ECONOMIC FACTORS AND SOLVENCY

BY

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ABSTRACT

We study solvency of insurers in a practical model where in addition to basic insurance claims and premiums, economic factors like inflation, real growth and returns on the investments affect the capital developments of the companies. The objective is to give qualitative descriptions of risks by means of crude estimates for finite time ruin probabilities. In our setup, the economic factors have a dominant role in the estimates. In addition to this theoretical view, we will focus on applied interpretations of the results by means of discussions and examples.

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KEYWORDS

Ruin probability, inflation, real growth, investment, cycle, large deviation.

1. INTRODUCTION

Solvency of insurance companies is one of the main concerns in actuarial practice and theory. In order to continue the business, the companies have to show a reasonable capacity to survive, that is, to meet their obligations. An appropriate requirement is that the survival probability within a given time horizon must be above a predescribed high level. For regulatory purposes, the time horizon is typically small, for example, one or two years. From the viewpoint of company management, longer time horizons are also of interest.

To get quantitative estimates for the solvency position of the company, it is necessary to build up mathematical models for claims, premiums, returns on the investments etc. Practical models tend to be complicated and therefore, simulation is a popular tool in the estimation of the survival probabilities.

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The purpose of the present paper is to take a more theoretical look at the problem. Our results should be understood as qualitative descriptions of risks associated with the company but not, for example, as competitors for simulation in the implementation of a solvency test. We will study a comprehensive model which is largely based on Pentikäinen and Rantala (1982). We also refer the reader to Pentikäinen et al. (1989) and Daykin et al. (1994) for further developments in modelling and for other practical aspects of actuarial problems. For empirical observations concerning causes of solvency problems, we refer to the report of The Conference of Insurance Supervisory Services of the Member States of the European Union (2002).

To describe our interest in detail, let $u > 0$ be the initial capital of the company, and let U_n be the capital at the end of the year n for $n = 1, 2, \ldots$. Instead of survival probabilities, it is equivalent to study ruin probabilities. Define the time of ruin T_u by

$$T_u = \begin{cases} \inf\{n \in \mathbb{N};\ U_n < 0\} \\ \infty \text{ if } U_n \geq 0 \text{ for every } n. \end{cases} \tag{1.1}$$

We take the most common approach seen in theoretical studies by considering limits of ruin probabilities as u tends to infinity. The limiting procedure directs the focus to small probabilities which is motivated in solvency considerations. For appropriate fixed $x > 0$, we will show that in our model,

$$\lim_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) = -R(x) \tag{1.2}$$

where $R(x)$ is a specific parameter. More intuitively, if $R(x)$ is finite then (1.2) means that for every given $\varepsilon > 0$,

$$u^{-(R(x)+\varepsilon)} \leq \mathbb{P}(T_u \leq x \log u) \leq u^{-(R(x)-\varepsilon)} \tag{1.3}$$

for sufficiently large u. The time horizon in the estimate increases slowly with u and hence, our study may be viewed to focus on solvency questions within moderate time horizons.

Approximation (1.3) is theoretical in the sense that it is crude and asymptotic. However, the result is also of applied interest. In our model, inflation, real growth of the business and the returns on the investments will completely determine $R(x)$. All these factors are connected with the general economy. The conclusion is that the economic factors determine the magnitude of the ruin probability for large u. In fact, if $R(x)$ is finite then (1.2) is equivalent to

$$\mathbb{P}(T_u \leq x \log u) = C_x(u)\, u^{-R(x)} \tag{1.4}$$

where the function C_x is such that

$$\lim_{u\to\infty} (\log u)^{-1} \log C_x(u) = 0. \tag{1.5}$$

Insurance risks only contribute the function C_x in (1.4) but they are by no means meaningless. As an example, limit (1.5) holds if $C_x(u) = \log u$ for large u. Moreover, it is not known how large u must be in (1.3), and the same view applies to limit (1.5). Nevertheless, in the asymptotic sense, the above limits indicate that a change in economic factors affects more drastically ruin probabilities than a change in the insurance side. As a concrete example, suppose that the company had a need to make its solvency position safer. This should be possible by cutting large losses in the investment side by means of appropriate options. The returns on the investments contribute the parameters $R(x)$, and we can expect that they would increase. Consequently, ruin probabilities would have a tendency to decrease. An alternative would be to cut large insurance claims by means of an excess of loss reinsurance contract. This should also decrease ruin probabilities but less drastically since the parameters $R(x)$ would remain unchanged.

A further application of approximation (1.2) is that it provides a quality control for nonasymptotic bounds for ruin probabilities. To explain this, suppose that it would be possible to show that

$$\mathbb{P}(T_u \le x \log u) \le \phi_x(u) \tag{1.6}$$

for every finite u where ϕ_x is a known function. These types of bounds are obviously of interest from the applied point of view, for example, in connection with solvency tests. To have a good upper bound for large initial capital, $\phi_x(u)$ should behave asymptotically similarly to $\mathbb{P}(T_u \le x \log u)$, that is, we should have

$$\lim_{u\to\infty} (\log u)^{-1} \log \phi_x(u) = -R(x). \tag{1.7}$$

If this is not the case then one can argue that the upper bound does not focus carefully on essential parts of the model, and consequently, relative errors are easily huge for large u. In this sense, (1.7) may be seen as a mimimal quality requirement for the upper bound.

In recent years, there has been a lot of interest in ruin probabilities for processes which include stochastic submodels for inflation and for the returns on the investments. It is generally understood that these factors have a crucial impact to ruin probabilities. An early observation in this direction is given by Schnieper (1983). The main part of the subsequent papers focus on asymptotic expressions for the infinite time ruin probability $\mathbb{P}(T_u < \infty)$. The background for the estimates is given in Goldie (1991) and Grey (1994). The conclusion is that

$$\mathbb{P}(T_u < \infty) \sim Cu^{-\tau}, \quad u \to \infty, \tag{1.8}$$

where C and $\mathfrak{r}$ are constants. The views given by the papers are different. Namely, the parameter $\mathfrak{r}$ is determined by the economic factors in Goldie (1991) and by the tail of claims distribution in Grey (1994). Roughly speaking, insurance risks dominate over economic risks in Grey (1994). An extension of (1.8) to the continuous time case is derived in Paulsen (2002). We also refer the reader to Paulsen (1993) where a general framework for the theory is developed and to Frolova et al. (2002) and Kalashnikov and Norberg (2002) where the focus is on the investment risk. Furthermore, all the above mentioned papers basically deal with random walk models. Collamore (2009) gives a natural extension by allowing a general Markovian dependence between the years. Finally, for a survey of the recent state of the theory, we refer the reader to Paulsen (2008).

A few of the papers in the area deals with finite time ruin probabilities. Approximation (1.2) as such has been studied in Nyrhinen (2001). The main extension of the present paper is that we allow real growth in the model. This leads to a multiplicative trend inside the insurance process and requires new mathematical arguments. We also discuss economic cycles and focus in general on applied aspects of the problem. Finite time ruin probabilities are much studied in more classical models where economic factors are not present. We refer the reader to Collamore (1998) and Chapter IV of Asmussen (2000). These studies provide a useful background for models where economic submodels *are* present. A heuristic explanation of connections can be found in Section 1 of Nyrhinen (2001).

The paper is organized as follows. Main results, discussions and examples are given in Section 2. Section 3 consists of the proofs.

2. Statement of results

We begin by describing the main variables and parameters of the model in our interest. Some variants and extensions will be discussed in Section 2.2 below. For the motivation and the background, we refer the reader to Pentikäinen and Rantala (1982) and Daykin et al. (1994)

Numbers of claims. Associated with the year n, write

N_n = the accumulated number of claims occured in the years $1, \ldots, n$,

λ = the initial level of the mean of the number of claims in the year,

g_n = the rate of real growth,

q_n = the structure variable describing short term fluctuations in the numbers of claims,

b_n = the variable describing cycles and other long term fluctuations in the numbers of claims.

Write further $N_0 = 0$ so that $N_n - N_{n-1}$ represents the number of claims occured in the year n. We assume that they have mixed Poisson distributions such that

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conditionally, given $b_1, \ldots, b_n$, $g_1, \ldots, g_n$ and $q_1, \ldots, q_n$, the variables $N_1 - N_0, \ldots, N_n - N_{n-1}$ are independent and $N_k - N_{k-1}$ has the Poisson distribution with the parameter

$$\lambda b_k (1 + g_1) \cdots (1 + g_k) q_k \tag{2.1}$$

for $k = 1, \ldots, n$.

