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GLOBAL ATTRACTIVITY IN DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract

Consider the forced differential equation with variable delay

$$x'(t) + b(t)x(t - \tau(t)) = f(t), \quad t \geq 0,$$

where

$$f \in C([0, \infty)) \quad \text{and} \quad b, \tau \in C([0, \infty), [0, \infty)).$$

We establish a sufficient condition for every solution to tend to zero. We also obtain a sharper condition for every solution to tend to zero when $\int_{t-\tau(t)}^{t} b(s)ds$ is asymptotically constant.

1. Introduction

Consider the forced delay differential equation

$$x'(t) + b(t)x(t - \tau(t)) = f(t), \quad t \geq 0, \tag{1.1}$$

where

$$f \in C([0, \infty), \mathbf{R}) \quad \text{and} \quad b, \tau \in C([0, \infty), [0, \infty))$$

with

$$\lim_{t\to\infty}[t - \tau(t)] = \infty. \tag{1.2}$$

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Our aim in this paper is to study the global attractivity of solutions of (1.1).

When $f(t) \equiv 0$, (1.1) reduces to

$$x'(t) + b(t)x(t - \tau(t)) = 0, \quad t \geq 0. \tag{1.3}$$

In the case of bounded delays, the asymptotic behaviour of solutions of (1.3), as well as more general cases, has been studied by several authors; see, for example, Burton and Haddock [1], Cooke [2, 3], Györi [7], Yorke [12] and the references cited therein. It is well known that if there are two positive numbers β and q such that

$$b(t) \leq \beta, \quad \tau(t) \leq q \quad \text{and} \quad \beta q < 3/2$$

for $t \geq 0$, then the trivial solution of (1.3) is uniformly stable. Furthermore, the upper bound $3/2$ is sharp in the sense that if $\beta q > 3/2$ there are equations with unbounded solutions. Yoneyama [11] removed the boundedness hypothesis on $b(t)$ and proved the following theorem.

THEOREM A. *If there is a positive number q such that*

$$\tau(t) \leq q \quad \textit{for } t \geq 0,$$

$$\inf_{t\geq 0} \int_t^{t+q} b(s)ds > 0 \quad \textit{and} \quad \sup_{t\geq 0} \int_t^{t+q} b(s)ds < 3/2,$$

then the trivial solution of (1.3) *is asymptotically stable.*

Recently, Kolmanovskii *et al.* [8] proved the following theorem which allows the delay function in (1.3) to be unbounded.

THEOREM B. *Assume that τ is differentiable and*

$$\tau'(t) \leq R < 1 \quad \textit{for } t \geq 0. \tag{1.4}$$

Suppose also that

$$\inf_{t\geq 0} b(t) = B > 0 \quad \textit{and} \quad \sup_{t\geq 0} \int_t^{g(t)} b(s)ds = \beta < 1,$$

where $g(t)$ is the inverse function of $t - \tau(t)$. Then the trivial solution of (1.3) *is uniformly stable. Moreover, if $b(t)$ is bounded, then the trivial solution of* (1.3) *is uniformly asymptotically stable.*

In the above theorem, although the delay function $\tau(t)$ is allowed to be unbounded, there is still a relatively strong restriction (see (1.4)) on its growth. In addition, the persistence and boundedness of $b(t)$ are also needed for the asymptotic stability. In the present paper, we establish a sufficient condition for the asymptotic stability of the trivial solution of (1.3) without using these hypotheses. In fact, we will establish this attractivity result for the more general forced equation (1.1). For the case that $\int_{t-\tau}^{t} b(s)ds$ is asymptotically constant, we obtain a sharper condition for the global attractivity of solutions of (1.1), which is an extention of the work of Ladas *et al.* [9].

For the last several decades, the asymptotic behaviour of various unforced delay equations has been investigated by many authors. However, results about the behavior of solutions of forced delay equations are relatively scarce. In general, the presence of the forcing term makes the problem more interesting as well as considerably more difficult. For recent studies of first-order forced differential equations, we refer the reader to Lim [10], Graef and Qian [6] and the references contained therein.

2. Global attractivity of (1.1)

THEOREM 1. *Assume that*

$$b(t) > 0, \quad \int_0^\infty b(t)dt = \infty \tag{2.1}$$

and

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} b(s)\, ds < 1. \tag{2.2}$$

