

ON SIMULTANEOUS RATIONAL APPROXIMATION TO A P -ADIC NUMBER AND ITS INTEGRAL POWERS, II

DZMITRY BADZIAHIN¹, YANN BUGEAUD² AND JOHANNES SCHLEISCHITZ³

¹ *School of Mathematics and Statistics, The University of Sydney, Camperdown, NSW 2006, Australia (dzmitry.badziahin@sydney.edu.au)*

² *Université de Strasbourg, Mathématiques, 7, rue René Descartes, Strasbourg 67084, France (bugeaud@math.unistra.fr)*

³ *Middle East Technical University, Northern Cyprus Campus, Kalkanlı, Güzelyurt, Turkey (johannes@metu.edu.tr)*

(Received 28 July 2020; first published online 7 May 2021)

Abstract Let p be a prime number. For a positive integer n and a real number ξ , let $\lambda_n(\xi)$ denote the supremum of the real numbers λ for which there are infinitely many integer tuples (x_0, x_1, \dots, x_n) such that $|x_0\xi - x_1|_p, \dots, |x_0\xi^n - x_n|_p$ are all less than $X^{-\lambda-1}$, where X is the maximum of $|x_0|, |x_1|, \dots, |x_n|$. We establish new results on the Hausdorff dimension of the set of real numbers ξ for which $\lambda_n(\xi)$ is equal to (or greater than or equal to) a given value.

Keywords: simultaneous approximation; transference theorem

2020 *Mathematics subject classification:* Primary 11J13

1. Introduction

Throughout, we let p denote a prime number. The present paper is concerned with the simultaneous approximation to successive integral powers of a given transcendental p -adic number ξ by rational numbers with the same denominator. It is partly motivated by a mistake found in [6], namely, the definition of the quantity $\lambda_n(\xi)$ given in [6, Definition 1.2] is not accurate and leads to the trivial result that $\lambda_n(\xi)$ is always infinite. Indeed, for any fixed non-zero integer y with $|y\xi|_p \leq 1$, the quantity $|y\xi - x|_p$ can be made arbitrarily small, by taking a suitable integer x . The authors overlooked the fact that, unlike in the real case, we can have simultaneously $|y\xi - x|_p$ very small and $|x|$ much larger than $|y|$. Fortunately, they (essentially) argue as they were using the correct definition of the exponent of approximation λ_n , which is the following one.

Definition 1.1. Let $n \geq 1$ be an integer and ξ a p -adic number. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ such that, for arbitrarily large real numbers X , the

inequalities

$$\max\{|x_0|, \dots, |x_n|\} \leq X, \quad 0 < \max_{1 \leq m \leq n} |x_0 \xi^m - x_m|_p \leq X^{-\lambda-1},$$

have a solution in integers x_0, \dots, x_n .

We will sometimes use that the above definition remains unchanged if we impose that the integers x_0, \dots, x_n have no common factor. To see this, it is sufficient to observe that, for every integers x_0, \dots, x_n and any positive integer k , if we have $|kx_0 \xi^m - kx_m|_p \leq (k \max |x_i|)^{-\lambda-1}$, for $m = 1, \dots, n$, then also $|x_0 \xi^m - x_m|_p \leq (\max |x_i|)^{-\lambda-1}$, for $m = 1, \dots, n$. This requires a short calculation. The p -adic version of the Dirichlet theorem (see e.g. [15]) implies that $\lambda_n(\xi) \geq 1/n$ for every irrational p -adic number ξ . Furthermore, it follows from the p -adic Schmidt Subspace Theorem that $\lambda_n(\xi) = \max\{1/n, 1/(d-1)\}$ for every positive integer n and every p -adic algebraic number ξ of degree $d \geq 2$. Moreover, almost every p -adic number ξ satisfies $\lambda_n(\xi) = 1/n$ for every $n \geq 1$. This follows from a classical result of Sprindžuk [20] combined with the p -adic analogue of the Khintchine transference theorem, established by Jarník [11].

In Definition 1.1, the inequalities $|x_0|, \dots, |x_n| \leq X$ replace the inequalities $0 < |x_0| \leq X$ occurring in [6, Definition 1.2]. All results of [6] hold for the exponents λ_n as in Definition 1.1, but some of the proofs have to be modified accordingly. Beside pointing out this mistake, we take the opportunity to considerably extend [6, Theorem 2.3] and [6, Lemma 5.1]. Our main motivation is the determination of the spectrum of λ_n , that is, the set of values taken by λ_n evaluated at transcendental p -adic numbers. We also aim at extending to λ_n the p -adic Jarník–Besicovich theorem [12, 14], which asserts that, for any $\lambda \geq 1$, we have

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_1(\xi) \geq \lambda\} = \dim\{\xi \in \mathbb{Q}_p : \lambda_1(\xi) = \lambda\} = \frac{2}{1 + \lambda}. \tag{1.1}$$

Here and below, \dim denotes the Hausdorff dimension. We do not solve completely these problems, however, we manage to establish the p -adic analogues of the results of [1, 18]. In the course of the proofs, we obtain new transference theorems. We also briefly investigate uniform simultaneous approximation and establish the p -adic analogues of results of Laurent [13] and Davenport and Schmidt [10]. As a consequence, we can slightly improve a claim of Teulié [21] on approximation to a p -adic number by p -adic algebraic numbers (respectively, integers).

Throughout this paper, $[\cdot]$ denotes the integer part function and $\lceil \cdot \rceil$ the ceiling function. The notation $a \gg_d b$ means that a exceeds b times a constant depending only on d . When \gg is written without any subscript, it means that the constant is absolute. We write $a \asymp b$ if both $a \gg b$ and $a \ll b$ hold.

2. Main results

Our first result is a p -adic analogue of [18, Corollary 1.8].

Theorem 2.1. *Let $n \geq 2$ be an integer and $\lambda > 1$ a real number. Then, we have*

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} = \dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1 + \lambda)}.$$

Theorem 2.1 was established under the much weaker condition $\lambda > n - 1$ in [6] (note that the correct definition of λ_n is used in the proof of [6, Theorem 2.3]). By adapting the arguments of [1] to the p -adic setting, if $n \geq 3$ we can relax the assumption $\lambda > 1$ in Theorem 2.1. The next results are the p -adic analogues of Theorems 2.1 to 2.3 of [1].

Theorem 2.2. *Let $n \geq 3$ be an integer. The spectrum of λ_n contains the interval $[(n + 4)/(3n), +\infty]$. Let $\lambda \geq (n + 4)/(3n)$ be a real number. Then, we have*

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1 + \lambda)}.$$

In particular, for any real number λ with $\lambda > 1/3$, there exists an integer n_0 such that

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1 + \lambda)},$$

for any integer n greater than n_0 .

We believe that Theorem 2.2 holds for $n = 2$ too. In view of Theorem 2.1, the conclusion of this theorem is satisfied for all $\lambda > 1 = (2 + 4)/(3 \cdot 2)$. Therefore, only the case of $n = 2, \lambda = 1$ remains open.

Theorem 2.3 shows that the assumption ‘ $\lambda > 1/3$ ’ in the last assertion of Theorem 2.2 is sharp.

Theorem 2.3. *For any integer $n \geq 2$, we have*

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq 1/3\} \geq \frac{2}{(n - 1)(1 + 1/3)}.$$

Theorems 2.2 and 2.3 show that there is a discontinuity at $1/3$ in the following sense. The function $n \mapsto n \dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) = \lambda\}$ is ultimately constant equal to $2/(1 + \lambda)$ for $\lambda > 1/3$, while the values taken by the function

$$n \mapsto n \dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) = 1/3\}$$

are all greater than $2/(1 + 1/3)$.

Theorems 2.2 and 2.3 are special cases of the following general statement.