Total claim amounts. Let

X_n = the total claim amount in the year n,
Z_j = the size of the jth claim in the inflation-free economy,
m_Z = the mean of the claim size in the inflation-free economy,
i_n = the rate of inflation in the year n.

We consider the model where

$$X_n = (1 + i_1) \cdots (1 + i_n) \sum_{j = N_{n-1}+1}^{N_n} Z_j . \tag{2.2}$$

Premiums. For the year n, write

P_n = the premium income,
s = the safety loading coefficient,
c_n = the variable describing long term fluctuations in the premiums.

We take

$$P_n = (1 + s) \lambda m_Z c_n (1 + g_1) \cdots (1 + g_n)(1 + i_1) \cdots (1 + i_n). \tag{2.3}$$

The transition rule. We next describe the development of the capital in time. Let

U_n = the capital at the end of the year n,
r_n = the rate of return on the investments in the year n.

Let $U_0 = u > 0$ be the deterministic initial capital of the company. We define

$$U_n = (1 + r_n)(U_{n-1} + P_n - X_n). \tag{2.4}$$

Technical specifications and assumptions. We end the description by specifying the dependence structure and other technical features of the model. All the random variables below are assumed to be defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We begin by giving a detailed mathematical description for the total claim amounts in the inflation-free economy. For the year n, denote this quantity by V_n, that is,

$$V_n = \sum_{j=N_{n-1}+1}^{N_n} Z_j. \tag{2.5}$$

The distributions of the N-variables depend on the b-, g- and q-variables. We take

$$(g,q),\ (g_1,q_1),\ (g_2,q_2),\ \ldots$$

to be an i.i.d. sequence of random vectors where the first one (g,q) is generic and is introduced for notational simplicity. We also assume that g and q are independent, and that the b-variables are independent of the g- and q-variables. We do not give a specific dependence structure for the sequence $\{b_n\}$. Instead of that, we just assume that

$$\mathbb{P}\big(b_n \in [\underline{b}, \bar{b}]\big) = 1 \quad \text{for every } n \tag{2.6}$$

where $\underline{b}$ and $\bar{b}$ are finite and positive constants. Let F_n^b be the joint distribution function of $(b_1, \ldots, b_n)$, and let F^g and F^q be the distribution functions of $1 + g$ and q, respectively. Let further F_n be the joint distribution function of the random vector

$$\xi_n := (b_1, \ldots, b_n,\ 1+g_1, \ldots, 1+g_n,\ q_1, \ldots, q_n). \tag{2.7}$$

Thus for every $y_1^b, \ldots, y_n^b,\ y_1^g, \ldots, y_n^g,\ y_1^q, \ldots, y_n^q \in \mathbb{R}$,

$$\begin{aligned} &F_n(y_1^b, \ldots, y_n^b,\ y_1^g, \ldots, y_n^g,\ y_1^q, \ldots, y_n^q) \\ &= F_n^b(y_1^b, \ldots, y_n^b) F^g(y_1^g) \cdots F^g(y_n^g) F^q(y_1^q) \cdots F^q(y_n^q). \end{aligned} \tag{2.8}$$

By these specifications, we assume that for every $h_1, \ldots, h_n \in \mathbb{N} \cup \{0\}$ and for every Borel set $C \subseteq \mathbb{R}^{3n}$,

$$\begin{aligned} &\mathbb{P}(N_1 - N_0 = h_1, \ldots, N_n - N_{n-1} = h_n, \xi_n \in C) \\ &= \int_C \prod_{k=1}^{n} e^{-\lambda y_k^b y_1^g \cdots y_k^g y_k^q} \frac{(\lambda y_k^b y_1^g \cdots y_k^g y_k^q)^{h_k}}{h_k!}\, dF_n(y_1^b, \ldots, y_n^b,\ y_1^g, \ldots, y_n^g,\ y_1^q, \ldots, y_n^q). \end{aligned} \tag{2.9}$$

The claim sizes $Z, Z_1, Z_2, \ldots$ are assumed to be i.i.d. (Z is again a generic variable). We also assume that they are independent of the numbers of claims in all respects. Let F^Z be the distribution function of Z, and let $(F^Z)^{h*}$ be the

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hth convolution power of F^Z. In precise terms, we assume that for every $h_1, \ldots, h_n \in \mathbb{N} \cup \{0\}$ and $y_1, \ldots, y_n \in \mathbb{R}$, and for every Borel set $C \subseteq \mathbb{R}^{3n}$,

$$\begin{aligned} &\mathbb{P}(V_1 \leq y_1, \ldots, V_n \leq y_n, N_1 - N_0 = h_1, \ldots, N_n - N_{n-1} = h_n, \xi_n \in C) \\ &= \mathbb{P}(N_1 - N_0 = h_1, \ldots, N_n - N_{n-1} = h_n, \xi_n \in C) \prod_{k=1}^{n} (F^Z)^{h_k *}(y_k). \end{aligned} \tag{2.10}$$

In words, the distribution of $(V_1, \ldots, V_n)$ is obtained by mixing distributions of n-dimensional random vectors with independent compound Poisson variables as components. We refer to Grandell (1997) for more information about mixed Poisson distributions and related topics.

Consider now the other parts of the model. We do not give a specific dependence structure for the fluctuation sequence $\{c_n\}$ associated with the premiums. Instead of that, we assume similarly to (2.6) that

$$\mathbb{P}\left(c_n \in \left[\underline{c}, \bar{c}\right]\right) = 1 \quad \text{for every } n \tag{2.11}$$

where $\underline{c}$ and $\bar{c}$ are finite and positive constants. We allow an arbitrary dependence structure between the c- and V-variables. As the model for inflation and the returns on the investments, we take

$$(i, r), (i_1, r_1), (i_2, r_2), \ldots \tag{2.12}$$

to be an i.i.d. sequence of random vectors, and these vectors are assumed to be independent of g- and V-variables.

Concerning the parameters of the model, we take λ, m_Z and s to be positive real numbers. For the supports of Z and q, we assume that

$$\mathbb{P}(Z \geq 0) = 1, \ \mathbb{P}(q > 0) = 1 \quad \text{and} \quad \mathbb{P}(q > (1+s)\,\bar{c}/\underline{b}) > 0, \tag{2.13}$$

and for the supports of the economic factors that

$$\mathbb{P}(i > -1) = 1, \ \mathbb{P}(g > -1) = 1 \quad \text{and} \quad \mathbb{P}(r > -1) = 1.$$

For the moments of the main variables, we assume that

$$\mathbb{E}\left((1+i)^{\alpha}\right), \ \mathbb{E}\left((1+g)^{\alpha}\right) \quad \text{and} \quad \mathbb{E}\left((1+r)^{\alpha}\right)$$

are all finite for every $\alpha \in \mathbb{R}$, and that $\mathbb{E}(q^{\alpha})$ and $\mathbb{E}(Z^{\alpha})$ are finite for every $\alpha > 0$. Finally, assume that $\mathbb{E}(q) = 1$ and that $\mathbb{E}(g) \geq 0$.

2.1. Estimates for ruin probabilities

Let the model be as described in the first part of Section 2, and let the time of ruin T_u be as in (1.1). Recall that $U_0 = u$ is the initial capital. Recall also that i, g and r are generic rates of inflation, real growth and the return on the investments, respectively, and that they are the main economic factors in the model. Our objective is to give the magnitude of the ruin probability $\mathbb{P}(T_u \le x \log u)$ for appropriate values of x and for large u. The impact of the economic factors will come via the variable

$$W = \frac{(1+i)(1+g)}{1+r}, \tag{2.14}$$

and in fact, W will be in the key role in our considerations. Roughly speaking, large W means that the risk of ruin is high. Thus an increase in inflation or in real growth increases the risk. This is natural since both of the changes basically mean that the volume of the insurance business increases. Thus in the long run, larger and larger yearly losses are possible. An increase in the returns on the investments decreases the risk which is even more natural.

