near-rings of the form $C(\mathscr{A};G)$ are of fundamental importance in the theory of near-rings, it is difficult to decide whether or not a given near-ring is isomorphic to $C(\mathscr{A};G)$ for some group G and group of automorphisms $\mathscr{A}$ ([2], [4]). In this paper a class $\mathscr{N}$ of near-rings is defined which contains all centralizer near-rings of the form $C(\mathscr{A};G)$ where $(\mathscr{A},G)$ is normal. It will be shown that many of the results on centralizer near-rings are true for the near-rings in $\mathscr{N}$. Moreover to decide whether or not a near-ring belongs to $\mathscr{N}$ is in general easier than deciding whether or not it is a centralizer near-ring $C(\mathscr{A};G)$.

2. Generalized centralizer near-rings

Let N be a near-ring with identity 1. An element $e\in N$ is idempotent if $e\neq 0$ and $e^2=e$. If e_i and e_j are idempotents in N we will let N_{ij} denote the set $N_{ij}=e_iNe_j=$

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$\{e_i n e_j | n \in N\}$, a subset of N. The idempotents e_i and e_j are orthogonal if $e_i e_j = e_j e_i = 0$. Finally an idempotent $e \in N$ is primitive if e is not the sum of two orthogonal idempotents in N.

Definition 2. Let N be a zero symmetric near-ring with 1 then N is a GC near-ring (generalized centralizer near-ring) if the following six axioms are satisfied.

(i) There is a finite number of idempotents $e_1, \ldots, e_s \in N$ such that $1 = e_1 + \cdots + e_s$, $e_i e_j = 0$ for all i, j with $i \neq j$ and $e_i + e_j = e_j + e_i$ for all i, j.

(ii) For $i = 1, \ldots, s$ the set $(e_i N e_i)^* = N_{ii}^* \equiv N_{ii} - \{0\}$ is a group under multiplication having identity e_i.

(iii) Let $n_{i_1 j_1} \in N_{i_1 j_1}, \ldots, n_{i_s j_s} \in N_{i_s j_s}$ with $\{j_1, \ldots, j_s\} = \{1, \ldots, s\}$, then for every $f \in N$, $f(n_{i_1 j_1} + \cdots + n_{i_s j_s}) = f n_{i_1 j_1} + \cdots + f n_{i_s j_s}$.

(iv) For every $f \in N$ and $n_{kj} \in N_{kj}$ then $f n_{kj}$ belongs to N_{tj} for some t (t depends on f and n_{kj}).

(v) For every $n_{ij} \in N_{ij}$ and $n_{kj} \in N_{kj}$ then $n_{ij} + n_{kj} \in N_{tj}$ for some t (t depends on n_{ij} and n_{kj}).

(vi) If $N_{ij} \neq \{0\}$ and $N_{jk} \neq \{0\}$ then $N_{ij} N_{jk} \neq \{0\}$.

For convenience we make the convention that $N = \{0\}$ is a GC near-ring.

Proposition 1. *Let N be a GC near-ring using idempotents $e_1, \ldots, e_s$. Then each e_i is a primitive idempotent.*

Proof. Suppose $e_i = f_1 + f_2$ where f_1 and f_2 are idempotents such that $f_1 f_2 = f_2 f_1 = 0$. We have $e_i f_1 = (f_1 + f_2) f_1 = f_1^2 + f_2 f_1 = f_1$. Similarly $e_i f_2 = f_2$. Also $(f_1 e_i)^2 = f_1 e_i f_1 e_i = f_1 f_1 e_i = f_1 e_i$, an element of N_{ii}. By axiom (ii) either $f_1 e_i = e_i$ or $f_1 e_i = 0$. Since $0 \neq f_1 = f_1^2 = e_i f_1 e_i f_1$ then $f_1 e_i \neq 0$, so $f_1 e_i = e_i$. A similar argument shows $f_2 e_i = e_i$. But then $e_i = e_i^2 = (f_1 + f_2) e_i = f_1 e_i + f_2 e_i = e_i + e_i$ which implies $e_i = 0$, a contradiction. So e_i is primitive.

Our first main result implies that the set of idempotents $\{e_1, \ldots, e_s\}$ in axiom (i) for a GC near-ring is unique.