Theorem 2.4. *Let k, n be integers with $1 \leq k \leq n$. Let λ be a real number with $\lambda \geq 1/n$. Then we have*

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \geq \frac{(k + 1)(1 - (k - 1)\lambda)}{(n - k + 1)(1 + \lambda)}. \tag{2.1}$$

If $\lambda > 1/\lfloor \frac{n+1}{2} \rfloor$, then, setting $m = 1 + \lfloor 1/\lambda \rfloor$, we have

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \leq \max_{1 \leq h \leq m} \left\{ \frac{(h + 1)(1 - (h - 1)\lambda)}{(n - 2h + 2)(1 + \lambda)} \right\}. \tag{2.2}$$

Theorem 2.3 corresponds to (2.1) applied with $\lambda = 1/3$ and $k = 2$. The deduction of Theorem 2.2 from Theorem 2.4 follows the same lines as in the real case. Thus, we

omit it and refer the reader to [1], and only remark that the special case $n = 2, \lambda = 1$ remains open, since the p -adic analogue of the metric results on planar curves obtained in [2, 22] has not yet been established. For $k \geq 2$, we remark that (2.1) is of interest for $\lambda < 1/(k - 1)$ only, otherwise the right-hand side is not positive.

A key ingredient for the proof of (2.1) is a transference inequality, which relates the exponents of Diophantine approximation λ_n and w_k , with $k \leq n$. The exponents w_n were introduced by Mahler [16] to measure how small an integer linear form in the first n powers of a given p -adic number can be.

Definition 2.5. Let $n \geq 1$ be an integer and ξ a p -adic number. We denote by $w_n(\xi)$ the supremum of the real numbers w such that, for arbitrarily large real numbers X , the inequalities

$$0 < |x_n \xi^n + \dots + x_1 \xi + x_0|_p \leq X^{-w-1}, \quad \max_{0 \leq m \leq n} |x_m| \leq X,$$

have a solution in integers x_0, \dots, x_n .

For similar reasons as in Definition 1.1 we can impose to the integers x_0, \dots, x_n in Definition 2.5 to be coprime. It follows from the Dirichlet Box Principle that $w_n(\xi) \geq n$ for every $n \geq 1$ and every transcendental p -adic number ξ . For an algebraic number ξ of degree d we have $w_n(\xi) = \min\{n, d - 1\}$. Finally, we remark that $w_1(\xi) = \lambda_1(\xi)$ for every ξ .

Theorem 2.6. Let k, n be integers with $1 \leq k \leq n$. Let ξ be a p -adic transcendental number. Then, we have

$$\lambda_n(\xi) \geq \frac{w_k(\xi) - n + k}{(k - 1)w_k(\xi) + n}. \tag{2.3}$$

The case $k = n$ of Theorem 2.6 has been established by Jarník [11] (this is the p -adic analogue of an inequality in Khintchine’s transference theorem).

For completeness, we state another transference inequality, whose real analogue is [1, Theorem 2.4].

Theorem 2.7. Let ξ be a p -adic transcendental number. For any positive integer k , we have

$$(k + 1)(1 + \lambda_{k+1}(\xi)) \geq k(1 + \lambda_k(\xi)),$$

with equality if $\lambda_{k+1}(\xi) > 1$. Consequently, for every integer n with $n \geq k$, we have

$$\lambda_n(\xi) \geq \frac{k\lambda_k(\xi) - n + k}{n},$$

with equality if $\lambda_n(\xi) > 1$.

Taking $k = 1$ and n arbitrary, Theorem 2.7 shows that, if $\lambda_n(\xi) > 1$, then $\lambda_1(\xi) = n\lambda_n(\xi) + n - 1$. Combined with (1.1), this gives an alternative proof of Theorem 2.1.

We take the opportunity of this paper to present new results on the exponents of uniform approximation $\widehat{\lambda}_n$ and \widehat{w}_n .

Definition 2.8. Let $n \geq 1$ be an integer and ξ a p -adic number. We denote by $\widehat{\lambda}_n(\xi)$ the supremum of the real numbers $\widehat{\lambda}$ such that, for every sufficiently large real number X , the inequalities

$$\max\{|x_0|, \dots, |x_n|\} \leq X, \quad 0 < \max_{1 \leq m \leq n} |x_0 \xi^m - x_m|_p \leq X^{-\widehat{\lambda}-1},$$

have a solution in integers x_0, \dots, x_n . We denote by $\widehat{w}_n(\xi)$ the supremum of the real numbers \widehat{w} such that, for every sufficiently large real number X , the inequalities

$$\max\{|x_0|, \dots, |x_n|\} \leq X, \quad 0 < |x_n \xi^n + \dots + x_1 \xi + x_0|_p \leq X^{-\widehat{w}-1},$$

have a solution in integers x_0, \dots, x_n .

By Dirichlet’s Theorem, for every p -adic number ξ , we have the relations

$$\lambda_n(\xi) \geq \widehat{\lambda}_n(\xi) \geq \frac{1}{n}, \quad w_n(\xi) \geq \widehat{w}_n(\xi) \geq n. \tag{2.4}$$

We remark that for the uniform exponents, and unlike in Definitions 1.1 and 2.5, we cannot impose that the integers x_0, \dots, x_n are coprime, without losing property (2.4). Theorem 2.9 is the p -adic analogue of a result of Schleichitz [19, Theorem 2.1].

Theorem 2.9. Let m, n be positive integers and ξ a p -adic transcendental number. Then

$$\widehat{\lambda}_{m+n-1}(\xi) \leq \max \left\{ \frac{1}{w_m(\xi)}, \frac{1}{\widehat{w}_n(\xi)} \right\}. \tag{2.5}$$

The case $m = 1$ of Theorem 2.9 implies the following result: For any integer $n \geq 1$ and any transcendental p -adic number ξ , we have

$$\widehat{\lambda}_n(\xi) \leq \max \left\{ \frac{1}{n}, \frac{1}{\lambda_1(\xi)} \right\}. \tag{2.6}$$

This is the p -adic analogue of [18, Theorem 1.12]. An alternative proof of (2.6) can be supplied by adapting to the p -adic setting the arguments given in [1, Section 5].

For $k \geq 1$, by applying Theorem 2.9 with $m = n = \lceil k/2 \rceil$ and noticing that $k \geq 2\lceil k/2 \rceil - 1$, we get

$$\widehat{\lambda}_k(\xi) \leq \widehat{\lambda}_{2\lceil k/2 \rceil - 1}(\xi) \leq \max \left\{ \frac{1}{w_{\lceil k/2 \rceil}(\xi)}, \frac{1}{\widehat{w}_{\lceil k/2 \rceil}(\xi)} \right\} \leq \frac{1}{\lceil \frac{k}{2} \rceil}.$$

In fact, the stronger claim holds that for some $c = c(k, p, \xi) > 0$ the system

$$0 < \max\{|x_0|, \dots, |x_k|\} \leq X, \quad \max_{1 \leq m \leq k} |x_0 \xi^m - x_m|_p \leq cX^{-\frac{1}{\lceil k/2 \rceil} - 1}, \tag{2.7}$$

has no integral solution for certain arbitrarily large X , thereby establishing the p -adic analogue of a result of Laurent [13] on uniform simultaneous approximation to the first n powers of a real number. The stronger version (2.7) is justified by the application of the

refined claims in Theorems 6.1 and 6.2 (that imply Theorem 2.9), we skip the details. Furthermore, by applying Theorem 2.9 with $m = n = k$, we get

$$\frac{1}{2k - 1} \leq \widehat{\lambda}_{2k-1}(\xi) \leq \max \left\{ \frac{1}{w_k(\xi)}, \frac{1}{\widehat{w}_k(\xi)} \right\} = \frac{1}{\widehat{w}_k(\xi)}.$$

Again a stronger claim involving a multiplicative constant, as in (2.7), can be proved, which is the p -adic analogue of [10, Theorem 2b]. We have thus established the following theorem.

Theorem 2.10. *For every positive integer n and every p -adic transcendental number ξ , we have*

$$\widehat{w}_n(\xi) \leq 2n - 1 \tag{2.8}$$

and

$$\widehat{\lambda}_n(\xi) \leq \left\lceil \frac{n}{2} \right\rceil^{-1}. \tag{2.9}$$

For even integers n , (2.9) has already been established by Teulié [21], who got the slightly weaker upper bound $1/\lfloor n/2 \rfloor$ for odd integers $n \geq 5$.