Define the generating functions $\Lambda : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ and $\Lambda_g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by

$$\Lambda(\alpha) = \log \mathbb{E}(W^\alpha) \tag{2.15}$$

and

$$\Lambda_g(\alpha) = \log \mathbb{E}((1+g)^\alpha). \tag{2.16}$$

Recall that $\mathbb{E}(g) \ge 0$ by our assumption so that either $\mathbb{P}(g > 0) > 0$ or $g \equiv 0$. Write

$$\mathfrak{r}_g = \begin{cases} \sup\{\alpha; \Lambda_g(\alpha) \le 0\} \text{ if } \mathbb{P}(g > 0) > 0 \\ 0 \text{ if } g \equiv 0, \end{cases}$$

and let

$$\underline{\mathfrak{r}} = \sup\{\alpha; \Lambda(\alpha) \le 0\}.$$

Write further

$$\mathfrak{r} = \max(\mathfrak{r}_g, \underline{\mathfrak{r}}) \in [0, \infty]. \tag{2.17}$$

Let $\Lambda^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the convex conjugate of Λ, that is,

$$\Lambda^*(x) = \sup\{\alpha x - \Lambda(\alpha);\ \alpha \in \mathbb{R}\}.$$

It is well-known that both Λ and Λ^* are convex functions. We refer the reader to Rockafellar (1970) and Dembo and Zeitouni (1998) for the background.

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Define the parameters μ and x_0 by

$$\mu = \begin{cases} 1/\Lambda'(\mathfrak{r}) \text{ if } \mathfrak{r} < \infty \text{ and } \Lambda'(\mathfrak{r}) \neq 0 \\ \infty \text{ otherwise,} \end{cases}$$

and

$$x_0 = \begin{cases} \inf\{1/\Lambda'(\alpha);\, \alpha > \mathfrak{r}\} \text{ if } \mathfrak{r} < \infty \\ \infty \text{ otherwise.} \end{cases}$$

Finally, define the function $R : (0, \mu) \to \mathbb{R} \cup \{\infty\}$ by

$$R(x) = x\Lambda^*(1/x). \tag{2.18}$$

We next state a simple lemma and the main result of the paper.

Lemma 2.1. *For the above parameters, we have $\mathfrak{r}_g < \infty$ and $x_0 \leq \mu$. Further, $R(x) = \infty$ for every $x \in (0, x_0)$, and R is finite and strictly decreasing on (x_0, μ). Let $\alpha_0 \in \mathbb{R}$ be such that $\Lambda'(\alpha_0) > 0$. Then*

$$\Lambda^*(1/x) = \sup\{\alpha/x - \Lambda(\alpha);\, \alpha \geq \alpha_0\} \tag{2.19}$$

whenever $0 < x \leq 1/\Lambda'(\alpha_0)$.

Theorem 2.1. *Let $x \in (0, \mu) \setminus \{x_0\}$ be arbitrary. Then*

$$\lim_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) = -R(x). \tag{2.20}$$

Remark 2.1. By Lemma 2.1, the right hand side of (2.20) is strictly increasing for $x \in (x_0, \mu)$. In particular, for large u, $\mathbb{P}(T_u \leq x \log u)$ is essentially smaller than $\mathbb{P}(T_u < \infty)$.

It is interesting to compare our model with the classical one where inflation, real growth and the return on the investments are not present. So let $i \equiv 0$, $g \equiv 0$ and $r \equiv 0$. Then $W \equiv 1$, $\mathfrak{r} = \infty$, $\mu = \infty$ and $x_0 = \infty$. Thus limit (2.20) holds for every $x > 0$ with $R(x) = \infty$, and hence, the magnitudes of ruin probabilities are asymptotically smaller than in general in the present paper. As an extension of the classical model, suppose that inflation and real growth are not present but that the return on the investments is always non-negative. Hence, $i \equiv 0$, $g \equiv 0$ and $r \geq 0$. Then $W \leq 1$. By Theorem 2.1, we still have (2.20) for every $x > 0$ with $R(x) = \infty$.

By the above discussion, the company could have a motivation to adjust its strategy such that $W \leq 1$ would hold. A problem here seems to be that it is difficult to control inflation. To illustrate this, take $g \equiv 0$, for simplicity. Then the target would be to have

$$\frac{1+i}{1+r} \leq 1. \tag{2.21}$$

In financial terms, this can be viewed as a superhedging against inflation by means of appropriate investments. It is not obvious that instruments can be found for the hedging, especially, because the company faces specific claim inflation instead of general inflation in the economy. We refer to Pentikäinen and Rantala (1982), Volume I, Section 2.5.

2.2. Discussion of conditions

The model we have studied is complicated but there is still applied motivation for generalizations. We briefly discuss in this section our conditions and some possible extensions.

Dependences between the years. Dependences between consequtive years in the model are caused, for example, by inflation, real growth and the returns on the investments. However, we assumed in (2.12) and elsewhere that the corresponding rates in different years are independent. It would be natural to allow at least a Markovian dependence. This type of extension has been generally possible in classical models in the case where the state space of the underlying Markov chain is finite. We refer to Asmussen (2000). We believe that a similar extension is possible here, especially, since we only consider crude estimates for ruin probabilities. More general Markovian structures could also be analyzed. We refer to Collamore (2009) for results in this direction.

Real growth. Real growth g is modelled as a part of the Poisson parameter in (2.1). A natural interpretation is that it describes changes in the numbers of insureds. We assumed that $\mathbb{E}(g) \geq 0$. This basically excludes the case where the insurance business will be stopped. A non-negative constant could be an appropriate model for real growth but it is also natural to allow some randomness in this part.

Short term fluctuations. We assumed for the structure variable q in (2.13) that

$$\mathbb{P}(q > (1+s)\bar{c}/\underline{b}) > 0. \tag{2.22}$$

This condition is satisfied at least in the popular Polya case where q has a gamma distribution. Roughly speaking, the condition means that in any circumstances, the yearly profit $P_n - X_n$ is negative with a moderate probability. Without the assumption, positive long-term real growth could make the probabilities very small. This is easiest understood by looking at a simple case where (2.22) does not hold. So take $\bar{c} = \underline{b}$ and $q \equiv 1$, and let g be a positive constant. Then by the law of large numbers, $\mathbb{P}(P_n - X_n < 0)$ tends to zero as n tends to infinity. It is not clear to us how ruin probabilities behave in this case.

Something like (2.22) seems to be necessary to end up with the conclusion of Theorem 2.1. We note, however, that it should be possible to relax the condition by specifying the fluctuation sequences $\{b_n\}$ and $\{c_n\}$ in more detail. Nevertheless, in the presence of real growth, also the short term fluctuation may be viewed as an essential risk factor in the model. Finally, it is worth to note that condition (2.22) can be dropped in the case where $g \equiv 0$.

Heavy tailed claim sizes. We assumed that $\mathbb{E}(Z^\alpha)$ is finite for every $\alpha > 0$. This excludes heavy tailed distributions as models for the claim sizes. If the assumption is relaxed then limit (2.20) may change but it may still be possible to specify it. We refer to Nyrhinen (2005), Example 3.4.

Economic cycles. Economic cycles may be included in the model by means of the b- and c-variables as it was described in the first part of Section 2. A possible interpretation is that cycles in the general economy induce cycles in the numbers of claims. This can be described by means of the b-variables. The affect to the premiums should be similar but it could come with a delay and in a smoothed way. This can be described by means of the c-variables.

It is intuitively clear that cycles increase the risk of ruin in a short time horizon, especially, if a bad period is just starting. Still their impact is not seen in the moderate time horizon of Theorem 2.1. We believe, however, that cyclicity associated with the *economic* factors would affect the limits of the theorem.

We did not give any specific model for the b- and c-variables but just assumed uniform bounds (2.6) and (2.11). These assumptions seem not to be very restrictive. An interpretation is that they just exclude extreme variations in the general economy. Cycles could be introduced by making use of appropriate autoregressive processes. As an example, let

$$\log b_n = a_1 \log b_{n-1} + \cdots + a_p \log b_{n-p} + \varphi_n \tag{2.23}$$

where $p \in \mathbb{N}$ and $a_1, \ldots, a_p \in \mathbb{R}$ are constants and $\{\varphi_n\}$ is an i.i.d sequence of random variables. We consider $\log b_n$ in (2.23) instead of b_n since it is necessary to guarantee that b_n is positive. An appropriate choice of the constants makes the process $\{\log b_n\}$ causal. Our condition (2.6) is then satisfied if

$$\mathbb{P}(\varphi_1 \in [-M, M]) = 1 \tag{2.24}$$

for some $M \in \mathbb{R}$. This is easily seen from the series representation for $\{\log b_n\}$. We refer to Brockwell and Davis (1991) for the background. Assumption (2.24) is rather strong but necessary in our proofs. It is plausible that it could be relaxed in this particular example.

Other variants. Some further changes in the model could be motivated from the applied point of view. For example, in the definition of the premium in (2.3), it could be natural to replace the last inflation rate i_n by an estimate. Also in the transition rule for the capital in (2.4), alternative models could be used for the investment return on the profit $P_n - X_n$ of the current year. The proofs indicate that small changes in these directions would not affect the limits of Theorem 2.1.

