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Moments of the Dedekind zeta function and other non-primitive L-functions

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Abstract

We give a conjecture for the moments of the Dedekind zeta function of a Galois extension. This is achieved through the hybrid product method of Gonek, Hughes and Keating. The moments of the product over primes are evaluated using a theorem of Montgomery and Vaughan, whilst the moments of the product over zeros are conjectured using a heuristic method involving random matrix theory. The asymptotic formula of the latter is then proved for quadratic extensions in the lowest order case. We are also able to reproduce our moments conjecture in the case of quadratic extensions by using a modified version of the moments recipe of Conrey et al. Generalising our methods, we then provide a conjecture for moments of non-primitive L-functions, which is supported by some calculations based on Selberg's conjectures.

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1. *Introduction and statement of results*

Let $\mathbb{K}$ be a number field of discriminant $d_{\mathbb{K}}$ and let $\zeta_{\mathbb{K}}(s)$ be its Dedekind zeta function. In this note we are interested in the asymptotic behaviour of the moments

$$I_k(T)=\frac{1}{T}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^{2k}dt \tag{1·1}$$

with k real. The only known asymptotic for $I_k(T)$ was given by Motohashi [18] in the case where $\mathbb{K}$ is quadratic and $k=1$. He showed that

$$I_1(T)\sim\frac{6}{\pi^2}L(1,\chi)^2\prod_{p|d_{\mathbb{K}}}\left(1+\frac{1}{p}\right)^{-1}\log^2 T, \tag{1·2}$$

where χ is the Kronecker character $(d_{\mathbb{K}}|\cdot)$. Other results concerning the mean values of $\zeta_{\mathbb{K}}(s)$ can be found in [1, 2, 10, 12, 19, 24].

Similarly to the Riemann zeta function, it is difficult to even form conjectures on the higher asymptotics of $I_k(T)$. In the paper [7], Conrey and Ghosh were able to provide a conjecture for the sixth moment of $\zeta(1/2+it)$. Later, Conrey and Gonek [9] described a method that could also give a conjecture for the eighth. Their methods involved mean values of long Dirichlet polynomials, and it seems these methods reach their limit with the eighth moment. It is only recently that believable conjectures have been made for all values $k > -1/2$. These were first given by Keating and Snaith [14] and took the form

$$\frac{1}{T}\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim \frac{a(k)g(k)}{\Gamma(k^2+1)} \log^{k^2} T, \tag{1·3}$$

where

$$a(k) = \prod_p \left(1-\frac{1}{p}\right)^{k^2} \sum_{j\geq 0} \frac{d_k(p^j)^2}{p^j} \tag{1·4}$$

and

$$\frac{g(k)}{\Gamma(k^2+1)} = \frac{G(k+1)^2}{G(2k+1)}, \tag{1·5}$$

where G is Barnes' G-function. Their main idea was to model the zeta function as a characteristic polynomial. This was motivated by the apparent similarities between the non-trivial zeros of the zeta function and eigenangles of matrices in the circular unitary ensemble. However, one drawback of their method was that the arithmetic factor had to be incorporated in an ad hoc fashion. Later, Gonek, Hughes and Keating [11] reproduced this conjecture in such a way that the arithmetic factor was included in a more natural way. In this paper we reproduce these results for the Dedekind zeta function.

The method of Gonek, Hughes and Keating first involves expressing the zeta function as a partial product over primes times a partial product over the zeros. This uses a smoothed form of the explicit formula due to Bombieri and Hejhal [3]. The equivalent for the Dedekind zeta function takes the following form:

THEOREM 1. *Let $X \geq 2$ and let l be any fixed positive integer. Let $u(x)$ be a real, non-negative, smooth function with mass 1 and compact support on $[e^{1-1/X}, e]$. Set*

$$U(z) = \int_0^\infty u(x) E_1(z \log x) dx,$$

where $E_1(z) = \int_z^\infty e^{-w}/w\, dw$. Then for $\sigma \geq 0$ and $|t| \geq 2$ we have

$$\zeta_{\mathbb{K}}(s) = P_{\mathbb{K}}(s, X) Z_{\mathbb{K}}(s, X)\left(1 + O\left(\frac{X^{l+2}}{(|s|\log X)^l}\right) + O(X^{-\sigma}\log X)\right), \tag{1·6}$$

where

$$P_{\mathbb{K}}(s, X) = \exp\left(\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}} \\ \mathfrak{N}(\mathfrak{a})\leq X}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})}\right) \tag{1·7}$$

with

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \mathfrak{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^m, \\ 0 & \text{otherwise,} \end{cases} \tag{1·8}$$

and

$$Z_{\mathbb{K}}(s, X) = \exp\left(-\sum_\rho U((s-\rho)\log X)\right), \tag{1·9}$$

where the sum is over all non-trivial zeros of $\zeta_{\mathbb{K}}(s)$.

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Following a similar reasoning to that in [**11**] we can view formula (1·6) as a hybrid of a truncated Euler product and a truncated Hadamard product. We can then make the equivalent of their *splitting conjecture* for the moments $I_k(T)$. This takes the form:

CONJECTURE 1. *Let $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then for $k > -1/2$, we have*

$$I_k(T) \sim \left(\frac{1}{T}\int_T^{2T} \left|P_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^{2k} dt\right) \times \left(\frac{1}{T}\int_T^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^{2k} dt\right). \quad (1\cdot10)$$

We plan to evaluate the moments of $P_{\mathbb{K}}$ by using the Montgomery–Vaughan mean value theorem [**16**]. Due to the nature of how primes split, or rather, how they are not known to split in some cases, we restrict ourselves to Galois extensions. It may be possible to remove this restriction given milder conditions on $\mathbb{K}$. In Section 3 we show:

THEOREM 2. *Let $\mathbb{K}$ be a Galois extension of degree n with Galois group $G = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ and for a given prime $\mathfrak{p}$ let $g_{\mathfrak{p}}$ denote the index of the decomposition group $G_{\mathfrak{p}}$ in G. Let $\epsilon > 0$, $k > 0$ and suppose that X and $T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then*

$$\frac{1}{T}\int_T^{2T} \left|P_{\mathbb{K}}\left(\tfrac{1}{2}+it,\ X\right)\right|^{2k} dt \sim a_{\mathbb{K}}(k)\chi_{\mathbb{K}}^{nk^2}(e^{\gamma}\log X)^{nk^2}, \quad (1\cdot11)$$

where $\chi_{\mathbb{K}}$ denotes the residue of $\zeta_{\mathbb{K}}(s)$ at $s = 1$ and

$$a_{\mathbb{K}}(k) = \prod_{\mathfrak{p}\subseteq\mathcal{O}_{\mathbb{K}}} \left(\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{nk^2}\left(\sum_{m\geq 0}\frac{d_{g_{\mathfrak{p}}k}(\mathfrak{p}^m)^2}{\mathfrak{N}(\mathfrak{p})^m}\right)^{1/g_{\mathfrak{p}}}\right) \quad (1\cdot12)$$

with $d_k(\mathfrak{p}^m) = d_k(p^m) = \Gamma(m+k)/(m!\Gamma(k))$.

In considering the moments of $Z_{\mathbb{K}}$ for Galois extensions we first express $\zeta_{\mathbb{K}}(s)$ as a product of Artin L-functions. For each individual L-function we then follow the heuristic argument given in [**11**, section 4]. This essentially allows us to write the moments of $Z_{\mathbb{K}}$ as an expectation over the unitary group. We then assume a certain quality of independence between the Artin L-functions, namely, that the matrices associated to the zeros of $L(s, \chi, \mathbb{K}/\mathbb{Q})$ at height T, act independently for distinct χ. This allows for a factorisation of the expectation and we are led to

CONJECTURE 2. *Let $\mathbb{K}$ be a Galois extension of degree n. Suppose that $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then for $k > -1/2$ we have*

$$\frac{1}{T}\int_T^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^{2k} dt \sim (e^{\gamma}\log X)^{-nk^2}\prod_{\chi}\frac{G(\chi(1)k+1)^2}{G(2\chi(1)k+1)}\left(\log\left(q(\chi)T^{d_\chi}\right)\right)^{\chi(1)^2k^2}, \quad (1\cdot13)$$

where the product is over the irreducible characters of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, G is the Barnes G-function, $q(\chi)$ is the conductor of $L(s, \chi, \mathbb{K}/\mathbb{Q})$ and d_χ is its degree.

We remark that for Artin L-functions the degree is simply the number of Gamma functions appearing in its completed form. We also note that we have kept the conductors in the statement of the conjecture. This is merely to emphasise our belief that, to leading order,

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the moment of a product of L-functions should factorise as the product of the moments. Although conductors are fixed throughout the paper, we retain this point of emphasis since it appears quite naturally in our arguments.

By combining Conjecture 2 with Theorem 2 and Conjecture 1 we see that the factors of $e^{\gamma}\log X$ cancel, as expected, and we acquire a full conjecture for the moments of $\zeta_{\mathbb{K}}(1/2+it)$ when $\mathbb{K}$ is Galois. We note that after using $\sum_{\chi}\chi(1)^2=|\mathrm{Gal}(\mathbb{K}/\mathbb{Q})|=n$ the resulting expression in this conjecture is $\sim c\log^{nk^2}T$ for some constant c. Now, in the paper [5], Conrey and Farmer express the idea that the mean square of $\zeta(s)^k$ should be a multiple of the sum $\sum_{n\leq T}d_k(n)^2n^{-1}$, and that this multiple is the measure of how many Dirichlet polynomials are needed to capture the full moment. Their reasoning is based on a combination of the Montgomery-Vaughan mean value Theorem and the form of the sixth and eighth moment conjectures given in [9]. Assuming this idea applies to other L-functions, we note a result of Chandrasekharan and Narasimhan [4]. They showed that for a Galois extension of degree n,

$$\sum_{m\leq T}f_{\mathbb{K}}(m)^2\sim cT\log^{n-1}T, \tag{1·14}$$

where $f_{\mathbb{K}}(m)$ is the number of integral ideals of norm m and c is some constant. Applying partial summation we thus gain a result which supports our conjecture, at least in the case $k=1$ (we note the results of [4] should easily extend to general k, and remain consistent with our conjecture). Alternatively, one could view our conjecture as adding support to the idea of Conrey and Farmer.

In this paper a particular emphasis is placed on quadratic extensions, so let us first fix our notation. We note that if $d_{\mathbb{K}}$ is the discriminant of a quadratic field and $\chi(n)=(d_{\mathbb{K}}|n)$ where $(\cdot\,|\,\cdot)$ is the Kronecker character, then χ is a real Dirichlet character mod

$$q=\begin{cases}4|d_{\mathbb{K}}| & \text{if } d_{\mathbb{K}}\equiv 2 \pmod 4,\\ |d_{\mathbb{K}}| & \text{otherwise}\end{cases} \tag{1·15}$$

and $\zeta_{\mathbb{K}}(s)=\zeta(s)L(s,\chi)$. In section 5 we prove Conjecture 2 in the lowest order case. That is, we prove

THEOREM 3. *Let $\mathbb{K}$ be a quadratic extension. Suppose that $X,T\to\infty$ with $X\ll(\log T)^{2-\epsilon}$. Then*

$$\frac{1}{T}\int_T^{2T}\left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it,X\right)\right|^2dt\sim\frac{\log T\cdot\log qT}{(e^{\gamma}\log X)^2}. \tag{1·16}$$

By combining this with Theorem 2 and then comparing with Motohashi's result (1·2), we see that Conjecture 1 is true for $k=1$ in the case of quadratic extensions.

