J. Plasma Phys. (2020), *vol.* 86, 905860521 © The Author(s), 2020.
Published by Cambridge University Press
doi:10.1017/S0022377820001154

Thermodynamic processes in dusty plasma

K. Avinash †

Sikkim University, 6th Mile, Samdur, P.O. Tadong-737102, Gangtok, Sikkim, India

(Received 23 March 2020; revised 1 September 2020; accepted 2 September 2020)

Here, we propose a thermodynamic model for dusty plasma, where the dust is confined in a small volume within a large plasma background by external fields. In this model, the parameters of dust, e.g. Helmholtz energy, pressure, entropy and enthalpy, etc. can be calculated for given dust density and temperature. The model is solved analytically in the mean field (gaseous) limit and various processes associated with the gaseous phase of dust, e.g. adiabatic/isothermal/constant internal energy expansion/compression, specific heat, free expansion within the plasma background, and the dispersion of novel acoustic waves are studied. Some predictions of the model, e.g. electrostatic pressure of the dust and the isothermal equation of state, have been earlier verified in experiments and numerical simulations. The model is compared with an earlier thermodynamic model of dusty plasma proposed by Hamaguchi and Farouki.

Key words: dusty plasmas, plasma simulation, strongly coupled plasmas

---

1. Introduction

One of the features of dusty plasma, which distinguishes it from the ordinary three component plasma, is that due to the heavy mass of the dust, there is a vast separation of time scales in the dynamics of the dust and that of the background plasma. This vast separation of time scales has an interesting consequence that the dust component can be confined in a state of thermodynamic equilibrium in a small volume within a sufficiently large plasma background, by external fields. This remarkable feature is shared by only one other system, i.e. non-neutral plasma which is confined in the state of thermodynamic equilibrium by external magnetic fields (Davidson 1990). In this paper, we construct a general thermodynamical model of dusty plasma where the dust is confined in a small volume within a sufficiently large plasma background, by external fields. The model is solved analytically in the mean field limit and various processes for the gaseous phase of dust, e.g. isothermal/adiabatic/constant internal energy expansion/compression, free expansion of dust, specific heat and dispersion of acoustic waves, etc. are studied.

While defining thermodynamic processes in dusty plasma, the role of plasma background must be carefully examined. The reason for this is that in these plasmas, the dust component is coupled with the background electrons and ions through quasi-neutral electric fields. Thus, when the dust component is compressed/expands or moves, for example in dust acoustic wave (DAW) or shocks, then the background plasma is also perturbed. The extra heat energy generated due to such perturbations of the background

†Email address for correspondence: khareavinash82@gmail.com

plasma must be properly accounted for while defining thermodynamic processes in dusty plasma. A somewhat truncated version of this model was described earlier to show the interconversion of plasma heat into work and *vice versa* via cold dust (Avinash 2010*a*,*b*; Avinash & Kaw 2014). However, since the dust temperature T_d was taken to be zero in the model, the thermodynamic processes involving the dust internal energy, pressure, enthalpy and other related thermodynamic parameters could not be described properly. In this paper, the thermodynamic model with finite dust temperature is discussed.

Hamaguchi & Farouki (1994) and Farouki & Hamaguchi (1994) in their seminal works, have also proposed a thermodynamic model of dusty plasma to calculate dust correlation effects and the solid–liquid melting boundary. In this model, the dust and the plasma background occupy the same volume. The negative charge of the dust is confined or neutralized by the cohesive plasma background. This, however, is not realistic. In dusty plasma experiments the dust is confined in a small volume within the plasma with the help of external electrostatic (ES) fields (Barkan & Merlino 1995; Trottenberg, Block & Piel 2006; Pilch *et al.* 2007; Thomas 2010). Our model takes into account this confinement of dust within the background plasma. There are some other important differences in the results of our model and those of the Hamaguchi–Farouki (HF) model which will be discussed later in the paper.

2. General formulation of the model

Our model consists of an ensemble of N_d identical point dust particles carrying a constant negative charge q_d, having pressure and temperature P_d, T_d, respectively. The dust cloud is assumed to be confined in a small volume V_d by external fields $P_{\text{ext}}(P_d = P_{\text{ext}})$, within a weakly coupled, statistically averaged, plasma background, having N_e and N_i numbers of electrons and ions, in volume V where $V \gg V_d$. Typically, in experiments $V_d/V \approx 10^{-3}-10^{-5}$ (Trottenberg *et al.* 2006; Fisher *et al.* 2013) which justifies the assumption of small V_d in the present model.

The electron and ion temperatures are denoted by T_e and T_i, respectively. In thermal equilibrium the electron and the ion densities are given by Boltzmann's relations $n_i = n_0\exp(-q\varphi/T_i)$, $n_e = n_0\exp(q\varphi/T_e)$ where φ is the electrostatic potential within the dust cloud. Sufficiently away from the cloud $\varphi \rightarrow 0$ and $n_i = n_e = n_0$. It should be noted that this assumption is asymptotically valid in the limit $V_d/V \rightarrow 0$, i.e. in the limit of dust cloud embedded in an asymptotically infinite plasma background. Another point to be noted is that the volume V and V_d can be varied independently in the present model, which is consistent with dusty plasma experiments (Barkan & Merlino 1995; Trottenberg *et al.* 2006; Pilch *et al.* 2007; Thomas 2010). This is different from the HF model where the dust density is assumed to be proportional to the average plasma density (Farouki & Hamaguchi 1994). In the last section, and further in appendix B, we will discuss these issues and the relationship of the present model and the HF model in detail.

The dust and the plasma are usually immersed in the background gas filling the plasma chamber. The collisions of the electrons, the ions and the dust with neutral atoms of the back ground gas help to regulate the temperatures of the dust and the plasma. Thus, the neutral background acts as the heat bath (Quinn & Goree 2000*a*). In dusty plasmas, the dust is heated by a number of processes, e.g. Brownian motion due to the neutral gas, fluctuations due to the electric field or the dust charge and it is cooled mainly due to the drag by neutrals (Quinn & Goree 2000*a*). The typical dust neutral collision frequency ν_{dn} ($\propto P_g$ neutral gas pressure) in dusty plasma experiments ranges from a few Hz at low gas pressures to $\geq$100 Hz at high pressures (Quinn & Goree 2000*b*). Thus, if $\tau_p > \tau_{\text{dn}}$ (τ_p is the time scale of the thermodynamic process and $\tau_{\text{dn}} = 1/\nu_{\text{dn}}$ is the time scale of the dust neutral collision), which would typically be true in experiments with high neutral

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

pressure (100–200 mTorr), then in such cases, the dust is in good thermal contact with the neutral bath. In this regime, the isothermal approximation for dust with $T_d \approx T_n$ (T_n is the temperature of neutrals) is appropriate (Thomas 2010). On the other hand, in experiments with low neutral pressures and relatively rare collisions with neutrals ($\leq$10 mTorr) τ_p typically is less than τ_{dn}. In such cases, the thermal contact of dust with the neutral bath is rather weak (Pilch *et al.* 2007). In this regime, the dust may be treated as isolated and an adiabatic approximation with variable temperature for the dust is appropriate. The thermal conductivity of electrons and ions in the background plasma is always very large. The ion heat is lost mainly through collisions with neutrals or due to the diffusive losses to the walls of the plasma vessel on time scale of approximately 1 to 10 µs. The electron heat is lost on an even shorter time scale. Thus, for the plasma background, on the time scale τ_p, we assume an isothermal approximation with fixed electron and ion temperature. Hence, apart from the neutral background, the dust is also coupled to the plasma background, with which it exchanges heat via electric fields.