2.3. Examples

We illustrate in this section Theorem 2.1 by means of two examples. It turns out that the crude description of the theorem is sufficient to confirm some intuitively natural viewpoints concerning the risk of ruin associated with the models in question. In both of the examples, the risk will be measured by $R(x)$ for *small* x. It is interesting that for large x, the conclusions may be different. We prefer to focus on short time horizons since they are probably more relevant from the applied point of view.

We will consider some financial instruments in the examples. For the background, the reader is referred to Chapters 1 and 2 of Panjer et al. (1998).

Example 2.1. We will compare two investment strategies in a model where inflation and real growth are not present, that is, $i \equiv 0$ and $g \equiv 0$. Suppose that there are a stock and an associated put option available in the financial market. Let S_n be the value of the stock at the end of the year n, and let κS_n be the strike price of the put option associated with the year $n + 1$ where $\kappa \in (0,1)$ is a constant. According to Theorem 2.1, assume that

$$\{S_{n+1} / S_n;\ n = 0, 1, 2, \ldots\}$$

is an i.i.d. sequence of random variables. The value of the option at the end of the year $n + 1$ is

$$\max(\kappa S_n - S_{n+1}, 0).$$

We assume that the price of the option at the beginning of the year $n + 1$ is $\pi(\kappa) S_n$ where $\pi(\kappa)$ is a constant. The above assumptions hold, for example, in the Black-Scholes model for the financial market.

Suppose first that the company invests all its money in the stock. Let ρ_s be the associated generic rate of return on the investments. Hence, $1 + \rho_s$ has the same distribution as S_{n+1} / S_n. Associated with this investment strategy, denote by Λ_s the function corresponding to (2.15). Thus

$$\Lambda_s(\alpha) = \log \mathbb{E}\Big((1 + \rho_s)^{-\alpha}\Big).$$

Let further $R_s(x)$ be the parameter corresponding to (2.18), that is, $R_s(x) = x\Lambda_s^*(1/x)$.

Consider an alternative investment strategy where the company cuts large losses in the investment side. This can be done by keeping the number of stocks and options equal in the portfolio. Define the variable ρ_a according to

$$1 + \rho_a = \frac{\max(1 + \rho_s, \kappa)}{1 + \pi(\kappa)}.$$

Then ρ_a describes the rate of return associated with the strategy. Corresponding to (2.15) and (2.18), write

$$\Lambda_a(\alpha) = \log \mathbb{E}\Big((1 + \rho_a)^{-\alpha}\Big) \quad \text{and} \quad R_a(x) = x\Lambda_a^*(1/x).$$

Let's compare ruin probabilities related to the above two strategies. Under the natural assumption that

$$\mathbb{P}\left(1 + \rho_s < \frac{\kappa}{1 + \pi(\kappa)}\right) > 0,$$

there exists $\alpha_0 \geq 0$ such that $\Lambda_s(\alpha) \geq \Lambda_a(\alpha)$ for every $\alpha \geq \alpha_0$. The point is that the derivative of Λ_a is bounded from above by $\log((1 + \pi(\kappa))/\kappa)$ but the derivative of Λ_s is not. See for example Bahadur and Zabell (1979), Theorem 2.4, and Rockafellar (1970), Corollary 26.4.1. We can further choose α_0 such that $\Lambda_s'(\alpha_0) > 0$. Let $0 < x \leq 1/\Lambda_s'(\alpha_0)$. By (2.19),

$$\begin{aligned}\Lambda_s^*(1/x) &= \sup\{\alpha/x - \Lambda_s(\alpha); \alpha \geq \alpha_0\} \\ &\leq \sup\{\alpha/x - \Lambda_a(\alpha); \alpha \geq \alpha_0\} \leq \Lambda_a^*(1/x).\end{aligned}$$

Thus $R_s(x) \leq R_a(x)$ for small x, and the inequality is often strict. If this is the case then the ruin probability within the time horizon $[0, x\log u]$ has a tendency to be smaller when the alternative strategy with options is used.

Example 2.2. We will focus in this example on the correlation between inflation and the returns on the investments. Suppose that there are a stock and a risk-free asset available in the financial market. Let i be the generic rate of inflation as earlier, and let ρ_s be the generic rate of return on the stock. We assume that the pair $(\log(1 + i), \log(1 + \rho_s))$ has a two-dimensional normal distribution. Let (m_i, m_s) be the mean and

$$\Sigma = \begin{pmatrix} \sigma_i^2 & \sigma_{is} \\ \sigma_{is} & \sigma_s^2 \end{pmatrix}$$

the covariance matrix of the distribution. Assume further that the rate of return on the risk-free asset is a fixed constant ρ_f. Write in short $m_f = \log(1 + \rho_f)$. Finally, let $g \equiv 0$.

We assume that $m_s > m_f > m_i$. This corresponds to the natural situation where in the mean, the returns on the investments suffice to compensate the affect of inflation, and the return on the stock is larger than the risk-free return. It is also natural to assume a positive correlation between inflation and the return on the stock. Hence, we take $\sigma_{is} > 0$.

Consider first the strategy where the company invests all its money in the stock. Let η be the variance of the variable $\log(1 + i) - \log(1 + \rho_s)$, that is,

$$\eta = \sigma_i^2 - 2\sigma_{is} + \sigma_s^2.$$

We assume that $\eta > 0$ which just excludes superhedging (2.21). Associated with this strategy, let Λ_s and R_s be the functions corresponding to Λ in (2.15) and R in (2.18). Then

$$\Lambda_s(\alpha) = (m_i - m_s)\alpha + \eta\alpha^2/2 \quad \text{and} \quad R_s(x) = \frac{x}{2\eta}\left(\frac{1}{x} + (m_s - m_i)\right)^2.$$

Consider an alternative strategy where the company invests its money in the risk-free asset only. Let R_a be the function corresponding to R in (2.18). Then

$$R_a(x) = \frac{x}{2\sigma_i^2}\left(\frac{1}{x} + (m_f - m_i)\right)^2.$$

Suppose first that $\sigma_s^2 - 2\sigma_{is} > 0$. It is easy to see that then $R_s(x) < R_a(x)$ for small $x > 0$. This indicates that by investing in the risk-free asset, the company ends up with smaller ruin probabilities than by investing in the stock. On the other hand, if $\sigma_s^2 - 2\sigma_{is} < 0$ then $R_a(x) < R_s(x)$ for small $x > 0$. This gives the signal that it is safer to invest in the risky asset in the case where the correlation between inflation and the return on the stock is high.

2.4. Simulation examples

The asymptotic estimate of Theorem 2.1 gives the magnitude of the ruin probability for large u. In this section, we will complement the view by means of simulation. We will study the speed of convergence in the theorem and compare the estimates with the classical case. We also vary economic and insurance parameters to get an idea of their impact to ruin probabilities.

We begin by fixing the model to be considered. Concerning the returns on the investments, we assume that $\log(1 + r)$ has a normal distribution. Denote by m_r and σ_r the mean and the standard deviation, respectively. The rates of inflation and real growth are constants. Write $m_i = \log(1 + i)$ and $m_g = \log(1 + g)$. We only consider cases where $m_r > m_i + m_g$. Then $x_0 = 0$ and $\mu = 1/(m_r - m_i - m_g)$ in Theorem 2.1. The model for the claims will be fixed in all respects. The initial level of the mean of the number of claims is $\lambda = 100$. The structure variable q has the distribution

$$\mathbb{P}(q = 0.8) = \mathbb{P}(q = 1.2) = 0.5,$$

and the long term fluctuations in the numbers of claims are dropped by taking $b_n \equiv 1$. The claim sizes in the inflation-free economy are exponentially distributed with mean $m_Z = 1$. The safety loading s is varied. We also drop the fluctuations in the premiums by taking $c_n \equiv 1$ in every example.