Recently, an alternative method for conjecturing moments of primitive L-functions was given by Conrey et al. in [6]. In section 6·1, we use a slight modification of the recipe to reproduce the full moments conjecture for quadratic extensions. This is given by

CONJECTURE 3. *Let $\mathbb{K}$ be a quadratic extension and let $a_{\mathbb{K}}(k)$ be given by (1·12). Then*

$$I_k(T)\sim a_{\mathbb{K}}(k)L(1,\chi)^{2k^2}\left(\frac{G(k+1)^2}{G(2k+1)}\right)^2(\log T\cdot\log qT)^{k^2}. \tag{1·17}$$

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Finally, in section 6·2 we attempt to generalise the main ideas of this paper to non-primitive L-functions. We restrict ourselves to reasonable L-functions, which is to say, we consider functions of the form

$$L(s) = \sum \alpha_L(n) n^{-s} = \prod_{j=1}^{m} L_j(s)^{e_j} \tag{1·18}$$

where $e_j \in \mathbb{N}$ and the $L_j(s)$ are distinct, primitive members of the Selberg class $\mathcal{S}$. We assume that we have the functional equation

$$\Lambda_{L_j}(s) := \gamma_{L_j}(s) L_j(s) = \epsilon_j \overline{\Lambda}_{L_j}(1-s) \tag{1·19}$$

where ϵ_j is some number of absolute value 1 and

$$\gamma_{L_j}(s) = Q_j^{s/2} \prod_{i=1}^{d_j} \Gamma(s/2 + \mu_{i,j}) \tag{1·20}$$

with the $\{\mu_{i,j}\}$ stable under complex conjugation. We assume the conductors Q_j to be distinct[1] for distinct j. We also require that the 'convolution' L-functions

$$M_j(s) = \sum_{n=1}^{\infty} \frac{|\alpha_{L_j}(n)|^2}{n^s} \tag{1·21}$$

behave reasonably, in particular, that they have an analytic continuation. We then claim

CONJECTURE 4. *With the notation as above, let $\alpha_{L,k}(n)$ be the Dirichlet coefficients of $L(s)^k$. Then for $k > -1/2$,*

$$\frac{1}{T}\int_0^T \left|L\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim b_L(k) \prod_{j=1}^{m} \frac{G^2(e_j k+1)}{G(2e_j k+1)} \left(\log\left(Q_j T^{d_j}\right)\right)^{(e_j k)^2}, \tag{1·22}$$

where

$$b_L(k) = \prod_p \left(1-\frac{1}{p}\right)^{n_L k^2} \sum_{n=0}^{\infty} \frac{|\alpha_{L,k}(p^n)|^2}{p^n} \tag{1·23}$$

with $n_L = \sum_{j=1}^m e_j^2$.

We remark that if $L(s) = \zeta_{\mathbb{K}}(s)$ with $\mathbb{K}$ Galois and we have a factorisation in terms of Dirichlet series, then the residue term $\chi_{\mathbb{K}}^{nk^2}$ of (1·11) is a factor of $b_L(k)$.

Note that the right-hand side of (1·22) is $\sim (b_L(k) g_L(k)/\Gamma(n_L k^2+1)) \log^{n_L k^2} T$ where

$$g_L(k) = \Gamma(n_L k^2+1) \prod_{j=1}^{m} \frac{G^2(e_j k+1)}{G(2e_j k+1)} d_j^{(e_j k)^2}. \tag{1·24}$$

As previously noted, one expects the mean square of $L(1/2+it)^k$ to be asymptotic to a multiple of the sum $\sum_{n\le T} |\alpha_{L,k}(n)|^2 n^{-1}$. On the assumption of Selberg's conjectures, we

[1] See section 6·2 for a discussion of the possible ambiguities here.

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give an argument showing that

$$\sum_{n\le T}\frac{|\alpha_{L,k}(n)|^2}{n}\sim\frac{b_L(k)}{(n_Lk^2)!}\log^{n_Lk^2}T, \tag{1·25}$$

which adds further support to our conjecture. We also note that for integral k,

$$g_L(k)=\binom{n_Lk^2}{(e_1k)^2,\,\ldots,\,(e_mk)^2}\prod_{j=1}^m g(e_jk)d_j^{(e_jk)^2}, \tag{1·26}$$

where the first factor is the multinomial coefficient and the function g is defined by $g(n)/n^2!=G(n+1)^2/G(2n+1)$. It is shown in [5] that $g(n)$ is an integer, and hence $g_L(k)$ is an integer for integral k.

2. *The hybrid product*

In this section we prove Theorem 1. For this we require a smoothed version of the explicit formula which is given by the following lemma.

LEMMA 1. *Let $u(x)$ be a real, nonnegative smooth function with compact support in $[1, e]$, and let u be normalized so that if*

$$v(t)=\int_t^\infty u(x)dx, \tag{2·1}$$

then $v(0)=1$. Let

$$\hat{u}(z)=\int_0^\infty u(x)x^{z-1}dx \tag{2·2}$$

be the Mellin transform of u. Then for s not a zero or a pole of $\zeta_{\mathbb{K}}(s)$ we have

$$\begin{aligned}-\frac{\zeta_{\mathbb{K}}'(s)}{\zeta_{\mathbb{K}}(s)}=&\sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}}\frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s}v(e^{\log\mathfrak{N}(\mathfrak{a})/\log X})-\sum_\rho\frac{\hat{u}(1-(s-\rho)\log X)}{s-\rho}\\&-(r_1+r_2)\sum_{m=1}^\infty\frac{\hat{u}(1-(s+2m)\log X)}{s+2m}\\&-r_2\sum_{j=0}^\infty\frac{\hat{u}(1-(s+2j+1)\log X)}{s+2j+1}-\chi_{\mathbb{K}}\frac{\hat{u}(1-(s-1)\log X)}{s-1},\end{aligned} \tag{2·3}$$

where $\Lambda(\mathfrak{a})$ is as in (1·8) and r_1, r_2 are, respectively, the number of real and complex embeddings $\mathbb{K}\to\mathbb{C}$.

This lemma is essentially due to Bombieri–Hejhal [3]. It is proved in the familiar way by first considering the integral

$$\frac{1}{2\pi i}\int_{(c)}\frac{\zeta_{\mathbb{K}}'(s+z)}{\zeta_{\mathbb{K}}(s+z)}\hat{u}(1+z\log X)\frac{dz}{z}$$

with $\Re(z)=c=\max\{2,\,2-\Re(s)\}$ and then shifting contours to the far left.

The support condition on u implies $v(e^{\log\mathfrak{N}(\mathfrak{a})/\log X})=0$ when $\mathfrak{N}(\mathfrak{a})>X$. Since there are at most n prime ideals above the rational prime p we see the sum over $\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}$ is finite.

Also, similarly to [11], we can show the sums over ρ, m and j converge absolutely so long as $s \neq \rho$, $s \neq -2m$ or $s \neq -(2j+1)$. We now turn to the proof of Theorem 1.

Let $f_{\mathbb{K}}(n)$ represent the number of ideals of $\mathcal{O}_{\mathbb{K}}$ with norm n. Then

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{f_{\mathbb{K}}(n)}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{f_{\mathbb{K}}(n)}{n^s} \tag{2·4}$$

and so $\zeta_{\mathbb{K}}(\sigma + it) \to 1$ as $\sigma \to \infty$ uniformly in t. Integrating (2·3) along the horizontal line from $s_0 = \sigma_0 + it_0$ to $+\infty$, with $\sigma_0 \geq 0$ and $|t_0| \geq 2$, we get on the left-hand side $-\log \zeta_{\mathbb{K}}(s_0)$. We can now follow the arguments in [11] to find

$$\zeta_{\mathbb{K}}(s) = \tilde{P}_{\mathbb{K}}(s, X) Z_{\mathbb{K}}(s, X) \left(1 + O\left(\frac{X^{l+2}}{(|s| \log X)^l}\right)\right), \tag{2·5}$$

where

$$\tilde{P}_{\mathbb{K}}(s, X) = \exp\left(\sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{K}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})} v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X})\right). \tag{2·6}$$

We note that this is not too different to $P_{\mathbb{K}}(s, X)$. Indeed, since $v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) = 1$ for $\mathfrak{N}(\mathfrak{a}) \leq X^{1-1/X}$ we have

$$\begin{aligned}
\tilde{P}_{\mathbb{K}}(s, X) &= P_{\mathbb{K}}(s, X) \exp\left(\sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{K}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})} (v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) - 1)\right) \\
&= P_{\mathbb{K}}(s, X) \exp\left(\sum_{X^{1-1/X} \leq \mathfrak{N}(\mathfrak{a}) \leq X} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})} (v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) - 1)\right) \\
&= P_{\mathbb{K}}(s, X) \exp\left(O\left(\sum_{X^{1-1/X} \leq p \leq X} p^{-\sigma}\right)\right) \\
&= P_{\mathbb{K}}(s, X) \exp\left(O\left(X^{-\sigma} \log X\right)\right) \\
&= P_{\mathbb{K}}(s, X)\left(1 + O(X^{-\sigma} \log X)\right),
\end{aligned}$$

where we have again used the fact that at most n prime ideals lie above the rational prime p.

To remove the restriction on s, we note that we may interpret exp(âĹŠ$U(z)$) to be asymptotic to Cz for some constant C as $z \to 0$, so both sides of (1·6) vanish at the zeros.

3. *Moments of the arithmetic factor*

In this section we prove Theorem 2. For a rational prime p we have the decomposition

$$p\mathcal{O}_{\mathbb{K}} = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i} \tag{3·1}$$

with

$$\mathfrak{N}(\mathfrak{p}_i) = p^{f_i}, \tag{3·2}$$

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where e_i and f_i are positive integers. Since $\mathbb{K}$ is Galois, $e_1 = e_2 = \cdots = e_g = e$ and $f_1 = f_2 = \cdots = f_g = f$, say. We then have the identity $efg = n$. Let $g_p(e)$ denote the number of prime ideals lying above p with ramification index e. Then

$$\begin{aligned} P_{\mathbb{K}}(s, X)^k &= \exp\left(k \sum_{\mathfrak{N}(\mathfrak{a}) \le X} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})}\right) = \exp\left(k \sum_{m\ge 1} \sum_{\mathfrak{N}(\mathfrak{p})^m \le X} \frac{1}{m\mathfrak{N}(\mathfrak{p})^{ms}}\right) \\ &= \exp\left(k \sum_{m \ge 1} \sum_{g|n} g \sum_{e | \frac{n}{g}} \sum_{\substack{p^{\frac{mn}{eg}} \le X \\ g_p(e)=g}} \frac{1}{mp^{(n/eg)ms}}\right) \qquad (3\cdot3) \\ &= \prod_{g|n} \prod_{e|\frac{n}{g}} \prod_{\substack{p^{\frac{n}{eg}} \le X \\ g_p(e)=g}} \exp\left(\log(1 - p^{-(n/eg)s})^{-gk} - gk \sum_{\substack{m \ge 2 \\ p^{\frac{mn}{eg}} > X}} \frac{1}{mp^{(n/eg)ms}}\right). \end{aligned}$$

We now write the innermost product as the Dirichlet series

$$\sum_{l \in \mathcal{L}_{e,g}(X)}^{\infty} \frac{\beta_{gk}(l)}{l^{(n/eg)s}}, \qquad (3\cdot4)$$

where $\mathcal{L}_{e,g}(X) = \{l \in \mathrm{Im}(\mathfrak{N}) : p|l \Rightarrow g_p = g \text{ and } p^{n/eg} \le X\}$. We see that $\beta_{gk}(l)$ is a multiplicative function of l, $0 \le \beta_{gk}(l) \le d_{gk}(l)$ for all l and $\beta_{gk}(p^m) = d_{gk}(p^m)$ if $p^m \le X$.

For an integer l, let $l_{e,g}$ denote the greatest factor of l composed of primes p for which $g_p(e) = g$. Now,

$$P_{\mathbb{K}}(s, X)^k = \prod_{g|n} \prod_{e|\frac{n}{g}} \left(\sum_{l \in \mathcal{L}_{e,g}(X)}^{\infty} \frac{\beta_{gk}(l)}{l^{(n/eg)s}}\right) = \sum_{l \in \mathcal{W}(X)}^{\infty} \frac{\gamma_k(l)}{l^s}, \qquad (3\cdot5)$$

where

$$\gamma_k(l) = \prod_{g|n} \prod_{e|\frac{n}{g}} \beta_{gk}(l_{e,g}^{eg/n}) \qquad (3\cdot6)$$

and $\mathcal{W}(X) = \{l \in \mathrm{Im}(\mathfrak{N}) : \mathfrak{N}(\mathfrak{p})|l \Rightarrow \mathfrak{N}(\mathfrak{p}) \le X\}$ and $\mathrm{Im}(\mathfrak{N})$ stands for the image of the norm map. The product representation of γ is made possible by the fact that for integers l, m belonging to different $\mathcal{L}_{e,g}(X)$, we have $(l, m) = 1$. This would not necessarily be the case for non-Galois extensions. For example, in a cubic extension we may have the factorisation $p\mathcal{O}_{\mathbb{K}} = \mathfrak{p}_1\mathfrak{p}_2$ and hence one of these ideals has norm p, whilst the other has norm p^2. We could then follow the previous reasoning whilst redefining the sets $\mathcal{L}$ with a consideration of this difference. However, we would then lose the coprimality condition.