We start our calculations by expressing the energy conservation for quasi-static ($\tau_p > \tau_{\mathrm{relax}}$ where τ_{relax} is relaxation time scale) work done by P_{ext} on V_d given by

$$\Delta Q_e + \Delta Q_i + \Delta Q_d = \Delta U + P_d \Delta V_d. \tag{2.1}$$

In this equation $\Delta Q_d = T_d \Delta S_d$, $\Delta Q_e = T_e \Delta S_e$, $\Delta Q_i = T_i \Delta S_i$ where ΔQ_d is the heat exchanged with dust heat bath, while ΔQ_i, ΔQ_e denote the heat exchanged with the plasma background and V is held constant. The entropies of the dust, electrons and ions are given by S_d, S_e and S_i, respectively, and U is the internal energy of the composite system of the dust and particles. The expressions for plasma entropies and U are given by (Hamaguchi & Farouki 1994; Avinash 2010*a*)

$$U = \frac{3}{2}(N_e T_e + N_i T_i + N_d T_d) + \frac{1}{2}\int \rho\varphi \,\mathrm{d}V - \frac{q_d^2}{8\pi\varepsilon_0}\sum_{j=1}^{N_d}\int \frac{\delta(\boldsymbol{r}-\boldsymbol{r_j})}{|\boldsymbol{r}-\boldsymbol{r_j}|}\,\mathrm{d}V, \tag{2.2}$$

$$\left.\begin{aligned} S_\alpha &= \tfrac{3}{2}N_\alpha - \textstyle\int_V n_\alpha[(\ln n_\alpha \Lambda_\alpha^3) - 1]\,\mathrm{d}V \\ \Lambda_\alpha &= \left(\frac{h^2}{2\pi m_\alpha T_\alpha}\right)^{1/2} \end{aligned}\right\}, \tag{2.3}$$

where α denotes electrons or ion, and ρ and φ are the local charge density and the electrostatic potential, while h denotes the Planck constant. In the isothermal equilibrium of the background and the approximation $q\phi/T_\alpha < 1$ (justified later), the electron and ion densities are given by the linearized Boltzmann relations $n_i = n_0(1 - q\varphi/T_i)$, $n_e = n_0(1 + q\varphi/T_e)$. The ES potential φ in the dust cloud can be obtained by solving the corresponding Poisson's equation given by

$$\varepsilon_0 \nabla^2 \varphi = \rho = -q_s \sum_{j=1}^{N_d} \delta(\boldsymbol{r}-\boldsymbol{r_j}) - \varepsilon_0 \varphi/\lambda_d^2, \tag{2.4}$$

where $1/\lambda_d^2 = (q^2 n_0/\varepsilon_0)(1/T_e + 1/T_i)$, and we have substituted linearized Boltzmann relations for electrons and ion densities. The solution of (2.4) is given by

$$\varphi = -\frac{q_d}{8\pi\varepsilon_0}\sum_j \frac{\exp(-|\boldsymbol{r}-\boldsymbol{r_j}|/\lambda_d)}{|\boldsymbol{r}-\boldsymbol{r_j}|}, \tag{2.5}$$

where the index $j = 1 \ldots N_d$. As shown in appendix A (Hamaguchi & Farouki 1994; Avinash 2010*a*) using (2.2)–(2.5) we derive the following expressions for the internal

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

energy and the heat exchanged with the isothermal plasma background:

$$\begin{aligned} U &= \frac{3}{2}(N_dT_d + N_eT_e + N_iT_i) - \frac{q_d^2 N_d\kappa_d}{8\pi\varepsilon_0} + \frac{q_d^2}{8\pi\varepsilon_0}\sum_i\sum_{j\neq i}\frac{\exp(-|\boldsymbol{r_i}-\boldsymbol{r_j}|/\lambda_d)}{|\boldsymbol{r_i}-\boldsymbol{r_j}|} \\ &\quad -\frac{q_d^2}{16\pi\varepsilon\lambda_d}\sum_i\sum_{j\neq i}\exp(-|\boldsymbol{r_i}-\boldsymbol{r_j}|/\lambda_d), \end{aligned} \tag{2.6}$$

$$\begin{aligned} T_iS_i + T_eS_e &= \frac{3}{2}(N_eT_e + N_iT_i) - \sum_\alpha T_\alpha N_\alpha[(\ln n_0\varLambda_\alpha^3) - 1] \\ &\quad -\frac{q_d^2}{16\pi\varepsilon\lambda_d}\sum_i\sum_{j\neq i}\exp(-|\boldsymbol{r_i}-\boldsymbol{r_j}|/\lambda_d), \end{aligned} \tag{2.7}$$

where $\kappa_d = 1/\lambda_d$ and indices $i, j = 1\ldots N_d$. The second term on the right-hand side of (2.7) is due to the uniform isothermal plasma background. The energy conservation equation for the dust component can be constructed from (2.1) as

$$\Delta Q_d = T_d\Delta S_d = \Delta U_d + P_d\Delta V_d, \quad U_d = U - T_eS_e - T_iS_i, \tag{2.8a,b}$$

where U_d is the effective internal energy of the dust component. The expression for U_d can be obtained by eliminating $-(T_iS_i + T_eS_e)$ and U from (2.6) and (2.7) to give

$$\begin{aligned} U_d &= \frac{3}{2}(N_dT_d) + \frac{q_d^2}{8\pi\varepsilon_0}\sum_i\sum_{j\neq i}\frac{\exp(-|\boldsymbol{r_i}-\boldsymbol{r_j}|/\lambda_d)}{|\boldsymbol{r_i}-\boldsymbol{r_j}|} \\ &\quad + \sum_\alpha T_\alpha N_\alpha[(\ln n_0\varLambda_\alpha^3) - 1] - \frac{q_d^2 N_d\kappa_d}{8\pi\varepsilon_0}. \end{aligned} \tag{2.9}$$

From (2.8a,b), the corresponding expressions for Helmholtz energy, pressure, entropy and enthalpy of the dust component can be calculated as

$$F_d = (U_d - T_dS_s), \quad P_d = -\left.\frac{\partial F}{\partial V_d}\right|_{T_d}, \quad S_d = -\left.\frac{\partial F_d}{\partial T_d}\right|_{V_d}, \quad H_d = (U_d + P_d V_d). \tag{2.10a–d}$$

From these expressions the thermodynamic variables of the dust can be calculated as follows. The effective internal energy of the dust U_d can be obtained from (2.9). The free energy F_d can be directly obtained from U_d by integrating the thermodynamic relation $U_d/T_d^2 = -\partial/\partial T_d(F_d/T_d)$. From F_d, the pressure, the entropy and the enthalpy of the dust can be calculated from (2.10a–d). Typically, these quantities will have the usual thermal component and an excess X_{ES} due to electrostatic contributions. Since the last two terms in (2.9) depend on T_e, T_i and V, which are constant, hence $U_d = U_d(T_d, V_d)$. Thus U_d and its X_{ES} can be calculated for given values of dust density and temperature from Molecular Dynamic (MD) simulations involving dust particles alone (Farouki & Hamaguchi 1994; Hamaguchi & Farouki 1994). In the next section, we show that in the thermodynamic limit, X_{ES} can be calculated analytically.

