Proceedings of the Edinburgh Mathematical Society (2000) **43**, 625–631 ©

ON THE BLOW-UP AND POSITIVE ENTIRE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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(Received 7 January 1999)

Abstract In this paper we consider the following semilinear elliptic equation

$$\Delta u + \frac{\beta}{(1+|x|)^{\mu}} u^p - \frac{\gamma}{(1+|x|)^{\nu}} u^q = 0, \quad \text{in } \mathbb{R}^n,$$

where $n \geqslant 3$, $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$, and $\beta \geqslant 0$, $\gamma \geqslant 0$, $q > p \geqslant 1$, μ and ν are real constants. We note that if $\gamma = 0$, $\beta > 0$ and $\mu \geqslant 2$, then the equation above is called the Matukuma-type equation. If $\beta = 0$, $\gamma > 0$ and $\nu > 2$, then the complete classification of all possible positive solutions had been conducted by Cheng and Ni. If $\beta > 0$, $\gamma > 0$ and $\mu \geqslant \nu > 2$, then some results about the maximal solution and positive solution structures can be found in Chern. The purpose of this paper is to discuss and investigate the blow-up and positive entire solutions of the equation above for the $\mu \geqslant 2 \geqslant \nu$ case.

Keywords: semilinear elliptic equations; blow-up; entire solutions

AMS 1991 *Mathematics subject classification:* Primary 35J60

1. Introduction

In this paper we consider the following semilinear elliptic equation

$$\Delta u + \frac{\beta}{(1+|x|)^{\mu}} u^p - \frac{\gamma}{(1+|x|)^{\nu}} u^q = 0, \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \geqslant 3$, $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$, and $\beta \geqslant 0$, $\gamma \geqslant 0$, $q > p \geqslant 1$, μ and ν are real constants. Equation (1.1) arises from physics and geometry. When $n = 3$, $\mu = 2$, $p > 1$ and $\gamma = 0$, (1.1) was proposed by Matukuma [**8**] in 1930 as a mathematical model to describe the dynamics of a globular cluster of stars. In this context, u represents the gravitational potential (therefore $u > 0$). Since the globular cluster has radial symmetry, positive radial entire solutions are of particular interest. For the solution structures and references of Matikuma-type equations, we refer to Li and Ni [**7**], Ni and Yotsutani [**11**] and Yanagida [**12**]. When $\beta = 0$, $\gamma > 0$ and $q = (n+2)/(n-2)$, (1.1) is then a conformal scalar curvature equation in $\mathbb{R}^n$. In this case, (1.1) becomes the following equation, without loss of generality we can assume $\gamma = 1$,

$$\Delta u - \frac{1}{(1+|x|)^{\nu}} u^q = 0, \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

The complete classification of all positive solutions of (1.2) for $q > 1$ had been conducted by Cheng and Ni [**4**]. From [**4**, Theorem II] they obtained Theorem 1.1.

Theorem 1.1. *If $\nu > 2$, then the following conclusions hold.*

(i) *For every positive constant α, equation (1.2) possesses a unique positive solution u_α such that $u_\alpha(x) \to \alpha$ as $x \to \infty$. Furthermore, u_α is radially symmetric.*

(ii) *Let u be a positive solution of equation (1.2). Then either $u \equiv U_0$ or $u \equiv u_\alpha$ for some $\alpha > 0$, where u_α is given by (i) above, and U_0 denotes the maximal solution:*

$$U_0(x) = \sup\{v(x) | v \text{ is a positive solution of equation (1.2)}\}. \tag{1.3}$$

(iii) *If $\alpha > \beta > 0$, then $U_0 > u_\alpha > u_\beta$ in $\mathbb{R}^n$. Furthermore, U_0 is radially symmetric and the asymptotic behaviour of U_0 near ∞ is given by*

$$U_0(x) \sim |x|^{\nu-2/q-1} \text{ near } \infty.$$

If $\beta > 0$ and $\gamma > 0$, then, without loss of generality, we can assume $\gamma = 1$ in (1.1) and consider the following equation:

$$\Delta u + \frac{\beta}{(1+|x|)^\mu} u^p - \frac{1}{(1+|x|)^\nu} u^q = 0, \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

Let

$$R_0 = U_0(0), \qquad T_\beta = \beta^{1/(q-p)}, \tag{1.5}$$

$$U = \sup\{v(x) | v \text{ is a positive solution of (1.4) with } u \geqslant T_\beta \text{ on } \mathbb{R}^n\}, \tag{1.6}$$

where U_0 is defined in (1.3) of Theorem 1.1. Then some results about the maximal solution U and positive solution structures of (1.4) can be found in Chern [**5**]. We proved Theorem 1.2.