Since we want to apply the mean value theorem for Dirichlet series we split the sum at T^θ where θ is to be chosen later and obtain

$$P_{\mathbb{K}}(s, X)^k = \sum_{\substack{l \in \mathcal{W}(X) \\ l \le T^\theta}} \frac{\gamma_k(l)}{l^s} + O\left(\sum_{\substack{l \in \mathcal{W}(X) \\ l > T^\theta}} \frac{\gamma_k(l)}{l^s}\right). \qquad (3\cdot7)$$

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Now for $\epsilon > 0$ and $\sigma \geq c$ the error term is

$$\ll T^{-\epsilon\theta} \sum_{l \in \mathcal{W}(X)} \frac{\prod_{g|n} \prod_{e|\frac{n}{g}} d_{gk}(l_{e,g}^{eg/n})}{n^{c-\epsilon}} = T^{-\epsilon\theta} \prod_{\mathfrak{N}(\mathfrak{p}) \leq X} (1 - \mathfrak{N}(\mathfrak{p})^{\epsilon - c})^{-k}$$
$$= T^{-\epsilon\theta} \exp\left(O\left(k \sum_{\mathfrak{N}(\mathfrak{p}) \leq X} \mathfrak{N}(\mathfrak{p})^{\epsilon-c}\right)\right) = T^{-\epsilon\theta} \exp\left(O\left(\frac{kX^{1-c+\epsilon}}{(1-c+\epsilon)\log X}\right)\right),$$

where in the last line we have used the prime ideal theorem. If we let $X \ll (\log T)^{1/(1-c+\epsilon)}$ then this is

$$\ll T^{-\epsilon\theta} \exp\left(O\left(k\frac{\log T}{\log\log T}\right)\right) \ll_k T^{-\epsilon\theta/2} \tag{3·8}$$

and hence

$$P_{\mathbb{K}}(s, X)^k = \sum_{\substack{l \in \mathcal{W}(X) \\ l \leq T^{\theta}}} \frac{\gamma_k(l)}{l^s} + O_k(T^{-\epsilon\theta/2}). \tag{3·9}$$

We now let $\theta = 1/2$ and apply the Montgomery–Vaughan mean value theorem [**16**] to give

$$\begin{aligned}\frac{1}{T}\int_T^{2T} \left|\sum_{\substack{l \in \mathcal{W}(X) \\ l \leq T^{1/2}}} \frac{\gamma_k(l)}{l^{\sigma+it}}\right|^2 dt &= (1 + O(T^{-1/2})) \sum_{\substack{l \in \mathcal{W}(X) \\ l \leq T^{1/2}}} \frac{\gamma_k(l)^2}{l^{2\sigma}} \\ &= (1 + O(T^{-1/2}))\left(\sum_{l \in \mathcal{W}(X)} \frac{\gamma_k(l)^2}{l^{2\sigma}} + O(T^{-\epsilon/4})\right) \\ &= (1 + O(T^{-\epsilon/4})) \sum_{l \in \mathcal{W}(X)} \frac{\gamma_k(l)^2}{l^{2\sigma}}.\end{aligned} \tag{3·10}$$

Therefore by (3·9) and the Cauchy–Schwarz inequality we have

$$\frac{1}{T}\int_T^{2T} |P_{\mathbb{K}}(\sigma + it, X)|^{2k} = (1 + O(T^{-\epsilon/4})) \sum_{l \in \mathcal{W}(X)} \frac{\gamma_k(l)^2}{l^{2\sigma}}. \tag{3·11}$$

We can now re-factorise the above Dirichlet series to give

$$\sum_{l \in \mathcal{W}(X)} \frac{\gamma_k(l)^2}{l^{2\sigma}} = \prod_{g|n} \prod_{e|\frac{n}{g}} \left(\sum_{l \in \mathcal{L}_{e,g}(X)}^{\infty} \frac{\beta_{gk}(l)^2}{l^{2(n/eg)\sigma}}\right). \tag{3·12}$$

For an individual series in the above product we can follow the arguments in [**11**] to find

$$\sum_{l \in \mathcal{L}_{e,g}(X)}^{\infty} \frac{\beta_{gk}(l)^2}{l^{2(n/eg)\sigma}} = (1 + O(X^{-1/2+\epsilon})) \prod_{\substack{p^{\frac{n}{eg}} \leq X \\ g_p(e) = g}} \sum_{m \geq 0} \frac{d_{gk}(p^m)^2}{p^{2m(n/eg)\sigma}}. \tag{3·13}$$

Now, the above product may be divergent as $X \to \infty$. In order to keep the arithmetic information, we factor out the divergent part and write it as

$$\prod_{\substack{p^{\frac{n}{eg}} \leq X \\ g_p(e) = g}} \left(\left(1 - p^{-2(n/eg)\sigma}\right)^{ngk^2} \sum_{m \geq 0} \frac{d_{gk}(p^m)^2}{p^{2m(n/eg)\sigma}}\right) \prod_{\substack{p^{\frac{n}{eg}} \leq X \\ g_p(e) = g}} \left(1 - p^{-2(n/eg)\sigma}\right)^{-ngk^2}. \tag{3·14}$$

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In terms of divergence, the worst case scenario is when $n/eg = 1$. If in this case $g_p(e) = g < n$, then p is ramified and hence the product is finite. Therefore, we only need consider the case $g = n$, for which the above equals

$$\begin{aligned}\prod_{\substack{p>X\\ g_p(e)=n}} \left((1-p^{-2\sigma})^{n^2k^2} \sum_{m\geq 0} \frac{d_{nk}(p^m)^2}{p^{2m\sigma}} \right) &= \prod_{\substack{p>X\\ g_p(e)=n}} \left(1 - n^2k^2p^{-2\sigma} + n^2k^2p^{-2\sigma} + O_k(p^{-4\sigma})\right)\\ &= \prod_{\substack{p>X\\ g_p(e)=n}} \left(1 + O_k(p^{-4\sigma})\right)\\ &= 1 + O_k(1/(X\log X)). \end{aligned} \tag{3·15}$$

It follows that we can extend the first product in (3·14) over all primes. Specialising to $\sigma = 1/2$ and using the product representation in (3·12) we see

$$\sum_{l\in\mathcal{W}(X)} \frac{\gamma_k(l)^2}{l} = a_{\mathbb{K}}(k) \prod_{\mathfrak{N}(\mathfrak{p})\leq X} (1-\mathfrak{N}(\mathfrak{p})^{-1})^{-nk^2}(1 + O_k(X^{-1/2+\epsilon})). \tag{3·16}$$

By a generalisation of Mertens theorem [21], we have

$$\prod_{\mathfrak{N}(\mathfrak{p})\leq X} (1-\mathfrak{N}(\mathfrak{p})^{-1})^{-nk^2} = \chi_{\mathbb{K}}^{nk^2}(e^{\gamma}\log X)^{nk^2}(1 + O(1/\log^2 X)) \tag{3·17}$$

and the result follows.

4. *Support for Conjecture 2*

Let $\mathbb{K}$ be a Galois extension of degree n with Galois group G. Then it is well known (see for example [20, chapter 7]) that

$$\zeta_{\mathbb{K}}(s) = \prod_{\chi} L(s,\chi,\mathbb{K}/\mathbb{Q})^{\chi(1)} \tag{4·1}$$

where the product is over the non-equivalent irreducible characters of G and $L(s,\chi,\mathbb{K}/\mathbb{Q})$ is the Artin L-function attached to χ. For each character χ, the associated L-function satisfies the functional equation

$$\Lambda(s,\chi) := q(\chi)^{s/2}\gamma(s,\chi)L(s,\chi) = W(\chi)\Lambda(1-s,\overline{\chi}) \tag{4·2}$$

where $W(\chi)$ is some complex number of modulus one and $q(\chi)$ is the conductor, for which we do not require an explicit expression. The gamma factor is given by

$$\gamma(s,\chi) = \pi^{-sd_\chi/2}\prod_{j=1}^{d_\chi}\Gamma\left(\frac{s+\mu_j}{2}\right) \tag{4·3}$$

with μ_j equal to 0 or 1. If we assume the Artin conjecture then $L(s,\chi)$ is an entire function for all non-trivial χ. If χ is the trivial character then $L(s,\chi)$ equals the Dedekind zeta function of the base field, which in our case is $\zeta(s)$. Under this assumption, these L-functions exhibit reasonable behaviour and the usual arguments (e.g. [13, theorem 5·8]) give the mean density of zeros of $L(\beta+it,\chi)$, $0\leq\beta\leq 1$, as

$$\frac{1}{\pi}\log\left(q(\chi)\left(\frac{t}{2\pi}\right)^{d_\chi}\right) =: \frac{1}{\pi}\mathcal{L}_\chi(t), \tag{4·4}$$

say. For each $L(s,\chi)$ in the product of equation (4·1), we associate to its zeros $\gamma_n(\chi)$ at height T, a unitary matrix $U(N(\chi))$ of size $N(\chi) = \lfloor \mathcal{L}_\chi(T)\rfloor$ chosen with respect to Haar measure, which we denote $d\mu(\chi)$. After rescaling, the zeros $\gamma_n(\chi)$ are conjectured [**23**] to share the same distribution as the eigenangles $\theta_n(\chi)$ of $U(N(\chi))$ when chosen with $d\mu(\chi)$.

In addition to the previous assumptions, we now also assume the extended Riemann hypothesis. Let $Z_{\mathbb{K}}(s,X)$ be given by (1·9). Since $\Re E_1(ix) = -\mathrm{Ci}(|x|)$ for $x\in\mathbb{R}$, where

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos w}{w}dw, \tag{4·5}$$

we see that

$$\begin{aligned}\frac{1}{T}\int_T^{2T} &\left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it,X\right)\right|^{2k}dt\\ &= \frac{1}{T}\int_T^{2T}\prod_{\gamma_n}\exp\left(2k\int_1^e u(y)\mathrm{Ci}(|t-\gamma_n|\log y\log X)\right)dydt\\ &= \frac{1}{T}\int_T^{2T}\prod_{\chi}\prod_{\gamma_n(\chi)}\exp\left(2k\chi(1)\int_1^e u(y)\mathrm{Ci}(|t-\gamma_n(\chi)|\log y\log X)\right)dydt,\end{aligned} \tag{4·6}$$

where $u(y)$ is a smooth, non-negative function supported on $[e^{1-1/X}, e]$ and of total mass one. We now replace the zeros with the eigenangles and argue that the above should be modeled by

$$\mathbb{E}\left[\prod_\chi\prod_{n=1}^{N(\chi)}\phi(k\chi(1),\theta_n(\chi))\right], \tag{4·7}$$

where

$$\phi(m,\theta) = \exp\left(2m\int_1^e u(y)\mathrm{Ci}(|\theta|\log y\log X)\right) \tag{4·8}$$

and the expectation is taken with respect to the product measure $\prod_\chi d\mu(\chi)$. We now assume that the matrices $U(N(\chi))$ can be chosen independently for any two distinct χ. This corresponds to a 'superposition' of ensembles; the behaviour of which is also shared by the distribution of zeros of a product of distinct L-functions [**15**]. With this assumption, the expectation factorises as

$$\prod_\chi \mathbb{E}\left[\prod_{n=1}^{N(\chi)}\phi(k\chi(1),\theta_n(\chi))\right]. \tag{4·9}$$

In [**11**] it is shown (Theorem 4) that for $k > -1/2$ and $X \geq 2$,

$$\mathbb{E}\left[\prod_{j=1}^{M}\phi(m,\theta_j)\right] \sim \frac{G(m+1)^2}{G(2m+1)}\left(\frac{M}{e^\gamma \log X}\right)^{m^2}\left(1+O_m\left(\frac{1}{\log X}\right)\right). \tag{4·10}$$

Therefore, by forming the product over χ and using $\sum_\chi \chi(1)^2 = |\mathrm{Gal}(\mathbb{K}/\mathbb{Q})| = n$ we are led to Conjecture 2.

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5. *The second moment of $Z_{\mathbb{K}}$ for quadratic extensions*

In this section we prove Theorem 3. For the most part, the remainder of this paper is concerned with quadratic extensions so we first state some useful facts whilst establishing our notation.