In the present model, the assumptions of constant dust charge q_d and $q\varphi/T_e, q\varphi/T_i < 1$ used in the linearization of the Boltzmann response are valid in the limit $q_dn_d/2qn_0 < 1$ (Havnes *et al.* 1987; Goertz 1989), which is well-satisfied for typical dusty plasma experimental parameters. The dust temperature T_d, on the other hand, is limited by the dust confinement. In dusty plasma experiments, the dust is electrostatically confined in a parabolic potential well $\varphi_{\mathrm{Ext}} \propto r^2$, which is nearly zero in the centre and rises monotonically to value φ_E at the edge of the confinement at $r = r_E$. Hence the condition for the confinement of dust in the ES potential well is $T_d < q_d\varphi_E$.

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

3. Mean field limit

In the mean field limit, the dust is in the rare density gaseous phase. In this phase, the correlations are exponentially weak and the dust can be smeared out into a homogenous uncorrelated fluid. As will be shown shortly, in this limit there is a mean electrostatic field within the dust cloud. The mean field limit is given by $q_d \to 0$, $N_d \to \infty$, $q_d N_d \neq 0$ and the double summation is replaced by smooth integrations as $\sum_i \sum_j = n_d N_d \int \mathrm{d}V_d$ in (2.6) and (2.7). Carrying out double summations according to this prescription we obtain (Avinash 2010*a*)

$$\begin{aligned} \frac{q_d^2}{8\pi\varepsilon_0} \sum_i \sum_{j\neq i} \frac{\exp(-\kappa_d |\boldsymbol{r_i} - \boldsymbol{r_j}|)}{|\boldsymbol{r_i} - \boldsymbol{r_j}|} &= \frac{q_d^2}{16\pi\varepsilon_0} \sum_i \sum_{j\neq i} \kappa_d \exp(-\kappa_d |\boldsymbol{r_i} - \boldsymbol{r_j}|) \\ &= \frac{q_d^2 N_d^2}{q^2 n} \frac{T_e T_i}{V_d (T_e + T_i)}. \end{aligned} \tag{3.1}$$

Thus the dust internal energy, in the thermodynamic limit, is given by

$$U_d = \frac{3}{2} N_d T_d + \frac{q_d^2 N_d^2 T_e T_i}{q^2 n V_d (T_e + T_i)} + \sum_\alpha T_\alpha N_\alpha [(\ln n_0 \varLambda_\alpha^3) - 1], \tag{3.2}$$

where $n = 2n_0$ is the pristine plasma density. In (3.2), the second term is due to the mean ES field in the dust cloud. Thus, in the thermodynamic limit, though the correlations are weak, dust particles still interact through the mean ES field. This is different from ideal gas where particles do not interact at all.

Substituting U_d in the thermodynamic relation $U_d/T_d^2 = -\partial/\partial T_d (F_d/T_d)$ and integrating gives the Helmholtz free energy, pressure, entropy of the dust as

$$F_d = T_d N_d [\ln(n_d \varLambda_d^3) - 1] + \frac{q_d^2 N_d^2 T_e T_i}{q^2 n V_d (T_e + T_i)} + \sum_\alpha T_\alpha N_\alpha [(\ln n_0 \varLambda_\alpha^3) - 1], \tag{3.3}$$

$$P_d = \frac{N_d T_d}{V_d} + \frac{q_d^2 N_d^2 T_e T_i}{q^2 n V_d^2 (T_e + T_i)}, \quad S_d = \tfrac{3}{2} N_d - N_d [\ln(n_d \varLambda_d^3) - 1]. \tag{3.4a,b}$$

In (3.4*a*), the second term gives the excess pressure P_{ES} due to mean electric fields in the confined dust cloud. This electrostatic pressure P_{ES} (Avinash 2010*a*,*b*) can be dominant for typical experimental parameters.

The dust pressure P_d was experimentally measured by Fisher *et al.* (2013). The value of P_d was found to be much greater than the thermal pressure $n_d T_d$. In fact, the experimentally measured value of P_d was found to be within a factor of order unity of the value predicted by the ES part of total pressure P_{ES} given in (3.4*a*). The ES dust pressure P_{ES} was further confirmed in the experiments by Williams (2019) and MD simulations (Shukla *et al.* 2017). From (3.4*b*) it is seen that in the gaseous limit, the ES contributions to the dust entropy are zero.

4. Thermodynamic processes

In this section we define thermodynamic processes, specific heat, dispersion of acoustic waves and free expansion involving dust.

4.1. *Isothermal process*

In the case where the time scale of the dust process (e.g. expansion/compression), τ_P is larger than the dust neutral collision time scale τ_{dn}, the dust is strongly coupled with

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

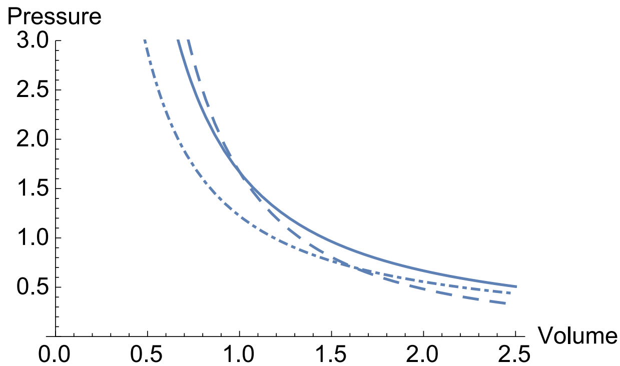

FIGURE 1. Isothermal process (solid), adiabatic (dash) and constant internal energy (dot-dash) processes of dusty plasma in the $P_d - V_d$ plane for $\Gamma_0 = 1$, $\kappa_0 = 1.5$, $\Gamma_0^* = 0.2$. In the limit of small V_d all the three processes become identical with $P_d \propto 1/V_d^2$.

neutrals. In this case, the dust temperature T_d is regulated by the neutral bath and can be taken to be constant $T_{d0}(\approx T_n)$. This would typically be true in experiments with high gas pressure (Thomas 2010). Thus, in the high gas pressure regime, the equation of state is given by

$$P_d = \frac{N_d T_{d0}}{V_d} + \frac{q_d^2 N_d^2 T_e T_i}{q^2 n (T_e + T_i) V_d^2}. \tag{4.1}$$

In this case, the thermal energy of the dust component $3N_d T_{d0}/2$ is constant. However, the internal energy U_d which contains ES contributions as well, and is defined in (3.2), is not constant. Processes where U_d is constant will be defined later. For quasi-static expansion/contraction of the dust volume ΔV_d, the energy conservation can be re-expressed as $T_d \Delta S_d + \Delta(T_i S_i + T_e S_e) = P_d \Delta V_d$, which shows that the heat taken from the plasma bath maintains an isothermal background with given electron and ion temperatures. The dust then expands/compresses at constant T_d in this isothermal background. The change in the dust entropy is given by $\Delta S_d = n_d \Delta V_d$ while the change in the plasma entropy (ES part), in the simple case where $T_e = T_i = T$ is $\Delta S = q_d^2 n_d^2 \Delta V_d / 2q^2 n$. In terms of coupling parameters Γ_0, κ_0 which, for given N_d, are related to the two constants T_{d0}, V_{do} by the relations $\Gamma_0 = q_d^2/4\pi\varepsilon_0 a_0 T_{d0}$, $\kappa_0 = a_0/\lambda_d$ where $a_0 = (3V_{d0}/4\pi N_d)^{1/3}$, the equation of state is given by

$$\bar{P}_d = \frac{1}{\bar{V}_d} + \frac{3}{2}\frac{\Gamma_0}{\kappa_0^2}\frac{1}{\bar{V}_d^2}, \tag{4.2}$$

where we have normalized pressure with $n_{d0}T_{d0}$ and V_d by V_{d0}. In the weak correlation regime the ratio of the ES potential energy to the average kinetic of dust is given by $\Gamma_0^* = \Gamma_0 \exp(-\kappa_0) < 1$. In figure 1 we show a dust isotherm for $\Gamma_0 = 1$, $\kappa_0 = 1.5$, $\Gamma_0^* = 0.2$. The isothermal equation of state given in (4.2) has been verified in recent MD simulations (Shukla *et al.* 2017). In the large V_d, limit, $\bar{P}_d \propto 1/\bar{V}_d$ as an ideal gas, however, for smaller dust clouds where ES pressure dominates, $\bar{P}_d \propto 1/\bar{V}_d^2$.

