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Existence and non-existence of the first eigenvalue of the perturbed Hardy–Sobolev operator

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In this paper we study the existence, non-existence and simplicity of the first eigenvalue of the perturbed Hardy–Sobolev operator $-\Delta - \frac{1}{4}(n-2)^2(q/|x|^2)$ under various assumptions on the perturbation q. We study the asymptotic behaviour of the first eigenfunction near the origin when the perturbation q is $q = s$, $0 < s < 1$. We will also establish the best constant in a Hardy–Sobolev inequality proved by Adimurthi *et al*.

1. Introduction

Recall the Hardy–Sobolev inequality, which states that, for $n \geqslant 3$ and for every $u \in H_0^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla u|^2 - \tfrac{1}{4}(n-2)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\,\mathrm{d}x \geqslant 0. \tag{1.1}$$

$\frac{1}{4}(n-2)^2$ is the best constant in (1.1) and is never achieved. Recently, there has been considerable interest in improving this inequality and one of the important improvements obtained is by Brezis and Vazques [3]. They showed that if Ω is a bounded domain with smooth boundary and $0 \in \Omega$, then there exists a constant $C > 0$ such that

$$\int_{\Omega} |\nabla u|^2 - \tfrac{1}{4}(n-2)^2 \int_{\Omega} \frac{u^2}{|x|^2} \geqslant C \int_{\Omega} u^2 \tag{1.2}$$

holds for every $u \in H_0^1(\Omega)$. Recently, Adimurthi *et al*. [1] have proved that if Ω is as before and $R \geqslant e \sup_{\Omega} |x|$, then there exists $C > 0$ such that

$$\int_{\Omega} |\nabla u|^2 - \tfrac{1}{4}(n-2)^2 \int_{\Omega} \frac{u^2}{|x|^2} \geqslant C \int \frac{u^2}{|x|^2(\log R/|x|)^2} \tag{1.2$'$}$$

holds for every $u \in H_0^1(\Omega)$. Furthermore, if $\lambda(\Omega)$ denotes the best choice for C in (1.2), then $\lambda(\Omega)$ is never achieved, i.e. the infimum in

$$\lambda(\Omega) = \inf_{u \in H_0^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 - \tfrac{1}{4}(n-2)^2 \int_{\Omega} \frac{u^2}{|x|^2} : \int_{\Omega} u^2 = 1 \right\} \tag{1.3}$$

is never achieved for any domain Ω, as before. This means that the following eigenvalue problem,

$$\left.\begin{aligned} -\Delta u - \tfrac{1}{4}(n-2)^2\frac{u}{|x|^2} &= \lambda u \quad \text{in } \Omega,\\ u > 0 \quad &\text{in } \Omega,\\ u \in H_0^1(\Omega),& \end{aligned}\right\} \tag{1.4}$$

does not have a solution for $\lambda = \lambda(\Omega)$. Cabré and Martel considered the problem

$$\left.\begin{aligned} -\Delta u - \tfrac{1}{4}(n-2)^2\nu\frac{u}{|x|^2} &= \lambda u \quad \text{in } \Omega,\\ u > 0 \quad &\text{in } \Omega,\\ u \in H_0^1(\Omega),& \end{aligned}\right\} \tag{1.5}$$

and showed that, for $0 < \nu < 1$, it admits a solution u_ν corresponding to the eigenvalue $\lambda_\nu(\Omega)$, and that when Ω is a ball centred at zero, u_ν behaves like $|x|^{(n-2)/2[-1+\sqrt{1-\nu}]}$ near zero.

Our interest in this direction is two fold. First, an existence/non-existence result for a perturbed form of (1.4), and secondly, the asymptotic study of u_ν near the origin, for general domain Ω. Observe that (1.4) has a similar phenomenon, as in the case of critical exponent problem, where the best Sobolev constant is never achieved in any $\Omega \neq \mathbb{R}^n$. Therefore, as in [2], our aim is to consider the perturbed form of (1.4),

$$\left.\begin{aligned} -\Delta u - \tfrac{1}{4}(n-2)^2\frac{qu}{|x|^2} &= \lambda u \quad \text{in } \Omega,\\ u > 0 \quad &\text{in } \Omega,\\ u \in H_0^1(\Omega),& \end{aligned}\right\} \tag{1.6}$$

where $0 \leqslant q(x) \leqslant 1$, and look for a necessary and sufficient condition on q so that (1.6) admits a solution. In this paper, we will give some conditions on q that assure the existence and non-existence of a solution to (1.6) and extend the result of Cabré and Martel [5] regarding the asymptotic behaviour of u_ν to general domains. Our work is motivated by the results of Brezis *et al*. [4] and the main results are the following.

Let q and η be such that

(H_1) $0 \leqslant q \leqslant 1$;

(H_2) $\eta \geqslant 0$, $\eta \in L^\infty(\Omega \setminus B(0,R))\ \forall R > 0$; and

(H_3)

$$\limsup_{x\to 0} |x|^2\biggl(\log\frac{1}{|x|}\biggr)^2\eta(x) = 0, \quad n \geqslant 3,$$

$$\limsup_{x\to 0} |x|^2\biggl(\log\frac{1}{|x|}\biggr)^2\biggl(\log\biggl(\log\frac{1}{|x|}\biggr)\biggr)^2\eta(x) = 0, \quad n = 2.$$

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Consider the problem

$$\left.\begin{aligned} -\Delta u - \tfrac{1}{4}(n-2)^2 \frac{qu}{|x|^2} &= \lambda\eta u \quad \text{in } \Omega, \\ u &> 0, \\ u &\in H_0^1(\Omega). \end{aligned}\right\} \tag{$\mathbb{P}_q$}$$

For q and η as before, define

$$\lambda(q) = \inf_{u \in H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 - \tfrac{1}{4}(n-2)^2 \int_\Omega \frac{qu^2}{|x|^2} : \int_\Omega \eta u^2 = 1 \right\}. \tag{1.7}$$

Note that, from (1.2′), $\lambda(q) > 0$.

THEOREM 1.1. *Let $n \geqslant 3$ and q, η satisfy (H_1)–(H_3). Then we have the following.*

(1) *Problem $(\mathbb{P}_q)$ has a solution for $\lambda = \lambda(q)$, provided q satisfies*

$$\liminf_{x\to 0} \left(\log \frac{1}{|x|}\right)^2 (1 - q(x)) > \frac{3}{(n-2)^2}. \tag{1.8}$$

(2) *Problem $(\mathbb{P}_q)$ does not have a solution for any $\lambda \in \mathbb{R}$ if q satisfies*

$$\sup_{0<|x|<R} \left(\log \frac{1}{|x|}\right)^2 (1 - q(x)) \leqslant \frac{3}{(n-2)^2} \quad \text{for some } R > 0. \tag{1.9}$$

(3) *Problem $(\mathbb{P}_q)$ always admits a distribution solution that lies in $W_0^{1,p}(\Omega)$ for every $1 \leqslant p < 2$.*

(4) *Let $q = \nu$, $0 < \nu < 1$, and let η satisfy the stronger assumption*

$$\limsup_{x\to 0} |x| \left(\log \frac{1}{|x|}\right)^{-1} \eta(x) = 0.$$

Let u_ν be the solution to $(\mathbb{P}_q)$ corresponding to $\lambda = \lambda(\nu)$. Then there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} C_1 &\leqslant \liminf_{x\to 0} |x|^{((n-2)/2)[1-\sqrt{1-\nu}]} u_\nu(x) \\ &\leqslant \limsup_{x\to 0} |x|^{((n-2)/2)[1-\sqrt{1-\nu}]} u_\nu(x) \\ &\leqslant C_2 \end{aligned}$$

and

$$\limsup_{x\to 0} |x|^{((n-2)/2)[1-\sqrt{1-\nu}]+1} |\nabla u_\nu(x)| \leqslant C_2.$$

REMARK 1.2. By taking

$$q = 1 - \frac{C}{(\log R/|x|)^\delta}, \quad \delta > 0, \quad R > \sup_{x\in\Omega} |x|,$$

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we can see that the problem

$$-\Delta u - \tfrac{1}{4}(n-2)^2\frac{u}{|x|^2} = \lambda u - \tfrac{1}{4}(n-2)^2 C\frac{u}{|x|^2(\log R/|x|)^\delta} \quad \text{in } \Omega,$$
$$u > 0 \quad \text{in } \Omega,$$
$$u \in H_0^1(\Omega)$$

has a solution if and only if $0 < \delta < 2$ or $\delta = 2$ and $C > 3/(n-2)^2$.