As mentioned in the introduction, $\zeta_{\mathbb{K}} = \zeta(s)L(s,\chi)$ where χ is the Kronecker character. We shall have occasion to work with more general (complex) characters χ mod $q > 1$ when the arguments in question work in such generalities, however, at some points we may specialise to the Kronecker character without mention. We also note in quadratic extensions the splitting of primes admits the following simple description:

$$\begin{aligned} p \text{ is split}: &\quad (p) = \mathfrak{p}_1\mathfrak{p}_2 &&\implies \mathfrak{N}(\mathfrak{p}_1) = \mathfrak{N}(\mathfrak{p}_2) = p \\ p \text{ is inert}: &\quad (p) = \mathfrak{p}_1 &&\implies \mathfrak{N}(\mathfrak{p}_1) = p^2 \\ p \text{ is ramified}: &\quad (p) = \mathfrak{p}_1^2 &&\implies \mathfrak{N}(\mathfrak{p}_1) = p. \end{aligned}$$

5·1. *The setup*

Our aim is to show

$$\frac{1}{T}\int_T^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^2 dt \sim \frac{\log T \cdot \log qT}{(e^\gamma \log X)^2} \tag{5·1}$$

for $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$ and $\mathbb{K}$ quadratic. Since $\zeta_{\mathbb{K}}(1/2+it)P_{\mathbb{K}}(1/2+it, X) = Z_{\mathbb{K}}(1/2+it, X)(1+o(1))$ for $t \in [T, 2T]$, it is enough to show that

$$\frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right) P_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)^{-1}\right|^2 dt \sim \frac{\log T \cdot \log qT}{(e^\gamma \log X)^2}. \tag{5·2}$$

To evaluate the left-hand side we first express $P_{\mathbb{K}}(1/2+it)^{-1}$ as a Dirichlet polynomial and then apply a formula given the author in [**12**].

By formula (3·9) we have

$$P_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)^{-1} = \sum_{\substack{n \in \mathcal{W}(X) \\ n \le T^\theta}} \frac{\gamma_{-1}(n)}{n^{1/2+it}} + O_k(T^{-\epsilon\theta/2}) \tag{5·3}$$

for any $\theta > 0$. Applying this, along with Cauchy–Schwarz and Motohashi's formula (1·2), gives

$$\begin{aligned} &\frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right) P_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)^{-1}\right|^2 dt \\ &= \left(1 + O\left(\frac{1}{\log X}\right)\right) \frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right) \sum_{\substack{n \in \mathcal{W}(X) \\ n \le T^\theta}} \frac{\gamma_{-1}(n)}{n^{1/2+it}}\right|^2 dt \end{aligned} \tag{5·4}$$

since $T^{-\delta} \ll 1/\log X$ for any $\delta > 0$.

The behaviour of the coefficients $\gamma_{-1}(n)$ may be determined from the Euler product given in (3·3). Indeed, the last line of equation (3·3) gives

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$$P_{\mathbb{K}}(s,X)^{-1}=\prod_{g|2}\prod_{e|\frac{2}{g}}\prod_{\substack{p^{\frac{2}{eg}}\le X\\ g_p(e)=g}}\left(1-p^{-(2/eg)s}\right)^g\exp\left(g\sum_{\substack{m\ge 2\\ p^{\frac{2m}{eg}}>X}}\frac{1}{mp^{(2/eg)ms}}\right)$$

$$=\prod_{g|2}\prod_{e|\frac{2}{g}}\prod_{\substack{p^{\frac{2}{eg}}\le\sqrt{X}\\ g_p(e)=g}}\left(1-p^{-(2/eg)s}\right)^g \qquad (5\cdot5)$$

$$\times\prod_{\substack{\sqrt{X}<p^{\frac{2}{eg}}\le X\\ g_p(e)=g}}\left(1-p^{-(2/eg)s}\right)^g\left(1+\frac{g}{2}p^{-(4/eg)s}+O(p^{-(6/eg)\sigma})\right).$$

Since we are considering the case $\sigma=1/2$ we may remove the term $O(p^{-(6/eg)\sigma})$ in the final product at the cost of a factor of $(1+O(1/\log X))$. By (5·4), we can clearly incur this with no real loss, and hence we may assume that $\gamma_{-1}(n)$ is supported on cube-free split/ramified primes and 6th power-free inert primes. Then, upon reading the behaviour of γ_{-1} from the Euler product we see that

$$\gamma_{-1}(p_{\rm s}^j)=\begin{cases}-2 & \text{if } j=1,\ p_{\rm s}\le X,\\ 1 & \text{if } j=2,\ p_{\rm s}\le\sqrt{X},\\ 2 & \text{if } j=2,\ \sqrt{X}<p_{\rm s}\le X,\\ 0 & \text{if } j\ge 3,\end{cases}\quad \gamma_{-1}(p_{\rm i}^{2j})=\begin{cases}-1 & \text{if } j=1,\ p_{\rm i}^2\le X,\\ 0 & \text{if } j=2,\ p_{\rm i}^2\le\sqrt{X},\\ \frac{1}{2} & \text{if } j=2,\ \sqrt{X}<p_{\rm i}^2\le X,\\ 0 & \text{if } j\ge 3\end{cases} \qquad (5\cdot6)$$

and

$$\gamma_{-1}(p_{\rm r}^j)=\begin{cases}-1 & \text{if } j=1,\ p_{\rm r}\le X,\\ 0 & \text{if } j=2,\ p_{\rm r}\le\sqrt{X},\\ \frac{1}{2} & \text{if } j=2,\ \sqrt{X}<p_{\rm r}\le X,\\ 0 & \text{if } j\ge 3,\end{cases} \qquad (5\cdot7)$$

where the notation $p_{\rm s}$, $p_{\rm i}$, $p_{\rm r}$ denotes split, inert and ramified primes respectively. We also note the bound $\gamma_{-1}(n)\ll d(n)$ for all $n\in\mathcal{W}(X)$.

We are now required to show that for $X,T\to\infty$ with $X\ll(\log T)^{2-\epsilon}$,

$$\frac{1}{T}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\sum_{\substack{n\in\mathcal{W}(X)\\ n\le T^\theta}}\frac{\gamma_{-1}(n)}{n^{1/2+it}}\right|^2=\frac{\log T\cdot\log qT}{(e^\gamma\log X)^2}\left(1+O\left(\frac{1}{\log X}\right)\right). \qquad (5\cdot8)$$

In order to state the formula given in [12] we must first establish some notation. So, let $\alpha,\beta,\gamma,\delta$ be complex numbers $\ll 1/\log T$ and let

$$A_{\alpha,\beta,\gamma,\delta}(s)=\zeta(1+\alpha+\gamma+s)\zeta(1+\beta+\delta+s)L(1+\beta+\gamma+s,\chi)$$

$$\times\frac{L(1+\alpha+\delta+s,\overline{\chi})}{\zeta(2+\alpha+\beta+\gamma+\delta+2s)}\prod_{p|q}\left(\frac{1-p^{-1-s-\beta-\delta}}{1-p^{-2-2s-\alpha-\beta-\gamma-\delta}}\right). \qquad (5\cdot9)$$

For integers h and k let

$$B_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi)=\prod_{p|hk}\frac{\sum_{j\ge0}f_{\alpha,\beta}(p^{k_p+j},\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})p^{-j(1+s)}}{\sum_{j\ge0}f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^j,\overline{\chi})p^{-j(1+s)}}, \qquad (5\cdot10)$$

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where

$$f_{\alpha,\beta}(n,\chi)=\sum_{n_1n_2=n} n_1^{-\alpha}n_2^{-\beta}\chi(n_2) \tag{5·11}$$

and where h_p and k_p are the highest powers of p dividing h and k respectively. Now let

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(s)=A_{\alpha,\beta,\gamma,\delta}(s)B_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi). \tag{5·12}$$

We must also define a slight variant of the above. For this we let

$$A'_{\alpha,\beta,\gamma,\delta}(s,\chi) = \frac{L(1+\alpha+\gamma+s,\chi)L(1+\beta+\delta+s,\chi)L(1+\alpha+\delta+s,\chi)L(1+\beta+\gamma+s,\chi)}{L(2+\alpha+\beta+\gamma+\delta+2s,\chi^2)} \tag{5·13}$$

and

$$B'_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi)=\prod_{p\mid hk}\frac{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^{k_p+j})\sigma_{\gamma,\delta}(p^{h_p+j})p^{-j(1+s)}}{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j(1+s)}}, \tag{5·14}$$

where

$$\sigma_{\alpha,\beta}(n)=\sum_{n_1n_2=n} n_1^{-\alpha}n_2^{-\beta}. \tag{5·15}$$

Now let

$$Z'_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi)=\overline{G(\chi)}A'_{\alpha,\beta,\gamma,\delta}(s,\chi)B'_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi), \tag{5·16}$$

where $G(\chi)$ is the Gauss sum associated to χ.

THEOREM 4 ([12]). *Let*

$$\begin{aligned} I(h,k)=\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta\left(\tfrac12+\alpha+it\right)L\left(\tfrac12+\beta+it,\chi\right)\\ \times\zeta\left(\tfrac12+\gamma-it\right)L\left(\tfrac12+\delta-it,\overline{\chi}\right)w(t)dt,\end{aligned} \tag{5·17}$$

where $w(t)$ is a smooth, nonnegative function with support contained in $[T/2,4T]$, satisfying $w^{(j)}(t)\ll_j T_0^{-j}$ for all $j=0,1,2,\ldots$, where $T^{1/2+\epsilon}\ll T_0\ll T$. Suppose $(h,k)=1$ and that $hk\leq T^{\frac{2}{11}-\epsilon}$. Then

$$\begin{aligned} I(h,k)=&\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}w(t)\Bigg(Z_{\alpha,\beta,\gamma,\delta,h,k}(0)+\frac{1}{q^{\beta+\delta}}Z_{-\gamma,-\delta,-\alpha,-\beta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta}\\ &+Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma}+\frac{1}{q^{\beta+\delta}}Z_{\alpha,-\delta,\gamma,-\beta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\beta-\delta}\\ &+\mathbf{1}_{q\mid h}\frac{\chi(k)}{q^{\delta}}Z'_{-\delta,\beta,\gamma,-\alpha,\frac{h}{q},k}(0,\chi)\left(\frac{t}{2\pi}\right)^{-\alpha-\delta}\\ &+\mathbf{1}_{q\mid k}\frac{\overline{\chi}(h)}{q^{\beta}}Z'_{\alpha,-\gamma,-\beta,\delta,h,\frac{k}{q}}(0,\overline{\chi})\left(\frac{t}{2\pi}\right)^{-\beta-\gamma}\Bigg)dt+E(T),\end{aligned} \tag{5·18}$$

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where

$$E(T) \ll T^{3/4+\epsilon}(hk)^{7/8+\epsilon}q^{1+\epsilon}(T/T_0)^{9/4}. \tag{5·19}$$

Now take a Dirichlet polynomial $M(s) = \sum_{n\le T^\theta} a(n)n^{-s}$ with $\theta \le 1/11 - \epsilon$ and let $w(t)$ satisfy the conditions of Theorem 4. Then, upon expanding, we have

$$\int_{-\infty}^{\infty} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^2 \left|M\left(\tfrac{1}{2}+it\right)\right|^2 w(t)dt$$
$$= \sum_{h,k\le T^\theta} \frac{a(h)\overline{a(k)}}{\sqrt{hk}}(h,k) \lim_{\alpha,\beta,\gamma,\delta\to 0} I(h_k, k_h), \tag{5·20}$$

where $h_k = h/(h,k)$. In order to evaluate this inner limit we express $Z_{\alpha,\beta,\gamma,\delta,h,k}(0)$ as a Laurent series and express the other terms as Taylor series. In doing this, the only real difficulty lies in calculating the derivatives of $B_{\alpha,\beta,\gamma,\delta,h,k}(0)$. For our purposes, which is to work over X-smooth numbers, we only need upper bounds however. The first order derivatives of $B_{\alpha,\beta,\gamma,\delta,h,k}(0)$ are

$$\ll B_{0,0,0,0,h,k}(0)\left(\log hk + \sum_{p|hk}\frac{\log p}{p}\right) \ll B_{0,0,0,0,h,k}(0)\big(\log hk + \log\log hk\big). \tag{5·21}$$

Similarly, one finds that the second order derivatives are

$$\ll B_{0,0,0,0,h,k}(0)\big(\log^2 hk + \log hk \log\log hk + \log^2 \log hk\big). \tag{5·22}$$

A short calculation gives

$$B_{0,0,0,0,h,k}(0) = \delta(h)\delta(k), \tag{5·23}$$

where

$$\delta(h) = \begin{cases} \prod_{\substack{p|h \\ p \text{ split}}}\left(1 + h_p\frac{1-p^{-1}}{1+p^{-1}}\right) & \text{if } h_{\mathrm{i}} \text{ is square} \\ 0 & \text{otherwise} \end{cases} \tag{5·24}$$

and h_{i} is the greatest factor of h composed solely of inert primes.