4.2. *Adiabatic process*

In the case where the expansion/compression time scale of dust τ_P is smaller than the dust neutral collision time scale τ_{dn}, the thermal contact of the dust with neutrals is weak. In these cases the dust can regarded as isolated with constant entropy ($\Delta S_d = 0$).

This would typically be true in experiments with low neutral gas pressure (Pilch *et al.* 2007). As stated earlier, the heat exchanges involved in local plasma perturbations must be taken into account while defining thermodynamic processes in our model. Accordingly, we define the quasi-static adiabatic process involving dust as 'the expansion/compression processes involving isolated dust that take place in an isothermal plasma background of given temperature'. Hence, in this process, while the dust is isolated and its entropy remains constant in expansion/compression, the background plasma is not and its entropy changes. The equation of state for this process can be calculated from the equation $0 = \Delta U_d + P_d \Delta V_d$. Eliminating U_d and P_d via (3.4*a*) we obtain the equation $\Delta T_d/T_d = -(2/3)\Delta V_d/V_d$ which can be integrated to give $T_d V_d^{2/3} = \text{const.}$, (like an ideal gas). Eliminating T_d using this relation in the expression for P_d in (3.4*a*) finally gives the adiabatic equation of state for gaseous dust in terms one arbitrary constant C (and given T_e and T_i) as

$$P_d = \frac{C}{V_d^{5/3}} + \frac{q_d^2 N_d^2 T_e T_i}{q^2 n (T_e + T_i) V_d^2}. \tag{4.3}$$

The adiabat/isentrope $P_d = P_d(V_d)$ can be plotted in the $P_d - V_d$ plane where C is determined by any point P_{d0}, V_{do} (alternately T_{d0}, V_{do}) on the adiabat. In terms of coupling parameters Γ_0, κ_0, defined earlier, the equation of state can be expressed thus

$$\bar{P}_d = \frac{1}{\bar{V}_d^{5/3}} + \frac{3}{2}\frac{\Gamma_0}{\kappa_0^2}\frac{1}{\bar{V}_d^2}. \tag{4.4}$$

In figure 1 we show a dust adiabat for $\Gamma_0 = 1$, $\kappa_0 = 1.5$, $\Gamma_0^* = 0.2$. If the dust volume expands against external pressure from $\bar{V}_{d1} \to \bar{V}_{d2}$, then the work done is

$$\bar{W} = \frac{3}{2}\left(\frac{1}{\bar{V}_{d1}^{2/3}} - \frac{1}{\bar{V}_{d2}^{2/3}}\right) + \frac{3}{2}\frac{\Gamma_0}{\kappa_0^2}\left(\frac{1}{\bar{V}_{d1}} - \frac{1}{\bar{V}_{d2}}\right). \tag{4.5}$$

This equation shows that in expanding from $\bar{V}_{d1} \to \bar{V}_{d2}$, dust does extra work (second term) compared with the corresponding ideal gas. The extra work is done by the ES pressure to extract requisite amount of heat from the plasma background to maintain uniform electron and ion temperatures. The changes in entropies of electrons and ions ΔS_e, ΔS_i can be calculated from expressions of electrons and ion entropies given in (2.3). In the simple case where $T_e = T_i = T$, the change in the ES part of the total plasma entropy is given by $\Delta S = q_d^2 n_d^2 \Delta V_d / 2q^2 n$.

4.3. *Constant internal energy process*

In addition to the two processes described above, we can define an additional new process where the internal energy of dust U_d is constant. This process will take place under the condition when the dust neutral collision time scale is of the same order as the time scale of the process, i.e. $\tau_{\mathrm{dn}} \approx \tau_p$. Under this condition, the rate at which the dust exchanges heat with the dust heat bath is equal to the work done by/on the dust against the external pressure so that $\Delta Q_d = P_d \Delta V_d$ and $\Delta U_d = 0$ in (3.2). The equation of state for this process can be calculated by using (3.2), (3.4*a*) and the condition $\Delta U_d = 0$, and is given by

$$T_d = T_{d0} - \frac{2 q_d^2 N_d T_e T_i}{3 q^2 n V_d (T_e + T_i)}, \quad P_d = \frac{N_d T_{d0}}{V_d} + \frac{q_d^2 N_d^2 T_e T_i}{3 q^2 V_d^2 n (T_e + T_i)}. \tag{4.6*a*,*b*}$$

Thus, in the constant U_d process, the dust temperature increases on expansion and decreases on compression. This is an interesting and somewhat counterintuitive

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

consequence which can be verified experimentally. However, the pressure, as usual, decreases with the volume as in the case of isothermal processes, except for a factor of 1/3 in the second term. In figure 1 we show the $P_d - V_d$ plot for constant U_d process. The change in the plasma entropy, as in earlier cases, is given by $\Delta S = q_d^2 n_d^2 \Delta V_d / 2q^2 n$, while the change in the dust entropy is given by $\Delta S_d = (1 + 3\Gamma/2\kappa^2) n_d \Delta V_d$. In the large $\bar{V}_d$ limit, the effects due to the mean field vanish and the constant U_d process becomes identical to isothermal process and $\Delta S_d = n_d \Delta V_d$.

4.4. *Specific heat of dust*

The specific heats of the dust at constant volume and constant pressure are defined as

$$C_{V_d} = \left.\frac{\partial Q_d}{\partial T_d}\right|_{V_d}, \quad C_{P_d} = \left.\frac{\partial Q_d}{\partial T_d}\right|_{P_d}. \tag{4.7a,b}$$

Eliminating ΔQ_d from (2.8*a*,*b*) and using (3.2) and (3.4*a*,*b*) we get

$$C_{V_d} = 3N_d/2, \quad C_{P_d} = 3N_d/2 + \frac{N_d T_d}{V_d} \left.\frac{\partial V_d}{\partial T_d}\right|_{P_d}. \tag{4.8a,b}$$

Eliminating the partial derivative in (4.8*a*,*b*) using (3.4*a*) we finally obtain the specific heat at constant pressure and the ratio of two specific heats γ as

$$C_{P_d} = \frac{3}{2} N_d + \frac{N_d}{\left(1 + \dfrac{3\Gamma}{2\kappa^2}\right)}, \quad \gamma = 1 + \frac{2/3}{\left(1 + \dfrac{3\Gamma}{2\kappa^2}\right)}. \tag{4.9a,b}$$

Equation (4.9*a*,*b*) shows that γ is no longer constant but a function of dust volume and temperature. In the ideal gas or weak coupling limit ($\Gamma \to 0$, $\kappa \to \infty$), $\gamma = 5/3$.