In addition, assume that

$$\lim_{t\to\infty} \frac{f(t)}{b(t)} = 0 \tag{2.3}$$

and

$$\int_0^\infty f(t)\, dt \quad \textit{converges.} \tag{2.4}$$

Then every solution of (1.1) *tends to zero as t tends to infinity.*

PROOF. First, assume that $x(t)$ is an eventually monotonic solution. Suppose that $x(t)$ is eventually positive; the proof for the case that $x(t)$ is eventually negative is similar and will be omitted. Let

$$l = \lim_{t\to\infty} x(t);$$

then $0 \le l \le \infty$. Clearly, it suffices to show that $l = 0$. Assume, for the sake of contradiction, that $l > 0$. Then there is a $T > 0$ such that

$$x(t - \tau(t)) \ge \min\{l/2, 1\} \quad \text{for } t \ge T. \tag{2.5}$$

Now, integrating both sides of (1.1) from T to t and letting $t \to \infty$, we have

$$l - x(T) + \int_T^\infty b(t)x(t - \tau(t))\,dt = \int_T^\infty f(t)\,dt.$$

Then, from (2.5), it follows that

$$l - x(T) + \min\{l/2, 1\} \int_T^\infty b(t)\,dt \le \int_T^\infty f(t)\,dt,$$

which contradicts (2.1) and (2.4). Hence, $l = 0$.

Next, assume that $x(t)$ is an eventually nonmonotonic solution. Then there is a sequence $\{t_n\}$ with

$$0 < t_1 < t_2 < \cdots < t_n < \cdots \quad \text{and} \quad \lim_{t\to\infty} t_n = \infty \tag{2.6}$$

such that $x(t)$ has relative extrema at t_n, $n = 1, 2, \ldots$. Since $x'(t_n) = 0$, it follows from (1.1) that

$$x(t_n - \tau(t_n)) = \frac{f(t_n)}{b(t_n)}, \quad n = 1, 2, \ldots.$$

Then, integrating both sides of (1.1) from $t_n - \tau(t_n)$ to t_n, we obtain

$$x(t_n) = \frac{f(t_n)}{b(t_n)} + \int_{t_n-\tau(t_n)}^{t_n} f(t)\,dt - \int_{t_n-\tau(t_n)}^{t_n} b(t)x(t - \tau(t))\,dt. \tag{2.7}$$

Clearly, (2.4) and (1.2) yield

$$\lim_{n\to\infty} \int_{t_n-\tau(t_n)}^{t_n} f(s)\,ds = 0.$$

By noting this fact, (2.2) and (2.3), we see that there are three positive numbers δ, μ and t_0 such that for $t \ge t_0$,

$$\left| \frac{f(t)}{b(t)} + \int_{t-\tau(t))}^{t} f(s)\,ds \right| < \delta, \tag{2.8}$$

$$\int_{t-\tau(t)}^{t} b(s)\,ds < \mu \tag{2.9}$$

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and

$$\delta + \mu < 1. \tag{2.10}$$

Since (1.2) and (2.6) hold, we can choose a subsequence $\{t_{N_k}\}$ of $\{t_n\}$ such that, for any positive integer m,

$$t - \tau(t) \geq t_{N_m} \quad \text{when} \quad t \geq t_{N_{m+1}} - \tau(t_{N_{m+1}})$$

and

$$\left|\frac{f(t)}{b(t)} + \int_{t-\tau(t))}^{t} f(s)\,ds\right| \leq \delta^m \quad \text{when} \quad t \geq t_{N_m}.$$

Let

$$M = \max_{0 \leq n \leq N_1} \{|x(t_n)|\}.$$

We claim that

$$|x(t_n)| \leq (\delta + \mu)^k (M + 1) \quad \text{for} \quad n \geq N_k. \tag{2.11}$$

In fact, from (2.7) we see that

$$\begin{aligned}
|x(t_{N_1})| &\leq \left|\frac{f(t_{N_1})}{b(t_{N_1})} + \int_{t_{N_1}-\tau(t_{N_1})}^{t_{N_1}} f(t)dt\right| + \int_{t_{N_1}-\tau(t_{N_1})}^{t_{N_1}} b(t)|x(t-\tau(t))|dt \\
&\leq \delta + M \int_{t_{N_1}-\tau(t_{N_1})}^{t_{N_1}} b(t)dt \\
&\leq \delta + M\mu \leq (\delta + \mu)(M + 1).
\end{aligned}$$

Now, assume that

$$|x(t_n)| \leq (\delta + \mu)(M + 1) \quad \text{for} \quad N_1 \leq n \leq m.$$

Then, noting that $\{x(t_n)\}$ is the sequence of relative extrema, it follows from (2.7), (2.8) and (2.9) that

$$\begin{aligned}
|x(t_{m+1})| &\leq \left|\frac{f(t_{m+1})}{b(t_{m+1})} + \int_{t_{m+1}-\tau(t_{m+1})}^{t_{m+1}} f(t)dt\right| + \int_{t_{m+1}-\tau(t_{m+1})}^{t_{m+1}} b(t)|x(t-\tau(t))|dt \\
&\leq \delta + \max\{M + 1, |x(t_{m+1})|\} \int_{t_{m+1}-\tau(t_{m+1})}^{t_{m+1}} b(t)dt \\
&\leq \delta + \max\{M + 1, |x(t_{m+1})|\}\mu.
\end{aligned} \tag{2.12}$$