When $n = 2$, the corresponding Hardy–Sobolev inequality is given by

$$\int_\Omega |\nabla u|^2 \geqslant C\int_\Omega \frac{u^2}{|x|^2(\log R/|x|)^2}\,\mathrm{d}x \quad \forall u \in H_0^1(\Omega), \tag{1.10}$$

where $R \geqslant e\sup_{x\in\Omega}\{|x|\}$. In general, we have, in $\mathbb{R}^n$ $\forall u \in W_0^{1,n}(\Omega)$,

$$\int_\Omega |\nabla u|^n \geqslant \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n(\log R/|x|)^n}, \tag{1.11}$$

where Ω is a domain in $\mathbb{R}^n$ containing the origin and $R \geqslant e^{2/n}\sup_{x\in\Omega}\{|x|\}$. As in the higher-dimensional case, we can get the following improvement in (1.11). Let Ω be as before, $R \geqslant e\sup_{x\in\Omega}\{|x|\}$ and $R_1 \geqslant (e^e)^{2/n}\sup_{x\in\Omega}\{|x|\}$. Then there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla u|^n \geqslant \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|u|^n}{|x|^n(\log R/|x|)^n} + C\int_\Omega \frac{|u|^n}{|x|^n(\log R/|x|)^n(\log(\log R_1/|x|))^n} \tag{1.11$'$}$$

(see [1, 6] for details (in these references, R is taken to be R_1, but it can be easily modified as (1.11$'$))).

Regarding (1.11), we have the following result.

Theorem 1.3. *$((n-1)/n)^n$ is the best constant in (1.11) and is never achieved.*

Remark 1.4. As a consequence of theorem 1.3, we obtain the best constant in (1.2$'$) as $C = \frac{1}{4}$.

Next we consider the two-dimensional analogue of $(\mathbb{P}_q)$,

$$\left.\begin{aligned} -\Delta u - \frac{1}{4}\frac{qu}{|x|^2(\log R/|x|)^2} &= \lambda\eta u \quad \text{in } \Omega,\\ u &> 0,\\ u &\in H_0^1(\Omega), \end{aligned}\right\} \tag{$\bar{\mathbb{P}}_q$}$$

where $R \geqslant e\sup_{x\in\Omega}\{|x|\}$. We also define $\lambda(q)$ in this case as

$$\lambda(q) = \inf_{u\in H_0^1(\Omega)}\left\{\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega \frac{qu^2}{|x|^2(\log R/|x|)^2} : \int_\Omega \eta u^2 = 1\right\}. \tag{1.12}$$

Again, $\lambda(q) > 0$, thanks to (1.11$'$).

THEOREM 1.5. *Let $n = 2$ and q, η satisfy (H_1)–(H_3). Then we have the following.*

(1) *Problem $(\bar{\mathbb{P}}_q)$ has a solution for $\lambda = \lambda(q)$, provided q satisfies*

$$\liminf_{x\to 0}\biggl(\log\biggl(\log\frac{R}{|x|}\biggr)\biggr)^2(1-q(x)) > 3. \tag{1.13}$$

(2) *Problem $(\bar{\mathbb{P}}_q)$ does not have a solution for any $\lambda \in \mathbb{R}$ if q satisfies*

$$\sup_{0<|x|<R_1}\biggl(\log\biggl(\log\frac{R}{|x|}\biggr)\biggr)^2(1-q(x)) \leqslant 3 \quad \text{for some } R_1 > 0. \tag{1.14}$$

(3) *Problem $(\bar{\mathbb{P}}_q)$ always admits a distribution solution that lies in $W_0^{1,q}(\Omega)$ for every $1 \leqslant q < 2$.*

(4) *Let $q = \nu$, $0 < \nu < 1$ and let η satisfy the stronger assumption*

$$\limsup_{x\to 0}|x|\biggl(\log\biggl(\log\frac{1}{|x|}\biggr)\biggr)^{-1}\eta(x) = 0.$$

Let u_ν be the solution to $(\bar{\mathbb{P}}_q)$ corresponding to $\lambda = \lambda(\nu)$. Then there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} C_1 &\leqslant \liminf_{x\to 0}\biggl(\log\frac{R}{|x|}\biggr)^{[-1+\sqrt{1-\nu}]/2}u_\nu(x) \\ &\leqslant \limsup_{x\to 0}\biggl(\log\frac{R}{|x|}\biggr)^{[-1+\sqrt{1-\nu}]/2}u_\nu(x) \\ &\leqslant C_2 \end{aligned} \tag{1.15}$$

and

$$\limsup_{x\to 0}\biggl(\log\frac{R}{|x|}\biggr)^{[-1+\sqrt{1-\nu}]/2}|x||\nabla u_\nu(x)| \leqslant C_2. \tag{1.16}$$

Finally, we have the following result, which establishes the simplicity of the first eigenvalue of $(\mathbb{P}_q)$ and $(\bar{\mathbb{P}}_q)$.

THEOREM 1.6. *Let Ω be a bounded domain with smooth boundary, let q, η satisfy (H_1) and (H_2) and let $u_1, u_2 \in H_0^1(\Omega)$ solve $(\mathbb{P}_q)$ (or $(\bar{\mathbb{P}}_q)$ when $n = 2$), with $\lambda = \lambda(q)$. Then $u_1 = mu_2$ for some $m > 0$.*

REMARK 1.7. As in [4], we can extend our results to more general equations of the form

$$-\operatorname{div}(p\nabla u) - \tfrac{1}{4}(n-2)^2\frac{qu}{|x|^2} = \lambda\eta u, \tag{1.17}$$

where p and q satisfy, $p \in C^2(\Omega)$, $p > 0$ in $\bar{\Omega}$ and

$$0 \leqslant Q = \frac{|x|^2}{p^2(n-2)^2}(2p\Delta p - |\nabla p|^2) - \frac{q}{p} \leqslant 1.$$

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This follows since if u satisfies (1.7), then $v = \sqrt{p}u$ satisfies

$$-\Delta v - \tfrac{1}{4}(n-2)^2 \frac{Qv}{|x|^2} = \frac{\lambda\eta}{p} v.$$

REMARK 1.8. In the sequel, we will extend our results to the p-Laplacian.

2. Preliminary lemmas

We start with a few lemmas needed in the construction of sub and supersolutions. Here we like to mention that Chaudhuri [6] has used a test function similar to the one used in lemma 2.1 to prove some non-existence results.

LEMMA 2.1. *Let $n \geqslant 3$ and*

$$u(x) = |x|^{-((n-2)/2)s}\left(\log \frac{1}{|x|}\right)^{\delta}, \quad 0 < s \leqslant 1, \quad \delta \in \mathbb{R}.$$

Then

(i) *for $0 < R < 1$, $u \in H^1(B(R))$ if and only if $s < 1$ or $s = 1$ and $\delta < -\frac{1}{2}$;*

(ii) *for $x \neq 0$,*

$$\Delta u(x) = \frac{u}{|x|^2}\left[\tfrac{1}{4}(n-2)^2 s(s-2) + \frac{\delta(n-2)(s-1)}{\log 1/|x|} + \frac{\delta(\delta-1)}{(\log 1/|x|)^2}\right]. \tag{2.1}$$

Proof. By direct calculation, for $x \neq 0$,

$$\nabla u(x) = \frac{u}{|x|^2}\left[-\tfrac{1}{2}(n-2)s - \frac{\delta}{\log 1/|x|}\right]x.$$

Hence $u \in H^1(B(R))$ if and only if

$$\int_{B(R)} |x|^{-(n-2)s-2}\left(\log\frac{1}{|x|}\right)^{2\delta} < \infty$$

and this happens if and only if $s < 1$ or $s = 1$ and $\delta < -\frac{1}{2}$. This proves (i).

Differentiating again, we obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} = {} & \frac{u}{|x|^2}\left[-\tfrac{1}{2}(n-2)s - \frac{\delta}{\log 1/|x|}\right] \\ & + 2\frac{u}{|x|^2}\left[\tfrac{1}{2}(n-2)s + \frac{\delta}{\log 1/|x|}\right]\frac{x_i^2}{|x|^2} \\ & + \frac{u}{|x|^2}\left[\tfrac{1}{2}(n-2)s + \frac{\delta}{\log 1/|x|}\right]^2 \frac{x_i^2}{|x|^2} \\ & - \frac{u}{|x|^2}\left[\frac{\delta}{(\log 1/|x|)^2}\frac{x_i^2}{|x|^2}\right].\end{aligned}$$

Hence

$$\Delta u(x) = \frac{u}{|x|^2}\left[\tfrac{1}{4}(n-2)^2 s(s-2) + \frac{\delta(n-2)(s-1)}{\log 1/|x|} + \frac{\delta(\delta-1)}{(\log 1/|x|)^2}\right].$$

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This proves the lemma. □

Next we will prove a two-dimensional version of lemma 2.1.