In addition to the two types of specific heats defined in (4.7*a*,*b*), we can define a third type of specific heat, i.e. the specific heat at constant internal energy C_{U_d} defined as

$$C_{U_d} = \left.\frac{\partial Q_d}{\partial T_d}\right|_{U_d} = P_d \left.\frac{\partial V_d}{\partial T_d}\right|_{U_d}, \tag{4.10}$$

where we have used $\Delta Q_d = P_d \Delta V_d$ for constant U_d. This specific heat has no ideal gas analogue, though it may have a real gas analogue. Evaluating the partial derivative and substituting for pressure from the condition in (4.6*a*,*b*), we get

$$C_{U_d} = \frac{3N_d}{2}\left(\frac{2\kappa^2}{3\Gamma} + 1\right). \tag{4.11}$$

In the limit $\Gamma \to 0$, $\kappa \to \infty$, $C_{U_d} \to \infty$.

4.5. *Dust acoustic waves*

The dust acoustic waves (DAW) are analogues of ion acoustic waves in electron–ion plasma where the inertia is due to the dust mass and the screening is due to electrons and ions. The dispersion relation of these waves can be obtained directly by using equation of states, derived above, in the fluid equations.

The total pressure of the dust P_d, which is the sum of the kinetic and the ES pressure, drives acoustic modes in dusty plasma. The dispersion of these acoustic modes driven by the total dust pressure is governed by following set of fluid equations:

$$\rho_d \frac{\mathrm{d}\boldsymbol{v}_d}{\mathrm{d}t} = -\nabla(P_d), \quad \frac{\partial \rho_d}{\partial t} + \nabla \cdot \rho_d \boldsymbol{v}_d = 0, \quad P_d = c n_d^{\gamma} + \frac{q_d^2 T_e T_i}{q^2 n (T_e + T_i)\delta} n_d^2. \quad (4.12a\text{–}c)$$

In these equations, P_d is the local dust pressure which is the sum of the dust kinetic and the ES pressure and ρ_d, v_d, are the local dust mass density and the fluid velocity, respectively. In (4.12*c*), $\gamma = 5/3$ for adiabatic processes, $\gamma = 1$ for isothermal or constant U_d processes, $\delta = 1$ for adiabatic or isothermal processes and $\delta = 3$ for constant internal energy processes. Performing standard linearization of these equations about a uniform and static equilibrium and using plane wave solutions where $\nabla = \mathrm{i}k$, $\partial/\partial t = -\mathrm{i}\omega$ in (4.12*a–c*), (ω and k are the frequency and the wave vector of DAW) we obtain following dispersion relation of acoustic waves:

$$\frac{\omega^2}{k^2} = C_{\mathrm{DAW}}^2 = \left[\frac{\gamma T_d}{m_d} + \frac{2 q_d^2 n_d T_e T_i}{q^2 n m_d (T_e + T_i)\delta}\right]. \quad (4.13)$$

In the low neutral gas pressure regime ($\nu_{\mathrm{dn}}/\omega \ll 1$), $\gamma = 5/3$ and $\delta = 1$ in (4.13) which corresponds to the adiabatic DAW; in the high neutral gas pressure regime ($\nu_{\mathrm{dn}}/\omega \gg 1$), $\gamma = 1$ and $\delta = 1$ which corresponds to the isothermal DAW. While in the intermediate neutral gas pressure regime ($\nu_{\mathrm{dn}}/\omega \approx 1$), $\gamma = 1$, $\delta = 3$ which corresponds to a new mode, i.e. the constant internal energy DAW.

As stated before, in DAW, the inertia is due to dust mass while the screening is due to both electrons and ions. In the usual dispersion relation of ion acoustic modes in electron–ion plasma given by $\omega^2/k^2 = C_{\mathrm{IA}}^2 = (\gamma T_i + T_e)/m_i$, the inertia is due to ions and the screening is due to electrons. If we consider screening due to electrons in the dispersion relation of DAW given in (4.13) by taking $T_i \gg T_e$ and the inertia due to ions by replacing dust with ions, i.e. $q_d \rightarrow q_i = q$, $m_d \rightarrow m_i$, $n_d \rightarrow n_i$, $T_d \rightarrow T_i$, $2n_i \approx n$ (n is plasma density of electron–ion plasma) the dispersion relation of DAW in (4.13) reduces to the dispersion relation of ion acoustic given above.

In dusty plasma experiments, DAW has been observed over a wide range of frequencies and neutral gas pressures. For example, in a 3PDX device (Thomas 2010), DAWs in the range from 7 Hz to 120 Hz were observed. The dust particle velocity distribution function was observed to be close to Maxwellian during the wave motion (Fisher & Thomas 2010; Thomas 2010), justifying the quasi-static assumption. The neutral pressure in this device was 72 mTorr and the corresponding dust neutral collision frequency was $\nu_{\mathrm{dn}} \approx 75$ Hz. Then, in this experiment, waves observed in the range $\omega < 75$ Hz will correspond to isothermal DAW, waves with $\omega > 75$ Hz will correspond to Adiabatic DAW, while waves with ω in the range $\approx$75 Hz will correspond to constant internal energy DAW as discussed here. Thus it should be possible to identify the constant U_d DAW in this experiment which has the novel and distinctive feature that in crests dust will be cooler and trough it will be hotter.

4.6. *Free expansion of dust*

In dusty plasma, the hydrodynamic free expansion of Coulomb balls and dust clouds (starting from an equilibrium state) has been examined via MD simulations (Piel & Goree 2013) and analytically using fluid equations (Ivlev 2013). In experiments (Barkan & Merlino 1995; Antonova *et al.* 2012), free expansion of dust in dusty plasma has been studied in the afterglow phase of the discharge. In this phase, the confining fields are removed by switching off the anode voltage causing the background plasma to decay.

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

The MD simulations show that at low gas pressure, the background plasma decays rapidly removing the shielding and the cloud explodes due to bare Coulomb repulsion. In contrast, at high gas pressure, the dust cloud expands gradually under the shielded Yukawa repulsion (Ivlev *et al.* 2003; Saxena, Avinash & Sen 2012) and may fission eventually (Merlino *et al.* 2016).

Here, in the last part of our paper, we describe the free, non-quasi-static expansion of isolated gaseous dust in a confined and steady isothermal plasma background. This is different from the free expansion of dust in an unconfined and decaying plasma background of the afterglow phase mentioned above. The non-quasi-static expansion is governed by the equation $\Delta Q_d = \Delta U_d + P_{\mathrm{ext}}\Delta V_d$. Since the dust is isolated $\Delta Q_d = 0$ and is allowed to expand freely, $P_{\mathrm{ext}} = 0$. Hence the condition for free expansion is given by $U_d =$ const. and the variation of temperature with volume is given by (4.6*a*), i.e.

$$T_d = T_{d0} - \frac{2q_d^2 N_d T_e T_i}{3q^2 n V_d (T_e + T_i)}. \tag{4.14}$$

This is an irreversible process hence the entropy increases. If the dust expands freely from state $(T_{d1}, V_{d1}) \rightarrow (T_{d2}, V_{d2})$ then the change in entropy is calculated by putting a reversible path between the two states. It is identical to change in entropy of the constant internal energy process and is given by $\Delta S_d = (1 + 3\varGamma/\kappa^2) n_d \Delta V_d$. In the ideal gas limit $(\Gamma \rightarrow 0,\ \kappa \rightarrow \infty)$, $\Delta S_d = n_d \Delta V_d$. Of course, as opposed to the constant internal energy process, the work done by dust in free expansion is zero. Since the dust is isolated, the free expansion must occur on time scale faster than the dust neutral time scale, i.e. $\tau_p \ll \tau_{\mathrm{dn}}$, which will be typically true at low neutral pressure. This neutral pressure condition is similar to the adiabatic process, however, the difference in the two processes is that for free expansion $P_{\mathrm{ext}} = 0$, while for adiabatic processes $P_{\mathrm{ext}} = P_d$. To calculate the rise in dust temperature for typical dusty plasma experimental parameters, we express (4.14) in the following dimensionless form:

$$\bar{T}_d = 1 + \frac{2Z_d p}{3} \frac{\bar{T}_e \bar{T}_i}{(\bar{T}_e + \bar{T}_i)} (1 - \bar{n}_d). \tag{4.15}$$