By applying a well-known transference theorem [10, 21], we deduce from (2.7) a slight improvement on Teulié’s results on approximation to a p -adic transcendental number by p -adic algebraic numbers (respectively, integers) of prescribed degree. As usual, throughout this paper, the height $H(P)$ of an integer polynomial $P(X)$ is the maximum of the absolute values of its coefficients and the height $H(\alpha)$ of a p -adic algebraic number α is the height of its minimal defining polynomial over \mathbb{Z} with coprime coefficients. Let ξ be a p -adic number. Let $n \geq 2$ be an integer. Then, there exists a positive constant $c = c(n, p, \xi)$ such that the inequality

$$|\xi - \alpha|_p \leq cH(\alpha)^{-\lceil \frac{n}{2} \rceil - 1} \tag{2.10}$$

has infinitely many solutions in p -adic algebraic numbers α of degree exactly n and also, if $|\xi|_p = 1$, in p -adic algebraic integers of degree exactly $n + 1$.

3. Proof of Theorem 2.1

We start with a general observation. It is clear that for every p -adic number ξ and every non-zero rational number r , any of the irrationality exponents $w_k, \widehat{w}_k, \lambda_k$ and $\widehat{\lambda}_k(\xi)$ takes the same value at the points ξ and $r\xi$. Since $\mathbb{Q}_p = \bigcup_{m \in \mathbb{Z}} p^m \mathbb{Z}_p$, for all $A \subset \mathbb{R}$ and ν in $\{w_k, \widehat{w}_k, \lambda_k, \widehat{\lambda}_k\}$, we have

$$\dim\{\xi \in \mathbb{Q}_p : \nu(\xi) \in A\} = \dim\{\xi \in \mathbb{Q}_p : |\xi|_p = 1 \text{ and } \nu(\xi) \in A\}.$$

Therefore, throughout all the proofs, we may assume without loss of generality that $|\xi|_p = 1$.

Proof of Theorem 2.1. Let $n \geq 2$ be an integer and ξ a p -adic number with $\lambda_n(\xi) > 1$. Let λ be a real number with $1 < \lambda < \lambda_n(\xi)$. Then, there are integers

q, p_1, \dots, p_n , with $Q = \max\{|q|, |p_1|, \dots, |p_n|\}$ arbitrarily large and $\gcd(q, p_1, \dots, p_n) = 1$, such that

$$|q\xi^j - p_j|_p < Q^{-\lambda-1}, \quad j = 1, \dots, n.$$

Since $|\xi|_p = 1$, we observe that p does not divide the product $qp_1 \cdots p_n$. Note that

$$|p_{j+1} - p_j\xi|_p = |p_{j+1} - q\xi^{j+1} - \xi(p_j - q\xi^j)|_p < Q^{-\lambda-1}, \quad j = 1, \dots, n-1.$$

Set $p_0 = q$. Observe that, for $j = 1, \dots, n-1$, we have

$$\Delta_j := p_{j-1}p_{j+1} - p_j^2 = p_{j-1}(p_{j+1} - p_j\xi) - p_j(p_j - p_{j-1}\xi),$$

thus, by the triangle inequality,

$$|\Delta_j|_p < Q^{-1-\lambda}.$$

Since $|\Delta_j| \leq 2Q^2$, we get that any non-zero Δ_j should satisfy $|\Delta_j|_p \geq (2Q^2)^{-1}$. Therefore, if Q is sufficiently large and $\lambda > 1$, we get that

$$\Delta_1 = \dots = \Delta_{n-1} = 0,$$

which implies that there exist coprime non-zero integers a, b such that

$$\frac{p_1}{q} = \frac{p_2}{p_1} = \dots = \frac{p_n}{p_{n-1}} = \frac{a}{b}.$$

We deduce at once that the point

$$\left(\frac{p_1}{q}, \dots, \frac{p_n}{q}\right) = \left(\frac{a}{b}, \dots, \left(\frac{a}{b}\right)^n\right)$$

lies on the Veronese curve $x \mapsto (x, x^2, \dots, x^n)$ and that q (respectively, p_n) is an integer multiple of b^n (respectively, of a^n). We remark that, since $\gcd(q, p_1, \dots, p_n) = 1$, we have in fact $(q, p_1, \dots, p_n) = \pm(b^n, b^{n-1}a, \dots, a^n)$. In particular, we get

$$|q\xi - p_1|_p = |b\xi - a|_p < Q^{-1-\lambda} \leq (\max\{|a|, |b|\})^{-n(1+\lambda)}. \tag{3.1}$$

This proves that

$$\lambda_n(\xi) > 1 \text{ implies } \lambda_1(\xi) \geq n(1 + \lambda_n(\xi)) - 1. \tag{3.2}$$

Furthermore, since Q (and, thus, $\max\{|a|, |b|\}$) is arbitrarily large, we deduce from (3.1) and the (easy half of the) p -adic Jarník-Besicovich theorem (see (1.1)) that

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \leq \frac{2}{n(1 + \lambda)}.$$

The reverse inequality is easier. Let $\lambda > \lambda_1(\xi)$ be a real number and a, b be (large) integers not divisible by p (recall that $|\xi|_p = 1$) such that

$$|b\xi - a|_p \leq \max\{|a|, |b|\}^{-1-\lambda}.$$

Then, for $j = 1, \dots, n$, we have

$$|b^j\xi^j - a^j|_p \leq \max\{|a|, |b|\}^{-1-\lambda} \quad \text{and} \quad |b^n\xi^j - a^j b^{n-j}|_p \leq \max\{|a|, |b|\}^{-1-\lambda},$$

giving that

$$\max_{1 \leq j \leq n} |b^n \xi^j - a^j b^{n-j}|_p \leq \max\{|a^n|, |b^n|\}^{-(1+\lambda)/n}.$$

This proves that

$$\lambda_n(\xi) \geq \frac{1 + \lambda_1(\xi)}{n} - 1, \tag{3.3}$$

which is the content of [6, Lemma 5.1]. Consequently,

$$\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \supset \{\xi \in \mathbb{Q}_p : \lambda_1(\xi) \geq n(\lambda + 1) - 1\},$$

and we conclude by (1.1). □

4. Proofs of the transference inequalities

Before proceeding with the proof of Theorem 2.6, we have to adapt to the p -adic setting the argument used by Champagne and Roy [9] to improve the transference inequality obtained in [1, Theorem 3.1]. We begin with a definition.

Definition 4.1. Let $n \geq 1$ be an integer and ξ a p -adic number. We denote by $w_n^{\text{lead}}(\xi)$ the supremum of the real numbers w such that, for arbitrarily large real numbers X , the inequalities

$$\max\{|x_0|, \dots, |x_n|\} \leq X, \quad 0 < |x_n \xi^n + \dots + x_1 \xi + x_0|_p \leq X^{-w-1},$$

have a solution in integers x_0, \dots, x_n with $|x_n| = \max\{|x_0|, \dots, |x_n|\}$ and $|x_n|_p = 1$.

The next lemma is a p -adic analogue to [9, Theorem].

Lemma 4.2. *Let $k \geq 1$ be an integer and r_0, \dots, r_k be distinct integers. There is an integer $M \geq 1$ such that, for each non-zero ξ in \mathbb{Q}_p , there exists at least one index i in $\{0, 1, \dots, k\}$ with $\xi \neq pr_i |\xi|_p^{-1}$, for which the p -adic number $\xi_i = 1/(M(|\xi|_p \xi - pr_i))$ satisfies $w_k^{\text{lead}}(\xi_i) = w_k(\xi_i) = w_k(\xi)$.*

Proof. If $\xi = a/b$ is rational and i is such that $\xi \neq pr_i |\xi|_p^{-1}$, then ξ_i is rational and $w_k(\xi) = w_k(\xi_i) = w_k^{\text{lead}}(\xi_i) = 0$.