LEMMA 2.2. *Let $n = 2$ and*

$$u(x) = \left(\log \frac{R}{|x|}\right)^{\delta_1} \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2}, \quad R > 0, \quad \delta_1, \delta_2 \in \mathbb{R}.$$

Then

(i) *for $0 < R_1 < e^{-1}R$, $u \in H^1(B(R_1))$ if and only if $\delta_1 < \frac{1}{2}$ or $\delta_1 = \frac{1}{2}$ and $\delta_2 < -\frac{1}{2}$;*

(ii) *for $0 < |x| < e^{-1}R$,*

$$\Delta u(x) = \frac{u}{|x|^2 (\log R/|x|)^2} \left[\delta_1(\delta_1 - 1) + \frac{\delta_2(2\delta_1 - 1)}{\log(\log R/|x|)} + \frac{\delta_2(\delta_2 - 1)}{(\log(\log R/|x|))^2}\right]. \tag{2.2}$$

Proof. By direct calculations, we obtain

$$\nabla u(x) = -\left(\log \frac{R}{|x|}\right)^{\delta_1 - 1} \left[\delta_1 \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2} + \delta_2 \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2 - 1}\right] \frac{x}{|x|^2}.$$

Hence

$$\int_{B(R)} |\nabla u|^2 < \infty$$

if and only if

$$\int_0^{R_1} \left(\log \frac{R}{r}\right)^{2\delta_1 - 2} \left(\log\left(\log \frac{R}{r}\right)\right)^{2\delta_2} \frac{\mathrm{d}r}{r}$$

is finite, and this happens if and only if $\delta_1 < \frac{1}{2}$ or $\delta_1 = \frac{1}{2}$ and $\delta_2 < -\frac{1}{2}$. This proves (i).

Differentiating again, we obtain

$$\begin{aligned}
\Delta u(x) = &-\sum_{i=1}^{2} \left(\log \frac{R}{|x|}\right)^{\delta_1 - 1} \left[\delta_1 \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2} \right. \\
&\qquad\qquad \left. + \delta_2 \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2 - 1}\right] \left(\frac{1}{|x|^2} - \frac{2x_i^2}{|x|^4}\right) \\
&+ \sum_{i=1}^{2} \left(\log \frac{R}{|x|}\right)^{\delta_1 - 2} \left[\delta_1(\delta_1 - 1) \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2} \right. \\
&\qquad\qquad + \delta_2(2\delta_1 - 1) \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2 - 1} \\
&\qquad\qquad \left. + \delta_2(\delta_2 - 1) \left(\log\left(\log \frac{R}{|x|}\right)\right)^{\delta_2 - 2}\right] \frac{x_i^2}{|x|^4} \\
= &\frac{u}{|x|^2 (\log R/|x|)^2} \left[\delta_1(\delta_1 - 1) + \frac{\delta_2(2\delta_1 - 1)}{\log(\log R/|x|)} + \frac{\delta_2(\delta_2 - 1)}{(\log(\log R/|x|))^2}\right].
\end{aligned}$$

This proves the lemma. □

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LEMMA 2.3. *Let $n \geqslant 3$ and $v(x) = |x|^{\alpha(x)}$, where*

$$\alpha(x) = \tfrac{1}{2}(n-2)[-1+\sqrt{1-s}] + |x| = \alpha_0 + |x|, \quad 0 < s < 1.$$

Then

(i) *for $0 < R < \infty$, $v \in H^1(B(R))$;*

(ii) *for $x \neq 0$,*

$$\Delta v(x) = v(x)\biggl[\frac{(n+(n-1)\log|x|)}{|x|} + \frac{(n-2)\alpha_0}{|x|^2} + (\log|x|)^2 + \frac{(\alpha_0+|x|)^2}{|x|^2} + \frac{2(\alpha_0+|x|)\log|x|}{|x|}\biggr]. \tag{2.3}$$

Proof. We have

$$v(x) = |x|^{\alpha(x)}.$$

Taking logs, we obtain

$$\log v = \alpha(x)\log|x|.$$

Also,

$$\nabla \log v = \frac{\nabla v}{v} \tag{2.4}$$

and

$$\Delta \log v = \frac{\Delta v}{v} - \frac{|\nabla v|^2}{v^2}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\Delta v = v[\Delta \log v + |\nabla \log v|^2]. \tag{2.6}$$

Now

$$\nabla \log v = \log|x|\nabla\alpha(x) + \alpha(x)\frac{x}{|x|^2} = (\log|x|)\frac{x}{|x|} + (\alpha_0+|x|)\frac{x}{|x|^2} \tag{2.7}$$

and

$$\Delta \log v = \frac{(n+(n-1)\log|x|)}{|x|} + \frac{(n-2)\alpha_0}{|x|^2}. \tag{2.8}$$

Hence (i) follows from (2.4) and (2.7). Now, substituting (2.7) and (2.8) in (2.6), we obtain

$$\Delta v = v\biggl[\frac{(n+(n-1)\log|x|)}{|x|} + \frac{(n-2)\alpha_0}{|x|^2} + (\log|x|)^2 + \frac{(\alpha_0+|x|)^2}{|x|^2} + \frac{2(\alpha_0+|x|)\log|x|}{|x|}\biggr].$$

This proves (ii) and hence the lemma. □

Again, we have the two-dimensional version of lemma 2.3.

LEMMA 2.4. *Let* $n = 2$ *and* $v(x) = (\log R/|x|)^{\alpha(x)}$, *where*

$$\alpha(x) = \tfrac{1}{2}[1 - \sqrt{1-s}] - |x| = \alpha_0 - |x|, \quad 0 < s < 1, \quad R > 0.$$

Then

(i) *for* $0 < R_1 < e^{-1}R$, $v \in H^1(B(R_1))$;

(ii) *for* $x \neq 0$,

$$\Delta v(x) = v(x)\left[\frac{\alpha_0^2 - \alpha_0}{|x|^2(\log R/|x|)^2} - \frac{\log(\log R/|x|)}{|x|}(1 + \circ(1))\right], \tag{2.9}$$

where $\circ(1)$ *goes to zero as* $|x| \to 0$.

Proof. We have

$$\log v = \alpha(x)\left(\log\left(\log\frac{R}{|x|}\right)\right).$$

By direct calculation,

$$\nabla \log v = \left[\frac{-\log(\log R/|x|)}{|x|} - \frac{\alpha(x)}{|x|^2(\log R/|x|)}\right]x, \tag{2.10}$$

$$\Delta \log v = \frac{-\log(\log R/|x|)}{|x|} + \frac{2}{|x|(\log R/|x|)} - \frac{\alpha(x)}{|x|^2(\log R/|x|)^2}. \tag{2.11}$$

Hence, substituting (2.10) and (2.11) in (2.6), we obtain

$$\begin{aligned}\Delta v &= v[\Delta \log v + |\nabla \log v|^2] \\ &= v(x)\left[\frac{-\log(\log R/|x|)}{|x|} + \frac{2}{|x|(\log R/|x|)} - \frac{\alpha(x)}{|x|^2(\log R/|x|)^2}\right. \\ &\qquad \left. + \left(\log\left(\log\frac{R}{|x|}\right)\right)^2 + \frac{(\alpha(x))^2}{|x|^2(\log R/|x|)^2} + \frac{2\alpha(x)(\log(\log R/|x|))}{|x|(\log R/|x|)}\right] \\ &= v(x)\left[\frac{\alpha_0^2 - \alpha_0}{|x|^2(\log R/|x|)^2} - (1 + \circ(1))\frac{\log(\log R/|x|)}{|x|}\right].\end{aligned}$$

This proves (2.9) and hence the lemma. □

LEMMA 2.5. *Let* $R > 1$ *and* $n \geqslant 2$. *Then*

$$\inf_{u \in W_0^{1,n}(B(1))}\left\{\int_{B(1)} |\nabla u|^n \bigg/ \int_{B(1)} \frac{|u|^n}{|x|^n(\log R/|x|)^n}\right\} \leqslant \left(\frac{n-1}{n}\right)^n, \tag{2.12}$$

where $B(1)$ *is the unit ball in* $\mathbb{R}^n$.