Theorem 1.2. *If $q > p \geqslant 1$ and $\mu \geqslant \nu > 2$, then the following conclusions hold.*

(a) *If $T_\beta < R_0$, then for each $\alpha \in [T_\beta, R_0)$, (1.4) possesses a positive radial solution v_α satisfying $v_\alpha \geqslant T_\beta$ on $\mathbb{R}^n$ and $\lim_{r\to\infty} v_\alpha(r) = C(\alpha)$, where $C(\alpha)$ is an increasing function in α. Furthermore, if $T_\beta \leqslant \alpha_1 < \alpha_2 < R_0$, then $v_{\alpha_1} < v_{\alpha_2} < U$ in $\mathbb{R}^n$, where U is given by (1.6).*

(b) *For every constant $c \geqslant T_\beta$, (1.4) possesses the type solution u_c satisfying $u_c \to c$ as $x \to \infty$. Furthermore, if $u_c \geqslant T_\beta$ in $\mathbb{R}^n$, then such a solution u_c is unique and radially symmetric, i.e. $u_c(x) = u_c(|x|)$ for all $x \in \mathbb{R}^n$.*

(c) *For every bounded positive solution u of (1.4), there exists a constant $c \geqslant 0$ such that $u \equiv u_c$, where u_c is given by (b) above. Furthermore, if $u \geqslant T_\beta$ in $\mathbb{R}^n$, then such a solution u_c is unique and radially symmetric.*

(d) *U is radially symmetric and satisfies the asymptotic behaviour*

$$U(r) \sim r^{\nu-2/q-1}, \quad \text{at } r=\infty.$$

(e) *Suppose that $T_\beta < R_0$ and u is a positive solution of (1.4) satisfying $u \geqslant T_\beta$ in $\mathbb{R}^n$. Then u is radially symmetric and either $u \equiv U$ or $u \equiv u_c$ for some $c > T_\beta$.*

We note that all the bounded positive solutions in both Theorems 1.1 and 1.2 tend to positive constants at infinity. The purpose of this paper is to discuss and investigate the blow-up and positive entire solutions of (1.4). These solution types are different from those mentioned in the above theorems. We will seek the radial solutions, i.e. $u = u(|x|)$. Let $r = |x|$. Then, in this case, (1.4) reduces to the following initial value problem:

$$\left.\begin{aligned} u'' + \frac{n-1}{r}u' + \frac{\beta}{(1+r)^\mu}u^p - \frac{1}{(1+r)^\nu}u^q = 0, \quad r>0, \\ u(0)=\alpha, \quad u'(0)=0. \end{aligned}\right\} \tag{1.7}$$

Now we state our results as follows.

Theorem 1.3. *If $q > p > 1$ and $\mu \geqslant 2 \geqslant \nu > 0$, then the following conclusions hold.*

(i) *For every $\alpha > \beta^{1/(q-p)} \geqslant 0$, there exists $R(\alpha) > 0$ such that the solution $u(r,\alpha)$ of (1.7) satisfies*

$$u(r,\alpha) > 0, \quad \text{on } [0, R(\alpha)) \quad \text{and} \quad \lim_{r\to R^-(\alpha)} u(r,\alpha) = \infty. \tag{1.8}$$

(ii) *There exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$, (1.7) possesses a positive solution u_α on $[0,\infty)$ satisfying*

$$u_\alpha(0) \leqslant \alpha \quad \text{and} \quad \lim_{r\to\infty} u_\alpha(r) = 0. \tag{1.9}$$

In addition, if $(n+2)/(n-2) > p > 1$, then (1.7) possesses a positive solution $u(r)$ on $[0,\infty)$ satisfying

$$u(r) = o(r^{2-n}), \quad \text{at } r = \infty. \tag{1.10}$$

We organize this paper as follows. In § 2, we prove the blow-up result of part (i) of Theorem 1.3. Finally, we give the complete proofs of the existence results of parts (ii) of Theorem 1.3 in § 3.

2. Proof of the blow-up solutions

In this section we give the proof of part (i) of Theorem 1.3.