In the first example, we study the ruin probability $\mathbb{P}(T_u \leq x \log u)$ with $x = 5$, and take $m_r = 0.1$, $\sigma_r = 0.5$ and $s = 0.05$. We consider two alternatives for inflation and real growth, namely, $(m_i, m_g) = (0.05, 0)$ and $(m_i, m_g) = (0, 0.05)$. In both of the cases, the distribution of $\log W$ is normal, and the mean and the standard deviation are -0.05 and 0.5, respectively. Further, $R(x) = 0.625$. Let $\hat{P}$ be the empirical estimate of $\mathbb{P}(T_u \leq x \log u)$ and let

$$\hat{R} = (\log u)^{-1} \log \hat{P}.$$

The results are given in Table 2.1. The estimates corresponding to the alternative values of (m_i, m_g) are rather close to each other which is not suprising. By Theorem 2.1, estimate $\hat{R}$ should be close to $R(x)$ for large u. The convergence seems not to be fast in this example. In the classical case where $i \equiv 0$, $g \equiv 0$ and $r \equiv 0$, the probability $\mathbb{P}(T_u \leq x \log u)$ is approximately 1.1×10^{-1} for $u = 10^2$, and it is less than 10^{-7} for $u = 10^3$. In comparison with the estimates of Table 2.1, the relative difference is huge for large u. The ruin probabilities in the table are larger that 10^{-4} even for $u = 10^6$.

TABLE 2.1

$m_r = 0.1$ AND $s = 0.05$

u	Time horizon	$(m_i, m_g) = (0.05, 0)$		$(m_i, m_g) = (0, 0.05)$	
		$\hat{P}$	$\hat{R}$	$\hat{P}$	$\hat{R}$
10^2	23	2.2×10^{-1}	0.330	1.9×10^{-1}	0.355
10^3	34	3.9×10^{-2}	0.472	3.7×10^{-2}	0.478
10^4	46	1.0×10^{-2}	0.497	8.1×10^{-3}	0.523
10^5	57	1.9×10^{-3}	0.543	1.6×10^{-3}	0.558
10^6	69	4.1×10^{-4}	0.564	3.5×10^{-4}	0.577

We next increase the mean of the returns on the investments, and also make a similar change in the insurance side by increasing the safety loading. In Table 2.1, we had $(m_r, s) = (0.1, 0.05)$. Now consider the cases where $(m_r, s) = (0.15, 0.05)$ and $(m_r, s) = (0.1, 0.1)$. We also fix $(m_i, m_g) = (0.05, 0)$, and take again $x = 5$ and $\sigma_r = 0.5$. The resulting estimates are given in Table 2.2.

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TABLE 2.2

$m_i = 0.05$ AND $m_g = 0$

u	Time horizon	$(m_r, s) = (0.15, 0.05)$		$(m_r, s) = (0.1, 0.1)$	
		$\hat{P}$	$\hat{R}$	$\hat{P}$	$\hat{R}$
10^2	23	1.5×10^{-1}	0.415	1.0×10^{-1}	0.493
10^3	34	1.3×10^{-2}	0.627	1.6×10^{-2}	0.603
10^4	46	1.9×10^{-3}	0.682	4.0×10^{-3}	0.599
10^5	57	2.2×10^{-4}	0.730	8.6×10^{-4}	0.613
10^6	69	2.4×10^{-5}	0.770	1.9×10^{-4}	0.619

In comparison with Table 2.1, both of the changes led to smaller ruin probabilities. For large u, the change in the economic parameter m_r reduced them more than the change in the safety loading s. This should be the case since $R(x) = 0.9$ when $m_r = 0.15$ and $R(x) = 0.625$ when $m_r = 0.1$.

3. PROOFS

We begin by giving an intuitive description of the proof of Theorem 2.1. Conditionally, for given values of economic factors, the mean of the number of claims in the year n can be approximated by

$$(1 + g_1) \cdots (1 + g_n), \tag{3.1}$$

and the mean of the discounted value of the claim size is approximately

$$\frac{1 + i_1}{1 + r_1} \cdots \frac{1 + i_n}{1 + r_n}.$$

The approximation concerning the number of claims is best motivated in the case where (3.1) is large. If ruin occurs in the year n then we can expect that the discounted value of the total claim amount is of the same magnitude as the initial capital u. This means that

$$\sum_{k=1}^{n} \log\left(\frac{(1 + i_k)(1 + g_k)}{1 + r_k}\right)$$

is approximately $\log u$. By taking n close to $x \log u$, the probability of the event can be estimated by making use of standard large deviations techniques. This leads to the estimate of the theorem.

The structure of the proof of Theorem 2.1 is largely the same as in Nyrhinen (2001). However, new technicalities arise because we allow real growth in the model. To explain this, let's drop inflation, returns on the investments, and cycles from the model. By the notations of Section 2, this means that the profit of the company in the year n is

$$P_n - X_n = (1+s)\lambda m_Z(1+g_1)\cdots(1+g_n) - \sum_{j=N_{n-1}+1}^{N_n} Z_j. \tag{3.2}$$

Real growth affects the premium P_n in a simple multiplicative way. If the same would be true for the total claim amount X_n then everything would be very similar to Nyrhinen (2001). This is not the case in the present model since real growth affects the number of claims $N_n - N_{n-1}$, and there seems to be no simple multiplicative connection between the distributions of N_n and N_{n-1}.

In the proof of upper bounds, we need new arguments to estimate moments of the total claim amounts V_n in (2.5). To this end, we state Lemma 3.1 below concerning compound Poisson distributions. For the lower bounds, the main new task is to study the interplay between real growth and short term fluctuations. The conclusion will be lower bound (3.28) below. All these things must be put together with the other parts of the model.

We need some basic facts from the theory of convex functions. They will be used throughout the proofs. The background can be found in Rockafellar (1970). Let $f:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ be a proper convex function. The convex conjugate f^* of f is a function $\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ defined by

$$f^*(x) = \sup\{tx - f(t);\ t\in\mathbb{R}\}.$$

Also f^* is convex. Assume henceforth that $f(0) = 0$. Then $f^*(x) \geq 0$ for every x. If $x = f'(t_x)$ for some $t_x \in \mathbb{R}$ then $f^*(x) = t_x x - f(t_x)$. In particular, if $f'(0)$ exists then $f^*(f'(0)) = 0$. In this case, f^* attains its global minimum at $f'(0)$, and so f^* is increasing on $(f'(0),\infty)$. Assume further that f is differentiable on some interval $[t_0,\infty)$, and write $z = \lim_{t\to\infty} f'(t)$. Then

$$f^*(x) = \sup\{tx - f(t);\ t \geq t_0\} \tag{3.3}$$

for $x \geq f'(t_0)$. Further,

$$f^*(x) < \infty \ \text{ for } \ x \in (f'(t_0), z) \ \text{ and } \ f^*(x) = \infty \ \text{ for } \ x > z. \tag{3.4}$$

Proof of Lemma 2.1. To see that $r_g < \infty$, it is sufficient to consider the case where $\mathbb{P}(g > 0) > 0$. It is clear that then $\Lambda_g(\alpha)$ tends to infinity as α tends to infinity. Thus $r_g < \infty$. The function R is strictly decreasing on (x_0, μ) by Martin-Löf (1983) and Nyrhinen (1998). The other results follow immediately from the convexity of Λ and from (3.3) and (3.4). □

Before the proof of Theorem 2.1, we will give an asymptotic result concerning the moments of compound Poisson distributions. The proof of the result will be given at the end of the section. Let $\mathcal{Z}$, $\mathcal{Z}_1$, $\mathcal{Z}_2$, ... be an i.i.d. sequence of non-negative random variables, and assume that $\mathbb{P}(\mathcal{Z} > 0) > 0$. Let $\mathcal{N}_\nu$ be a Poisson distributed random variable with the parameter ν. Assume that $\mathcal{N}_\nu$ is independent of the $\mathcal{Z}$-variables so that

$$\mathcal{X}_\nu := \mathcal{Z}_1 + \cdots + \mathcal{Z}_{\mathcal{N}_\nu} \tag{3.5}$$

has a compound Poisson distribution.