Proof. For $0 < l < 1$ and $(n-1)/n < \delta < 1$, define

$$u(x) = u_{l,\delta}(x) = \begin{cases}(\log R/l)^\delta - (\log R)^\delta, & 0 \leqslant |x| \leqslant l, \\ (\log R/|x|)^\delta - (\log R)^\delta, & l \leqslant |x| \leqslant 1.\end{cases}$$

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Then u is well defined, $u=0$ on $\partial B(1)$ and

$$\nabla u(x)=\begin{cases}0, & 0\leqslant |x|<l,\\ -\delta(\log R/|x|)^{\delta-1}(x/|x|^2), & l<|x|<1.\end{cases}$$

Hence, by the choice of δ, $u\in W_0^{1,n}(B(1))$. Now

$$\begin{aligned}\int_{B(1)}|\nabla u|^n &= w_{n-1}\delta^n\int_l^1\frac{(\log R/r)^{n\delta-n}}{r}\,\mathrm{d}r\\ &= w_{n-1}\frac{\delta^n}{(n\delta-n+1)}\biggl[\biggl(\log\frac{R}{l}\biggr)^{n\delta-n+1}-(\log R)^{n\delta-n+1}\biggr]. \end{aligned}\tag{2.13}$$

Here, w_{n-1} denotes the surface area of an $(n-1)$-dimensional sphere,

$$\begin{aligned}&\int_{B(1)}\frac{|u|^n}{|x|^n(\log R/|x|)^n}\\ &\quad= w_{n-1}\biggl(\biggl(\log\frac{R}{l}\biggr)^{\delta}-(\log R)^{\delta}\biggr)^n\int_0^l\frac{1}{r(\log R/r)^n}\,\mathrm{d}r\\ &\qquad+w_{n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\log R)^{k\delta}\int_l^1\frac{(\log R/r)^{(n\delta-n-\delta k)}}{r}\,\mathrm{d}r\\ &\quad= w_{n-1}\sum_{k=0}^{n}\frac{(-1)^k}{(n-1)}\binom{n}{k}(\log R)^{k\delta}\biggl(\log\frac{R}{l}\biggr)^{n\delta-k\delta-n+1}\\ &\qquad+w_{n-1}\sum_{k=0}^{n}\frac{(-1)^k}{(n\delta-k\delta-n+1)}\binom{n}{k}(\log R)^{k\delta}\\ &\qquad\qquad\times\biggl[\biggl(\log\frac{R}{l}\biggr)^{n\delta-n-\delta k+1}-(\log R)^{n\delta-n-\delta k+1}\biggr]\\ &\quad= w_{n-1}\biggl[\frac{n\delta}{(n-1)(n\delta-n+1)}\biggl(\log\frac{R}{l}\biggr)^{n\delta-n+1}\\ &\qquad+\sum_{k=1}^{n}C_{n,k,\delta}(\log R)^{k\delta}\biggl(\log\frac{R}{l}\biggr)^{n\delta-k\delta-n+1}+C_{n,\delta}(\log R)^{n\delta-n+1}\biggr],\end{aligned}\tag{2.14}$$

where $c_{n,k,\delta}$ and $C_{n,\delta}$ are finite constants bounded uniformly with respect to δ close to $(n-1)/n$. Now, from (2.13) and (2.14), we obtain

$$\begin{aligned}&\int|\nabla u|^n\bigg/\int_{B(1)}\frac{|u|^n}{|x|^n\log(\log R/|x|)^n}\\ &\quad=\biggl(\frac{n-1}{n}\biggr)\delta^{n-1}\biggl[1-\biggl(\frac{\log R}{\log R/l}\biggr)^{n\delta-n+1}\biggr]\\ &\qquad\times\biggl[1+\tilde{C}_{n,\delta}\biggl(\frac{\log R}{\log R/l}\biggr)^{n\delta-n+1}+\sum_{k=1}^{n}\tilde{C}_{n,k,\delta}\biggl(\frac{\log R}{\log R/l}\biggr)^{k\delta}\biggr]^{-1},\end{aligned}\tag{2.15}$$

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where $\tilde{C}_{n,\delta}$, $\tilde{C}_{n,k,\delta}$ are finite constants bounded uniformly with respect to δ close to $(n-1)/n$. Now, by choosing δ close to $(n-1)/n$ and then l close to zero, we can make the right-hand side of (2.15) as close as we want to $((n-1)/n)^n$. This proves the lemma. □

3. Proof of theorems

3.1. Proof of theorem 1.1

STEP 1 (existence). Let q satisfy (1.8), $0 < s < 1$ and $\lambda(sq)$, $\lambda(q)$ be as in (1.7). Since the operator $-\Delta - \frac{1}{4}(n-2)^2(sq/|x|^2)$ defined on $H_0^1(\Omega)$ is coercive, there exists $u_s \in H_0^1(\Omega)$ satisfying

$$\left.\begin{aligned} -\Delta u_s - \tfrac{1}{4}(n-2)^2\frac{squ_s}{|x|^2} &= \lambda(sq)\eta(x)u_s \quad \text{in } \Omega, \\ u_s > 0 \quad &\text{in } \Omega, \\ \|u_s\|_{H_0^1(\Omega)} &= 1. \end{aligned}\right\} \tag{3.1}$$

We will prove the existence of a solution to $(\mathbb{P}_q)$ by showing that u_s converges in $H_0^1(\Omega)$ to u_1 (say) and u_1 satisfies $(\mathbb{P}_q)$. We proceed as follows.

Let

$$u = |x|^{-(n-2)/2}\left(\log\frac{1}{|x|}\right)^{-\delta_1},$$

where δ_1 is chosen so that

$$\tfrac{1}{2} < \delta_1 \quad \text{and} \quad \liminf_{x\to 0}\left(\log\frac{1}{|x|}\right)^2(1-q(x)) > \frac{4\delta_1(\delta_1+1)}{(n-2)^2}.$$

Then, from (2.1), we obtain

$$\begin{aligned} &-\Delta u - \tfrac{1}{4}(n-2)^2\frac{qu}{|x|^2} - \lambda(sq)\eta(x)u \\ &\quad = \tfrac{1}{4}(n-2)^2\frac{u}{|x|^2(\log 1/|x|)^2} \\ &\qquad\qquad \times\left[(1-q(x))\left(\log\frac{1}{|x|}\right)^2 - \frac{4\delta_1(\delta_1+1)}{(n-2)^2}\right. \\ &\qquad\qquad\qquad \left. - \eta(x)\lambda(sq)\frac{4}{(n-2)^2}\left(\log\frac{1}{|x|}\right)^2|x|^2\right] \\ &\quad \geqslant 0, \end{aligned}$$

if $|x|$ is small enough, because of the choice of δ_1, and (H_3). That is, there exists an $R > 0$ such that $B(R) \subset \Omega$ and

$$-\Delta u - \tfrac{1}{4}(n-2)^2\frac{qu}{|x|^2} - \lambda(sq)\eta(x)u \geqslant 0 \quad \text{in } B(R). \tag{3.2}$$

Now, using a standard elliptic estimate, we can find an $M > 0$ such that $u_s \leqslant Mu$ on $|x| = R$ and $\forall s \in (0,1)$. Let $w_s = u_s - Mu$. Then $w_s^+ \in H_0^1(B(R))$ and,

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from (3.1) and (3.2), we obtain

$$-\Delta w_s - \tfrac{1}{4}(n-2)^2 \frac{sqw_s}{|x|^2} - \lambda(sq)\eta(x)w_s \leqslant -\tfrac{1}{4}(1-s)M(n-2)^2 \frac{qu}{|x|^2} \quad \text{in } B(R).$$

Testing the above relation against w_s^+, we obtain

$$\int_{B(R)} |\nabla w_s^+|^2 - \tfrac{1}{4}(n-2)^2 s \int_{B(R)} \frac{q}{|x|^2}(w_s^+)^2 - \lambda(sq)\int_{B(R)} \eta (w_s^+)^2 \leqslant -\tfrac{1}{4}(1-s)M(n-2)^2 \int_{B(R)} \frac{quw_s^+}{|x|^2} \leqslant 0$$

Hence, from the definition of $\lambda(sq)$, we get that the function

$$W_s = \begin{cases} w_s^+ & \text{in } B(R), \\ 0 & \text{in } \Omega \setminus B(R), \end{cases}$$

is an eigenfunction of (3.1). Hence, by the strong maximum principle,

$$w_s^+ = 0 \quad \text{in } B(R),$$

i.e.

$$u_s \leqslant Mu \quad \text{in } B(R). \tag{3.3}$$

Since u_s is bounded in $H_0^1(\Omega)$, by passing to a subsequence if necessary, we may assume that u_s converges to u_1 weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwise almost everywhere in Ω, as $s \to 1$. We know from (3.3) that

$$\frac{u_s^2}{|x|^2} \leqslant M^2 \frac{u^2}{|x|^2} \quad \text{and} \quad \int_{B(R)} \frac{u^2}{|x|^2} < \infty.$$

Hence, by the dominated convergence theorem,

$$\int_\Omega \frac{qu_s^2}{|x|^2} \to \int_\Omega \frac{qu_1^2}{|x|^2} \quad \text{as } s \to 1 \tag{3.4}$$

and

$$\int_\Omega \eta u_s^2 \to \int_\Omega \eta u_1^2 \quad \text{as } s \to 1. \tag{3.5}$$

Now, from (3.1), we have

$$\int_\Omega |\nabla u_s|^2 - \tfrac{1}{4}(n-2)^2 s \int_\Omega \frac{qu_s^2}{|x|^2} = \lambda(sq)\int_\Omega \eta u_s^2. \tag{3.6}$$