Let ξ be an irrational p -adic number. Define $\xi' = |\xi|_p \xi$ and observe that $|\xi'|_p = 1$ and $w_k(\xi) = w_k(\xi')$. We follow the proof of [9], with suitable modifications. By Lagrange Interpolation Formula, there exists a positive constant C_1 such that every integer polynomial $P(X)$ of degree at most k satisfies

$$H(P) \leq C_1 \max\{|P(pr_i)| : 0 \leq i \leq k\}.$$

There also exists a positive constant C_2 such that

$$\max\{H(P(X + pr_i)) : 0 \leq i \leq k\} \leq C_2 H(P).$$

Let M be an integer greater than $C_1 C_2$ and not divisible by p . By definition of $w_k(\xi')$, there exists a sequence of polynomials $(P_j)_{j \geq 1}$ in $\mathbb{Z}[X]$ of degree at most k such that

$H(P_{j+1}) > H(P_j)$ for $j \geq 1$ and

$$\lim_{j \rightarrow \infty} -\frac{\log |P_j(\xi')|_p}{\log H(P_j)} = w_k(\xi) + 1. \tag{4.1}$$

Furthermore, since $|\xi'|_p = 1$, we may assume that, for $j \geq 1$, the constant coefficient of $P_j(X)$ is not divisible by p . In particular, $P_j(pr_i)$ is not divisible by p , for $j \geq 1$ and $0 \leq i \leq k$.

There exist i in $\{0, 1, \dots, k\}$ and an infinite set \mathcal{S} of positive integers such that $H(P_j) \leq C_1|P_j(pr_i)|$ for every j in \mathcal{S} . Let j be in \mathcal{S} . Set

$$Q_j(X) = (MX)^k P_j\left(\frac{1}{MX} + pr_i\right) \in \mathbb{Z}[X].$$

The absolute value of the coefficient of X^k in $Q_j(X)$ is $M^k|P_j(pr_i)|$, while its other coefficients have absolute value at most

$$M^{k-1}H(P_j(X + pr_i)) \leq C_2M^{k-1}H(P_j) \leq C_1C_2M^{k-1}|P_j(pr_i)| \leq M^k|P_j(pr_i)|.$$

Consequently, the absolute value of the leading coefficient of $Q_j(X)$ is equal to the height of $Q_j(X)$ and is not divisible by p . We also have $|pr_i - \xi'|_p = 1$ and, setting

$$\xi_i = \frac{1}{M(\xi' - pr_i)},$$

we check that $|\xi_i|_p = 1$ and

$$|Q_j(\xi_i)|_p = |M\xi_i^k|_p |P_j(\xi')|_p = |P_j(\xi')|_p.$$

Since the quotient $H(Q_j)/H(P_j)$ is bounded from above and from below by positive constants, we deduce from (4.1) that

$$\lim_{j \rightarrow \infty, j \in \mathcal{S}} -\frac{\log |Q_j(\xi_i)|_p}{\log H(Q_j)} = w_k(\xi) + 1.$$

This shows that $w_k^{\text{lead}}(\xi_i) \geq w_k(\xi)$. As ξ_i is the image of ξ' by a linear fractional transformation with rational coefficients, we have $w_k(\xi_i) = w_k(\xi') = w_k(\xi)$. Since $w_k(\xi_i) \geq w_k^{\text{lead}}(\xi_i)$, this proves the lemma. \square

Proof of Theorem 2.6. The case $k = 1$ has been established in [11]. Let k, n be integers with $2 \leq k \leq n$. Let ξ be a transcendental p -adic number. Let ε be a positive real number.

Assume first that $w_k(\xi) = w_k^{\text{lead}}(\xi)$ and that $w_k(\xi)$ is finite. For arbitrarily large integers H , there exist integers a_0, a_1, \dots, a_k , not all zero, such that $H = |a_k| =$

$\max\{|a_0|, |a_1|, \dots, |a_k|\}$ and p does not divide a_k , and

$$H^{-w_k(\xi)-1-\varepsilon} \leq |a_k \xi^k + \dots + a_1 \xi + a_0|_p \leq H^{-w_k(\xi)-1+\varepsilon}. \tag{4.2}$$

Take such an integer H and set

$$\rho := a_k \xi^k + \dots + a_1 \xi + a_0.$$

Observe that the absolute value of the determinant of the $(n + 1) \times (n + 1)$ matrix

$$M := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ a_0 & a_1 & a_2 & \dots & a_{k-1} & a_k & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{k-2} & a_{k-1} & a_k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & a_{k-1} & a_k \end{pmatrix}.$$

is equal to H^{n-k+1} . It then follows from Satz 1 of Mahler [15] that there exist integers v_0, \dots, v_n , not all zero, such that

$$\begin{aligned} |v_0 \xi^j - v_j|_p &\leq |\rho|_p, \quad 1 \leq j \leq k - 1, \\ |a_0 v_i + a_1 v_{i+1} + \dots + a_k v_{i+k}| &\leq p^{-1}, \quad 0 \leq i \leq n - k, \\ |v_i| &\leq (pH)^{(n-k+1)/k} |\rho|_p^{-(k-1)/k}, \quad 0 \leq i \leq k - 1. \end{aligned}$$

Since the a_j 's and v_j 's are integers, we get that

$$a_0 v_i + a_1 v_{i+1} + \dots + a_k v_{i+k} = 0, \quad 0 \leq i \leq n - k.$$

Taking $i = 0$ above and recalling that $|a_k| = \max\{|a_0|, |a_1|, \dots, |a_k|\}$, we get that $|v_k| \leq k(pH)^{(n-k+1)/k} |\rho|_p^{-(k-1)/k}$ and, by increasing i step by step, in a similar way we estimate

$$|v_{k+1}|, \dots, |v_n| \leq k^{n-k} (pH)^{(n-k+1)/k} |\rho|_p^{-(k-1)/k}.$$

Furthermore, for $i = 0, \dots, n - k$, we have

$$\begin{aligned} |a_k v_0 \xi^{i+k} - a_k v_{i+k}|_p &= |v_0 (a_{k-1} \xi^{i+k-1} + \dots + a_1 \xi^{i+1} + a_0 \xi^i - \rho \xi^i) \\ &\quad - (a_0 v_i + a_1 v_{i+1} + \dots + a_{k-1} v_{i+k-1})|_p. \end{aligned}$$

Since $|a_k|_p = 1$, we derive inductively that

$$|v_0 \xi^{i+k} - v_{i+k}|_p \ll_{p,n,\xi} |\rho|_p, \quad i = 0, \dots, n - k. \tag{4.3}$$

Recall that

$$\max\{|v_0|, |v_1|, \dots, |v_n|\} \ll_{p,n} H^{(n-k+1)/k} |\rho|_p^{-(k-1)/k}. \tag{4.4}$$

Since ε can be taken arbitrarily close to 0, we deduce at once from (4.2)–(4.4) that

$$\lambda_n(\xi) \geq \frac{w_k(\xi) - n + k}{(k - 1)w_k(\xi) + n}.$$

An inspection of the proof shows that it yields $\lambda_n(\xi) \geq 1/(k - 1)$ when $w_k(\xi)$ is infinite, so (2.3) holds in all cases, provided that $w_k(\xi) = w_k^{\text{lead}}(\xi)$.

If $w_k(\xi) > w_k^{\text{lead}}(\xi)$, then we apply (2.3) with ξ replaced by a p -adic number ξ_i satisfying the conclusion of Lemma 4.2. Since $\lambda_n(\xi) = \lambda_n(\xi_i)$, we deduce that (2.3) holds in all cases. □

Proof of Theorem 2.7.. Write $\lambda_k = \lambda_k(\xi)$. Assume that λ_k is finite (otherwise, it follows from (3.2) and (3.3) that $\lambda_{k+1}(\xi)$ is infinite and we are done). Let ε be a positive real number. There exist integers q, v_1, \dots, v_k , with $Q = \max\{|q|, |v_1|, \dots, |v_k|\}$ arbitrarily large, such that

$$Q^{-\lambda_k - 1 - \varepsilon} \leq \max\{|q\xi - v_1|_p, \dots, |q\xi^k - v_k|_p\} \leq Q^{-\lambda_k - 1 + \varepsilon}, \tag{4.5}$$

where $Q = \max\{|q|, |v_1|, \dots, |v_k|\}$. Take such an integer Q . In particular, q is non-zero. It follows from Siegel’s lemma (see [5, Lemma 2.9.1]) that there exist integers a_0, a_1, \dots, a_k , not all zero, such that

$$a_0q + a_1v_1 + \dots + a_kv_k = 0$$

and

$$1 \leq H := \max\{|a_0|, |a_1|, \dots, |a_k|\} \ll_k Q^{1/k}.$$

We may assume $a_k \neq 0$, otherwise we replace k by the largest index j with $a_j \neq 0$ in the argument below. Then, we derive from (4.5) that

$$\begin{aligned} &|q(a_k\xi^k + \dots + a_1\xi + a_0)|_p \\ &= |q(a_k\xi^k + \dots + a_1\xi + a_0) - (a_kv_k + \dots + a_1v_1 + a_0q)|_p \leq Q^{-\lambda_k - 1 + \varepsilon}. \end{aligned} \tag{4.6}$$