Lemma 3.1. *Assume that* $\mathbb{E}(\mathcal{Z}) < \infty$. *If* $\mathbb{E}(\mathcal{Z}^\alpha) < \infty$ *for* $\alpha > 0$ *then*

$$\lim_{\nu \to \infty} (\log \nu)^{-1} \log \mathbb{E}(\mathcal{X}_\nu^\alpha) = \alpha. \tag{3.6}$$

We now turn to proof of Theorem 2.1. It is convenient to consider a discounted version of the process $\{U_n\}$. For $n \in \mathbb{N}$, write

$$A_n = \frac{1 + i_n}{1 + r_n}$$

and

$$B_n = V_n - (1 + s)\lambda\, m_Z\, c_n (1 + g_1) \cdots (1 + g_n) \tag{3.7}$$

where V_n is as in (2.5). Let further

$$Y_n = \sum_{k=1}^{n} A_1 \cdots A_{k-1} (1 + i_k) B_k. \tag{3.8}$$

By dividing each U_n by $(1 + r_1) \cdots (1 + r_n)$, it is seen that the time of ruin T_u can be expressed as

$$T_u = \begin{cases} \inf\{n \in \mathbb{N};\ Y_n > u\} \\ \infty \text{ if } Y_n \le u \text{ for every } n. \end{cases} \tag{3.9}$$

Define the generating functions Λ_i and Λ_A by

$$\Lambda_i(\alpha) = \log \mathbb{E}\left((1 + i)^\alpha\right) \tag{3.10}$$

and

$$\Lambda_A(\alpha) = \log \mathbb{E}\left(\left(\frac{1 + i}{1 + r}\right)^\alpha\right) \tag{3.11}$$

for $\alpha \in \mathbb{R}$. Let Λ and Λ_g be as in (2.15) and in (2.16). By our assumptions,

$$\Lambda = \Lambda_A + \Lambda_g. \tag{3.12}$$

Proof of Theorem 2.1. We begin by showing that for every $x \in (0, \mu)$,

$$\limsup_{u \to \infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) \leq -R(x). \tag{3.13}$$

Let V_n be as in (2.5), and let

$$\bar{Y}_n = 1 + \sum_{k=1}^{n} A_1 \cdots A_{k-1} (1 + i_k) V_k. \tag{3.14}$$

Then $\bar{Y}_n \geq 1$, $\bar{Y}_n \geq Y_n$, and $\{\bar{Y}_n\}$ is an increasing process. Hence,

$$\mathbb{P}(T_u \leq x \log u) \leq \mathbb{P}(\bar{Y}_{\lceil x \log u \rceil} \geq u) \tag{3.15}$$

where $\lceil a \rceil$ denotes the smallest integer $\geq a$.

We will apply the Gärtner-Ellis theorem to the sequence $\{\log \bar{Y}_n\}$. We refer to Dembo and Zeitouni (1998) for the background. To apply the theorem, define the function $\Gamma : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\Gamma(\alpha) = \limsup_{n \to \infty} n^{-1} \log \mathbb{E}\left(\bar{Y}_n^{\alpha}\right).$$

Then Γ is convex. The first step is to show that

$$\Gamma(\alpha) \leq \begin{cases} 0 \text{ for } \alpha \leq 0 \\ \Lambda(\alpha) \text{ for } \alpha \geq \mathrm{r}. \end{cases} \tag{3.16}$$

For $\alpha \leq 0$, (3.16) holds since $\bar{Y}_n \geq 1$. Consider now an arbitrary $\alpha > 0$, and let $\varepsilon > 0$ and $k \in \mathbb{N}$. By our model assumptions,

$$\mathbb{E}(V_k^{\alpha}) = \mathbb{E}\left(e^{-\lambda b_k (1+g_1) \cdots (1+g_k) q_k} \sum_{h=0}^{\infty} \frac{(\lambda b_k (1+g_1) \cdots (1+g_k) q_k)^h}{h!} \mathbb{E}((Z_1 + \cdots + Z_h)^{\alpha})\right).$$

Take $\mathcal{Z} = Z$ in Lemma 3.1 and choose large $M > 0$ such that $\mathbb{E}(\mathcal{X}_{\nu}^{\alpha}) \leq \nu^{\alpha+\varepsilon}$ whenever $\nu > M$. Write

$$G_M = \{\lambda b_k (1+g_1) \cdots (1+g_k) q_k > M\}.$$

Then

$$\begin{aligned} \mathbb{E}(V_k^{\alpha} \mathbf{1}(G_M)) &\leq \mathbb{E}\left((\lambda b_k (1+g_1) \cdots (1+g_k) q_k)^{\alpha+\varepsilon}\right) \\ &\leq (\lambda \bar{b})^{\alpha+\varepsilon} \mathbb{E}(q^{\alpha+\varepsilon}) e^{k \Lambda_g(\alpha+\varepsilon)}. \end{aligned}$$

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Concerning the complement of G_M, we have

$$\mathbb{E}\Big(V_k^\alpha \mathbf{1}(G_M^c)\Big) \le e^M \mathbb{E}(X_M^\alpha).$$

By the above estimates, there exists a constant C_1 such that

$$\mathbb{E}(V_k^\alpha) \le C_1 \max\Big(1, e^{k\Lambda_g(\alpha+\varepsilon)}\Big)$$

for every $k \in \mathbb{N}$. By (3.12), there exists a constant C_2 such that

$$\mathbb{E}((A_1 \cdots A_{k-1}(1+i_k)V_k)^\alpha) \le C_2 e^{(k-1)\Lambda(\alpha)} e^{-(k-1)\Lambda_g(\alpha)} \max(1, e^{k\Lambda_g(\alpha+\varepsilon)}) \quad (3.17)$$

for every k.

Assume now that $\mathfrak{r} < \infty$, and let $\alpha \ge \mathfrak{r}$. Then $\Lambda(\alpha)$, $\Lambda_g(\alpha)$ and $\Lambda_g(\alpha+\varepsilon)$ are all non-negative. For $\alpha > 1$, apply Minkowski's inequality to conclude by (3.17) that there exists a constant C such that for every n,

$$\mathbb{E}(\bar{Y}_n^\alpha) \le C e^{n\Lambda(\alpha)} e^{n(\Lambda_g(\alpha+\varepsilon)-\Lambda_g(\alpha))}. \quad (3.18)$$

By the continuity of Λ_g, the estimate implies (3.16) for $\alpha \ge \mathfrak{r}$ in the case where $\alpha \ge 1$. A similar proof applies to the case where $\alpha \in (0,1)$. Instead of Minkowski's inequality, we now make use of the inequality

$$(x+y)^\alpha \le x^\alpha + y^\alpha \quad (3.19)$$

for $x, y \ge 0$.

We need an additional estimate for Γ in the case where $\mathfrak{r} = \infty$. Then $\Lambda(\alpha) \le 0$ for every $\alpha \ge 0$. Since $\Lambda_g(\alpha)$ is non-negative for $\alpha \ge \mathfrak{r}_g$ we conclude as above that

$$\Gamma(\alpha) \le 0 \quad \text{for} \quad \alpha \ge \mathfrak{r}_g. \quad (3.20)$$

Let $\varepsilon > 0$. By the above estimates and by convexity, Γ is finite everywhere. By the Gärtner-Ellis theorem,

$$\begin{aligned} &\limsup_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(\bar{Y}_{\lceil x\log u\rceil} \ge u) \\ &\le \limsup_{u\to\infty} \frac{\lceil x\log u\rceil}{\log u} \big(\lceil x\log u\rceil\big)^{-1} \log \mathbb{P}\left(\frac{\log \bar{Y}_{\lceil x\log u\rceil}}{\lceil x\log u\rceil} \ge \frac{1}{x} - \varepsilon\right) \quad (3.21) \\ &\le -x \inf\Big\{\Gamma^*(v); v \ge \frac{1}{x} - \varepsilon\Big\}. \end{aligned}$$

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Consider first the case where $\mathfrak{r} = \infty$. Then (3.20) holds and consequently, $\Gamma^*(v) = \infty$ for every $v > 0$. Thus (3.15) and (3.21) imply (3.13). Let now $\mathfrak{r} < \infty$. It follows from (3.3) and (3.16) that for $v > \Lambda'(\mathfrak{r})$,

$$\begin{aligned}\Gamma^*(v) &= \sup\{\alpha v - \Gamma(\alpha);\, \alpha \in \mathbb{R}\}\\ &\geq \sup\{\alpha v - \Lambda(\alpha);\, \alpha \geq \mathfrak{r}\} = \Lambda^*(v)\,.\end{aligned}$$

Further, Λ^* is increasing on $(\Lambda'(0), \infty)$, and hence, on $(\Lambda'(\mathfrak{r}), \infty)$. Recall that $x < \mu$. By the above discussion, it is seen that for small ε, (3.21) is at most $-x\Lambda^*(1/x - \varepsilon)$. Finally, $x \neq x_0$ so that $x\Lambda^*(1/x - \varepsilon)$ tends to $R(x)$ as ε tends to zero. Thus (3.15) implies (3.13).

To complete the proof, we have to show that for every $x \in (x_0, \mu)$,

$$\liminf_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) \geq -R(x). \tag{3.22}$$

Namely, by Lemma 2.1, (3.22) is trivial for $x \in (0, x_0)$. In particular, we can assume that $x_0 < \mu$ and $\mathfrak{r} < \infty$. This implies that Λ is strictly convex. Recall the definitions of Λ_g, Λ_i and Λ_A from (2.16), (3.10) and (3.11).