Taking the limit as $s \to 1$ in (3.6) and using (3.4), (3.5), the weak lower semicontinuity of $H_0^1(\Omega)$ norm and the fact that $\lambda(sq) \to \lambda(q)$ as $s \to 1$, we obtain

$$\int_\Omega |\nabla u_1|^2 - \tfrac{1}{4}(n-2)^2 \int_\Omega \frac{qu_1^2}{|x|^2} \leqslant \lambda(q)\int_\Omega \eta u_1^2.$$

Therefore, from the definition of $\lambda(q)$, we obtain

$$\int_\Omega |\nabla u_1|^2 - \tfrac{1}{4}(n-2)^2 \int_\Omega \frac{qu_1^2}{|x|^2} = \lambda(q)\int_\Omega \eta u_1^2. \tag{3.7}$$

Hence, from (3.4), (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}\lim_{s\to 1}\int_\Omega |\nabla u_s|^2 &= \lim_{s\to 1}\left\{\tfrac{1}{4}(n-2)^2 s\int_\Omega \frac{qu_s^2}{|x|^2} + \lambda(sq)\int_\Omega \eta u_s^2\right\}\\ &= \tfrac{1}{4}(n-2)^2\int_\Omega \frac{qu_1^2}{|x|^2} + \lambda(q)\int_\Omega \eta u_1^2\\ &= \int_\Omega |\nabla u_1|^2,\end{aligned}$$

i.e. $u_s \rightharpoonup u_1$ in $H_0^1(\Omega)$ and $\|u_s\|_{H_0^1(\Omega)} \to \|u\|_{H_0^1(\Omega)} \Rightarrow u_s$ converges strongly in $H_0^1(\Omega)$ to u_1. Hence u_1 solves

$$\begin{gathered}-\Delta u_1 - \tfrac{1}{4}(n-2)^2\frac{qu_1}{|x|^2} = \lambda(q)\eta(x)u_1 \quad \text{in } \Omega,\\ u_1 \geqslant 0 \quad \text{in } \Omega,\\ u_1 \in H_0^1(\Omega).\end{gathered}$$

Also, by the strong maximum principle, $u_1 > 0$ in Ω. This proves the first part of theorem 1.1.

Step 2 (non-existence). We will prove the non-existence by contradiction. Let q satisfy (1.9) and assume that $(\mathbb{P}_q)$ has a solution $u_1 \in H_0^1(\Omega)$, for some λ. Then, by Hardy's inequality (1.1), λ is greater than or equal to zero. We claim that there exist $m > 0$ and $R > 0$ such that

$$u_1 \geqslant m|x|^{-(n-2)/2}\left(\log\frac{1}{|x|}\right)^{-1/2} \quad \text{in } B(R).$$

This gives a contradiction, since, by Hardy's inequality,

$$\frac{u_1^2}{|x|^2} \in L^1(B(R)), \quad \text{but} \int_{B(R)} |x|^{-n}\left(\log\frac{1}{|x|}\right)^{-1} = \infty.$$

Proof of claim. We cannot use $|x|^{-(n-2)/2}(\log 1/|x|)^{-1/2}$ as a test function because it is not in $H^1(B(R))$, for any $R > 0$. So, for $s > 1$, let us define

$$\phi_s(x) = |x|^{-(n-2)/2}\left(\log\frac{1}{|x|}\right)^{-s/2}.$$

Let R be as in (1.9). Then $\phi_s \in H^1(B(R))$ and, using (2.1), we obtain

$$\begin{aligned}&-\Delta\phi_s - \tfrac{1}{4}(n-2)^2\frac{q\phi_s}{|x|^2}\\ &\quad = \tfrac{1}{4}(n-2)^2\frac{\phi_s}{|x|^2(\log 1/|x|)^2}\left[(1-q(x))\left(\log\frac{1}{|x|}\right)^2 - \frac{s(s+2)}{(n-2)^2}\right] \leqslant 0. \qquad (3.8)\end{aligned}$$

Now, using the strong maximum principle, we can find an $m > 0$ such that

$$u_1 \geqslant m\phi_s \quad \text{on } |x| = R \quad \text{for } 1 < s < 2.$$

Let $\psi_s = m\phi_s - u_1$. Then $\psi_s^+ \in H_0^1(B(R))$ and, by (3.8) and $(\mathbb{P}_q)$,

$$-\Delta\psi_s - \tfrac{1}{4}(n-2)^2\frac{q\psi_s}{|x|^2} \leqslant 0 \quad \text{in } B(R).$$

Testing the above relation against ψ_s^+, we obtain

$$\int_{B(R)} |\nabla\psi_s^+|^2 - \tfrac{1}{4}(n-2)^2\int_{B(R)}\frac{q(\psi_s^+)^2}{|x|^2} \leqslant 0.$$

Therefore,

$$\begin{aligned}\int_{B(R)} |\nabla\psi_s^+|^2 &\leqslant \tfrac{1}{4}(n-2)^2\int_{B(R)}\frac{q(\psi_s^+)^2}{|x|^2}\\ &\leqslant \tfrac{1}{4}(n-2)^2\int_{B(R)}\frac{(\psi_s^+)^2}{|x|^2}.\end{aligned}$$

Hence, by Hardy's inequality,

$$\int_{B(R)} |\nabla\psi_s^+|^2 = \tfrac{1}{4}(n-2)^2\int_{B(R)}\frac{(\psi_s^+)^2}{|x|^2},$$

and this implies $\psi_s^+ = 0$ in $B(R)$, because equality is never achieved in Hardy's inequality, i.e.

$$u_1 \geqslant m\phi_s \quad \text{in } B(R) \quad \forall s \in (1,2).$$

Taking the limit as $s \to 1$, we obtain

$$u_1 \geqslant m|x|^{-(n-2)/2}\biggl(\log\frac{1}{|x|}\biggr)^{-1/2} \quad \text{in } B(R).$$

This proves the claim and hence the second part of theorem 1.1. □

Step 3 (existence of $W_0^{1,p}$ solution). Let $0 < \nu < 1$ and v_ν satisfy $(\mathbb{P}_{\nu q})$, with $\int_\Omega v_\nu^2 = 1$. As mentioned before, v_ν exists because $-\Delta - \frac{1}{4}(n-2)^2(\nu q/|x|^2)$ is coercive on $H_0^1(\Omega)$. We will show that $v_\nu \to v_1$ in $W_0^{1,p}(\Omega)$ $\forall p < 2$. First, we will prove the following estimates on v_s and ∇v_s.

Fix δ such that $0 < \delta < 1$. Then there exists an $R > 0$ such that, for $\nu \in (\frac{1}{2}, 1)$,

$$v_\nu \leqslant M_1|x|^{-((n-2)/2)\nu}\biggl(\log\frac{1}{|x|}\biggr)^{\delta} \quad \text{in } B(R), \tag{3.9}$$

$$|\nabla v_\nu| \leqslant M_2|x|^{-((n-2)/2)\nu-1}\biggl(\log\frac{1}{|x|}\biggr)^{\delta} \quad \text{in } B(R), \tag{3.10}$$

where M_1 and M_2 are constants independent of ν.

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Proof of (3.9). Let $\xi_\nu = |x|^{-((n-2)/2)\nu}(\log 1/|x|)^\delta$. Then, for R small enough, we have, from (2.1),

$$\begin{aligned}
&-\Delta\xi_\nu - \tfrac{1}{4}(n-2)^2\nu\frac{q\xi_\nu}{|x|^2} - \lambda(\nu q)\eta\xi_\nu \\
&\quad = \frac{\xi_\nu}{|x|^2}\biggl[\tfrac{1}{4}(n-2)^2\nu(2-\nu-q) + \frac{\delta(n-2)(1-\nu)}{\log 1/|x|} + \frac{\delta(1-\delta)}{(\log 1/|x|)^2} - \eta\lambda(\nu q)|x|^2\biggr] \\
&\quad \geqslant 0 \quad \text{in } B(R) \quad \forall \nu \in (0,1).
\end{aligned}$$

Fix $R > 0$ such that the above relation holds and then, using the elliptic estimate, choose $M_1 > 0$ so that $z_\nu = u_\nu - M_1\xi_\nu \leqslant 0$ on $|x| = R$ for $\frac{1}{2} < \nu < 1$. Then $z_\nu^+ \in H_0^1(B(R))$ and

$$-\Delta z_\nu - \tfrac{1}{4}(n-2)^2\nu\frac{qz_\nu}{|x|^2} - \lambda(\nu q)\eta z_\nu \leqslant 0 \quad \text{in } B(R).$$

Then, as before, we obtain

$$0 \leqslant \int_{B(R)} |\nabla z_\nu^+|^2 - \tfrac{1}{4}(n-2)^2\nu\int_{B(R)}\frac{q(z_\nu^+)^2}{|x|^2} - \lambda(\nu q)\int_{B(R)}\eta(z_\nu^+)^2 \leqslant 0.$$