Theorem 1. *Suppose N is a near-ring satisfying axioms* (i)–(v) *using idempotents in the set $E = \{e_1, \ldots, e_s\}$, and N also satisfies axioms* (i)–(v) *using idempotents in the set $F = \{f_1, \ldots, f_t\}$, then $E = F$.*

Proof. For $k = 1, \ldots, t$ we have $f_k = f_k(e_1 + \cdots + e_s) f_k = f_k e_1 f_k + \cdots + f_k e_s f_k$ using axioms (i) and (iii). Since $f_k \neq 0$ then $f_k e_j f_k \neq 0$ for some j. By axiom (iv) there is an i such that $e_j f_k \in N'_{ik} \equiv f_i N f_k$, and since $f_k e_j f_k \neq 0$ then $e_j f_k \in N'_{kk}$, that is $i = k$. We now have $f_k e_j f_k = e_j f_k$. The element $e_j f_k$ is idempotent since $(e_j f_k)^2 = e_j f_k e_j f_k = e_j e_j f_k = e_j f_k$. Since $e_j f_k \in N'_{kk}$ axiom (ii) implies $e_j f_k = f_k$.

We have now shown that for each $k, k = 1, \ldots, t$, there is a j, depending on k, such that $f_k e_j f_k = e_j f_k = f_k$. Moreover this j is unique, for if $f_k e_i f_k = e_i f_k = f_k$ then $f_k = e_j f_k = e_i f_k$ and

$i=j$. Similarly for each $j, j=1,\ldots,s$, there exists a unique $c=c(j)$ such that $e_jf_ce_j=f_ce_j=e_j$. Hence $s=t$. The maps are universes of each other since $f_k=f_ke_jf_k=f_kf_ce_jf_k$ implies $k=c$. We may reorder the e_i's if necessary so that $1=f_1+\cdots+f_s=e_1+\cdots+e_s$ with $e_if_i=f_i$ and $f_ie_i=e_i$. For $i=1,\ldots,s$ we have $f_i=f_i(e_1+\cdots+e_s)=f_if_1e_1+\cdots+f_if_se_s=f_if_ie_i=e_i$. This shows $E=F$.

Proposition 2. *Let N be a GC near-ring with respect to the set of idempotents $E=\{e_1,\ldots,e_s\}$. Suppose I is an ideal of N which is a near-ring with identity f. Then $f=e_{i_1}+\cdots+e_{i_t}$ where $\{e_{i_1},\ldots,e_{i_t}\}\subseteq E$.*

Proof. Suppose that $fe_i\neq 0$ where $e_i\in E$. Then $fe_i\in I$ since I is an ideal. Also $(fe_i)^2=fe_ife_i=fe_ie_i=fe_i$ using the fact that f is the identity on I. By axiom (iv), fe_i belongs to N_{ki} for some k. Since fe_i is idempotent then $k=i$ and $fe_i=e_i$ using axiom (ii). So if $fe_i\neq 0$ then $fe_i=e_i\in I$. Let $e_{i_1},\ldots,e_{i_t}$ be the idempotents in E such that $fe_{i_j}\neq 0$. We have $f=f(e_1+\cdots+e_s)=fe_1+\cdots+fe_s=e_{i_1}+\cdots+e_{i_t}$.

The following theorem indicates that the class $\mathcal{N}$ of GC near-rings is a "large" class with nice properties.

Theorem 2. *Let $\mathcal{N}$ be the class of GC near-rings. The following are properties of $\mathcal{N}$.*

(a) *$\mathcal{N}$ contains all near-rings of the form $C(\mathscr{A};G)$ where $(\mathscr{A},G)$ is normal.*

(b) *If $N\in\mathcal{N}$ and I is an ideal of N which is a near-ring with identity then $I\in\mathcal{N}$.*

(c) *If $N_1,N_2\in\mathcal{N}$ then $N_1\oplus N_2\in\mathcal{N}$.*

(d) *If $N\in\mathcal{N}$ and if I is an ideal of N such that $I\in\mathcal{N}$, then $N/I\in\mathcal{N}$.*

Proof. (a) Suppose $N=C(\mathscr{A};G)$ with nonzero $\mathscr{A}$-orbits $\theta_1,\ldots,\theta_s$. For $i=1,\ldots,s$ let $e_i\colon G\to G$ be the function which is the identity on orbit θ_i and zero elsewhere. Then $e_i\in N$ and e_i is idempotent. We have $1=e_1+\cdots+e_s$ and it is straightforward to check that axioms (i)–(vi) are satisfied.