Upon taking the limit as $\alpha, \beta, \gamma, \delta \to 0$ and taking smooth approximations to the characteristic function of the interval $[T, 2T]$ with $T_0 = T^{1-\epsilon}$ we get the following:

PROPOSITION 1. *Let $M(s) = \sum_{n\le T^\theta} a(n)n^{-s}$ with $\theta \le 1/11 - \epsilon$. Then,*

$$\frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^2 \left|M\left(\tfrac{1}{2}+it\right)\right|^2 dt$$
$$= \sum_{h,k\le T^\theta} \frac{a(h)\overline{a(k)}}{hk}(h,k)\left[\sum_{n=0}^{2} c_n(h,k,T) + O\left(T^{-\frac{1}{4}+\epsilon}(h_k k_h)^{7/8+\epsilon}\right)\right]. \tag{5·25}$$

The leading order term is given by

$$c_2(h,k,T) = \frac{6}{\pi^2}L(1,\chi)^2 \prod_{p|d_{\mathbb{K}}}\left(1+\frac{1}{p}\right)^{-1}$$
$$\times\, \delta(h_k)\delta(k_h)\big[\log T \cdot \log qT + O(\log T \log h_k k_h)\big]. \tag{5·26}$$

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For the lower order terms we have

$$c_1(h,k,T) \ll \delta(h_k)\delta(k_h)\log T \log\log h_k k_h \tag{5·27}$$

and

$$\begin{aligned} c_0(h,k,T) = c_0'(h,k,T) &+ \mathbf{1}_{q|h_k}\chi(k_h)Z'_{0,0,0,0,\frac{h_k}{q},k_h}(0,\chi) \\ &+ \mathbf{1}_{q|k_h}\chi(h_k)Z'_{0,0,0,0,h_k,\frac{k_h}{q}}(0,\overline{\chi}) \end{aligned} \tag{5·28}$$

with

$$c_0'(h,k,T) \ll \delta(h_k)\delta(k_h)(\log\log h_k k_h)^2. \tag{5·29}$$

The Z' terms may be written as

$$Z'_{0,0,0,0,m,n}(0,\chi) = \overline{G(\chi)}\frac{L(1,\chi)^4}{L(2,\chi^2)}\delta'(m)\delta'(n), \tag{5·30}$$

where

$$\delta'(m) = \prod_{\substack{p|m \\ p \text{ split}}}\left(1 + m_p\frac{p-1}{p+1}\right)\prod_{\substack{p|m \\ p \text{ inert}}}\left(1 + m_p\frac{p+1}{p-1}\right). \tag{5·31}$$

5·2. *Evaluating the main term*

PROPOSITION 2. *Let $c_2(h,k,T)$ be given by (5·26). Suppose $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then*

$$\sum_{\substack{h,k \le T^\theta \\ h,k \in \mathcal{W}(X)}} \frac{\gamma_{-1}(h)\gamma_{-1}(k)c_2(h,k,T)}{hk}(h,k) = (1+o(1))\frac{\log T \cdot \log qT}{(e^\gamma \log X)^2}. \tag{5·32}$$

Proof. Inputting the formula for $c_2(h,k,T)$ we see that we are required to show

$$\begin{aligned} S_0 := &\sum_{\substack{h,k \le T^\theta \\ h,k \in \mathcal{W}(X)}} \frac{\gamma_{-1}(h)\gamma_{-1}(k)\delta(h_k)\delta(k_h)}{hk}(h,k)\Big[\log T \cdot \log qT + O(\log T \log h_k k_h)\Big] \\ = &(1+o(1))\frac{\pi^2}{6}L(1,\chi)^{-2}\prod_{p|d_{\mathbb{K}}}\left(1+\frac{1}{p}\right)\frac{\log T \cdot \log qT}{(e^\gamma \log X)^2}. \end{aligned} \tag{5·33}$$

We first group together the terms for which $(h,k) = g$. Replacing h by hg and k by kg we obtain

$$S_0 = \sum_{\substack{g \le Y \\ g \in \mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k \le Y/g \\ k \in \mathcal{W}(X)}} \frac{\gamma_{-1}(kg)\delta(k)}{k} \sum_{\substack{h \le Y/g \\ h \in \mathcal{W}(X) \\ (h,k)=1}} \frac{\gamma_{-1}(hg)\delta(h)}{h} \times \Big[\log T \cdot \log qT + O(\log T \log hk)\Big], \tag{5·34}$$

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where $Y = T^{\theta}$. Let us first estimate the error term. We have

$$\begin{aligned}
&\sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\gamma_{-1}(kg)\delta(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \frac{\gamma_{-1}(hg)\delta(h)}{h} \log(hk)\\
&\ll \sum_{g\in\mathcal{L}(X)} \frac{d(g)^2}{g} \sum_{h,k\in\mathcal{L}(X)} \frac{d(k)^2 d(h)^2}{hk} \log hk \qquad (5{\cdot}35)\\
&\ll \sum_{g\in\mathcal{L}(X)} \frac{d(g)^2}{g} \left(\sum_{m\in\mathcal{L}(X)} \frac{d(m)^2 \log m}{m} \right)^2.
\end{aligned}$$

Writing $f(\sigma) = \sum_{m\in\mathcal{L}(X)} d(m)^2 m^{-\sigma}$ the inner sum is $-f'(1)$. Since $f(\sigma) = \prod_{p\le X}(1 - p^{-\sigma})^{-4}(1-p^{-2\sigma})$ we see $f'(1) \ll f(1)\sum_{p\le X} \log p/(p-1) \ll \log^5 X$ and hence the above sum is $\ll \log^{14} X$. We can now turn to the main term and consider

$$S := \sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\gamma_{-1}(kg)\delta(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \frac{\gamma_{-1}(hg)\delta(h)}{h}. \qquad (5{\cdot}36)$$

We define the function $\mu' : \mathrm{Im}(\mathfrak{N}) \to \mathbb{C}$, $\mathfrak{N}(\mathfrak{a}) \mapsto \mu(\mathfrak{a})$ where μ is the extension of the usual möbius function to ideals given by

$$\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = \mathcal{O}_{\mathbb{K}},\\ (-1)^r & \text{if } \mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2\ldots\mathfrak{p}_r,\\ 0 & \text{otherwise.}\end{cases} \qquad (5{\cdot}37)$$

So basically; for split and ramified primes $\mu'(p) = -1$ and $\mu'(p^j) = 0$ for $j \ge 2$; for inert primes $\mu(p^2) = -1$ and $\mu(p^{2j}) = 0$ for $j \ge 2$, and μ' is multiplicative. Similarly to the usual möbius function we now have

$$\sum_{\substack{d|h\\ d|k\\ d\in\mathrm{Im}(\mathfrak{N})}} \mu'(d) = \begin{cases} 1 & \text{if } (h,k) = 1\\ 0 & \text{otherwise}\end{cases} \qquad (5{\cdot}38)$$

for $h, k \in \mathrm{Im}(\mathfrak{N})$. Substituting this into the sum over h in S we see

$$\begin{aligned}
S &= \sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\gamma_{-1}(kg)\delta(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)}} \left(\sum_{\substack{d|h\\ d|k\\ d\in\mathrm{Im}(\mathfrak{N})}} \mu'(d) \right) \frac{\gamma_{-1}(hg)\delta(h)}{h}\\
&= \sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{l\le Y/g\\ l\in\mathcal{W}(X)}} \frac{\mu'(l)}{l^2} \left(\sum_{\substack{m\le Y/gl\\ m\in\mathcal{W}(X)}} \frac{\gamma_{-1}(glm)\delta(lm)}{m} \right)^2.
\end{aligned} \qquad (5{\cdot}39)$$

Manipulating the sums in this way allows us to avoid the rather technical and lengthy calculations involved in [**11**].

We wish to extend these sums over all $\mathcal{W}(X)$ and this requires some estimates. These will follow in a similar fashion to that found between (3·7) and (3·9). Throughout we use $\gamma_{-1}(m), \delta(m) \ll d(m)$ and $d(mn) \le d(m)d(n)$. First, let b be positive and small, then

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$$\frac{1}{d(g)d(l)^2}\sum_{\substack{m>Y/lg\\ m\in\mathcal{W}(X)}}\frac{\gamma_{-1}(glm)\delta(lm)}{m} \ll \sum_{\substack{m>Y/lg\\ m\in\mathcal{W}(X)}}\frac{d(m)^2}{m} \ll \Big(\frac{Y}{lg}\Big)^{-b}\sum_{m\in\mathcal{W}(X)}\frac{d(m)^2}{m^{1-b}}$$
$$\ll \Big(\frac{Y}{lg}\Big)^{-b}\prod_{p\le X}\left(1-p^{-1+b}\right)^{-4}\left(1-p^{-2(1-b)}\right) \tag{5·40}$$
$$\ll \Big(\frac{Y}{lg}\Big)^{-b} e^{8X^b/\log X} \ll (lg)^b T^{-b\theta/2}.$$

Second,

$$\sum_{m\in\mathcal{W}(X)}\frac{\gamma_{-1}(glm)\delta(lm)}{m} \ll d(g)d(l)^2\sum_{m\in\mathcal{W}(X)}\frac{d(m)^2}{m} \ll d(g)d(l)^2\log^4 X. \tag{5·41}$$

From these it follows that the square of the sum over m in (5·39) is

$$\Big(\sum_{m\in\mathcal{W}(X)}\frac{\gamma_{-1}(glm)\delta(lm)}{m}\Big)^2 + O\left(d(g)^2d(l)^4(lg)^{2b}T^{-b\theta/4}\right). \tag{5·42}$$

Similarly we find

$$\sum_{l\in\mathcal{W}(X)}\frac{\mu'(l)d(l)^4}{l^{2-2b}} \ll 1, \qquad \sum_{\substack{l>Y/g\\ l\in\mathcal{W}(X)}}\frac{\mu'(l)d(l)^4}{l^{2-2b}} \ll g^cT^{-c\theta}, \tag{5·43}$$

for some small $c>0$, and

$$\sum_{g\in\mathcal{W}(X)}\frac{d(g)^2}{g^{1-2b-c}} \ll T^{\epsilon}, \qquad \sum_{\substack{g>Y\\ g\in\mathcal{W}(X)}}\frac{d(g)^2}{g^{1-2b-c}} \ll T^{-d\theta} \tag{5·44}$$

for some small $d>0$. The above estimates give

$$S = \Big(\sum_{g\in\mathcal{W}(X)} - \sum_{\substack{g>Y\\ g\in\mathcal{W}(X)}}\Big)\frac{1}{g}\Big(\sum_{l\in\mathcal{W}(X)} - \sum_{\substack{l>Y/g\\ l\in\mathcal{W}(X)}}\Big)\frac{\mu'(l)}{l^2}$$
$$\times\left[\Big(\sum_{m\in\mathcal{W}(X)}\frac{\gamma_{-1}(glm)\delta(lm)}{m}\Big)^2 + O\left(d(g)^2d(l)^4(lg)^{2b}T^{-b\theta/4}\right)\right] \tag{5·45}$$
$$= (1+o(1))\sum_{g\in\mathcal{W}(X)}\frac{1}{g}\sum_{l\in\mathcal{W}(X)}\frac{\mu'(l)}{l^2}\Big(\sum_{m\in\mathcal{W}(X)}\frac{\gamma_{-1}(glm)\delta(lm)}{m}\Big)^2.$$

Now, since all coefficients in S are multiplicative we may expand the sum into an Euler product:

$$S = (1+o(1))\prod_{\substack{p\le X\\ p\ \text{split}}}G(p)\prod_{\substack{p\le\sqrt{X}\\ p\ \text{inert}}}G(p^2)\prod_{\substack{p\le X\\ p\ \text{ramified}}}G(p) \tag{5·46}$$