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Proof of part (i) of Theorem 1.3. Let $\alpha > \beta^{1/(q-p)}$ and $u(r,\alpha)$ be the positive solution of (1.7). Then, from $q > p > 1$ and $\mu \geqslant \nu$, there exists $r_1 > 0$ such that $u^{q-p}(r,\alpha) - \beta(1+r)^{\nu-\mu} > 0$ $\forall r \in [0, r_1]$. From (1.7) we can easily obtain

$$u'(r,\alpha) = \frac{1}{r^{n-1}} \int_0^r s^{n-1}(u^{q-p}(s,\alpha) - \beta(1+s)^{\nu-\mu}) \frac{1}{(1+s)^\nu} u^p(s,\alpha)\,\mathrm{d}s > 0, \quad \text{for all } r \in (0, r_1]. \tag{2.1}$$

Hence, we obtain that $u(r,\alpha)$ is increasing on $(0, r_1]$, and

$$u'(r,\alpha) > 0 \quad \text{and} \quad u(r,\alpha) > \alpha, \quad \text{for all } r \text{ in the interval in which } u \text{ can be defined.} \tag{2.2}$$

We shall prove that there exists $R(\alpha) > 0$ such that $u(r,\alpha) \to \infty$ as $r \to R^-(\alpha)$.

Suppose that this is not true. Then $u(r,\alpha)$ is defined on the entire domain $[0,\infty)$. From (2.1) and (2.2), it is easy to see that

$$u(r,\alpha) \geqslant \alpha + \frac{1}{n-2} \int_0^r s\left[1 - \left(\frac{s}{r}\right)^{n-2}\right] [\alpha^{q-p} - \beta] \frac{1}{(1+s)^\nu} u^p(s,\alpha)\,\mathrm{d}s. \tag{2.3}$$

Since $2 \geqslant \nu$, it follows that there exists $R_1 > 0$ such that

$$\frac{1}{(1+r)^\nu} \geqslant \frac{1}{r^2}, \quad \forall r \geqslant R_1.$$

Let $r > R_1$. Then, from (2.3), we have

$$\begin{aligned} u(r,\alpha) &\geqslant \alpha + \frac{\alpha^{q-p} - \beta}{n-2} \int_{R_1}^r s\left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \frac{1}{s^2} u^p(s,\alpha)\,\mathrm{d}s \\ &\geqslant \alpha + \frac{C_0}{n-2} \cdot \left[1 - \left(\frac{1}{2}\right)^{n-2}\right] \cdot \int_{R_1}^{r/2} \frac{1}{s}\,\mathrm{d}s \\ &\geqslant C_1 \log r, \end{aligned} \tag{2.4}$$

for $r \geqslant R_2 \geqslant 2R_1$ and for some $C_1 > 0$.

Now the rest of the argument is the same as that in the proof of Theorem 2.1 in [**2**]. We can get a contradiction. We omit the details. This completes the proof of part (i) in Theorem 1.3. □

3. Proofs of existence results

In this section we give the complete proofs of part (ii) of Theorem 1.3.

Proof of the first result of part (ii). First we consider the equation

$$\left.\begin{aligned} v'' + \frac{n-1}{r} v' + \frac{\beta}{(1+r)^\mu} v^p = 0, \quad r > 0, \\ v(0) = \alpha, \quad v'(0) = 0. \end{aligned}\right\} \tag{3.1}$$

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Since $\mu \geqslant 2$ and $p > 1$, from [**11**, Theorem 3], there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$, (3.1) possesses a solution $v(r, \alpha) > 0$ on $[0, \infty)$ and v is monotonically decreasing in r, which satisfies

$$\lim_{r \to \infty} v(r, \alpha) = 0.$$

It is easy to see that v is a super-solution of (1.4).

Now, we choose

$$0 < \varepsilon \leqslant \min\left\{1, \left[\frac{\beta}{2^\mu}\left(\frac{8}{n+8}\right)^{5p} \frac{1}{(10n+1)}\right]^{(q-1)/(q-p)}\right\},$$

and define

$$v_\varepsilon(x) = v_\varepsilon(|x|) = \begin{cases} \varepsilon^k(\varepsilon^{-1} - \varepsilon^{-2}|x|^2)^5, & \text{for } 0 \leqslant |x| \leqslant \varepsilon^{1/2}, \\ 0, & \text{for } |x| \geqslant \varepsilon^{1/2}, \end{cases} \tag{3.2}$$

where $k = (5q-4)/(q-1)$. Then $v_\varepsilon \in C^2(\mathbb{R}^n)$ and

$$\begin{aligned} \Delta v_\varepsilon + \frac{\beta}{(1+r)^\mu} v_\varepsilon^p - \frac{1}{(1+r)^\nu} v_\varepsilon^q \\ = 10\varepsilon^{k-4}(\varepsilon^{-1} - \varepsilon^{-2}r^2)^3[(n+8)r^2 - n\varepsilon] + \frac{\beta}{(1+r)^\mu}\varepsilon^{pk}(\varepsilon^{-1} - \varepsilon^{-2}r^2)^{5p} \\ - \frac{1}{(1+r)^\nu}\varepsilon^{qk}(\varepsilon^{-1} - \varepsilon^{-2}r^2)^{5q}. \end{aligned} \tag{3.3}$$

In order to show that v_ε is a sub-solution of (1.4), we estimate (3.3) by two cases as follows.