This shows that the function

$$Z_s := \begin{cases} z_s^+ & \text{in } B(R), \\ 0 & \text{in } \Omega \setminus B(R) \end{cases}$$

is an eigenfunction of $-\Delta - \frac{1}{4}(n-2)^2\nu(q/|x|^2) = \lambda\eta$. Hence, by the strong maximum principle, $z_s \equiv 0$ in Ω, i.e. $z_s^+ = 0$ in $B(R)$. This proves (3.9). □

Proof of (3.10). To prove our estimate on $|\nabla v_s|$, we proceed as in [4]. Fix $x \in B(\frac{1}{2}R)$, where R is as in (3.9). Let $r = \frac{1}{2}|x|$ and define

$$\tilde{v}_\nu(y) = v_\nu(x + ry), \quad y \in B(1).$$

Then $\tilde{v}_\nu$ satisfies

$$-\Delta\tilde{v}_\nu(y) = c_\nu(y)\tilde{v}_\nu(y) \quad \text{in } B(1),$$

where $|c_\nu(y)| \leqslant C$ $\forall y \in B(1)$ and $\nu \in (\frac{1}{2}, 1)$ and C is independent of x. Hence, by the standard elliptic estimate,

$$\begin{aligned}
|\nabla\tilde{v}_\nu(0)| &\leqslant C_1(\|\tilde{v}_\nu\|_{L^\infty(B(1))} + \|\Delta\tilde{v}_\nu\|_{L^\infty(B(1))}) \\
&\leqslant C_1(1 + C)\|\tilde{v}_\nu\|_{L^\infty(B(1))},
\end{aligned}$$

where C_1 is independent of ν and x. Writing $\nabla\tilde{v}_\nu$ in terms of ∇v_ν and using (3.9), we obtain, for $x \in B(\frac{1}{2}R)$,

$$\nabla v_\nu(x) \leqslant M_2|x|^{((n-2)/2)s-1}\biggl(\log\frac{1}{|x|}\biggr)^\delta.$$

This proves (3.10). □

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From the above two estimates (3.9) and (3.10), it follows that v_ν is bounded in $W_0^{1,p}(\Omega)$ $\forall p \in [1,2)$. Hence we can find a subsequence $\nu_n \to 1$ such that v_{ν_n} converges weakly to v_1 (say) in $W_0^{1,p}(\Omega)$ for all $p \in (1,2)$ and pointwise almost everywhere. Also, by the standard elliptic estimate, we can assume that

$$v_{\nu_n} \to v_1 \quad \text{in } C^1_{\text{loc}}(\bar{\Omega} \setminus 0).$$

Let $1 < p < 2$ and choose p_1 and q_1 so that $p_1, q_1 > 1$, $pp_1 < 2$ and $1/p_1 + 1/q_1 = 1$. Then

$$\int_{B(r)} (|v_\nu|^p + |\nabla v_\nu|^p) \leqslant |B(r)|^{1/q_1} \biggl(\biggl(\int_{B(r)} |v_\nu|^{pp_1} \biggr)^{1/p_1} + \biggl(\int_{B(r)} |\nabla v_\nu|^{pp_1} \biggr)^{1/p_1} \biggr).$$

Combining this fact with the $C^1_{\text{loc}}(\bar{\Omega} \setminus 0)$ convergence of v_{ν_n} to v_1, we obtain $\|v_{\nu_n}\|_{W_0^{1,p}} \to \|v_1\|_{W_0^{1,p}}$, and hence $v_{\nu_n} \to v_1$ strongly in $W_0^{1,p}(\Omega)$. Since v_{ν_n} satisfies equation $(\mathbb{P}_{\nu q})$ with $\int v_{\nu_n}^2 = 1$, passing to the limits as $n \to \infty$, we obtain

$$\begin{gathered} -\Delta v_1 - \tfrac{1}{4}(n-2)^2 \frac{q v_1}{|x|^2} = \lambda(q) \eta v_1 \quad \text{in } \Omega, \\ v_1 \geqslant 0 \quad \text{in } \Omega, \\ \int v_1^2 = 1. \end{gathered}$$

Since $\int v_1^2 = 1$, v_1 is not identically equal to zero. Hence, by the strong maximum principle, $v_1 > 0$.

Step 4 (asymptotic behaviour). Let $0 < \nu < 1$ and u_ν be as in the statement of theorem 1.1. Define $\phi_\nu^1 = |x|^{(n-2)[-1+\sqrt{1-\nu}]/2}$. Then, from (2.1), it follows that

$$-\Delta \phi_\nu^1 - \tfrac{1}{4}(n-2)^2 \nu \frac{\phi_\nu^1}{|x|^2} = 0. \tag{3.11}$$

Let $0 < R < \operatorname{dist}(0, \partial\Omega)$. Choose $C_1 > 0$ so that $u_\nu \geqslant C_1 \phi_\nu^1$ on $|x| = R$. As before, define

$$w_\nu = C_1 \phi_\nu^1 - u_\nu.$$

Then $w_\nu^+ \in H_0^1(B(R))$ and

$$-\Delta w_\nu - \tfrac{1}{4}(n-2)^2 \nu \frac{w_\nu}{|x|^2} = -\lambda(\nu) \eta u_\nu \quad \text{in } B(R).$$

Hence

$$\begin{aligned} 0 &\leqslant \int_{B(R)} |\nabla w_\nu^+|^2 - \tfrac{1}{4}(n-2)^2 \nu \int_{B(R)} \frac{(w_\nu^+)^2}{|x|^2} \\ &= -\lambda(\nu) \int_{B(R)} \eta u_\nu w_\nu^+. \end{aligned}$$

Therefore, $w_\nu^+ = 0$ in $B(R)$, i.e.

$$C_1 \phi_\nu^1 \leqslant u_\nu \quad \text{in } B(R),$$

i.e.

$$C_1 \leqslant \liminf_{x\to 0} |x|^{(n-2)[1-\sqrt{1-\nu}]/2} u_\nu(x).$$

Now, to prove the other inequality, we define

$$\phi_\nu^{(2)}(x) = |x|^{\alpha(x)},$$

where

$$\alpha(x) = \tfrac{1}{2}(n-2)[-1+\sqrt{1-\nu}] + |x| = \alpha_0 + |x|.$$

Then, using identity (2.3), we obtain

$$\begin{aligned}
&-\Delta\phi_\nu^{(2)} - \tfrac{1}{4}(n-2)^2\nu\frac{\phi_\nu^{(2)}}{|x|^2} - \eta\lambda(\nu)\phi_\nu^{(2)} \\
&\quad = \frac{\phi_\nu^{(2)}}{|x|^2}\biggl[-(\alpha_0+|x|)^2 - (n-2)\alpha_0 - \tfrac{1}{4}(n-2)^2\nu \\
&\qquad\qquad + |x|\biggl((n-1)\log\frac{1}{|x|} - n\biggr) - |x|^2(\log|x|)^2 \\
&\qquad\qquad\qquad + 2|x|(\alpha_0+|x|)\log\frac{1}{|x|} - \lambda(\nu)\eta|x|^2\biggr] \\
&\quad = \frac{\phi_\nu^{(2)}}{|x|^2}\biggl[-(\alpha_0^2 + (n-2)\alpha_0 + \tfrac{1}{4}(n-2)^2\nu) \\
&\qquad\qquad + |x|\biggl((n-1)\log\frac{1}{|x|} - n - 2\alpha_0 + 2\alpha_0\log\frac{1}{|x|}\biggr) - \lambda(\nu)\eta|x|^2\biggr] \\
&\quad = \frac{\phi_\nu^{(2)}}{|x|}\biggl[(1+(n-2)\sqrt{1-s})\log\frac{1}{|x|} - (n+2\alpha_0) - \lambda(\nu)\eta|x|\biggr] \\
&\quad \geqslant 0 \quad \text{in } B(R) \quad \text{for } R \text{ small enough.}
\end{aligned}$$

Now, by choosing $C_2 > 0$ so that $u_\nu \leqslant C_2\phi_\nu^{(2)}$ on $|x| = R$ and proceeding exactly as we did in the proof of (3.9), we obtain

$$u_\nu \leqslant C_2\phi_\nu^{(2)} \quad \text{in } B(R),$$

i.e.

$$\limsup_{x\to 0} |x|^{(n-2)[1-\sqrt{1-\nu}]/2} u_\nu(x) \leqslant C_2.$$

Now we can estimate $|\nabla u_\nu(x)|$ for x near zero exactly, as in the proof of equation (3.10), using the fact that $u_\nu \leqslant c_2\phi_\nu^{(2)}$ in $B(R)$, to obtain

$$\limsup_{x\to 0} |x|^{(n-2)[1-\sqrt{1-\nu}]/2+1}|\nabla u_\nu(x)| \leqslant C_3 \quad \text{for some } C_3 > 0.$$

This proves the fourth part of theorem 1.1 and hence theorem 1.1.