Using triangle inequalities, we get from (4.5) and (4.6) that

$$\begin{aligned} &|a_kq\xi^{k+1} + a_{k-1}v_k + a_{k-2}v_{k-1} + \dots + a_1v_2 + a_0v_1|_p \\ &\leq \max\{|q(a_k\xi^{k+1} + \dots + a_0\xi)|_p, |q(a_{k-1}\xi^k + \dots + a_0\xi) - a_{k-1}v_k - \dots - a_0v_1|_p\} \\ &\ll_\xi \max\{|q(a_k\xi^k + \dots + a_1\xi + a_0)|_p, Q^{-\lambda_k - 1 + \varepsilon}\} \\ &\ll_\xi Q^{-\lambda_k - 1 + \varepsilon}. \end{aligned} \tag{4.7}$$

Since $a_k \neq 0$, it now follows from

$$|a_{k-1}v_k + a_{k-2}v_{k-1} + \dots + a_1v_2 + a_0v_1| \ll_k Q^{1+1/k},$$

(4.5), and (4.7) that

$$1 + \lambda_{k+1}(\xi) \geq \frac{\lambda_k(\xi) + 1 - \varepsilon}{1 + 1/k}.$$

As ε can be chosen arbitrarily close to 0, we deduce that

$$(k + 1)(1 + \lambda_{k+1}(\xi)) \geq k(1 + \lambda_k(\xi)).$$

This concludes the proof of the first inequality of the theorem. By iterating it, we immediately get the last inequality. In particular, we obtain

$$(k + 1)(\lambda_{k+1}(\xi) + 1) \geq \lambda_1(\xi) + 1 \quad \text{and} \quad k(\lambda_k(\xi) + 1) \geq \lambda_1(\xi) + 1. \tag{4.8}$$

Assume now that $\lambda_{k+1}(\xi) > 1$. Then, we also have $\lambda_k(\xi) > 1$ and we get from (3.2) that $\lambda_1(\xi) \geq (k + 1)(1 + \lambda_{k+1}(\xi)) - 1$ and $\lambda_1(\xi) \geq k(1 + \lambda_k(\xi)) - 1$. Combined with (4.8), this gives at once

$$(k + 1)(1 + \lambda_{k+1}(\xi)) = k(1 + \lambda_k(\xi)).$$

By iterating this equality, we obtain that the last inequality of the theorem is an equality when $\lambda_n(\xi)$ exceeds 1. □

5. Proof of Theorem 2.4

The following result, established by Bernik and Morotskaya [4, 17] (see also [3, Section 6.3]), is a key ingredient in the proof of Theorem 2.4.

Theorem 5.1. *For every positive integer n and every real number w with $w \geq n$, we have*

$$\dim\{\xi \in \mathbb{Q}_p : w_n(\xi) \geq w\} = \frac{n + 1}{w + 1}. \tag{5.1}$$

We observe that Theorem 5.1 extends (1.1).

Proof of the first assertion of Theorem 2.4.. Let k, n be integers with $1 \leq k \leq n$. For $k = 1$, inequality (2.1) follows from combining (1.1) with the inclusion

$$\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \supset \{\xi \in \mathbb{Q}_p : \lambda_1(\xi) \geq n\lambda + n - 1\}$$

that follows from the consequence $\lambda_n(\xi) \geq (\lambda_1(\xi) - n + 1)/n$ of Theorem 2.7. For $k \geq 2$ and λ in $[1/n, 1/(k - 1))$, inequality (2.3) implies that

$$\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \supset \left\{ \xi \in \mathbb{Q}_p : w_k(\xi) \geq \frac{(\lambda + 1)n - k}{1 - \lambda(k - 1)} \right\}. \tag{5.2}$$

By Theorem 5.1, this yields (2.1). Finally, the assertion is trivial if $\lambda \geq 1/(k - 1)$. □

The proof of the second assertion of Theorem 2.4 requires more work. We keep the same steps as in the proof of its real analogue. Instead of appealing to a result of Davenport and Schmidt [10] as in [1], we make use of its p -adic analogue, which was proved by Teulié [21, Lemme 3].

We consider the $(n + 1)$ -tuples $\mathbf{p} := (q, p_1, p_2, \dots, p_n)$ of integers which approximate at least one point $(\xi, \xi^2, \dots, \xi^n)$ on the Veronese curve, that is, which satisfy

$$|q\xi^i - p_i|_p \ll_{\xi, n} Q^{-\lambda-1}, \quad i = 1, \dots, n, \quad \text{with } Q = \max\{|q|, |p_1|, \dots, |p_n|\}. \tag{5.3}$$

For convenience, we will often write p_0 instead of q .

Throughout this section, we extensively make use of matrices of the form

$$\Delta_{m,k} := \begin{pmatrix} p_{k-m+1} & p_{k-m+2} & \cdots & p_k \\ p_{k-m+2} & p_{k-m+3} & \cdots & p_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_k & p_{k+1} & \cdots & p_{k+m-1} \end{pmatrix}.$$

Observe that $\Delta_{m,k}$ is an $m \times m$ matrix with p_k in its antidiagonal. The matrices $\Delta_{2,k}$ have been used in the proof of Theorem 2.1.

Proposition 5.2. *Assume that a tuple $\mathbf{p} = (p_0, \dots, p_n)$ in \mathbb{Z}^{n+1} satisfies (5.3) for some p -adic number ξ . Then, we have*

$$|p_i \xi - p_{i+1}|_p \ll_{\xi,n} Q^{-\lambda-1}, \quad \text{for } i \in \{0, \dots, n-1\} \tag{5.4}$$

and

$$|\det(\Delta_{2,i})|_p \ll_{\xi,n} Q^{-\lambda-1}, \quad \text{for } i \in \{1, \dots, n-1\}. \tag{5.5}$$

Conversely, if an integer tuple \mathbf{p} in \mathbb{Z}^{n+1} with $|p_0|_p, \dots, |p_n|_p \gg 1$ satisfies (5.5), then there exists a p -adic number ξ for which (5.3) is true.

Proof. For the first two statements, see the beginning of the proof of Theorem 2.1 and observe that the assumption $|\xi|_p = 1$ can be removed.

For the last statement, consider an integer tuple \mathbf{p} which satisfies (5.5). Then, for $i = 1, \dots, n-1$, we have

$$|p_i^2 - p_{i-1}p_{i+1}|_p \ll_{\xi,n} Q^{-1-\lambda}.$$

Setting $\xi := p_1/p_0$, these inequalities yield, for i in $\{0, 1, \dots, n-1\}$,

$$\left| \xi - \frac{p_{i+1}}{p_i} \right|_p \ll_{\xi,n} Q^{-1-\lambda}, \quad \text{thus } |p_i \xi - p_{i+1}|_p \ll_{\xi,n} Q^{-1-\lambda}.$$

Now we use induction on i . For $i = 0$ and setting $q = p_0$, the statement $|q\xi - p_1|_p \ll_{\xi,n} Q^{-1-\lambda}$ follows from the last estimate. Assuming that (5.3) is true for i in $\{0, 1, \dots, n-1\}$, we deduce from

$$|q\xi^{i+1} - p_{i+1}|_p = |(q\xi^i - p_i)\xi + p_i\xi - p_{i+1}|_p \ll_{\xi,n} Q^{-1-\lambda},$$

that it is also true for $i + 1$. □

The next proposition is the p -adic analogue of [1, Proposition 4.2].