In this equation, the dust temperature, electron and ion temperature are all normalized with initial dust temperature T_{d0}, the dust density by the initial dust density n_{d0}, $Z_d = q_d/q$ and, p is given by $p = Z_d n_{d0}/n$. Typical parameters of dusty plasma experiments are $T_e \approx 3$ eV, $T_i \approx 0.025$ eV, $n \approx 5 \times 10^{13}\ \mathrm{m}^{-3}$, $n_d \approx 5 \times 10^9\ \mathrm{m}^{-3}$, $T_d \approx 100$ eV, while for micron sized dust $Z_d \approx 3 \times 10^3$ (Thomas 2010). Then, if we take the values of dust and plasma density given above as initial values, and if in the experiment the dust volume increases (freely) by two times so that $\bar{n}_d = 1/2$, then $\overline{T}_d = 1.15$ in (4.15), i.e. the dust temperature increases by 15 % of its initial value. In some dusty plasma experiments higher ion temperatures, $T_i \approx 0.1$ eV, are reported (Trottenberg *et al.* 2006). In such cases dust temperature increases by 50 %, i.e. $\overline{T}_d = 1.5$ for $\bar{n}_d = 1/2$.

The reason for heating of dust particles in free expansion is the repulsive nature of the interparticle force. In free expansion of isolated dust particles, the positive potential energy decreases which is converted into thermal energy. The reverse would be true in cases of attractive force between particles. This is similar to the case of free expansion of real gases (sometimes called Joule expansion) where heating observed above inversion temperature T_{inv} is attributed to the repulsive part, and cooling below T_{inv} is attributed to the attractive part of the Lennard–Jones potential (Goussard & Roulet 1993). For example, hydrogen gas with low inversion temperature (~ 200 K) shows heating on free expansion at normal

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

temperatures. For purely repulsive potential like the present case of negatively charged dust, $T_{\mathrm{inv}} \rightarrow 0$, and temperature always increases in free expansion.

In the case of Joule–Thompson effect as well, where the gas is throttled through a constricted passage, the heating observed in the case $T > T_{\mathrm{inv}}$ is due to the repulsive part of the interparticle force. In a separate paper, we will consider a Joule–Thompson-like throttling experiment for dusty plasma where dust particles are electrostatically throttled through a negatively biased mesh. This is expected to further enhance the dust temperature increase.

5. Summary and discussions

In this paper, we have proposed a thermodynamic model of dusty plasma for the case where dust is confined in a small volume within a large plasma background by external fields. The model takes into account the heat exchanged with the plasma background during the quasi-static motion of the dust. It is solved analytically in the mean field limit and various processes of gaseous phase of dust, e.g. adiabatic, isothermal and constant internal energy expansion/contraction, specific heat, dispersion of acoustic waves and free expansion of dust in the plasma background are studied.

Next, we compare our model with the HF model. The basic difference in the two models is due to the difference in the dust charge neutralization by the uniform plasma background. In the HF model where $V_d/V = 1$, the dust charge is completely neutralized by the uniform plasma background. In the present model where $V_d/V \ll 1$, the neutralization of dust charge by the plasma is vanishingly small. This is reflected in two different charge densities used in the HF and the present model. To show this we start with the charge density of the HF model given by (appendix B)

$$\rho = -q_d \sum_{i=1}^{N_d} \delta(\boldsymbol{r} - \boldsymbol{r}_i) + q_d(\overline{n_i} - \overline{n_e}) - \varepsilon_0 \varphi/\lambda_d^2, \tag{5.1}$$

where $\bar{n}_\alpha = (1/V) \int n_\alpha \, \mathrm{d}V = N_\alpha/V$ (α represents electrons and ions). Using overall charge neutrality in V, i.e. $q_d N_d = q(N_i - N_e)$ we may express (5.1) in the form

$$\rho = -q_d \sum_{i=1}^{N_d} \delta(\boldsymbol{r} - \boldsymbol{r}_i) + q_d n_d \frac{V_d}{V} - \varepsilon_0 \varphi/\lambda_d^2. \tag{5.2}$$

From this common expression, we can obtain charge densities of both models as follows. In case $V = V_d$, we obtain the charge density of HF model. Further, using $\int_V \rho \, \mathrm{d}V = 0$, we can show that the first term cancels with the second, implying thereby that the dust charge is completely neutralized (or confined) by the second term due to the uniform plasma background and the mean ES potential $\bar{\varphi}$ is zero; there is no need of external confinement. In contrast, in case if we take the limit of large plasma volume, i.e. $V_d/V \rightarrow 0$, (for given $q_d n_d$), then the second term in (5.2) drops out and we obtain the charge density of the present model given in (2.4). In this case the neutralization of dust charge by the uniform plasma background is asymptotically small. Further, in the dust cloud there is a non-zero mean ES potential given by $\bar{\varphi} = -q_d n_d \lambda_d^2/\varepsilon_0$. This negative potential tries to expel dust particles from V_d requiring an external field for the dust confinement. Substituting $\bar{\varphi} = -q_d n_d \lambda_d^2/\varepsilon_0$ in $P_{\mathrm{ES}} = \varepsilon_0 \bar{\varphi}^2/2\lambda_d^2$ (Avinash 2010*b*) we obtain the ES part of the total dust pressure P_{ES} given by the second term in (3.4*a*). If we retain small terms of order V_d/V,

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

HF model	Present model
Plasma and dust in the same volume $V_d/V = 1$	Dust volume much smaller than plasma volume $V_d/V \ll 1$
No external confinemen	Dust confinement within the plasma by external field
Dust confined by the neutralizing plasma background	
ES field and the ES pressure zero in the mean field limit	Mean ES field and pressure finite in the mean field limit
Dust pressure is purely thermal	Dust pressure is the sum of thermal and ES contributions
Dust density proportional to plasma density	Dust and the plasma density can be varied independently

TABLE 1. Comparison of the present model with the HF model.

then P_{ES} is given by (appendix B)

$$P_{\mathrm{ES}} = \frac{q_d^2 N_d^2 T_e T_i}{q^2 n V_d^2 (T_e + T_i)} \left(1 - \frac{V_d}{V}\right). \tag{5.3}$$

From the above discussion it follows that the HF model is valid in the limit $V_d/V \to 1$ while the present model is valid in the limit $V_d/V \to 0$. In dusty plasma experiments $V_d/V \approx 10^{-3}$ to 10^{-5} (Barkan & Merlino 1995; Trottenberg *et al.* 2006; Pilch *et al.* 2007; Thomas 2010) hence neutralization of the dust charge by the plasma background is vanishingly small and an external field is used in these experiments for the dust confinement.