(b) Let N satisfy axioms (i)–(vi) using idempotents in the set $E=\{e_1,\ldots,e_s\}$. By Proposition 2 the identity element f of I may be written $f=e_{i_1}+\cdots+e_{i_t}$ where $F=\{e_{i_1},\ldots,e_{i_t}\}$ is a subset of E. The near-ring I is a GC near-ring using F.

(c) Since $N_1\in\mathcal{N}$ there are idempotents $e_1,\ldots,e_s\in N_1$ such that axioms (i)–(vi) are true. Similarly for N_2 using idempotents $e'_1,\ldots,e'_t$. In $N_1\oplus N_2$ we have $1=e_1+\cdots+e_s+e'_1+\cdots+e'_t$ and axioms (i)–(vi) are easily verified.

(d) If N satisfies axioms (i)–(vi) using idempotents in $E=\{e_1,\ldots,e_s\}$ then Proposition 2 implies I is a GC near-ring using $F=\{e_{i_1},\ldots,e_{i_t}\}\subseteq E$. Without loss of generality we may assume $F=\{e_1,\ldots,e_t\}$. We claim that N/I is a GC near-ring using idempotents $\bar{E}=\{\bar{e}_{t+1},\ldots,\bar{e}_s\}$ where $\bar{e}_i=e_i+I$. Axioms (i)$-$(v) are obviously true. To check axiom (vi) suppose $\bar{N}_{ij}\neq\{\bar{0}\}$ and $\bar{N}_{jk}\neq\{\bar{0}\}$ where $\bar{N}_{ij}=\bar{e}_i\bar{N}\bar{e}_j$, etc. In N we have $N_{ij}N_{jk}\neq\{0\}$ by axiom (vi). Since $\{0\}\neq N_{ij}N_{jk}\subseteq N_{ik}$ and $N_{ik}\cap I=\{0\}$ then $\bar{N}_{ij}\bar{N}_{jk}\neq\{\bar{0}\}$ as desired.

Theorem 3. *A GC near-ring N is a ring if and only if N is a (finite) direct sum of division rings.*

Proof. Suppose N is a ring which is also a GC near-ring using idempotents $e_1,\dots,e_s$. We will show that if $i\neq j$ then $N_{ij}=\{0\}$. Assume by way of contradiction that $N_{ij}\neq\{0\}$ and select a nonzero element $n_{ij}\in N_{ij}$. By axiom (v) $n_{ij}+e_j$ belongs to N_{tj} for some t. We have $n_{ij}+e_j=e_t(n_{ij}+e_j)=e_tn_{ij}+e_te_j$ since N is a ring. If $t=i$ then $n_{ij}+e_j=n_{ij}$ and $e_j=0$, impossible. If $t=j$ then $n_{ij}+e_j=e_j$ and $n_{ij}=0$, a contradiction. Finally, if $t\neq i,j$ then $n_{ij}+e_j=0$ and $n_{ij}=-e_j\in N_{jj}$ which is not possible since $N_{ij}\cap N_{jj}=\{0\}$. Hence $N_{ij}=\{0\}$ whenever $i\neq j$ and using axiom (ii), N is a direct sum of division rings.

It is clear that a division ring is a GC near-ring and by Theorem 2 $\mathcal{N}$ contains all finite direct sums of division rings.

We note that $\mathcal{N}$ is a larger class of near-rings than the class of centralizer near-rings $C(\mathscr{A};G)$ with $(\mathscr{A},G)$ normal, for it was shown in [**4**] that not every direct sum of fields is a centralizer near-ring $C(\mathscr{A};G)$.

It is trivial to verify that every nonzero homomorphic image of a near-ring N satisfying axioms (i)–(v) also satisfies axioms (i)–(v). However, a nonzero homomorphic image of a GC near-ring need not be a GC near-ring as the following example shows. Thus the hypotheses in Theorem 2, part (d) are necessary.