We will construct a subset of $\{T_u \leq x \log u\}$ which is large enough to lead to (3.22). Consider first the case where g is not identically zero. Define the continuous time processes

$$\left\{z_n^A(t);\, 0 < t < \infty\right\},\ \left\{z_n^{A,i}(t);\, 0 < t < \infty\right\} \text{ and } \left\{z_n^g(t);\, 0 < t < \infty\right\}$$

by

$$\begin{aligned} z_n^A(t) &= \left(\log A_1 + \cdots + \log A_{[tn]}\right)/n,\\ z_n^{A,i}(t) &= \left(\log A_1 + \cdots + \log A_{[tn]-1} + \log(1 + i_{[tn]})\right)/n\end{aligned}$$

and

$$z_n^g(t) = \left(\log(1 + g_1) + \cdots + \log(1 + g_{[tn]})\right)/n.$$

Fix $p > 0$ and small $\varepsilon > 0$, and let x_1 and x_2 be such that $0 < x_1 < x_2 < x$. Write

$$\begin{aligned}\mathcal{H}_n^A(\varepsilon) &= \left\{\sup_{0<t\leq x} |\, z_n^A(t) - (1-p)\,t/x_1\,| \leq \varepsilon\right\},\\ \mathcal{H}_n^{A,i}(\varepsilon) &= \left\{\sup_{0<t\leq x} |\, z_n^{A,i}(t) - (1-p)\,t/x_1\,| \leq \varepsilon\right\}\end{aligned}$$

and

$$\mathcal{H}_n^g(\varepsilon) = \left\{\sup_{0<t\leq x} |\, z_n^g(t) - pt/x_1\,| \leq \varepsilon\right\}. \tag{3.23}$$

Write further

$$\mathcal{H}_n^B(\varepsilon) = \left\{B_k \geq \varepsilon(1+g_1) \cdots (1+g_k),\ \forall k \in \left[x_1 n, xn\right]\right\}$$

where B_k is as in (3.7). By Mogulskii's theorem,

$$\liminf_{n\to\infty} n^{-1} \log \mathbb{P}\left(\mathcal{H}_n^A(\varepsilon/4)\right) \geq -x\Lambda_A^*\left(\frac{1-p}{x_1}\right). \tag{3.24}$$

We refer to Dembo and Zeitouni (1998) and Martin-Löf (1983). For $\alpha > 0$, we have by Chebycheff's inequality,

$$\mathbb{P}\left(\log(1+i)/n > \varepsilon/4\right) \leq e^{-n\alpha\varepsilon/4 + \Lambda_i(\alpha)}.$$

A similar estimate holds for the probability $\mathbb{P}(\log(1+i)/n < -\varepsilon/4)$ so that

$$\limsup_{n\to\infty} n^{-1} \log \mathbb{P}\left(|\log(1+i_k)/n| > \varepsilon/4 \text{ for some } 1 \leq k \leq \lceil xn \rceil\right)$$
$$\leq \limsup_{n\to\infty} n^{-1} \log\left(\lceil xn \rceil\left(e^{-n\alpha\varepsilon/4 + \Lambda_i(\alpha)} + e^{-n\alpha\varepsilon/4 + \Lambda_i(-\alpha)}\right)\right) = -\alpha\varepsilon/4.$$

By the same arguments, it is seen that

$$\limsup_{n\to\infty} n^{-1} \log \mathbb{P}\left(|\log A_k/n| > \varepsilon/4 \text{ for some } 1 \leq k \leq \lceil xn \rceil\right) \leq -\alpha\varepsilon/4.$$

Since α is arbitrary we conclude by (3.24) that

$$\liminf_{n\to\infty} n^{-1} \log \mathbb{P}\left(\mathcal{H}_n^{A,i}(\varepsilon)\right) \geq -x\Lambda_A^*\left(\frac{1-p}{x_1}\right). \tag{3.25}$$

Similarly, by making use of Mogulskii's theorem, it is seen that

$$\liminf_{n\to\infty} n^{-1} \log \mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\right) \geq -x\Lambda_g^*\left(\frac{p}{x_1}\right).$$

Let $\mathcal{X}_v$ be a compound Poisson variable as in (3.5), and take $\mathcal{Z} = Z$. Let $m_Z = \mathbb{E}(Z)$ as earlier. Fix $v_0 > 0$, and write

$$\gamma = \gamma(v_0) = \inf\left\{\mathbb{P}(\mathcal{X}_v \geq v m_Z);\ v \geq v_0\right\}. \tag{3.26}$$

Obviously, γ is strictly positive. Denote

$$\mathbb{R}_+^k = \left\{(y_1, \ldots, y_k);\ y_1 > 0, \ldots, y_k > 0\right\}.$$

Corresponding to $\mathcal{H}_n^g(\varepsilon)$ in (3.23), define the subset of $\mathbb{R}_+^{[xn]}$ by

$$H_n^g(\varepsilon) = \left\{\left(y_1^g,\ldots,y_{[xn]}^g\right)\in\mathbb{R}_+^{[xn]};\ \sup_{0<t\le x}\left|\left(\log y_1^g+\cdots+\log y_{[tn]}^g\right)/n-pt/x_1\right|\le\varepsilon\right\}.$$

Choose $a>(1+s)\bar{c}/\underline{b}$ such that $\mathbb{P}(q>a)>0$. This is possible by assumption (2.13). Recall the definition of the distribution function F_n from (2.7) and (2.8). By (2.9) and (2.10),

$$\begin{aligned}\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\right)&\ge\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\cap\left\{q_k>a\text{ for every }\left\lceil x_1n\right\rceil\le k\le\left[xn\right]\right\}\right)\\&\ge\int_{C_n}\prod_{k=\lceil x_1n\rceil}^{[xn]}\mathbb{P}\left(X_{\lambda y_k^b y_1^g\cdots y_k^g y_k^q}\ge((1+s)\lambda m_Z\bar{c}+\varepsilon)y_1^g\cdots y_k^g\right)\\&\qquad dF_{[xn]}\left(y_1^b,\ldots,y_{[xn]}^b,y_1^g,\ldots,y_{[xn]}^g,y_1^q,\ldots,y_{[xn]}^q\right)\end{aligned}$$

where

$$\begin{aligned}C_n=&\left\{\left(y_1^b,\ldots,y_{[xn]}^b,\ y_1^g,\ldots,y_{[xn]}^g,\ y_1^q,\ldots,y_{[xn]}^q\right)\in\mathbb{R}_+^{3[xn]};\right.\\&\left.\left(y_1^g,\ldots,y_{[xn]}^g\right)\in H_n^g(\varepsilon),\ y_k^q>a,\ \forall k\in\left[\lceil x_1n\rceil,[xn]\right]\right\}.\end{aligned}$$

Recall that $b_k\ge\underline{b}>0$ for every k. Take $v_0=1$ in (3.26) to see that for small ε and large n,

$$\begin{aligned}\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\right)&\ge\int_{C_n}\prod_{k=\lceil x_1n\rceil}^{[xn]}\mathbb{P}\left(X_{\lambda y_k^b y_1^g\cdots y_k^g y_k^q}\ge\lambda y_k^b\,y_1^g\cdots y_k^g\,y_k^q\,m_Z\right)\\&\qquad dF_{[xn]}\left(y_1^b,\ldots,y_{[xn]}^b,\ y_1^g,\ldots,y_{[xn]}^g,\ y_1^q,\ldots,y_{[xn]}^q\right)\\&\ge\gamma^{[xn]-\lceil x_1n\rceil+1}\,\mathbb{P}(q>a)^{[xn]-\lceil x_1n\rceil+1}\,\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\right).\end{aligned}\tag{3.27}$$

Consequently,

$$\begin{aligned}&\liminf_{n\to\infty}n^{-1}\log\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\right)\\&\ge(x-x_1)\left(\log\gamma+\log\mathbb{P}\left(q>a\right)\right)-x\Lambda_g^*\left(\frac{p}{x_1}\right).\end{aligned}\tag{3.28}$$