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with

$$G(p)=\sum_{i,j,u,v\geq 0}\frac{\mu'(p^j)\gamma_{-1}(p^{i+j+u})\gamma_{-1}(p^{i+j+v})\delta(p^{j+u})\delta(p^{j+v})}{p^{i+2j+u+v}}. \tag{5·47}$$

Performing the various sums whilst using the support conditions of γ_{-1} and μ' we see

$$G(p)=1+\frac{2\gamma_{-1}(p)\delta(p)+\gamma_{-1}(p)^2}{p}+\frac{2\gamma_{-1}(p^2)\delta(p^2)+\gamma_{-1}(p^2)^2+2\gamma_{-1}(p)\gamma_{-1}(p^2)\delta(p)}{p^2}. \tag{5·48}$$

Recall that for a split prime p we have $\delta(p^r)=1+r(p-1)/(p+1)$ and hence $\delta(p)=2p/(p+1)$ and $\delta(p^2)=2\delta(p)-1$. We also have $\gamma_{-1}(p)=-2$ for all $p\leq X$, $\gamma_{-1}(p^2)=1$ for $p\leq\sqrt{X}$ and $\gamma_{-1}(p^2)=2$ for $\sqrt{X}<p\leq X$. A straightforward calculation now gives

$$\begin{aligned}\prod_{\substack{p\leq X\\ p\text{ split}}}G(p)&=\prod_{\substack{p\leq\sqrt{X}\\ p\text{ split}}}\left(\frac{(1-1/p)^4}{1-1/p^2}\right)\prod_{\substack{\sqrt{X}<p\leq X\\ p\text{ split}}}\left(\frac{(1-1/p)^4}{1-1/p^2}+O\left(\frac{1}{p^2}\right)\right)\\ &=\prod_{\substack{p\leq X\\ p\text{ split}}}\left(\frac{(1-1/p)^4}{1-1/p^2}\right)\prod_{\substack{\sqrt{X}<p\leq X\\ p\text{ split}}}\left(1+O\left(\frac{1}{p^2}\right)\right)\\ &=(1+o(1))\prod_{\substack{p\leq X\\ p\text{ split}}}\left(1-\frac{1}{p}\right)^4\prod_{p\text{ split}}\left(1-\frac{1}{p^2}\right)^{-1}.\end{aligned} \tag{5·49}$$

In evaluating the remaining products in (5·46) we note that γ_{-1} behaves the same on square inert primes as it does on ramified primes. The same goes for δ since the number 1 varies little. We describe the ramified case since the inert case is simply handled by replacing p with p^2.

For a ramified prime p we have $\delta(p)=\delta(p^2)=1$, $\gamma_{-1}(p)=-1$ for all $p\leq X$, $\gamma_{-1}(p^2)=0$ for $p\leq\sqrt{X}$ and $\gamma_{-1}(p^2)=1/2$ for $\sqrt{X}<p\leq X$. With this information we see

$$\begin{aligned}\prod_{\substack{p\leq X\\ p\text{ ramified}}}G(p)&=\prod_{\substack{p\leq\sqrt{X}\\ p\text{ ramified}}}\left(1-\frac{1}{p}\right)\prod_{\substack{\sqrt{X}<p\leq X\\ p\text{ ramified}}}\left(1-\frac{1}{p}+\frac{1}{4p^2}\right)\\ &=\prod_{\substack{p\leq X\\ p\text{ ramified}}}\left(1-\frac{1}{p}\right)\prod_{\substack{\sqrt{X}<p\leq X\\ p\text{ ramified}}}\left(1+O\left(\frac{1}{p^2}\right)\right)\\ &=(1+o(1))\prod_{\substack{p\leq X\\ p\text{ ramified}}}\left(1-\frac{1}{p}\right).\\ &=(1+o(1))\prod_{\substack{p\leq X\\ p\text{ ramified}}}\left(1-\frac{1}{p}\right)^2\prod_{p\text{ ramified}}\left(1+\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}.\end{aligned} \tag{5·50}$$

In the last line here we have used the fact that a prime is ramified if and only if it divides $d_{\mathbb{K}}$ and hence $\sum_{X<p|d_{\mathbb{K}}}1/p=o(1)$. Similarly, for inert primes we find

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$$\prod_{\substack{p\le\sqrt{X}\\ p\text{ inert}}} G(p^2)=(1+o(1))\prod_{\substack{p\le\sqrt{X}\\ p\text{ inert}}}\left(1-\frac{1}{p^2}\right)$$
$$=(1+o(1))\prod_{\substack{p\le\sqrt{X}\\ p\text{ inert}}}\left(1-\frac{1}{p^2}\right)^2\prod_{p\text{ inert}}\left(1-\frac{1}{p^2}\right)^{-1}. \tag{5·51}$$

Collecting the infinite products in (5·49), (5·50) and (5·51) we acquire the factor

$$\frac{\pi^2}{6}\prod_{p|d_{\mathbb{K}}}\left(1+\frac{1}{p}\right). \tag{5·52}$$

The remaining terms are then given by

$$(1+o(1))\prod_{\mathfrak{N}(\mathfrak{p})\le X}\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^2=(1+o(1))(L(1,\chi)e^{\gamma}\log X)^{-2}. \tag{5·53}$$

5·3. *Estimating the lower order terms*

By virtue of the upper bounds (5·27), (5·29) and Proposition 2 we are only required to evaluate the sum of the 'big O' and Z' terms of formula (5·25). For the 'big O' term we have

$$T^{-\frac{1}{4}+\epsilon}\sum_{\substack{h,k\le T^{\theta}\\ h,k\in\mathcal{W}(X)}}\frac{\gamma_{-1}(h)\gamma_{-1}(k)(h,k)}{hk}\left(\frac{hk}{(h,k)^2}\right)^{7/8+\epsilon}$$
$$\ll T^{-\frac{1}{4}+\epsilon}\left(\sum_{n\le T^{\theta}}d(n)\right)^2\ll T^{2\theta-\frac{1}{4}+\epsilon} \tag{5·54}$$

and so taking $\theta\le 1/11-\varepsilon$ the error term is $o(1)$.

We now estimate the sums involving the Z' terms. By (5·28) and (5·30) we see that we must consider sums of the form

$$S':=\sum_{\substack{h,k\le Y\\ h,k\in\mathcal{W}(X)}}\frac{\gamma_{-1}(h)\gamma_{-1}(k)}{hk}(h,k)\mathbf{1}_{q|h_k}\chi(k_h)\delta'(h_k/q)\delta'(k_h)$$
$$=\sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}}\frac{1}{g}\sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}}\frac{\chi(k)\gamma_{-1}(kg)\delta'(k)}{k}\sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}}\mathbf{1}_{q|h}\frac{\gamma_{-1}(hg)\delta'(h/q)}{h}, \tag{5·55}$$

where $Y=T^{\theta}$. The innermost sum is given by

$$\sum_{\substack{h\le Y/qg\\ qh\in\mathcal{W}(X)\\ (qh,k)=1}}\frac{\gamma_{-1}(qhg)\delta'(h)}{qh}\ll\sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}}\frac{\gamma_{-1}(hg)\delta'(h)}{h}, \tag{5·56}$$

where we have used $|\gamma_{-1}(qm)|\le\gamma_{-1}(m)$ which follows from (1·15) and the definition of γ_{-1}. We deduce that S' is $\ll$ a sum of the form (5·36) with δ replaced by δ'. Using the bound $\delta'(n)\le d(n^2)$ we may follow the analysis of Proposition 2 to see that

$$S'\ll(1+o(1))\prod_{\substack{p\le X\\ p\text{ split}}}G'(p)\prod_{\substack{p\le\sqrt{X}\\ p\text{ inert}}}G'(p^2)\prod_{\substack{p\le X\\ p\text{ ramified}}}G'(p), \tag{5·57}$$

where

$$G'(p) = 1 + \frac{2\gamma_{-1}(p)\delta'(p) + \gamma_{-1}(p)^2}{p} + \frac{2\gamma_{-1}(p^2)\delta'(p^2) + \gamma_{-1}(p^2)^2 + 2\gamma_{-1}(p)\gamma_{-1}(p^2)\delta'(p)}{p^2}. \tag{5·58}$$

For split and ramified primes we have $\delta'(p^r) = \delta(p^r)$ and so we only need evaluate G at the inert primes. For inert p we have $\delta(p^2) = 1 + 2(p+1)/(p-1) \leq 5$ and hence

$$G'(p^2) = 1 + O\left(\frac{1}{p^2}\right). \tag{5·59}$$

For the sake of argument we write

$$\prod_{\substack{p \leq \sqrt{X} \\ p \text{ inert}}} G'(p^2) = (1 + o(1)) \prod_{\substack{p \leq \sqrt{X} \\ p \text{ inert}}} \left(1 - \frac{1}{p^2}\right)^2 \tag{5·60}$$

and then combine this with the products over split and ramified primes given by (5·49) and (5·50). This gives

$$S' \ll (\log X)^{-2}. \tag{5·61}$$

6. *The moments recipe for non-primitive L-functions*

6·1. *Conjecture 3 via the recipe*

In this section we use the moments recipe given in [**6**] to reproduce Conjecture 3. The recipe in question is concerned with primitive L-functions, so cannot be applied directly to our situation without some slight modification. Our modifications are in-keeping with the reasoning of the original recipe, with which we assume some familiarity.

Consider the shifted product

$$Z(s, \boldsymbol{\alpha}, \boldsymbol{\beta}) = Z_\zeta(s, \boldsymbol{\alpha}) Z_L(s, \boldsymbol{\beta}), \tag{6·1}$$

where

$$Z_\zeta(s, \boldsymbol{\alpha}) = \zeta(s + \alpha_1) \cdots \zeta(s + \alpha_k)\zeta(1 - s - \alpha_{k+1}) \cdots \zeta(1 - s - \alpha_{2k}) \tag{6·2}$$

and

$$Z_L(s, \boldsymbol{\beta}) = L(s + \beta_1, \chi) \cdots L(s + \beta_k, \chi) L(1 - s - \beta_{k+1}, \overline{\chi}) \cdots L(1 - s - \beta_{2k}, \overline{\chi}). \tag{6·3}$$

We first substitute the respective approximate functional equations, which we write as

$$\zeta(s) = \sum_m \frac{1}{m^s} + \varkappa_\zeta(s) \sum_n \frac{1}{n^{1-s}}, \tag{6·4}$$

$$L(s, \chi) = \sum_m \frac{\chi(m)}{m^s} + \varkappa_L(s) \sum_n \frac{\overline{\chi}(n)}{n^{1-s}}. \tag{6·5}$$

We have (after an application of Stirling's formula)

$$\varkappa_\zeta(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-s} e^{it + i\pi/4} \left(1 + O\left(\frac{1}{t}\right)\right) \tag{6·6}$$

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and

$$\varkappa_L(s) = \frac{G(\chi)}{i^{\mathfrak{a}}\sqrt{q}} \left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-s} e^{it+i\pi/4}\left(1+O\left(\frac{1}{t}\right)\right), \tag{6·7}$$

where $G(\chi)$ is the Gauss sum of χ and $\mathfrak{a}$ is defined by the equation $\chi(-1) = (-1)^{\mathfrak{a}}$.

We now expand the resulting expression and throw away any oscillatory terms. This includes any terms involving t^{it}, but also any terms that may solely involve q^{it} since this still oscillates mildly. The effect is to retain only the terms with an equal number of $\varkappa_\zeta(s)$ and $\varkappa_\zeta(1-s)$ factors and an equal number of $\varkappa_L(s)$ and $\varkappa_L(1-s)$ factors. This gives a sum of $\binom{2k}{k}^2$ terms. We then retain only the diagonals in each term and extend them over all positive integers. We denote the resulting expression by $M(s, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and conjecture that

$$\int_{-\infty}^{\infty} Z\left(\tfrac{1}{2}+it, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) w(t)dt \sim \int_{-\infty}^{\infty} M\left(\tfrac{1}{2}+it, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) w(t)dt \tag{6·8}$$

for any ‘reasonable’ w.