(i) For $\varepsilon^{1/2} \geqslant r \geqslant (n/(n+8))^{1/2}\varepsilon^{1/2}$:

$$\begin{aligned} (3.3) &\geqslant \frac{1}{(1+r)^\mu}\varepsilon^{pk}(\varepsilon^{-1} - \varepsilon^{-2}r^2)^{5p}[\beta - (1+r)^{\mu-\nu}\varepsilon^{k(q-p)}(\varepsilon^{-1} - \varepsilon^{-2}r^2)^{5(q-p)}] \\ &\geqslant \frac{1}{(1+r)^\mu}\varepsilon^{pk}(\varepsilon^{-1} - \varepsilon^{-2}r^2)^{5p}[\beta - 2^\mu \varepsilon^{(q-p)/(q-1)}] \\ &\geqslant 0 \qquad \left(\text{since } \varepsilon \leqslant \left(\beta \cdot \frac{1}{2^\mu}\right)^{q-1/q-p} \text{ and } q > p > 1\right). \end{aligned}$$

(ii) For $0 \leqslant r \leqslant (n/(n+8))^{1/2}\varepsilon^{1/2}$:

$$\begin{aligned} (3.3) &\geqslant -10n\varepsilon^{k-4} + \frac{\beta}{(1+r)^\mu}\left(\frac{8}{n+8}\right)^{5p}\varepsilon^{pk-5p} - \frac{1}{(1+r)^\nu}\varepsilon^{qk-5q} \\ &\geqslant -(10n+1)\varepsilon^{q/(q-1)} + \frac{\beta}{2^\mu}\left(\frac{8}{n+8}\right)^{5p}\varepsilon^{p/(q-1)} \\ &= \varepsilon^{p/(q-1)}\left[\frac{\beta}{2^\mu}\left(\frac{8}{n+8}\right)^{5p} - (10n+1)\varepsilon^{(q-p)/(q-1)}\right] \\ &\geqslant 0 \qquad \left(\text{since } \varepsilon \leqslant \left[\frac{\beta}{2^\mu}\left(\frac{8}{n+8}\right)^{5p}\frac{1}{(10n+1)}\right]^{(q-1)/(q-p)} \text{ and } q > p > 1\right) \end{aligned}$$

Then we conclude that

$$\Delta v_\varepsilon + \frac{\beta}{(1+r)^\mu} v_\varepsilon^p - \frac{1}{(1+r)^\nu} v_\varepsilon^q \geqslant 0, \quad \text{for all } r = |x| \geqslant 0.$$

Hence, v_ε is a sub-solution of (1.4).

Moreover, since $v(r,\alpha)$ is monotonically decreasing in r, we can make sure that

$$v_\varepsilon(r) \leqslant v(r,\alpha), \quad \text{for all } r \geqslant 0 \text{ if } \varepsilon \text{ is sufficiently small.}$$

Therefore, (1.4) possesses a radial solution $u_\alpha = u_\alpha(r)$ satisfying

$$v_\varepsilon(r) \leqslant u_\alpha(r) \leqslant v(r,\alpha), \quad \text{for all } r \geqslant 0. \tag{3.4}$$

Then it is easy to see that $\lim_{r\to\infty} u_\alpha(r) = 0$ and

$$\alpha \geqslant u_\alpha(0) \geqslant \varepsilon^{1/(q-1)}, \quad u_\alpha(r) > 0, \quad \forall r \geqslant 0. \tag{3.5}$$

This proves the first result of part (ii) in Theorem 1.3. □

Proof of the second result of part (ii). Since $(n+2)/(n-2) > p > 1$ and $\mu \geqslant 2$, by [**1**, Theorem 1.1], there exists a positive radial solution $w = w(r)$ of (3.1), $r = |x|$, such that

$$\lim_{r\to\infty} r^{n-2} w(r) = c_0, \quad \text{for some positive constant } c_0. \tag{3.6}$$

Furthermore, $w(r)$ is strictly decreasing in r and a super-solution of (1.4).

Hence, (1.4) possesses a positive radial solution $u = u(r)$ and we have

$$v_\varepsilon(r) \leqslant u(r) \leqslant w(r), \quad \text{for all } r \geqslant 0 \text{ and for } \varepsilon \text{ sufficiently small,} \tag{3.7}$$

where v_ε is defined in (3.2). From (3.2), (3.6) and (3.7), we obtain

$$u(r) = o(r^{2-n}), \quad \text{at } r = \infty.$$

This proves the second result and completes the proof of part (ii) of Theorem 1.3. □

Acknowledgements. Work partly supported by the National Science Council of the Republic of China.

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