3.2. Proof of theorem 1.3

First note that if $u \in W_0^{1,n}(B(R))$, then the function u_R defined by $u_R(x) = u(Rx)$ is in $W_0^{1,n}(B(1))$ (when $B(1)$ and $B(R)$ denotes balls in $\mathbb{R}^n$ with centre at

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zero and radius 1 and R, respectively) and

$$\int_{B(R)} |\nabla u|^n \Big/ \int_{B(R)} \frac{|u|^n}{|x|^n(\log eR/|x|)^n} = \int_{B(1)} |\nabla u_R|^n \Big/ \int_{B(1)} \frac{|u_R|^n}{|x|^n(\log e/|x|)^n}.$$

This shows that the best constant in (1.11) is same for all balls $B(R)$ and, from lemma 2.5 and Hardy's inequality (1.11), it is clear that this constant is $(n-1)/n)^n$. This proves the theorems when Ω is a ball. Now consider the general case when Ω is not a ball. Let R be as in (1.11). Choose $R_1 > 0$ so that $B(R_1) \subset \Omega$. Then, by using the scaling argument as before, we obtain

$$\begin{aligned}
&\inf_{\substack{u\in W_0^{1,n}(\Omega)\\ u\neq 0}} \left[\int_\Omega |\nabla u|^n \Big/ \int_\Omega \frac{|u|^n}{|x|^n(\log R/|x|)^n}\right] \\
&\qquad \leqslant \inf_{\substack{u\in W_0^{1,n}(B(R_1))\\ u\neq 0}} \left[\int_{B(R_1)} |\nabla u|^n \Big/ \int_{B(R_1)} \frac{|u|^n}{|x|^n(\log R/|x|)^n}\right] \\
&\qquad = \inf_{\substack{u\in W_0^{1,n}(B(1))\\ u\neq 0}} \left[\int_{B(1)} |\nabla u|^n \Big/ \int_{B(1)} \frac{|u|^n}{|x|^n(\log(R/R_1)/|x|)^n}\right] \\
&\qquad = \left(\frac{n-1}{n}\right)^n \quad \text{because we can use (2.12), as } R/R_1 \text{ is greater than one.}
\end{aligned}$$

This, together with Hardy's inequality (1.11), proves that the best constant in (1.11) is $((n-1)/n)^n$. Also, it follows from (1.11′) that the best constant is never achieved. This completes the proof of theorem 1.3.

3.3. Proof of theorem 1.5

This proof is very much similar to that of theorem 1.1. We will be using lemmas 2.2 and 2.4 in place of lemmas 2.1 and 2.3, respectively. We will be using the same notations for sub and supersolutions.

STEP 5 (existence). Let q satisfy (1.13), with $\lambda(q)$, $\lambda(sq)$ as defined in (1.12) and $R = \sup_{x\in\Omega}\{|x|\}e$. As in the proof of theorem 1.1, choose $u_s \in H_0^1(\Omega)$ such that

$$\left.\begin{aligned}
-\Delta u_s - \tfrac{1}{4}s\frac{qu_s}{|x|^2(\log R/|x|)^2} &= \lambda(sq)\eta u_s \quad \text{in } \Omega,\\
u_s &> 0,\\
\|u_s\|_{H_0^1(\Omega)} &= 1.
\end{aligned}\right\} \tag{3.12}$$

We will prove the existence of a solution to $(\bar{\mathbb{P}}_q)$ by showing that $u_s \to u_1$ strongly in $H_0^1(\Omega)$.

Let

$$u = \left(\log\frac{R}{|x|}\right)^{1/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{-\delta},$$

where δ is chosen so that $\delta > \frac{1}{2}$ and

$$\liminf_{x\to 0}\left(\log\left(\log\frac{R}{|x|}\right)\right)^2(1-q(x)) > 4\delta(\delta+1).$$

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Then, for R small, from (2.2), we obtain in $B(R)$

$$\begin{aligned} &-\Delta u - \frac{1}{4}\frac{qu}{|x|^2(\log R/|x|)^2} - \lambda(sq)\eta u \\ &\quad = \frac{1}{4}\frac{u}{|x|^2(\log R/|x|)^2}\left[(1-q) - \frac{4\delta(\delta+1)}{(\log(\log R/|x|))^2} - 4\lambda(sq)\eta|x|^2\left(\log\frac{R}{|x|}\right)^2\right] \\ &\quad \geqslant 0. \end{aligned} \tag{3.13}$$

Now, proceeding exactly as in the case of theorem 1.1, we obtain, for all $s \in (0,1)$,

$$u_s \leqslant Mu \quad \text{in } B(R) \tag{3.14}$$

for some $M > 0$. Using the boundedness of u_s in $H_0^1(\Omega)$, choose $u_1 \in H_0^1(\Omega)$ such that u_s (or a subsequence if necessary) converges to u_1 weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwise almost everywhere in Ω as $s \to 1$. Now, using the dominated convergence theorem, and with the help of (3.14), we obtain

$$\int_\Omega \frac{qu_s^2}{|x|^2(\log R/|x|)^2} \to \int_\Omega \frac{qu_1^2}{|x|^2(\log R/|x|)^2} \quad \text{as } s \to 1 \tag{3.15}$$

and

$$\int_\Omega \eta u_s^2 \to \int_\Omega \eta u_1^2 \quad \text{as } s \to 1. \tag{3.16}$$

Now, multiplying (3.13) by u_s, integrating by parts and passing to the limit as $s \to 1$, and using (3.15) and (3.16) as in the case of $n \geqslant 3$, we obtain

$$\lim_{s\to 1}\int_\Omega |\nabla u_s|^2 \to \int_\Omega |\nabla u_1|^2,$$

and hence $u_s \to u_1$ strongly in $H_0^1(\Omega)$. Now the existence of a solution to $(\bar{\mathbb{P}}_q)$ for $\lambda = \lambda(q)$ follows by passing to the limit as $s \to 1$ in (3.12) and using the strong maximum principle.

Step 6 (non-existence). Let q satisfy (1.14) and assume that $(\bar{\mathbb{P}}_q)$ has a solution $u_1 \in H_0^1(\Omega)$ for some λ. Then, by Hardy's inequality (1.10), $\lambda \geqslant 0$. We claim that

$$u_1 \geqslant m\left(\log\frac{R}{|x|}\right)^{1/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{-1/2} \quad \text{in } B(R_1)$$

for some $m > 0$, and $R_1 > 0$. This gives a contradiction because, by (1.10),

$$\frac{u_1^2}{|x|^2(\log R/|x|)^2} \in L^1(B(R_1)) \quad \text{for } R_1 < R,$$

but

$$\int_{B(R_1)}\left(\log\frac{R}{|x|}\right)^{-1}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{-1}|x|^{-2} = \infty.$$

Proof of claim. Define

$$\phi_s(x) = \left(\log\frac{R}{|x|}\right)^{1/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{-s/2}, \quad s > 1.$$

Then, for $R_1 < R$, $\phi_s \in H_0^1(B(R_1))$.

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Let $R_1 > 0$ be as in (1.14). We can assume that $R_1 < R$. Now, using (2.2), we obtain in $B(R_1)$

$$-\Delta\phi_s - \frac{1}{4}\frac{q\phi_s}{|x|^2(\log R/|x|)^2} = \frac{1}{4}\frac{\phi_s}{|x|^2(\log R/|x|)^2}\left[(1-q) - \frac{s(s+2)}{(\log(\log R/|x|))^2}\right] \leqslant 0.$$

Now, choosing $m > 0$ such that $u_1 \geqslant m\phi_s$ on $|x| = R_1$ for $1 < s < 2$, and proceeding as in the case of $n \geqslant 3$, we obtain

$$u_1 \geqslant m\phi_s \quad \text{in } B(R_1) \quad \forall s \in (1,2).$$

Taking the limit as $s \to 1$, we obtain

$$u_1 \geqslant m\left(\log\frac{R}{|x|}\right)^{1/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{-1/2} \quad \text{in } B(R_1).$$

This proves our claim and hence the non-existence. □

Step 7 (Existence of $W_0^{1,p}$ solution). Let $0 < \nu < 1$ and let v_ν satisfy $(\bar{\mathbb{P}}_{\nu q})$ for $\lambda = \lambda(\nu q)$, with $\int_\Omega v_\nu^2 = 1$. We will show that $v_\nu \to v_1$ in $W_0^{1,p}(\Omega)$ $\forall p < 2$. First, we will prove the following estimate on v_ν and ∇v_ν.

Fix $0 < \delta < 1$. Then there exists an $R_1 > 0$ such that, for $\nu \in (\frac{1}{2}, 1)$,

$$v_\nu \leqslant M_1\left(\log\frac{R}{|x|}\right)^{\nu/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{\delta} \quad \text{in } B(R_1), \tag{3.17}$$

$$|\nabla v_\nu| \leqslant M_2|x|^{-1}\left(\log\frac{R}{|x|}\right)^{\nu/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{\delta} \quad \text{in } B(R_1), \tag{3.18}$$

where M_1 and M_2 are constants independent of ν.