Example. Let $G=Z_8$, the additive group of integers modulo 8. Let $\mathscr{A}=\{\alpha_1,\alpha_3,\alpha_5,\alpha_7 \mid \alpha_i:G\to G$ defined by $\alpha_i(a)=ia\}$, a group of automorphisms of G. Finally let $N=C(\mathscr{A};G)$. The nonzero $\mathscr{A}$-orbits of G are $\theta(1)=\{1,3,5,7\}$, $\theta(2)=\{2,6\}$, and $\theta(4)=\{4\}$. The set $I=\{f\in N \mid f(1)\in\{0,4\}, f(2)=0, f(4)=0\}$ is an ideal of N.

We claim that $\bar N=N/I$ is not a GC near-ring. Let e_1,e_2,e_4 be the idempotents in N that are the identity on $\theta(1)$, $\theta(2)$, $\theta(4)$ respectively and zero elsewhere. Then N is a GC near-ring using e_1,e_2,e_4 and $\bar N$ satisfies axioms (i)–(v) using $\bar e_1,\bar e_2,\bar e_4$. By Theorem 1 $\{\bar e_1,\bar e_2,\bar e_4\}$ is the only set of idempotents for which $\bar N$ satisfies axioms (i)–(v). It suffices now to show that $\bar N$ does not satisfy (vi) using $\{\bar e_1,\bar e_2,\bar e_4\}$. Let $n_{42}\in N_{42}\equiv e_4Ne_2$ and $n_{21}\in N_{21}\equiv e_2Ne_1$, then $n_{42}n_{21}\in I$ so $N_{42}N_{21}\subseteq I$. Also $N_{42}\cap I=\{0\}$ and $N_{21}\cap I=\{0\}$. In $\bar N$ we have $\bar N_{42}=\bar e_4N\bar e_2\neq\{\bar 0\}$, $\bar N_{21}\neq\{\bar 0\}$ and yet $\bar N_{42}\bar N_{21}=\{\bar 0\}$, violating axiom (vi).

3. Structure theorems

In this section we investigate the structure of GC near-rings. In particular we determine when a GC near-ring N is simple and when it is semi-simple. Moreover we present a decomposition theorem for left ideals in N. These results generalize results known for centralizer near-rings ([**1**], [**5**], [**6**] and [**7**]).

Theorem 4. *A GC near-ring N is simple if and only if $N_{ij}\neq\{0\}$ for all i and j.*

Proof. Assume $N_{ij}\neq\{0\}$ for all i,j where N is a GC near-ring using $E=\{e_1,\dots,e_s\}$. Let I be a nonzero ideal of N and select $g\neq 0$ in I. Using axioms (i) and (iii) we have $g=(e_1+\cdots+e_s)g(e_1+\cdots+e_s)=\sum e_jge_i$. Since $g\neq 0$ then $e_jge_i\neq 0$ for some j,i. Also $e_jge_i\equiv n_{ji}\in I\cap N_{ji}$. By axiom (vi) $N_{ij}N_{ji}\neq\{0\}$ so there exists $m_{ij}\in N_{ij}$ such that $0\neq m_{ij}n_{ji}\in I\cap N_{ii}$. This means $e_i\in I$. For each k and for each $n_{ki}\in N_{ki}$ we have $n_{ki}e_i=n_{ki}\in I\cap N_{ki}$. As above this implies $e_k\in I$. So $e_1+\cdots+e_s=1\in I$ and $I=N$.

For the converse assume $N_{ij}=\{0\}$ for some i,j where $i\neq j$. Let $I=\text{Ann}(Ne_j)$, the annihilator of Ne_j. The set I is a left ideal of N since it is the annihilator of a set of elements in N. Since $N\cdot Ne_j\subseteq Ne_j$, I is an ideal of N. We have $I\neq\{0\}$ since $e_i\in I$ and $I\neq N$ since $e_j\notin I$. So N is not simple.