The expressions for the Helmholtz free energy, the dust internal energy and the dust pressure of HF model, in the mean field limit, are obtained in appendix B and are given by

$$F_d = T_d N_d [\ln(n_d \Lambda_d^3) - 1] + \sum_\alpha T_\alpha N_\alpha [(\ln n_0 \Lambda_\alpha^3) - 1], \tag{5.4}$$

$$U_d = \tfrac{3}{2}(N_d T_d) + \sum_\alpha T_\alpha N_\alpha [(\ln n_0 \Lambda_\alpha^3) - 1], \tag{5.5}$$

$$P_d = \frac{N_d T_d}{V_d} = n_d T_d. \tag{5.6}$$

Clearly, in the HF model, the ES contributions are zero in the mean field limit, i.e. the mean field limit contains only thermal terms; ES contributions arise solely due to finite dust correlation effects. In contrast, in our model given in (2.3) and (2.4), thermodynamic parameters have a dominant ES contribution even in absence of dust correlations (or in the mean field limit).

The ES part of the dust pressure P_{ES} has been measured in experiments (Fisher *et al.* 2013; Williams 2019) where, especially for smaller clouds, it is found to be substantially greater than the thermal dust pressure predicted by the HF model in (5.6). In fact, it is of the same order as the pressure predicted by our model in (3.4*a*) which has substantial ES contribution (Fisher *et al.* 2013). Additionally, in the HF model the dust density is assumed to be proportional to average plasma density for which there is no experimental justification. In experiments, dust and plasma density can be varied independently

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

(Barkan & Merlino 1995; Thomas 2010) as in our model. In table 1 we show the comparison of the two models.

Finite dust correlation effects are contained in the double summation term of the effective dust internal energy expression given in (2.9). In the present paper we have appropriately approximated the double summation term to construct the thermodynamic limit of dusty plasma. A theory of finite correlation effects, e.g. melting/freezing phase transitions, sound propagation in correlated medium etc. can be constructed analytically by suitable approximation of the double summation term or by calculating this term via MD simulations. This will be the subject of future publications.

Acknowledgements

The author is grateful to one of the referees for thoughtful comments. This work is dedicated to the memory of P.K. Kaw, my teacher and mentor, who greatly appreciated and advocated the use of simple thermodynamic arguments in plasma physics pioneered by early workers in the field.

Editor Edward Thomas thanks the referees for their advice in evaluating this article.

Declaration of interests

The author reports no conflict of interest.

Appendix A. Derivation of internal energy

The expression for the composite internal energy of the system of plasma and dust is given by

$$U = \frac{3}{2}(N_eT_e + N_iT_i + N_dT_d) + \frac{1}{2}\int \rho\varphi\, \mathrm{d}r - \frac{q_d^2}{8\pi\varepsilon_0}\sum_{j=1}^{N_d}\int \frac{\delta(\boldsymbol{r}-\boldsymbol{r_j})}{|\boldsymbol{r}-\boldsymbol{r_j}|}\,\mathrm{d}r, \tag{A 1}$$

where ρ and φ are given by

$$\rho = -q_s\sum_{i=1}^{N_d}\delta(\boldsymbol{r}-\boldsymbol{r_i}) - \varepsilon_0\varphi/\lambda_d^2, \quad \varphi = -\frac{q_d}{8\pi\varepsilon_0}\sum_j \frac{\exp(-|\boldsymbol{r}-\boldsymbol{r_j}|/\lambda_d)}{|\boldsymbol{r}-\boldsymbol{r_j}|}. \tag{A 2a,b}$$

Eliminating ρ and φ through (A 2) in the second term of (A 1), and performing the integration with delta function (after subtracting the singular term) we obtain

$$U = \frac{3}{2}(N_dT_d + N_iT_i + N_eT_e) - \left(\frac{q_d^2N_d\kappa_d}{8\pi\varepsilon_0}\right) + \left[\frac{q_d^2}{8\pi\varepsilon_9}\sum_i\sum_{j\neq i}\left(\frac{\exp(-\kappa_d|\boldsymbol{r_i}-\boldsymbol{r_j}|)}{|\boldsymbol{r_i}-\boldsymbol{r_j}|}\right)\right]$$
$$-\frac{\varepsilon_0\kappa_d^2}{2}\int\varphi^2\mathrm{d}^3r. \tag{A 3}$$

To evaluate the last integral in (A 3), we eliminate φ in the φ^2 integral in (A 3) by (A 2*b*), which gives

$$\frac{\varepsilon_0\kappa_d^2}{2}\int\varphi^2\mathrm{d}^3r = \frac{\varepsilon_0\kappa_d^2}{2}\left(\frac{q_d}{4\pi\varepsilon_0}\right)^2\int\sum_i\frac{\exp(-\kappa_d|\boldsymbol{r}-\boldsymbol{r_i}|)}{|\boldsymbol{r}-\boldsymbol{r_i}|}\sum_j\frac{\exp(-\kappa_d|\boldsymbol{r}-\boldsymbol{r_j}|)}{|\boldsymbol{r}-\boldsymbol{r_j}|}\,\mathrm{d}^3r. \tag{A 4}$$

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

The integral is evaluated using spherical polar coordinates (Avinash 2010*a*) to give

$$\sum_i \sum_j \int \frac{\exp(-\kappa_d|\boldsymbol{r}-\boldsymbol{r}_i|)}{|\boldsymbol{r}-\boldsymbol{r}_i|} \frac{\exp(-\kappa_d|\boldsymbol{r}-\boldsymbol{r}_j|)}{|\boldsymbol{r}-\boldsymbol{r}_j|} \, \mathrm{d}^3 r = \frac{2\pi}{\kappa_d} \sum_i {\sum_j}' \exp(-\kappa_d|\boldsymbol{r}_i-\boldsymbol{r}_j|). \tag{A 5}$$

Eliminating the integral in (A 3) using (A 4) and (A 5) gives the (2.6) of the main text.

To derive (2.7) of the text, we start with expression of electron and ion entropy in (2.3) of the text and eliminate the electron and ion densities through linearized Boltzmann response. The resulting expression is expanded in powers of $q\varphi/T_e \approx, q\varphi/T_i < 1$ and terms proportional to φ^2 are retained to obtain following expression (Hamaguchi & Farouki 1994):

$$T_e S_e + T_i S_i = \frac{3}{2} \sum_\alpha N_\alpha T_\alpha - \sum_\alpha T_\alpha N_\alpha (\ln n \Lambda_\alpha^3 - 1) - \frac{\varepsilon_0 \kappa_d^2}{2} \int \varphi^2 \, \mathrm{d}^3 r. \tag{A 6}$$

Eliminating the last integral in (A 6) through (A 4) and (A 5) we obtain (2.7) of the main text.

Appendix B. Homogenous limit of the Hamaguchi–Faouki model

In the HF model the background electron–ion plasma and dust particles occupy the same volume V.