Note that before retaining the diagonals, we could essentially apply the first two steps of the recipe to the product of zeta functions and then to the product of L-functions independently, since the result is the same. Consequently, the combinatorics are not much more complicated than that of the original recipe when applied to primitive L-functions. Indeed, as in [6], let $\Xi \subset S_{2k}$ denote the set of permutations τ for which

$$\tau(1) < \ldots < \tau(k), \quad \tau(k+1) < \ldots < \tau(2k). \tag{6·9}$$

Then we see that

$$\begin{aligned} M(s, \boldsymbol{\alpha}, \boldsymbol{\beta}) = &\left(\frac{t}{2\pi}\right)^{(-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})/2} \left(\frac{qt}{2\pi}\right)^{(-\beta_1-\cdots-\beta_k+\beta_{k+1}+\cdots+\beta_{2k})/2}, \\ &\times \sum_{\tau,\tau'\in\Xi} W(s, \boldsymbol{\alpha}, \boldsymbol{\beta}, \tau, \tau') \end{aligned} \tag{6·10}$$

where

$$\begin{aligned} W(s, \boldsymbol{\alpha}, \boldsymbol{\beta}, \tau, \tau') = &\left(\frac{t}{2\pi}\right)^{(\alpha_{\tau(1)}+\cdots+\alpha_{\tau(k)}-\alpha_{\tau(k+1)}-\cdots-\alpha_{\tau(2k)})/2} \\ &\times \left(\frac{qt}{2\pi}\right)^{(\beta_{\tau'(1)}+\cdots+\beta_{\tau'(k)}-\beta_{\tau'(k+1)}-\cdots-\beta_{\tau'(2k)})/2} \\ &\times S(\sigma; \alpha_{\tau(1)}, \ldots, \alpha_{\tau(2k)}; \beta_{\tau'(1)}, \ldots, \beta_{\tau'(2k)}) \end{aligned} \tag{6·11}$$

with

$$\begin{aligned} &S(\sigma; \alpha_1, \ldots, \alpha_{2k}; \beta_1, \ldots, \beta_{2k}) \\ &= \sum_{\substack{m_1\cdots m_k m_1'\cdots m_k' = \\ n_1\cdots n_k n_1'\cdots n_k'}} \chi(m_1')\cdots\chi(m_k')\overline{\chi(n_1')\cdots\chi(n_k')}\Big[m_1^{\sigma+\alpha_1}\cdots m_k^{\sigma+\alpha_k} \\ &\times m_1'^{\,\sigma+\beta_1}\cdots m_k'^{\,\sigma+\beta_k} n_1^{1-\sigma-\alpha_{k+1}}\cdots n_k^{1-\sigma-\alpha_{2k}} n_1'^{\,1-\sigma-\beta_{k+1}}\cdots n_k'^{\,1-\sigma-\beta_{2k}}\Big]^{-1}. \end{aligned} \tag{6·12}$$

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Since the condition $m_1 \cdots m_k m_1' \cdots m_k' = n_1 \cdots n_k n_1' \cdots n_k'$ is multiplicative we have

$$
\begin{aligned}
&S(\sigma; \alpha_1, \ldots, \alpha_{2k}; \beta_1, \ldots, \beta_{2k}) \\
&= \prod_p \sum_{\substack{\sum_{j=1}^k e_j + e_j' = \\ \sum_{j=1}^k e_{j+k} + e_{j+k}'}} \chi(p^{e_1'}) \cdots \chi(p^{e_k'}) \overline{\chi(p^{e_{k+1}'}) \cdots \chi(p^{e_{2k}'})} \Big[p^{e_1(\sigma+\alpha_1)+\cdots+e_k(\sigma+\alpha_k)} \\
&\quad \times p^{e_1'(\sigma+\beta_1)+\cdots+e_k'(\sigma+\beta_k)+\cdots+e_{k+1}(1-\sigma-\alpha_{k+1})+\cdots+e_{2k}(1-\sigma-\alpha_{2k})} \\
&\quad \times p^{e_{k+1}'(1-\sigma-\beta_{k+1})+\cdots+e_{2k}'(1-\sigma-\beta_{2k})} \Big]^{-1} \\
&= A_k(\sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{i,j=1}^k \zeta(1+\alpha_i - \alpha_{k+j}) L(1+\beta_i - \beta_{k+j}, |\chi|^2) \\
&\quad \times L(1+\beta_i - \alpha_{j+k}, \chi) L(1+\alpha_i - \beta_{j+k}, \overline{\chi}) \\
&= A_k(\sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{i,j=1}^k \zeta(1+\alpha_i - \alpha_{k+j}) \zeta(1+\beta_i - \beta_{k+j}) \\
&\quad \times L(1+\beta_i - \alpha_{j+k}, \chi) L(1+\alpha_i - \beta_{j+k}, \overline{\chi}) \Big(\prod_{p|q} \big(1 - p^{-1-\beta_i+\beta_{k+j}}\big) \Big),
\end{aligned}
\tag{6·13}
$$

where A_k is an Euler product that is absolutely convergent for $\sigma > 1/4$.

Now, denote the holomorphic part of $S(1/2, \boldsymbol{\alpha}, \boldsymbol{\beta})$ by

$$
\begin{aligned}
A_k'(1/2, \boldsymbol{\alpha}, \boldsymbol{\beta}) = &A_k(1/2, \boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{i,j=1}^k L(1+\beta_i - \alpha_{j+k}, \chi) L(1+\alpha_i - \beta_{j+k}, \overline{\chi}) \\
&\times \Big(\prod_{p|q} \big(1 - p^{-1-\beta_i+\beta_{k+j}}\big) \Big).
\end{aligned}
\tag{6·14}
$$

As in [6], we may express the combinatorial sum over Ξ in equation (6·10) as a multiple contour integral. Applying [6, Lemma 2·5·1] twice to (6·10) we see that $M(1/2+it, \mathbf{0}, \mathbf{0})$ is given by

$$
\begin{aligned}
&\left(\frac{(-1)^k}{k!^2 (2\pi i)^{2k}} \right)^2 \oint \cdots \oint A_k'(1/2, u_1, \ldots, u_{2k}, v_1, \ldots, v_{2k}) \\
&\times \prod_{i,j=1}^k \zeta(1+u_i - u_{k+j}) \zeta(1+v_i - v_{k+j}) \frac{\Delta^2(u_1, \ldots, u_{2k})}{\prod_{j=1}^{2k} u_j^{2k}} \frac{\Delta^2(v_1, \ldots, v_{2k})}{\prod_{j=1}^{2k} v_j^{2k}} \\
&\times e^{\frac{1}{2}\mathcal{L} \sum_{j=1}^k u_j - u_{k+j}} e^{\frac{1}{2}\mathcal{L}_q \sum_{j=1}^k v_j - v_{k+j}} du_1 \cdots du_{2k} dv_1 \cdots dv_{2k},
\end{aligned}
\tag{6·15}
$$

where $\mathcal{L} = \log(t/2\pi)$, $\mathcal{L}_q = \log(qt/2\pi)$, and Δ is the Vandermonde determinant. Since $A_k'(1/2, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is holomorphic in the neighbourhood of $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mathbf{0}, \mathbf{0})$ after a change of variables this becomes

$$\left(\frac{(-1)^k}{k!^2(2\pi i)^{2k}}\right)^2 \oint\cdots\oint A_k'\left(1/2, \frac{u_1}{\mathcal{L}/2},\ldots,\frac{u_{2k}}{\mathcal{L}/2},\frac{v_1}{\mathcal{L}_q/2},\ldots,\frac{v_{2k}}{\mathcal{L}_q/2}\right)$$
$$\times \prod_{i,j=1}^{k}\zeta\left(1+\frac{u_i-u_{k+j}}{\mathcal{L}/2}\right)\zeta\left(1+\frac{v_i-v_{k+j}}{\mathcal{L}_q/2}\right)\frac{\Delta^2(u_1,\ldots,u_{2k})}{\prod_{j=1}^{2k}u_j^{2k}}$$
$$\times \frac{\Delta^2(v_1,\ldots,v_{2k})}{\prod_{j=1}^{2k}v_j^{2k}} e^{\sum_{j=1}^k u_j-u_{k+j}} e^{\sum_{j=1}^k v_j-v_{k+j}} du_1\cdots du_{2k}dv_1\cdots dv_{2k}.$$

$$= A_k'(1/2,\mathbf{0},\mathbf{0})\,\mathcal{L}^{k^2}\mathcal{L}_q^{k^2}\left(1+O\left(\frac{1}{\mathcal{L}}\right)\right)\left(\frac{(-1)^k}{2^{k^2}k!^2(2\pi i)^{2k}}\right)^2\oint\cdots\oint \tag{6·16}$$
$$\times \frac{\Delta^2(u_1,\ldots,u_{2k})}{\prod_{i,j=1}^k(u_i-u_{k+j})\prod_{j=1}^{2k}u_j^{2k}}\frac{\Delta^2(v_1,\ldots,v_{2k})}{\prod_{i,j=1}^k(v_i-v_{k+j})\prod_{j=1}^{2k}v_j^{2k}}$$
$$\times e^{\sum_{j=1}^k u_j-u_{k+j}} e^{\sum_{j=1}^k v_j-v_{k+j}} du_1\cdots du_{2k}dv_1\cdots dv_{2k}.$$

$$\sim A_k'(1/2,\mathbf{0},\mathbf{0})\,\mathcal{L}^{k^2}\mathcal{L}_q^{k^2}\left(\frac{(-1)^k}{2^{k^2}k!^2(2\pi i)^{2k}}\oint\cdots\oint\right.$$
$$\left.\times\frac{\Delta^2(u_1,\ldots,u_{2k})}{\prod_{i,j=1}^k(u_i-u_{k+j})\prod_{j=1}^{2k}u_j^{2k}} e^{\sum_{j=1}^k u_j-u_{k+j}} du_1\cdots du_{2k}\right)^2.$$

It is shown in [6, section 2·7] that the quantity in parentheses is given by $G(k+1)^2/G(2k+1)$ and so it only remains to show that $A_k'(1/2,\mathbf{0},\mathbf{0}) = a_{\mathbb{K}}(k)L(1,\chi)^{2k^2}$ where $a_{\mathbb{K}}(k)$ is given by (1·12). Since,

$$A_k'(1/2,\mathbf{0},\mathbf{0}) = A_k(1/2,\mathbf{0},\mathbf{0})\,L(1,\chi)^{2k^2}\prod_{p|q}(1-p^{-1})^{k^2} \tag{6·17}$$

we only need show that $a_{\mathbb{K}}(k) = A_k(1/2,\mathbf{0},\mathbf{0})\prod_{p|q}(1-p^{-1})^{k^2}$. In the case of quadratic extensions, $a_{\mathbb{K}}(k)$ is the product of the following three factors

$$\prod_{p\text{ split}}\left(1-\frac{1}{p}\right)^{4k^2}\sum_{m=0}^{\infty}\frac{d_{2k}(p^m)^2}{p^m}, \tag{6·18}$$

$$\prod_{p\text{ inert}}\left(1-\frac{1}{p^2}\right)^{2k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^{2m}}, \tag{6·19}$$

$$\prod_{p\text{ ramified}}\left(1-\frac{1}{p}\right)^{2k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m} \tag{6·20}$$

and from (6·13) we see that

$$A_k(1/2,\mathbf{0},\mathbf{0})\prod_{p|q}(1-p^{-1})^{k^2} = \prod_p\left(1-\frac{1}{p}\right)^{2k^2}\left(1-\frac{\chi(p)}{p}\right)^{k^2}\left(1-\frac{\overline{\chi}(p)}{p}\right)^{k^2}$$
$$\times \sum_{\substack{\sum_{j=1}^k e_j+e_j'=\\ \sum_{j=1}^k e_{j+k}+e_{j+k}'}} \chi(p^{e_1'})\cdots\chi(p^{e_k'})\overline{\chi(p^{e_{k+1}'})\cdots\chi(p^{e_{2k}'})}\,p^{-\frac{1}{2}\sum_{j=1}^{2k}(e_j+e_j')}. \tag{6·21}$$

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Since $\chi(p) = 1$ for split primes, the relevant factor in (6·21) is given by

$$\prod_{p \text{ split}} \left(1 - \frac{1}{p}\right)^{4k^2} \sum_{\substack{\sum_{j=1}^{k} e_j + e'_j = \\ \sum_{j=1}^{k} e_{j+k} + e'_{j+k}}} p^{-\frac{1}{2}\sum_{j=1}^{2k}(e_j + e'_j)}. \tag{6·22}$$

Clearly, for each $m \geq 0$ there are $d_{2k}(p^m)$ choices of e_j, e'_j such that $\sum_1^k e_j + e'_j = m$. For each such choice there are another $d_{2k}(p^m)$ choices of e_{j+k}, e'_{j+k} such that $\sum_1^k e_{j+k} + e'_{j+k} = m$. Hence the total number of choices is $d_{2k}(p^m)^2$ and so the above product is seen to equal (6·18).