Proposition 5.3. *Let \mathbf{p} be in \mathbb{Z}^{n+1} which satisfies (5.5). Let ξ be given by the last assertion of Proposition 5.3. Then, for any positive integers m, k with $k - m + 1 \geq 0$ and $k + m - 1 \leq n$, we have*

$$|\det(\Delta_{m,k})|_p \ll_{\xi,n} Q^{-(m-1)(\lambda+1)}.$$

Proof. For integers i, j with $0 \leq j < i \leq n - 1$, observe that $|p_j \xi^{i-j} - p_i|_p$ is, up to a factor that depends on ξ only, at most equal to

$$\max\{|p_j \xi^{i-j} - p_{j+1} \xi^{i-j-1}|_p, |p_{j+1} \xi^{i-j-1} - p_{j+2} \xi^{i-j-2}|_p, \dots, |p_{i-1} \xi - p_i|_p\},$$

thus, by (5.4), we have

$$|p_j \xi^{i-j} - p_i|_p \ll_{\xi, n} Q^{-(\lambda+1)}.$$

To compute the determinant of the matrix $\Delta_{m, k}$, we replace its h th column by ξ^{m-h} times this column minus the last column. Then, we expand the determinant as a sum of $m!$ products of m terms. Each of this product is the product of an integer by $m - 1$ terms which are p -adically $\ll_{\xi, n} Q^{-(\lambda+1)}$. This proves the proposition. \square

The proof of Proposition 5.3 can easily be adapted to show the next proposition, which is more general.

Proposition 5.4. *Let \mathbf{p} be in \mathbb{Z}^{n+1} which satisfies (5.5) and m a positive integer. Let ξ be given by the last assertion of Proposition 5.3. For $i = 0, \dots, n - m + 1$, let \mathbf{y}_i denote the vector $(p_i, p_{i+1}, \dots, p_{i+m-1})$. Then, for any sequence c_1, c_2, \dots, c_m of integers in $\{0, \dots, n - k + 1\}$, the determinant $d(c_1, \dots, c_m)$ of the $m \times m$ matrix composed of the vectors $\mathbf{y}_{c_1}, \mathbf{y}_{c_2}, \dots, \mathbf{y}_{c_m}$ satisfies*

$$|d(c_1, \dots, c_m)|_p \ll_{\xi, n} Q^{-(m-1)(\lambda+1)}.$$

The next auxiliary result is [21, Lemme 3].

Theorem 5.5. *Let a_0, a_1, \dots, a_h be integers with no common factor throughout. Assume that, for some non-negative integers t, k with $k + h - 1 \leq t$ and $t + h \leq n$, the integers $p_k, p_{k+1}, \dots, p_{t+h}$ are related by the recurrence relation*

$$a_0 p_i + a_1 p_{i+1} + \dots + a_h p_{i+h} = 0, \quad k \leq i \leq t.$$

Let Z be the maximum of the absolute values of all the $h \times h$ determinants formed from any h of the vectors $\mathbf{y}_i := (p_i, p_{i+1}, \dots, p_{i+h-1})$, $i = k, \dots, t + 1$. Let $p^{-\rho}$ with $\rho > 0$ be the maximum of the p -adic absolute values of all the $h \times h$ determinants formed from any h of the vectors $\mathbf{y}_i := (p_i, p_{i+1}, \dots, p_{i+h-1})$, $i = k, \dots, t + 1$. If Z is non-zero, then

$$\max\{|a_0|, |a_1|, \dots, |a_h|\} \ll_{\xi, n} (Z p^{-\rho})^{1/(t-k-h+2)}.$$

Proof of the second assertion of Theorem 2.4.. Let $\lambda > 1/\lfloor (n + 1)/2 \rfloor$ be a real number and set $m = 1 + \lfloor 1/\lambda \rfloor$. Let ξ be a transcendental p -adic number such that $\lambda_n(\xi) \geq \lambda$ and consider an $(n + 1)$ -tuple \mathbf{p} for which (5.3) is satisfied and Q is large enough.

Let h be the smallest non-negative integer number such that the matrix

$$P_h := \begin{pmatrix} p_0 & p_1 & \cdots & p_{n-h-1} & p_{n-h} \\ p_1 & p_2 & \cdots & p_{n-h} & p_{n-h+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_h & p_{h+1} & \cdots & p_{n-1} & p_n \end{pmatrix}.$$

has rank at most h . Obviously, $h \leq \lceil \frac{n+1}{2} \rceil$, because for $\ell = \lceil \frac{n+1}{2} \rceil$ the matrix P_ℓ has more rows than columns and its rank is at most ℓ . Also, we have $h \geq 1$ since \mathbf{p} is not the zero vector. On the other hand, for $q = p_0$ large enough, we get $h \leq m$. Indeed, consider $m + 1$ arbitrary columns of the matrix P_m . By Proposition 5.4, the determinant of the integer matrix formed from these columns has p -adic absolute value at most $cQ^{-m(\lambda+1)}$ for some positive constant $c = c(n, \xi)$. Since all its entries are integers of absolute value at most Q , the p -adic absolute value of this determinant is either 0, or at least $Q^{-(m+1)}$. Consequently, since $\lambda > 1/m$, for Q large enough, this determinant is zero. Since $\lambda > 1/\lfloor (n + 1)/2 \rfloor$, we have

$$h \leq m \leq \left\lfloor \frac{n + 1}{2} \right\rfloor. \tag{5.6}$$

By construction of the matrix P_h , there exist integers a_0, a_1, \dots, a_h with no common factor such that

$$a_0 p_i + a_1 p_{i+1} + \cdots + a_h p_{i+h} = 0, \quad 0 \leq i \leq n - h. \tag{5.7}$$

Note that the matrix P_{h-1} has rank h and therefore the value of Z , defined in Theorem 5.5, is non-zero. Moreover, with ρ defined as in Theorem 5.5, Propositions 5.3 and 5.4 imply that

$$Zp^{-\rho} \ll_{\xi, n} Q^h Q^{-(h-1)(\lambda+1)}.$$

From inequality (5.6), we have $h - 1 \leq n - h$ and thus all the assumptions of Theorem 5.5 are satisfied. Applied with $k = 0$ and $t = n - h$, it yields

$$H := \max\{|a_0|, |a_1|, \dots, |a_h|\} \leq (Zp^{-\rho})^{1/(n-2h+2)} \ll_{\xi, n} Q^{\frac{h-(h-1)(\lambda+1)}{n-2h+2}}.$$

Consider the relation (5.7) for $i = 0$ and divide it by $p_0 = q$. Then, the condition (5.3) implies that

$$|a_h \xi^h + a_{h-1} \xi^{h-1} + \cdots + a_0|_p \ll_{\xi, n} |q|_p^{-1} Q^{-1-\lambda} \ll_{\xi, n} Q^{-1-\lambda} \ll_{\xi, n} H^{-\frac{(1+\lambda)(n-2h+2)}{h-(h-1)(\lambda+1)}}.$$

Here, in the second inequality, we used that $|q|_p \gg_{\xi, n} 1$, by (5.3). Indeed, if otherwise $|q|_p < \min\{|\xi|_p^{-n}, 1\}$, then, since p does not divide all p_i , we get $|\xi^i q|_p < 1 = |p_i|_p$ and thus $|\xi^i q - p_i|_p = \max\{|\xi^i q|_p, |p_i|_p\} = 1$ for some i , in contradiction to (5.3). See also the proof of Theorem 2.2. For the moment assume $h - (h - 1)(1 + \lambda) > 0$. By applying Theorem 5.1, we conclude that

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) \geq \lambda\} \leq \max_{1 \leq h \leq m} \left\{ \frac{(h + 1)(h - (h - 1)(\lambda + 1))}{(n - 2h + 2)(1 + \lambda)} \right\},$$

which gives the expected result. Finally, in the special case $h - (h - 1)(1 + \lambda) = 0$, we get $h = m$ and $\lambda = 1/(m - 1)$. We can then let λ tend to $1/(m - 1)$ from below and use monotonicity and a limit argument to derive (2.2) in this case as well. \square

6. Proof of Theorem 2.9

Theorem 2.9 directly follows from the combination of Theorems 6.1 and 6.2.