The linearized Boltzmann response of HF model is given by $n_\alpha = \bar{n}_\alpha(1 - q_\alpha \varphi/T_\alpha)$ where $\bar{n}_\alpha = (1/V)\int n_\alpha \, \mathrm{d}V = (N_\alpha/V)$, $\int \varphi \, \mathrm{d}V = 0$ and α denotes ions or electrons. With this linearized response, and using the overall charge neutrality in V given by $q_d n_d = q(\bar{n}_i - \bar{n}_e)$, the net charge density is given by

$$\rho = -q_d \sum_{i=1}^{N_d} \delta(\boldsymbol{r}-\boldsymbol{r}_i) + q_d n_d - \varepsilon_0 \varphi/\lambda_d^2. \tag{B 1}$$

The corresponding expression for the dust internal energy U_d of the HF model, without periodic boundary condition terms, is given by eliminating ρ from (B 1) in (2.2) of the text, which gives

$$\begin{aligned} U_d = {} & \frac{3}{2}(N_d T_d) + \frac{q_d^2}{8\pi\varepsilon_0} \sum_i \sum_j \frac{\exp(-|\boldsymbol{r}_i-\boldsymbol{r}_j|/\lambda_d)}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} + \sum_\alpha T_\alpha N_\alpha [(\ln n_0 \Lambda_\alpha^3) - 1] \\ & - \frac{q_d^2 N_d^2 T_e T_i}{q^2 V_d (T_e + T_i)} - \frac{N_d q_d^2 \kappa_d}{8\pi\varepsilon_0}. \end{aligned} \tag{B 2}$$

The second last term (which is absent in our model) arises due to $q_d n_d$ in (B 1). It corresponds to a cohesive field in the uniform plasma background which, as stated earlier, neutralizes or confines the negative dust charge. If now we take the homogenous limit in (B 2), then the second term cancels with the second last term. The corresponding expressions for U_d and the Helmholtz free energy in the homogenous limit, which now

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

contain only thermal terms, are

$$U_d = \tfrac{3}{2}(N_d T_d) + \sum_\alpha T_\alpha N_\alpha[(\ln n_0 \Lambda_\alpha^3) - 1], \tag{B 3}$$

$$F_d = T_d N_d[\ln(n_d \Lambda_d^3) - 1] + \sum_\alpha T_\alpha N_\alpha[(\ln n_0 \Lambda_\alpha^3) - 1], \tag{B 4}$$

where α denotes electrons and ions. The dust pressure is given by

$$P_d = \frac{N_d T_d}{V_d}. \tag{B 5}$$

To obtain P_{ES} with small corrections of order $V_d/V \ll 1$, we calculate $\bar{\varphi}$ from (5.2), retaining the V_d/V corrections (middle term) to give

$$\bar{\varphi} = -\frac{q_d n_d \lambda_d^2}{\varepsilon_0}\left(1 - \frac{V_d}{V}\right). \tag{B 6}$$

It has been shown earlier that P_{ES} can be obtained directly from the expression $P_{ES} = \int q_d n_d \, \mathrm{d}\bar{\varphi}$ (Avinash 2010*b*). Eliminating $q_d n_d$ in the integral from (B 6) we obtain

$$P_{ES} = \frac{q_d^2 N_d^2 T_e T_i}{q^2 n V_d^2 (T_e + T_i)}\left(1 - \frac{V_d}{V}\right), \tag{B 7}$$

where we have eliminated Debye length using $1/\lambda_d^2 = (q^2 n_0/\varepsilon_0)(1/T_e + 1/T_i)$.

REFERENCES

ANTONOVA, T., DU, C.-R., IVLEV, A. V., ANNARATONNE, B. M., HOU, L.-J., KOMPANEETS, R., THOMAS, H. M. & MORFILL, G. E. 2012 Microparticles deep in the plasma sheath: Coulomb 'explosion'. *Phys. Plasmas* **19**, 093709-1–093709-6.

AVINASH, K. 2010*a* Thermodynamics of the interconversion of heat and work via plasma electric fields. *Phys. Plasmas* **17**, 123710-1–123710-6.

AVINASH, K. 2010*b* Plasma heat pump and heat engine. *Phys. Plasmas* **17**, 082105-1–082105-8.

AVINASH, K. & KAW, P. K. 2014 Plasma heating by electric field compression. *Phys. Rev. Lett.* **112**, 185002-1–185002-4.

BARKAN, A. & MERLINO, R. L. 1995 Confinement of dust particles in a double layer. *Phys. Plasmas* **2**, 3261–3265.

DAVIDSON, R. C. 1990 *Physics of Non Neutral Plasmas*. Addison-Wesley.

FAROUKI, R. T. & HAMAGUCHI, S. 1994 Thermodynamic of strongly-coupled Yukawa systems near the one-component plasma limit. II. Molecular dynamic simulations. *J. Chem. Phys.* **101**, 9885–9893.

FISHER, R., AVINASH, K., THOMAS, E., MERLINO, R. & GUPTA, V. 2013 Thermal energy density of dust in dusty plasma: experiment and theory. *Phys. Rev.* E **88**, 031101-1–031101-5.

FISHER, R. & THOMAS, E. 2010 Thermal properties of a dusty plasma in the presence of dust acoustic waves. *IEEE Trans. Plasma Sci.* **38**, 833–837.

GOERTZ, C. K. 1989 Dusty plasma in the solar system. *Rev. Geophys.* **27**, 271–292.

GOUSSARD, J.-O. & ROULET, B. 1993 Free expansion for real gases. *Am. J. Phys.* **61**, 845–848.

HAMAGUCHI, S. & FAROUKI, R. T. 1994 Thermodynamic of strongly-coupled Yukawa systems near the one-component plasma limit. I. Derivation of the excess energy. *J. Chem. Phys.* **101**, 9876–9884.

HAVNES, O., GOERTZ, C. K., MORFILL, G. E., GRUN, E. & IP, W. 1987 Dust charge, cloud potential and instabilities in a dust cloud embedded in a plasma. *J. Geophys. Res.* **92**, 2281–2288.

IVLEV, A. V. 2013 Coulomb expansion: analytical solutions. *Phys. Rev.* E **87**, 025102-1–025102-3.

https://doi.org/10.1017/S0022377820001154 Published online by Cambridge University Press

IVLEV, A. V., KRETSCHMER, M., ZUZIC, M., MORFILL, G. E., ROTHERMEL, H., THOMAS, H. M., FORTOV, V. E., MOLOTKOV, V. I., NEFEDOV, A. P., LIPAEV, A. M., *et al.* 2003 Decharging of complex plasma: first kinetic observations. *Phys. Rev. Lett*. **90**, 055003-1–055003-4.

MERLINO, R. L., MEYER, J. K., AVINASH, K. & SEN, A. 2016 Coulomb fission of a dusty plasma. *Phys. Plasmas* **23**, 064506-1–064506-4.

PIEL, A. & GOREE, J. A. 2013 Collisional and collision less expansion of Yukawa balls. *Phys. Rev.* E **88**, 063103-1–063103-11.

PILCH, I., PIEL, A., TROTTENBERG, T. & KOEPKE, M. E. 2007 Dynamics of small dust clouds trapped in a magnetized anodic plasma. *Phys. Plasmas* **14**, 123704-1–123704-8.

QUINN, R. A. & GOREE, J. 2000*a* Single particle Langevin model of particle temperature in dusty plasma. *Phys. Rev.* E **61**, 3033–3041.

QUINN, R. A. & GOREE, J. 2000*b* Experimental investigation of particle heating in strongly coupled dusty plasma. *Phys. Plasmas* **7**, 3904–3911.

SAXENA, V., AVINASH, K. & SEN, A. 2012 Dust cluster explosion. *Phys. Plasmas* **19**, 093706-1–093706-5.

SHUKLA, M., AVINASH, K., MUKHERJEE, R. & GANESH, R. 2017 Isothermal equation of state of three dimensional Yukawa gas. *Phys. Plasmas* **24**, 113704-1–113704-6.

THOMAS, J. E. JR. 2010 Driven dust acoustic wave with thermal effects: comparison of experiment to fluid theory. *Phys. Plasmas* **17**, 043701-1–043701-8.

TROTTENBERG, T., BLOCK, D. & PIEL, A. 2006 Dust confinement and dust acoustic waves in weakly ionized anodic plasma. *Phys. Plasmas* **13**, 042105-1–042105-10.

WILLIAMS, J. 2019 Measurements of thermal effects in the dust acoustic wave. *IEEE Trans. Plasma Sci*. **47**, 3107–3112.