On the event $\mathcal{H}_n^{A,i}(\varepsilon)\cap\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)$, we have for large n,

$$Y_{\lceil x_1n\rceil}\ge-D_1e^{n(1+2\varepsilon)}$$

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and

$$Y_{\lfloor x_2 n \rfloor} - Y_{\lfloor x_1 n \rfloor} \geq D_2 e^{n(x_2/x_1 - 2\varepsilon)}$$

where D_1 and D_2 are positive constants. Choose ε, x_1 and x_2 in an appropriate way to see that $Y_{\lfloor x_2 n \rfloor} > e^n$ for large n on the event. By (3.25) and (3.28),

$$\begin{aligned} &\liminf_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) \\ &\geq \liminf_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(Y_{\lfloor x_2 \lceil \log u \rceil \rfloor} > u) \\ &\geq \liminf_{u\to\infty} (\log u)^{-1} \log \mathbb{P}\Big(\mathcal{H}^{A,i}_{\lceil \log u \rceil}(\varepsilon) \cap \mathcal{H}^{g}_{\lceil \log u \rceil}(\varepsilon) \cap \mathcal{H}^{B}_{\lceil \log u \rceil}(\varepsilon)\Big) \\ &\geq -x\left(\Lambda_A^*\left(\frac{1-p}{x_1}\right) + \Lambda_g^*\left(\frac{p}{x_1}\right)\right) + o(1) \end{aligned} \tag{3.29}$$

where $o(1)$ tends to zero as x_1 tends to x from the left. Choose x_1 close to x such that $x_1 \in (x_0, \mu)$. Then $\Lambda'(\alpha_1) = 1/x_1$ for some $\alpha_1 > \mathfrak{r}$ and hence,

$$\Lambda^*(1/x_1) = \alpha_1/x_1 - \Lambda(\alpha_1).$$

We now choose p such that $\Lambda_g'(\alpha_1) = p/x_1$. Then $p > 0$ and it is easy to see by (3.12) that

$$\Lambda^*\left(\frac{1}{x_1}\right) = \Lambda_A^*\left(\frac{1-p}{x_1}\right) + \Lambda_g^*\left(\frac{p}{x_1}\right).$$

By (3.29),

$$\liminf_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) \geq -x\Lambda^*\left(\frac{1}{x_1}\right) + o(1). \tag{3.30}$$

Now Λ^* is continuous at $1/x$ since $x \in (x_0, \mu)$. Thus (3.22) follows from (3.30).

Consider finally the case where $g \equiv 0$. We now choose $p = 0$ in the definitions of the sets $\mathcal{H}_n^A(\varepsilon)$ and $\mathcal{H}_n^g(\varepsilon)$. Then $\mathbb{P}(\mathcal{H}_n^g(\varepsilon)) = 1$ for every n. Similarly to the case $p > 0$, it is seen by choosing $v_0 = \lambda \underline{b} a$ in (3.26) that

$$\begin{aligned} &\liminf_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T_u \leq x \log u) \\ &\geq -x\Lambda_A^*\left(\frac{1}{x_1}\right) + o(1) = -x\Lambda^*\left(\frac{1}{x_1}\right) + o(1). \end{aligned}$$

This implies (3.22). □

Proof of Lemma 3.1. Define the function $L_{\mathcal{N}} : (0,\infty) \to \mathbb{R}$ by

$$L_{\mathcal{N}}(t) = \limsup_{v\to\infty} (\log v)^{-1} \log \mathbb{E}\left(\mathcal{N}_v^t\right). \tag{3.31}$$

We first show that $L_{\mathcal{N}}(t) = t$ and that (3.31) holds as the limit for each t.

We have $\mathbb{E}(\mathcal{N}_v) = v$, and by convention, let $\mathbb{E}(\mathcal{N}_v^0) = 1$. It is easy to see that then

$$\mathbb{E}\left(\mathcal{N}_v^k\right) = \sum_{h=0}^{k-1} \binom{k-1}{h} v\,\mathbb{E}\left(\mathcal{N}_v^h\right) \tag{3.32}$$

for $k = 2, 3, \ldots$. This shows that

$$\lim_{v\to\infty} (\log v)^{-1} \log \mathbb{E}\left(\mathcal{N}_v^t\right) = t \tag{3.33}$$

for every $t \in \mathbb{N}$. Let $t \in (0,1)$. Apply Jensen's inequality to conclude that

$$\mathbb{E}\left(\mathcal{N}_v^t\right) \le \mathbb{E}\left(\mathcal{N}_v\right)^t = v^t$$

so that $L_{\mathcal{N}}(t) \le t$. Now by Hölder's inequality, $L_{\mathcal{N}}$ is convex so that necessarily, $L_{\mathcal{N}}(t) = t$ for every $t > 0$. It remains to show that (3.31) holds as the limit. Assume on the contrary that there would exist a sequence $v_j \to \infty$ and $t_0 > 0$ such that

$$\lim_{j\to\infty} (\log v_j)^{-1} \log \mathbb{E}\left(\mathcal{N}_{v_j}^{t_0}\right) < t_0. \tag{3.34}$$

Write

$$\underline{L}_{\mathcal{N}}(t) = \limsup_{j\to\infty} (\log v_j)^{-1} \log \mathbb{E}\left(\mathcal{N}_{v_j}^t\right) \tag{3.35}$$

for $t > 0$. By the first part of the proof, $\underline{L}_{\mathcal{N}}(t) = t$ for every $t \in \mathbb{N}$, and $\underline{L}_{\mathcal{N}}(t) \le t$ for every $t \in (0,1)$. By (3.34), $\underline{L}_{\mathcal{N}}(t_0) < t_0$. This is a contradiction since also $\underline{L}_{\mathcal{N}}$ is convex. It follows that (3.31) holds as the limit for every $t > 0$.

Consider now (3.6). Let first $\alpha \ge 1$. By Minkowski's inequality,

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}_v^\alpha\right) &= \sum_{h=1}^{\infty} e^{-v} \frac{v^h}{h!} \mathbb{E}\left(\left(\mathcal{Z}_1 + \cdots + Z_h\right)^\alpha\right) \\ &\le \sum_{h=1}^{\infty} e^{-v} \frac{v^h}{h!} h^\alpha\, \mathbb{E}\left(\mathcal{Z}^\alpha\right) = \mathbb{E}\left(\mathcal{N}_v^\alpha\right) \mathbb{E}\left(\mathcal{Z}^\alpha\right). \end{aligned} \tag{3.36}$$

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By (3.26) and Jensen's inequality,

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}_\nu^\alpha\right) &\geq \sum_{h=1}^{\infty} e^{-\nu} \frac{\nu^h}{h!} h^\alpha \, \mathbb{E}(\mathcal{Z})^\alpha \\ &= \mathbb{E}\left(\mathcal{N}_\nu^\alpha\right) \mathbb{E}(\mathcal{Z})^\alpha. \end{aligned} \tag{3.37}$$

By the above estimates and the first part of the proof, (3.6) holds if $\alpha \geq 1$.
Let now $\alpha \in (0,1)$. By (3.36) and Jensen's inequality,

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}_\nu^\alpha\right) &\leq \sum_{h=1}^{\infty} e^{-\nu} \frac{\nu^h}{h!} h^\alpha \, \mathbb{E}(\mathcal{Z})^\alpha \\ &= \mathbb{E}\left(\mathcal{N}_\nu^\alpha\right) \mathbb{E}(\mathcal{Z})^\alpha. \end{aligned}$$

To get an appropriate lower bound, let $M > 0$ be such that $\mathbb{P}(\mathcal{Z} \in (0, M)) > 0$, and let $\underline{\mathcal{Z}}_k = \min(\mathcal{Z}_k, M)$ for $k \in \mathbb{N}$. Write

$$\underline{\mathcal{X}}_\nu = \underline{\mathcal{Z}}_1 + \cdots + \underline{\mathcal{Z}}_{N_\nu}.$$

Then also $\underline{\mathcal{X}}_\nu$ has a compound Poisson distribution. Write

$$\underline{L}_{\mathcal{X}}(t) = \limsup_{\nu \to \infty} (\log \nu)^{-1} \log \mathbb{E}\left(\underline{\mathcal{X}}_\nu^t\right) \tag{3.38}$$

for $t > 0$. By the first part of the proof, $\underline{L}_{\mathcal{X}}(t) = t$ for $t \geq 1$, and $\underline{L}_{\mathcal{X}}(t) \leq t$ for $t \in (0,1)$. Also $\underline{L}_{\mathcal{X}}$ is convex so that by making use of arguments similar to the first part of the proof, it is seen that $\underline{L}_{\mathcal{X}}(t) = t$ for every $t > 0$, and further, that (3.38) holds as the limit for every t. The desired lower bound now follows since $\mathcal{X}_\nu \geq \underline{\mathcal{X}}_\nu$. □

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