For inert primes we have $\chi(p) = -1$ and so the relevant factor is

$$\prod_{p \text{ inert}} \left(1 - \frac{1}{p^2}\right)^{2k^2} \sum_{\substack{\sum_{j=1}^{k} e_j + e'_j = \\ \sum_{j=1}^{k} e_{j+k} + e'_{j+k}}} (-1)^{\sum_{j=1}^{k}(e'_j + e'_{j+k})} p^{-\frac{1}{2}\sum_{j=1}^{2k}(e_j + e'_j)}. \tag{6·23}$$

In this case the quickest argument is analytic. Applying the orthogonality of exponential characters to detect the condition $\sum_1^k e_j + e'_j = \sum_1^k e_{j+k} + e'_{j+k}$, we see that, after a short manipulation, the inner sum is given by

$$\int_0^1 \left(1 - \frac{e^{4\pi i\theta}}{p}\right)^{-k} \left(1 - \frac{e^{-4\pi i\theta}}{p}\right)^{-k} d\theta.$$

Expanding the integrand as two geometric series, pushing the integral through, and applying orthogonality shows the equivalence with (6·19).

Finally, for ramified primes, or equivalently the primes dividing q, we have $\chi(p) = 0$. Here, the relevant factor is given by

$$\prod_{p|q} \left(1 - \frac{1}{p}\right)^{2k^2} \sum_{\substack{\sum_{j=1}^{k} e_j = \\ \sum_{j=1}^{k} e_{j+k}}} p^{-\frac{1}{2}\sum_{j=1}^{2k} e_j} \tag{6·24}$$

which is easily seen to be equal to (6·20).

6·2. *Moments of general non-primitive L-functions*

A key point in both derivations of Conjecture 3 was that, aside from the arithmetic factor, the leading term in the moment of $\zeta(1/2+it)L(1/2+it, \chi)$ was given by the product of the leading terms of the moments of $\zeta(1/2+it)$ and $L(1/2+it, \chi)$. We believe this should be the case for general non-primitive L-functions too. Indeed, by applying our modified moments recipe to non-primitive L-functions this idea becomes more apparent.

The recipe for general non-primitive L-functions goes as follows. Suppose we have the product $L(s) = \prod_{j=1}^{m} L_j(s)^{e_j}$ where the $L_j(s)$ are distinct, primitive members of the Selberg class $\mathcal{S}$. Suppose for each $L_j(s)$ we have the functional equation

$$\xi_{L_j}(s) = \gamma_{L_j}(s) L_j(s) = \epsilon_j \overline{\xi}_{L_j}(1-s), \tag{6·25}$$

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where

$$\gamma_{L_j}(s) = Q_j^{s/2} \prod_{i=1}^{d_j} \Gamma(s/2 + \mu_{i,j}), \tag{6·26}$$

with the $\{\mu_{i,j}\}$ stable under complex conjugation, and where ϵ_j is the so-called root number which satisfies $|\epsilon_j| = 1$. We then have the approximate functional equations

$$L_j(s) = \sum_n \frac{\alpha_{L_j}(m)}{m^s} + \epsilon \varkappa_{L_j}(s) \sum_n \frac{\overline{\alpha_{L_j}(n)}}{n^s}, \tag{6·27}$$

where

$$\varkappa_{L_j}(s) = \frac{\overline{\gamma_{L_j}(1-s)}}{\gamma_{L_j}(s)} = Q_j^{1/2-s} \prod_{i=1}^{d_j} \frac{\Gamma((1-s)/2 + \overline{\mu_{i,j}})}{\Gamma(s/2 + \mu_{i,j})}. \tag{6·28}$$

Similarly to before, if we apply the original recipe we encounter terms of the form $(Q_j/Q_{j'})^{-it}$ which are oscillating. We can prevent the occurrence of these terms by applying the first step of the recipe to each $L_j(s)$ separately. We then continue as in the original recipe. It should be clear that when the resulting expression is written as a contour integral, the same manipulations used on (6·16) will allow for a factorisation of the main term.

On a side note, the term $(Q_j/Q_{j'})^{-it}$ is oscillating only if we assume that the conductors Q_j and $Q_{j'}$ are distinct when $j \neq j'$. Although it seems likely that distinct L-functions of the above form should have distinct conductors, the author is not aware of such a result (Note that if we generalised the above form by replacing the gamma functions with $\Gamma(w_{i,j}s + \mu_{i,j})$ then one can apply the duplication formula to show that $Q_j, w_{i,j}, \mu_{i,j}$ are not uniquely defined for a given $L_j(s)$).

In terms of the random matrix theory, let us assume that we have a hybrid product for $L(s)$. Since it is believed that when the $L_j(s)$ are distinct their zeros are uncorrelated [**15**], their associated matrices should act independently. Hence, when the moment of the product over zeros is considered as an expectation, it will factorise.

As we have already seen, this phenomenon occurs when considering $\zeta(s)L(s, \chi)$. Let us restate the conjecture in the more descriptive form

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \left|L\left(\tfrac{1}{2}+it, \chi\right)\right|^{2k} dt \sim c(k)\frac{G(k+1)^2}{G(2k+1)} \log^{k^2} T \cdot \frac{G(k+1)^2}{G(2k+1)} \log^{k^2} qT, \tag{6·29}$$

with

$$c(k) = \prod_p \left(1 - \frac{1}{p}\right)^{2k^2} \sum_{m \geq 0} \frac{|F_{\chi,k}(p^m)|^2}{p^m}, \tag{6·30}$$

$$F_{\chi,k}(n) = \sum_{n_1 n_2 = n} d_k(n_1) d_k(n_2) \chi(n_2). \tag{6·31}$$

The coefficients $F_{\chi,k}(n)$ are, of course, the Dirichlet coefficients of $\zeta(s)^k L(s, \chi)^k$.

As another example, we state a result from an unpublished note of the author and Caroline Turnage–Butterbaugh. Here it is established, by an application of Theorem 4, that

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right) L\left(\tfrac{1}{2}+it,\chi\right)\sum_{n\le T^\theta}\frac{1}{n^{1/2+it}}\right|^2 dt \sim \gamma(1)\log^4 T\cdot\log qT\left(\frac{4\theta^3-3\theta^4}{12}\right) \quad (6{\cdot}32)$$

where

$$\gamma(1)=\prod_p\left(1-\frac{1}{p}\right)^5\sum_{m\ge 0}\frac{|H_\chi(p^m)|^2}{p^m},\qquad H_\chi(n)=\sum_{n_1n_2=n} d(n_1)\chi(n_2) \quad (6{\cdot}33)$$

and $\theta<1/11-\epsilon$. It is expected that Theorem 4 remains valid for $\theta=1$, in which case the Dirichlet polynomial is a good approximation to $\zeta(1/2+it)$ and the above relation reads as

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4\left|L\left(\tfrac{1}{2}+it,\chi\right)\right|^2 dt \sim \frac{\gamma(1)}{12}\log^4 T\cdot\log qT = b(1)\cdot\frac{G(3)^2}{G(5)}\log^4 T\cdot\frac{G(2)^2}{G(3)}\log qT. \quad (6{\cdot}34)$$

In terms of the T behaviour, this can be thought of as the product of the fourth moment of zeta times the second moment of L. Again, this is consistent with our random matrix theory/moments recipe reasoning. Guided by these examples we are led to Conjecture 4 which, after ignoring the conductors, we restate as

$$\frac{1}{T}\int_0^T \left|L\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim \frac{b_L(k)g_L(k)}{\Gamma(n_Lk^2+1)}\log^{n_Lk^2} T, \quad (6{\cdot}35)$$

where $n_L=\sum_{j=1}^m e_j^2$,

$$g_L(k)=\Gamma(n_Lk^2+1)\prod_{j=1}^m\frac{G^2(e_jk+1)}{G(2e_jk+1)}d_j^{(e_jk)^2}, \quad (6{\cdot}36)$$

and

$$b_L(k)=\prod_p\left(1-\frac{1}{p}\right)^{n_Lk^2}\sum_{n=0}^\infty\frac{|\alpha_{L,k}(p^n)|^2}{p^n}. \quad (6{\cdot}37)$$

Let us cast this in the light of some of the Selberg's conjectures. First, we note that the integer n_L is the same integer appearing in Selberg's 'regularity of distribution' conjecture:

$$\sum_{p\le x}\frac{|\alpha_L(p)|^2}{p}=n_L\log\log x+O(1). \quad (6{\cdot}38)$$

This is not so surprising since one expects the mean square of $L(1/2+it)$ to be asymptotic to a multiple of the sum $\sum_{n\le T}|\alpha_L(n)|^2n^{-1}$. The implication of (6·38) is that this sum is in fact $\sim(b_L(1)/n_L!)\log^{n_L} T$.

For general k, we outline a verification of this last assertion. We assume the following two conjectures of Selberg [26]: For primitive $F\in\mathcal{S}$ we have

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$$\sum_{p\le x}\frac{|\alpha_F(p)|^2}{p}=\log\log x+O(1), \tag{6·39}$$

and for two distinct and primitive $F, G\in\mathcal{S}$ we have

$$\sum_{p\le x}\frac{\alpha_F(p)\overline{\alpha_G(p)}}{p}=O(1). \tag{6·40}$$

We also require that the functions

$$M_j(s)=\sum_{n=1}^{\infty}\frac{|\alpha_{L_j}(n)|^2}{n^s} \tag{6·41}$$

behave 'reasonably', in particular, that they posses an analytic continuation.

Now, given the factorisation $L(s)=\prod_{j=1}^m L_j(s)^{e_j}$ into primitive functions we have

$$\begin{aligned}\sum_{p\le x}\frac{|\alpha_{L,k}(p)|^2}{p}&=\sum_{p\le x}k^2\left(\sum_{j=1}^m e_j^2|\alpha_{L_j}(p)|^2+\sum_{i\ne j}e_ie_j\alpha_{L_i}(p)\overline{\alpha_{L_j}(p)}\right)p^{-1}\\&=n_Lk^2\log\log x+O(1).\end{aligned} \tag{6·42}$$

If $M(s)=\sum|\alpha_{L,k}(n)|^2n^{-s}$, then the above equation implies a factorisation of the form

$$M(s)=U_k(s)\prod_{j=1}^m M_j(s)^{(e_jk)^2}, \tag{6·43}$$

where $U_k(s)$ is some Euler product that is absolutely convergent for $\sigma>1/2$. Therefore, we may analytically continue $M(s)$ to $\sigma>1/2$. Also, by applying partial summation to (6·42) we see

$$\sum_p\frac{|\alpha_{L,k}(p)|^2}{p^{s+1}}=n_Lk^2\int_2^\infty\frac{dx}{x^{s+1}\log x}+\cdots=-n_Lk^2\log s+\cdots, \tag{6·44}$$

for small $\sigma>0$. If we write

$$\begin{aligned}M(s+1)&=\prod_p\left(1+\frac{|\alpha_{L,k}(p)|^2}{p^{s+1}}+\frac{|\alpha_{L,k}(p^2)|^2}{p^{2(s+1)}}+\cdots\right)\\&=\prod_p\left(\exp\left(\frac{|\alpha_{L,k}(p)|^2}{p^{s+1}}\right)+E_k(p,s)\right)\\&=\exp\left(\sum_p\frac{|\alpha_{L,k}(p)|^2}{p^{s+1}}\right)\prod_p(1+F_k(p,s)),\end{aligned} \tag{6·45}$$

where $E_k(p,s)$ and $F_k(p,s)$ are both $\ll p^{-2(\sigma+1)+\epsilon}$, we see that $M(s+1)$ has a pole of order n_Lk^2 at $s=0$. It is shown in [8] that on the assumption of Selberg's conjectures, if $F\in\mathcal{S}$ has a pole of order m at $s=1$ then $\zeta(s)^m$ divides $F(s)$. Consequently, the residue of $M(s+1)$ at $s=0$ is given by $b_L(k)$. The usual argument involving Perron's formula now gives

$$\sum_{n\le T}\frac{|\alpha_{L,k}(n)|^2}{n}\sim\frac{b_L(k)}{(n_Lk^2)!}\log^{n_Lk^2}T. \tag{6·46}$$

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