For integers ℓ, n with $n \geq 2$ and $1 \leq \ell \leq n + 1$, we write $w_{n,\ell}(\xi)$ for the supremum of w for which there are arbitrarily large integers H such that the system

$$0 < |a_0 + a_1\xi + \dots + a_n\xi^n|_p \leq H^{-w-1}, \quad \max_{0 \leq i \leq n} |a_i| \leq H,$$

has ℓ linearly independent solutions in integer $(n + 1)$ -tuples (a_0, a_1, \dots, a_n) . Note that $w_{n,1}(\xi) = w_n(\xi)$. The following theorem is the p -adic analogue of a result obtained in the course of the proof of [19, Theorem 2.1]. Its proof is very similar.

Theorem 6.1. *Let m, n be positive integers and ξ in \mathbb{Q}_p . We have*

$$w_{m+n-1,m+n}(\xi) \geq \min\{w_m(\xi), \widehat{w}_n(\xi)\}.$$

If $m = n = \widehat{w}_n(\xi)$, then the following slightly stronger claim holds: For arbitrarily large H , there exist $m + n = 2n$ linearly independent solutions to

$$|P(\xi)|_p \leq c(\xi, n)H^{-n-1},$$

in integer polynomials P of degree at most $m + n - 1 = 2n - 1$ and height at most H , where $c(\xi, n)$ is a suitable positive number depending only on n and ξ .

The second auxiliary result is a transference theorem.

Theorem 6.2. *Let $n \geq 1$ be an integer and ξ in \mathbb{Q}_p . We have*

$$\widehat{\lambda}_n(\xi) \leq \frac{1}{w_{n,n+1}(\xi)}. \tag{6.1}$$

More precisely, if for some positive real number c_1 and for arbitrarily large integers H , the system

$$H(P) \leq H, \quad |P(\xi)|_p \leq c_1H^{-w-1} \tag{6.2}$$

has $n + 1$ linearly independent solutions in integer polynomials P of degree at most n , then, there are a positive constant c_2 and arbitrarily large integers Z such that the system

$$1 \leq \max_{0 \leq j \leq n} |y_j| \leq Z, \quad \max_{1 \leq j \leq n} |y_0\xi^j - y_j|_p \leq c_2Z^{-\frac{1}{w}-1} \tag{6.3}$$

has no solution in integers y_0, y_1, \dots, y_n .

As in the real case, there is in fact equality in (6.1); the reverse inequality has been proved in the course of the proof of [21, Théorème 3].

Recall that Gelfond’s Lemma (see e.g. [7, Lemma A.3]) asserts that there exists a real number $K(n) > 1$, depending only on the positive integer n , such that all integer polynomials P, Q of degree at most n satisfy

$$\frac{1}{K(n)}H(P)H(Q) \leq H(PQ) \leq K(n)H(P)H(Q). \tag{6.4}$$

Proof of Theorem 6.1.. Let ℓ be the smallest integer in $\{1, 2, \dots, m\}$ such that $w_\ell(\xi) = w_m(\xi)$. We will show that

$$w_{\ell+n-1, \ell+n}(\xi) \geq \min\{w_\ell(\xi), \widehat{w}_n(\xi)\} = \min\{w_m(\xi), \widehat{w}_n(\xi)\}. \tag{6.5}$$

Then, the claim follows since, if $\ell < m$, it is sufficient to consider products of the involved polynomials by the powers $1, X, \dots, X^{m-\ell}$ of their variable X .

Assume first that $w_\ell(\xi)$ is finite. Let ε be in $(0, 1)$. By Gelfond’s Lemma and the definition of ℓ , for arbitrarily large H , there exist *irreducible* polynomials $P(X)$ of degree exactly ℓ that satisfy

$$H(P) = H, \quad |P(\xi)|_p \leq H^{-w_\ell(\xi)-1+\varepsilon}. \tag{6.6}$$

With $K(n)$ as in (6.4), set

$$\widetilde{H} = \frac{H}{2K(n)}.$$

Setting $w = \min\{\widehat{w}_n(\xi), w_\ell(\xi)\}$, by definition of $\widehat{w}_n(\xi)$, there exists a polynomial Q_0 in $\mathbb{Z}[X]$ of degree at most n such that

$$H(Q_0) \leq \widetilde{H}, \quad 0 < |Q_0(\xi)|_p \leq \widetilde{H}^{-w-1+\varepsilon}.$$

Set $Q(X) = X^{n-d}Q_0(X)$, where d denotes the degree of Q_0 . Since w is finite and $\varepsilon < 1$, we have

$$H(Q) \leq \widetilde{H}, \quad |Q(\xi)|_p \leq K_1 \cdot H^{-w-1+\varepsilon}, \tag{6.7}$$

for some positive K_1 .

By (6.4), the polynomial Q_0 (and, thus, Q) cannot be a multiple of P . Hence, since P is irreducible, the polynomials P and Q have no common factor over $\mathbb{Z}[X]$. Moreover, the pair (P, Q) satisfies the properties

$$\max\{H(P), H(Q)\} = H, \quad \max\{|P(\xi)|_p, |Q(\xi)|_p\} \leq K_1 \cdot H^{-w-1+\varepsilon}.$$

Consider the set \mathcal{P} of $\ell + n$ polynomials

$$\mathcal{P} = \{P, XP, X^2P, \dots, X^{n-1}P, Q, XQ, \dots, X^{\ell-1}Q\}. \tag{6.8}$$

The determinant of the $(\ell + n) \times (\ell + n)$ matrix whose rows are given by the coefficients of the polynomials in \mathcal{P} is the resultant of P and Q , which is non-zero since P and Q have no common factor. Hence, the elements of \mathcal{P} are linearly independent and span the space of polynomials of degree at most $\ell + n - 1$. Furthermore, it follows from (6.6) and (6.7) that any R in \mathcal{P} satisfies

$$H(R) \leq H, \quad |R(\xi)|_p \ll_{n, \xi, p} H^{-\min\{\widehat{w}_n(\xi), w_\ell(\xi)\}-1+\varepsilon} \tag{6.9}$$

This shows (6.5). If $w_\ell(\xi)$ is infinite, then the same type of argument gives (6.5), we omit the details.

Finally, if $m = n = \widehat{w}_n(\xi)$, in place of (6.6), we may use a variant of Dirichlet’s Theorem stating that for every $\xi \in \mathbb{Q}_p$ and every large H the system

$$H(P) \leq H, \quad |P(\xi)|_p \ll_{n,\xi} H^{-n-1}$$

has a solution in non-zero integer polynomials P of degree at most n . Consequently, again by Gelfond’s Lemma, the real number ε in (6.6) can be taken equal to 0 upon introduction of a multiplicative constant. Thus, from the method above, we get the last assertion of the theorem. □

Definition 6.3. For a convex body $K \subseteq \mathbb{R}^{n+1}$ and a lattice $\Lambda \subseteq \mathbb{R}^{n+1}$, for any $j = 1, \dots, n + 1$, the j th of the successive minima of K with respect to Λ , denoted by $\tau_j(K, \Lambda)$, is the smallest λ such that the convex body $\lambda K = \{\lambda k : k \in K\}$ contains j linearly independent points of Λ .

Proof of Theorem 6.2.. We follow the initial steps of the proof of [21, Théorème 3], where we apply the shift $n \rightarrow n + 1$ in the notation.