Theorem 5. *A GC near-ring N is J_2-semisimple if and only if whenever $N_{ij}=\{0\}$ then $N_{ji}=\{0\}$.*

Proof. Assume N is semisimple. Suppose $N_{ij}=\{0\}$ but $N_{ji}\neq\{0\}$. Select a nonzero element $n_{ji}\in N_{ji}$ and let $M=Nn_{ji}$, an N-subgroup of N. For $g\in N$ we have $gn_{ji}\in M\cap N_{ki}$ for some k using axiom (iv). We claim the product of any two elements in M is 0. Clearly $(fn_{ji})(gn_{ji})=0$ if $k\neq i$. If $k=i$ then $gn_{ji}=e_ign_{ji}e_i=e_ige_jn_{ji}e_i$ and $e_ige_j=0$ since $N_{ij}=\{0\}$. Hence $gn_{ji}=0$ when $k=i$. This shows M is nilpotent and N is not semisimple since the J_2-radical of N contains all nilpotent N-subgroups ([**6**], page 153).

For the converse we may assume $N_{ij}=\{0\}$ for some $i\neq j$. (For if $N_{ij}\neq\{0\}$ for all i,j then N is simple by Theorem 4.) Let $S_j=\{k\,|\,N_{kj}\neq\{0\}\}$, and if $S_j=(k_1,\dots,k_t\}$ let $I=Ne_{k_1}+\cdots+Ne_{k_t}$, a left ideal of N. We claim that I is an ideal of N. To prove the claim it suffices to prove that $(n_1e_{k_1}+\cdots+n_te_{k_t})n_{kl}\in I$ for all $n_1,\dots,n_t\in N$ and all k, l. If $(n_1e_{k_1}+\cdots+n_te_{k_t})n_{kl}\neq 0$ then $k\in S_j$ and $n_{kl}\neq 0$. Since $n_{kl}\neq 0$ then $N_{kl}\neq\{0\}$ and $N_{lk}\neq\{0\}$. So $N_{lk}N_{kj}\neq[0\}$ and $l\in S_j$. Since $l\in S_j$, I contains every element in N_{il}, $i=1,2,\dots,s$ and I contains $(n_1e_{k_1}+\cdots+n_te_{k_t})n_{kl}=n_1e_{k_1}n_{kl}+\cdots+n_te_{k_t}n_{kl}$. Hence I is an ideal of N.

Let $\bar{S}_j=\{1,\dots,s\}-S_j$ and let $\bar{I}=Ne_{i_1}+\cdots+Ne_{i_v}$ where $\bar{S}_j=\{i_1,\dots,i_v\}$. The set $\bar{I}$ is a left ideal of N and we want to show it is an ideal. As before if $(n_1e_i+\cdots+n_ve_i)n_{kl}\neq 0$ then $k\in\bar{S}_j$ and $n_{kl}\neq 0$. If $l\in S_j$ then $N_{lj}\neq\{0\}$ and $\{0\}\neq N_{kl}N_{lj}\subseteq N_{kj}$. But $N_{kj}\neq\{0\}$ implies $k\in S_j$, a contradiction. Hence $l\in\bar{S}_j$ and $(n_1e_{i_1}+\cdots+n_ve_{i_v})n_{kl}\in\bar{I}$. So $\bar{I}$ is an ideal of N with $\bar{I}\cap I=\{0\}$. If $\bar{I}$ is not a simple near-ring the above process may be repeated. Ultimately we obtain N as a direct sum of finitely many simple near-rings, and N is semisimple.

We next establish a decomposition theorem for left ideals in a *GC* near-ring. This theorem is established with the aid of two propositions which are similar to results found in [7] for centralizer near-rings. The proofs are different and seem less technical than those for centralizer near-rings. In what follows N is a *GC* near-ring using idempotents $e_1,\dots,e_s$.

Proposition 3. *Let L be a left ideal of a GC near-ring N. If $f\in L$ with $fe_j\in N_{kj}$, $k\neq j$, and if $fe_j+e_j\in N_{kj}$ then $e_j\in L$.*

Proof. We have $e_kfe_j=fe_j$ and $e_kfe_j+e_j=fe_j+e_j\in N_{kj}$. Since $e_kf\in L$ we may assume $e_kf=f$. Using the left ideal property of L we have $e_k(f+e_j)-e_ke_j=e_k(f+e_j)\in L$. Let $g=e_k(f+e_j)$ then $ge_j=e_k(fe_j+e_j)=fe_j+e_j$ since $fe_j+e_j\in N_{kj}$. If $i\neq j$ then $ge_i=e_kfe_i=fe_i$. We have $-f+g\in L$ and $-f+g=(-f+g)(e_1+\cdots+e_s)=(-fe_1+ge_1)+\cdots+(-fe_s+ge_s)=0+\cdots+(-fe_j+fe_j+e_j)+\cdots+0=e_j$.