Proof of (3.17). Let

$$\xi_\nu(x) = \left(\log\frac{R}{|x|}\right)^{\nu/2}\left(\log\left(\log\frac{R}{|x|}\right)\right)^{\delta}.$$

Then, for R_1 small enough and $\nu \in (0,1)$, we have, from (2.2),

$$\begin{aligned} -\Delta\xi_\nu - \tfrac{1}{4}\nu\frac{q\xi_\nu}{|x|^2(\log R/|x|)^2} - \lambda(\nu q)\eta\xi_\nu \\ &= \frac{\xi_\nu}{|x|^2(\log R/|x|)^2}\left[\tfrac{1}{4}\nu(2-\nu-q) + \frac{\delta(1-\nu)}{\log(\log R/|x|)}\right. \\ &\qquad \left. + \frac{\delta(1-\delta)}{(\log(\log R/|x|))^2} - \lambda(\nu q)\eta|x|^2\left(\log\frac{R}{|x|}\right)^2\right] \\ &\geqslant 0 \quad \text{in } B(R_1). \end{aligned}$$

Fix such an R_1. Now, choosing $M_1 > 0$ so that $v_\nu \leqslant M_1\xi_\nu$ on $|x| = R_1$ for $\nu \in (\frac{1}{2}, 1)$, and proceeding as in the case $n \geqslant 3$, we obtain

$$v_\nu \leqslant M_1\xi_\nu \quad \text{in } B(R_1).$$

This proves (3.17). □

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Proof of (3.18). Let $R_1 > 0$ be as in (3.18). Fix $x \in B(\frac{1}{2}R_1)$. Let $r = \frac{1}{2}|x|$ and define

$$\tilde{v}_\nu(y) = v_\nu(x + ry), \quad y \in B(1).$$

Then, as in the proof of (3.10) we obtain for $\nu \in (\frac{1}{2}, 1)$

$$|\nabla \tilde{v}_\nu(0)| \leqslant C \|\tilde{v}_\nu\|_{L^\infty(B(1))},$$

where C is independent of ν and x. Using (3.18) in the above estimate, we obtain

$$|\nabla v_\nu(x)| \leqslant M_2 |x|^{-1} \biggl(\log \frac{R}{|x|}\biggr)^{\nu/2} \biggl(\log\biggl(\log \frac{R}{|x|}\biggr)\biggr)^{\delta}$$

for some $M_2 > 0$. This proves (3.18). □

From the above two estimates (3.17) and (3.18), it follows that v_ν is bounded in $W_0^{1,p}(\Omega)$ $\forall p \in [1, 2)$. Now, arguing exactly as in the case $n \geqslant 3$, we can show that, for a subsequence of ν converging to 1, $v_\nu \to v_1$ (say) in $W_0^{1,p}(\Omega)$ $\forall p \in [1, 2)$ and v_1 is a distributional solution to our equation $(\bar{\mathbb{P}}_q)$. This proves the third part of theorem 1.5.

STEP 8 (asymptotic behaviour). Let $0 < \nu < 1$ and u_ν be as in theorem 1.5. Define

$$\phi_\nu^1 = \biggl(\log \frac{R}{|x|}\biggr)^{[1-\sqrt{1-\nu}]/2}.$$

Then, from (2.2), we have

$$-\Delta \phi_\nu^1 - \tfrac{1}{4}\nu \frac{\phi_\nu^1}{|x|^2 (\log R/|x|)^2} = 0.$$

Let $0 < R_1 < \operatorname{dist}(0, \partial\Omega)$. Choosing $C_1 > 0$ such that $u_\nu \geqslant C_1 \phi_\nu^1$ on $|x| = R_1$ and proceeding as in the case of $n \geqslant 3$, we obtain

$$C_1 \phi_\nu^1 \leqslant u_\nu \quad \text{in } B(R_1),$$

i.e.

$$C_1 \leqslant \liminf_{x \to 0} \biggl(\log \frac{R}{|x|}\biggr)^{[-1+\sqrt{1-\nu}]/2} u_\nu(x).$$

Now define

$$\phi_\nu^{(2)}(x) = \biggl(\log \frac{R}{|x|}\biggr)^{[1-\sqrt{1-\nu}]/2 - |x|} = \biggl(\log \frac{R}{|x|}\biggr)^{\alpha_0 - |x|}$$

Then, from (2.9), we obtain, for R_1 small enough,

$$\begin{aligned}
&- \Delta \phi_\nu^{(2)} - \tfrac{1}{4}\nu \frac{\phi_\nu^{(2)}}{|x|^2 (\log R/|x|)^2} - \lambda(\nu)\eta \phi_\nu^{(2)} \\
&\qquad = \phi_\nu^{(2)} \biggl[\frac{-(\alpha_0^2 - \alpha_0 + \frac{1}{4}\nu)}{|x|^2 (\log R/|x|)^2} + \frac{\log(\log R/|x|)}{|x|}(1 + \circ(1)) - \lambda(\nu)\eta\biggr] \\
&\qquad = \phi_\nu^{(2)} \biggl[(1 + \circ(1)) \log\biggl(\log \frac{R}{|x|}\biggr) - \lambda(\nu)\eta |x|\biggr] \\
&\qquad \geqslant 0 \quad \text{in } B(R_1).
\end{aligned}$$

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Now fix R_1 small enough so that the above inequality holds and then choose $C_2 > 0$ such that $u_\nu \leqslant C_2\phi_\nu^{(2)}$ on $|x| = R_1$. Proceeding as in the case $n \geqslant 3$, we obtain

$$u_\nu \leqslant C_2\phi_\nu^{(2)} \quad \text{in } B(R_1),$$

i.e.

$$\limsup_{x\to 0}\biggl(\log\frac{R}{|x|}\biggr)^{[-1+\sqrt{1-\nu}]/2} u_\nu(x) \leqslant C_2$$

This proves (1.15).

Now, estimating $|\nabla u_\nu(x)|$ as in the proof of (3.18) and using the above proved estimate on u_ν, we obtain

$$\limsup_{x\to 0}\biggl(\log\frac{R}{|x|}\biggr)^{[-1+\sqrt{1-\nu}]/2} |x|\cdot|\nabla u_\nu(x)| \leqslant C_3$$

for some $C_3 > 0$. This proves (1.16) and hence theorem 1.5.

3.4. Proof of theorem 1.6

We will prove the theorem when $n \geqslant 3$. For $n = 2$, the proof goes word by word. Let u_1, u_2 be as in theorem 1.6. Then, by the strong maximum principle,

$$m := \min_{x\in\partial\Omega}\frac{\partial u_1/\partial\nu(x)}{\partial u_2/\partial\nu(x)} \tag{3.19}$$

and there exists a point $x_0 \in \partial\Omega$ such that

$$m = \frac{\partial u_1/\partial\nu(x_0)}{\partial u_2/\partial\nu(x_0)}. \tag{3.20}$$

We claim that $u_1 = mu_2$. Suppose not. Then the function u, defined as

$$u = u_1 - mu_2,$$

is in $H_0^1(\Omega)$ and satisfies

$$-\Delta u - \tfrac{1}{4}(n-2)^2\frac{qu}{|x|^2} = \lambda(q)\eta u \quad \text{in } \Omega. \tag{3.21}$$

Since u is not identically equal to zero, at least one of u^+ or u^- is not identically equal to zero. Let it be u^+. Then, testing (3.21) against u^+, we obtain

$$\int_\Omega |\nabla u^+|^2 - \tfrac{1}{4}(n-2)^2\int_\Omega\frac{q(u^+)^2}{|x|^2} - \lambda(q)\int_\Omega \eta(u^+)^2 = 0.$$

Hence, from the definition of $\lambda(q)$, u^+ satisfies

$$\left.\begin{gathered} -\Delta u^+ - \tfrac{1}{4}(n-2)^2\frac{qu^+}{|x|^2} = \lambda(q)\eta u^+ \quad \text{in } \Omega,\\ u^+ \geqslant 0 \quad \text{in } \Omega,\\ u^+ \in H_0^1(\Omega). \end{gathered}\right\} \tag{3.22}$$

Hence, by the strong maximum principle, $u^+ > 0$ in Ω, and consequently $u = u^+ > 0$ in Ω. Similarly, when u^- is not identically equal to zero, we obtain $u < 0$ in Ω. Thus we have either $u > 0$ or $u < 0$ in Ω. Therefore, u or $-u$ solves $(\mathbb{P}_q)$. Again, by the strong maximum principle, $\partial u/\partial\nu(x)$ is not zero for any $x \in \partial\Omega$. But we have, from (3.20), $\partial u/\partial\nu(x_0) = 0$, which is a contradiction. Hence $u_1 = mu_2$ and finishes the proof of theorem 1.6.

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