Let λ be a real number with

$$1 > \lambda > \frac{1}{1 + \widehat{\lambda}_n(\xi)}.$$

By definition of $\widehat{\lambda}_n(\xi)$, the system

$$\begin{aligned} \max_{0 \leq i \leq n} |z_i| &\leq p^{(n+1)\lambda h}, \\ \max_{1 \leq j \leq n} |z_0 \xi^j - z_j|_p &\leq p^{-(n+1)h} \end{aligned} \tag{6.10}$$

has a solution in non-zero integer vectors (z_0, \dots, z_n) for every large enough h . Equivalently, for every sufficiently large positive integer h the system

$$\begin{aligned} \max_{0 \leq i \leq n} |y_i| &\leq p^{-(n+1)(1-\lambda)h}, \\ \max_{1 \leq j \leq n} |y_0 \xi^j - y_j|_p &\leq 1 \end{aligned} \tag{6.11}$$

has a non-zero solution (y_0, \dots, y_n) in the \mathbb{Z} -module $p^{-(n+1)h} \mathbb{Z}^{n+1}$. We may assume that $|\xi|_p = 1$. The solutions to (6.11) in $p^{-(n+1)h} \mathbb{Z}^{n+1}$ form a lattice generated by

$$\begin{aligned} &(p^{-h(n+1)}, p^{-h(n+1)}b_1, \dots, p^{-h(n+1)}b_n) \\ &(0, 1, 0, 0 \dots, 0), \\ &(0, 0, 1, 0, 0 \dots, 0), \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &(0, 0, 0, 0, \dots, 1), \end{aligned}$$

for a given set of integers b_1, \dots, b_n . Denote this lattice by $\Lambda(h)$. Its dual lattice $\Lambda^*(h)$, defined by

$$\Lambda^*(h) = \{y \in \mathbb{R}^{n+1} : \forall x \in \Lambda, x \cdot y \in \mathbb{Z}\},$$

where \cdot is the usual scalar product in \mathbb{R}^{n+1} , is generated by the vectors

$$\begin{aligned} &(p^{h(n+1)}, 0, 0, \dots, 0), \\ &(-b_1, 1, 0, 0, \dots, 0), \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ &(-b_n, 0, 0, 0, \dots, 1). \end{aligned}$$

In other words, $\Lambda^*(h)$ is the sublattice of \mathbb{Z}^{n+1} consisting of the vectors (x_0, \dots, x_n) in \mathbb{Z}^{n+1} that satisfy

$$|x_0 + x_1\xi + \dots + x_n\xi^n|_p \leq p^{-h(n+1)}.$$

Let K be the convex body formed by the vectors (x_0, \dots, x_n) in \mathbb{R}^{n+1} that satisfy $\max_{0 \leq i \leq n} |x_i| \leq 1$, and K^* its dual convex body defined by

$$K^* := \{y \in \mathbb{R}^{n+1} : \forall x \in K, |x \cdot y| \leq 1\}.$$

Note that K is almost equal to K^* , more precisely $c^{-1}K^* \subseteq K \subseteq cK^*$ for some $c = c(n)$. Consequently, for $j = 1, \dots, n + 1$, the j th of the successive minima $\tau_j(K^*, \Lambda^*(h))$ and $\tau_j(K, \Lambda^*(h))$ satisfy

$$\tau_j(K^*, \Lambda^*(h)) \asymp \tau_j(K, \Lambda^*(h)).$$

Consider the first of the successive minima $\tau_1(K, \Lambda(h))$ of K with respect to $\Lambda(h)$ and the $(n + 1)$ th successive minima $\tau_{n+1}^*(K^*, \Lambda^*(h))$ of K^* with respect to $\Lambda^*(h)$. Then, by [8, Theorem VI, p. 219], we have

$$\tau_1(K, \Lambda(h)) \cdot \tau_{n+1}^*(K^*, \Lambda^*(h)) \gg 1.$$

We remark that Teulié [21, (2.5)] used the reverse estimate (which is true as well). Since the system (6.11) has a non-zero solution in the \mathbb{Z} -module $p^{-(n+1)h}\mathbb{Z}^{n+1}$ for every sufficiently large integer h , there exists an integer h_0 such that

$$\tau_1(K, \Lambda(h)) \leq p^{-(n+1)(1-\lambda)h}, \quad h \geq h_0.$$

Hence, recalling that $c^{-1}K \subseteq K \subseteq cK^*$, we get

$$\tau_{n+1}^*(K^*, \Lambda^*(h)) \gg \tau_{n+1}(K^*, \Lambda^*(h)) \gg p^{(n+1)(1-\lambda)h}, \quad h \geq h_0.$$

Consequently, there exists $c > 0$ such that, for $h > h_0$, the system

$$0 < \max\{|x_0|, \dots, |x_n|\} \leq cp^{(n+1)(1-\lambda)h}, \quad |x_0 + x_1\xi + \dots + x_n\xi^n|_p \leq p^{-h(n+1)},$$

does not have $n + 1$ linearly independent integer vector solutions (x_0, \dots, x_n) . We have shown that

$$w_{n,n+1}(\xi) \leq \frac{1}{1-\lambda} - 1 = \frac{\lambda}{1-\lambda}.$$

Since λ can be chosen arbitrarily close to $(1 + \widehat{\lambda}_n(\xi))^{-1}$, we see that the right-hand side can be arbitrarily close to $\widehat{\lambda}_n(\xi)^{-1}$. This completes the proof of (6.1). Analysing the proof, a very similar argument upon taking the contrapositive of the claim gives the implication of (6.3) from (6.2). We omit the details. \square

Acknowledgments. The authors are very grateful to the referee for an extremely careful reading and many corrections.

References

1. D. BADZIAHIN AND Y. BUGEAUD, On simultaneous rational approximation to a real number and its integral powers II, *N. Y. J. Math.* **26** (2020), 362–377.
2. V. BERESNEVICH, D. DICKINSON AND S. VELANI, Diophantine approximation on planar curves and the distribution of rational points. With an Appendix II by R. C. Vaughan, *Ann. Math. (2)* **166**(2) (2007), 367–426.
3. V. I. BERNIK AND M. M. DODSON, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, **Volume 137** (Cambridge University Press, 1999).
4. V. I. BERNIK AND I. L. MOROTSKAYA, Diophantine approximations in \mathbb{Q}_p and Hausdorff dimension, *Vesti Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk* **3** (1986), 3–9, 123 (in Russian).
5. E. BOMBIERI AND W. GUBLER, *Heights in diophantine geometry* (New Mathematical Monographs, **Volume 4** (Cambridge University Press, 2006)).
6. N. BUDARINA, Y. BUGEAUD, D. DICKINSON AND H. O'DONNELL, On simultaneous rational approximation to a p -adic number and its integral powers, *Proc. Edinb. Math. Soc.* **54** (2011), 599–612.
7. Y. BUGEAUD, *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics (Cambridge University Press, 2004).
8. J. W. S. CASSELS, *An introduction to the geometry of numbers* (Springer Verlag, 1959).
9. J. CHAMPAGNE AND D. ROY, A transference inequality for rational approximation to points in geometric progression, *Hardy–Ramanujan J.* **42** (2019), 70–72.
10. H. DAVENPORT AND W. M. SCHMIDT, Approximation to real numbers by algebraic integers, *Acta Arith.* **15** (1969), 393–416.
11. V. JARNÍK, Über einen p -adischen Übertragungssatz, *Monatsh. Math. Phys.* **48** (1939), 277–287.
12. V. JARNÍK, Sur les approximations diophantiques des nombres p -adiques, *Rev. Ci. (Lima)* **47** (1945), 489–505.
13. M. LAURENT, Simultaneous rational approximation to the successive powers of a real number, *Indag. Math.* **11** (2003), 45–53.
14. É. LUTZ, *Sur les approximations diophantiennes linéaires p -adiques*, Actualités Sci. Ind., Volume 1224, 106 p. (Paris, Hermann & Cie, 1955).
15. K. MAHLER, Über Diophantische Approximationen in Gebiete der p -adische Zahlen, *Jahresbericht d. Deutschen Math. Verein.* **44** (1934), 250–255.
16. K. MAHLER, Über eine Klassen-einteilung der p -adischen Zahlen, *Math. Leiden* **3** (1935), 177–185.

17. I. L. MOROTSKAYA, Hausdorff dimension and Diophantine approximations in \mathbb{Q}_p , *Dokl. Akad. Nauk BSSR* **31** (1987), 597–600, 668 (in Russian).
18. J. SCHLEISCHITZ, On the spectrum of Diophantine approximation constants, *Mathematika* **62** (2016), 79–100.
19. J. SCHLEISCHITZ, On uniform approximation to successive powers of a real number, *Indag. Math.* **28** (2017), 406–423.
20. V. G. SPRINDŽUK, *Mahler's problem in metric number theory* (American Mathematical Society, 1969).
21. O. TEULIÉ, Approximation d'un nombre p -adique par des nombres algébriques, *Acta Arith.* **102** (2002), 137–155.
22. R. C. VAUGHAN AND S. VELANI, Diophantine approximation on planar curves: the convergence theory, *Invent. Math.* **166**(1) (2006), 103–124.