Proposition 4. *Let L be a left ideal of a GC near-ring N. Suppose $f\in L$ such that $e_jfe_i\neq 0$ and there exists an $m_{ij}\in N_{ij}$ with $m_{ij}e_jfe_i=e_i$, then $e_i\in L$.*

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Proof. Let $n_{ji}=e_jfe_i$, then $m_{ij}f\in L$, $e_i(m_{ij}f)=m_{ij}f$, and $m_{ij}fe_i=m_{ij}e_jfe_i=e_i$. So we may assume $i=j$ and that $f\in L$ is such that $e_if=f$ and $fe_i=e_i$. We may also assume $fe_k\neq 0$ for some $k\neq i$ or else $f=e_i\in L$ and we are done.

Among those $f\in L$ with $e_if=f, fe_i=e_i$ select one such that the number of k with $fe_k\neq 0$ is minimal. Now let k be such that $k\neq i$ and $fe_k\neq 0$.

Case 1. Suppose there exists a $g\in N$ such that $e_ige_k=0$ and $fe_k+ge_k\notin N_{ik}$. Let $h=e_i(f+ge_k)-e_ige_k=e_i(f+ge_k)$. We have $h\in L$, $he_i=e_i$ and $e_ih=h$. Moreover if $j\neq k$ then $he_j=fe_j$ and $he_k=0$. This contradicts the minimality of f.

Case 2. Assume $fe_k+ge_k\in N_{ik}$ for all g such that $e_ige_k=0$. Let $g=e_k$. Then $f\in L$, $fe_k\in N_{ik}$ and $fe_k+ge_k=fe_k+e_k\in N_{ik}$. By Proposition 3, $e_k\in L$. This means $f-fe_k\equiv h\in L$. We have $he_i=(f-fe_k)e_i=fe_i=e_i$. Also $e_ih=e_i(f-fe_k)(e_1+\cdots+e_k+\cdots+e_s)=e_i(fe_1-0)+\cdots+e_i(fe_k-fe_k)+\cdots+e_i(fe_s-0)=e_ife_1+\cdots+0+\cdots+e_ife_s=fe_1+\cdots+0+\cdots+fe_s$, and $h=h(e_1+\cdots+e_k+\cdots+e_s)=he_1+\cdots+he_k+\cdots+he_s=fe_1+\cdots+0+\cdots+fe_s$. This shows $e_ih=h$. We also have $he_k=0$ and this contradicts the minimality of f. So $e_i\in L$ as desired.

Theorem 6. *Let L be a left ideal of N, where N is a GC near-ring using idempotents $e_1,\dots,e_s$. Then for each i, $Le_i\subseteq L$. Also $L=Le_1+\cdots+Le_s$.*

Proof. For the first part we need to show that if $f\in L$ then $fe_i\in L$, $i=1,\dots,s$. We have $fe_i\in N_{ki}$ for some k and so $e_kfe_i=fe_i$. This shows we may assume $e_kf=f$. If $fe_i=f$ we are done so we may assume $fe_j\neq 0$ for some $j\neq i$. If we can show that $f-fe_j\in L$ then we have $f^{(1)}\equiv f-fe_j\in L$ with $f_1^{(1)}e_i=fe_i$ and $f^{(1)}e_j=0$. So $f^{(1)}$ has one fewer j such that $f^{(1)}e_j\neq 0$. This process can be continued until we have $f^{(t)}\in L$ such that $f^{(t)}e_i=fe_i$ and $f^{(t)}e_j=0$ for all $j\neq i$. Then $fe_i=f^{(t)}e_i=f^{(t)}(e_1+\cdots+e_s)=f^{(t)}\in L$.

So it remains to show that $f-fe_j\in L$. If $j=k$ then $0\neq e_kfe_j=e_kfe_k\in N_{kk}$ and Proposition 4 applies. So $e_j=e_k\in L$ and $f-fe_j\in L$.

If $j\neq k$ we have two cases to consider.

Case 1. Assume $e_k(f+e_j)e_j\neq 0$. By axiom (v) we have $(f+e_j)e_j=fe_j+e_j\in N_{tj}$ for some t. We must have $t=k$ and so $e_k(f+e_j)e_j=(f+e_j)e_j$. Let $g=e_k(f+e_j)-e_ke_j=e_k(f+e_j)$, an element of L. Note that $ge_j=e_k(f+e_j)e_j=(f+e_j)e_j=fe_j+e_j$ and if $l\neq j$ then $ge_l=fe_l$. We have $-f+g\in L$ and $(-f+g)=(-f+g)(e_1+\cdots+e_s)=-fe_j+fe_j+e_j=e_j\in L$. So $f-fe_j\in L$ as desired.

Case 2. Assume $e_k(f+e_j)e_j=0$. Let $g=e_k(f+e_j)-e_ke_j=e_k(f+e_j)$, an element in L. We have $ge_t=fe_t$ for $t\neq j$ and $ge_j=0$. Using axiom (i), $g=g(e_1+\cdots+e_j+\cdots+e_s)=ge_1+\cdots+ge_{j-1}+ge_{j+1}+\cdots+ge_s=fe_1+\cdots+fe_{j-1}+fe_{j+1}+\cdots+fe_s+fe_j-fe_j=f(e_1+\cdots+e_{j-1}+e_{j+1}+\cdots+e_s+e_j)-fe_j=f-fe_j$, an element in L.

We have now shown that $fe_i\in L$ for every $f\in L$. It remains to prove that $L=Le_1\oplus\cdots\oplus Le_s$. We note that Le_i is a left ideal of N since $Le_i=L\cap \operatorname{Ann}\{e_1,\dots,e_{i-1},e_{i+1},\dots,e_n\}$ and that $Le_i\cap Le_j=\{0\}$ if $i\neq j$ since $Le_i\cdot e_i=Le_i$ and $Le_j\cdot e_i=\{0\}$. Since $Le_i\subseteq L$ for each i, then $Le_1+\cdots+Le_s\subseteq L$. On the other hand if $g\in L$ then $g=g(e_1+\cdots+e_s)=ge_1+\cdots+ge_s$ which implies $L\subseteq Le_1+\cdots+Le_s$. So $L=Le_1\oplus\cdots\oplus Le_s$.

Our final theorem gives more information about left ideals in a *GC* near-ring and has relevance to Theorem 2.

Theorem 7. *Suppose N is a GC near-ring using idempotents $e_1,\dots,e_s$. The following are equivalent.*

(a) *N contains no nonzero nilpotent left ideals.*

(b) *Ne_i is a minimal left ideal for each i.*

(c) *Every nonzero left ideal of N is generated by an idempotent.*

(d) *If L is a nonzero left ideal then there exist idempotents $e_{i_1},\dots,e_{i_k}$ such that $L=Ne_{i_1}\oplus\cdots\oplus Ne_{i_k}$.*

Proof. We will prove a⇒b⇒d⇒c⇒a.

a⇒b. Assume Ne_i is not minimal, say $\{0\}\neq L\subset Ne_i$ where $L\neq Ne_i$. Then $L=Le_i$ by Theorem 6, and if $l\in L$ then $l=le_i\in N_{ki}$ for some k where $N_{ik}=\{0\}$. So if $l_1,l_2\in L$ then $l_1l_2=0$ and L is nilpotent, a contradiction. So Ne_i is minimal.

b⇒d. Let L be a nonzero left ideal of N. Then $L=Le_1\oplus\cdots\oplus Le_n$ where Le_i is a left ideal of N contained in Ne_i. Since Ne_i is minimal then either $Le_i=Ne_i$ or $Le_i=\{0\}$.

d⇒c. If L is a nonzero left ideal of N then $L=Ne_{i_1}\oplus\cdots\oplus Ne_{i_k}=N(e_{i_1}+\cdots+e_{i_k})$.

c⇒a. Obvious.

If N is a GC near-ring satisfying any of the conditions a–d in Theorem 7 then every ideal of N and every homomorphic image of N is a GC near-ring.

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