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We claim that

$$|x(t_{m+1})| \le M + 1. \tag{2.13}$$

Otherwise, $|x(t_{m+1})| > M + 1$. Then, from (2.12), we see that

$$|x(t_{m+1})| \le \delta + |x(t_{m+1})|\mu,$$

which implies

$$|x(t_{m+1})| \le \frac{\delta}{1-\mu}.$$

It follows that

$$\frac{\delta}{1-\mu} > M + 1,$$

which clearly contradicts (2.10). Hence, (2.13) holds and so it follows that

$$|x(t_{m+1})| \le \delta + (M+1)\mu \le (\delta + \mu)(M+1).$$

Therefore, by induction, (2.11) holds when $k = 1$. Next, assume that

$$|x(t_n)| \le (\delta + \mu)^m (M+1) \quad \text{for } n \ge N_m. \tag{2.14}$$

We are going to show that

$$|x(t_n)| \le (\delta + \mu)^{m+1} (M+1) \quad \text{for } n \ge N_{m+1}.$$

In fact, in view of (2.14), it follows from (2.7) that

$$\begin{aligned} |x(t_n)| &\le \left| \frac{f(t_n)}{b(t_n)} + \int_{t_n - \tau(t_n)}^{t_n} f(t)dt \right| + \int_{t_n-\tau(t_n)}^{t_n} b(t)|x(t-\tau(t))|dt \\ &\le \delta^{m+1} + (\delta+\mu)^m (M+1) \int_{t_n-\tau(t_n)}^{t_n} b(t)dt \\ &\le \delta^{m+1} + (\delta+\mu)^m \mu (M+1). \end{aligned}$$

Since

$$\delta^{m+1} + (\delta+\mu)^m \mu \le (\delta+\mu)^{m+1},$$

we see that

$$|x(t_n)| \le (\delta + \mu)^{m+1} (M+1) \quad \text{for } n \ge N_{m+1}.$$

Hence, by induction, we see that (2.11) holds for all $k = 1, 2, \ldots$. Clearly (2.11) implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

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The following result is a direct consequence of Theorem 1.

COROLLARY 1. *Consider* (1.3) *and assume that* (2.1) *and* (2.2) *hold. Then the trivial solution of* (1.3) *is asymptotically stable.*

EXAMPLE 1. Consider the delay differential equation

$$x'(t) + \frac{1}{2(1+t)} x\left(\frac{1}{3}t(2+\sin t)\right) = \frac{1}{(1+t)^2}, \quad t \geq 0. \tag{2.15}$$

Let

$$b(t) = \frac{1}{2(t+1)}, \quad \tau(t) = \frac{1}{3}t(1-\sin t) \quad \text{and} \quad f(t) = \frac{1}{(1+t)^2}.$$

Observe that

$$\int_0^\infty b(t)dt = \infty, \quad \lim_{t\to\infty} \frac{f(t)}{b(t)} = 0, \quad \int_0^\infty f(t)dt = 1$$

and

$$\begin{aligned}
\int_{t-\tau(t)}^t b(s)ds &= \int_{\frac{1}{3}t(2+\sin t)}^t \frac{1}{2(1+s)} ds \\
&\leq \int_{\frac{1}{3}t}^t \frac{1}{2(1+s)} ds \\
&= \frac{1}{2}\left(\ln(1+t) - \ln\left(1+\frac{1}{3}t\right)\right) \to \frac{1}{2}\ln 3 \quad \text{as } t \to \infty.
\end{aligned}$$

Hence, by Theorem 1, every solution of (2.15) tends to zero as t tends to infinity.

3. A sharper condition

In this section, we establish a sharper condition for the global attractivity of solutions of (1.1) when $\int_{t-\tau(t)}^t b(s)ds$ is asymptotically constant. Our work is motivated by a paper of Ladas *et al.* [9] for (1.3) with $\tau(t) \equiv \tau$, a constant.

Consider the autonomous delay differential equation

$$y'(t) + py(t-\tau_0) = 0, \quad t \geq t_0, \tag{3.1}$$

where

$$p, \tau_0 \in (0,\infty) \quad \text{and} \quad p\tau_0 < \pi/2.$$

It is known, see for example [4], that if $y(t, t_0, \phi)$ denotes a solution of (3.1) and

$$y(t) = \phi(t) \quad \text{with} \quad \phi \in C([t_0 - \tau, t_0], \mathbf{R}),$$

then there are two positive numbers M and γ such that

$$|y(t, t_0, \phi)| \le M \max_{t_0-\tau \le s \le t_0} \{|\phi(t)|\} e^{-\gamma t}, \quad t \ge t_0. \tag{3.2}$$

Furthermore, if $z(t, t_0, 0)$ is the solution of

$$z'(t) + pz(t - \tau_0) = h(t), \quad t \ge t_0,$$

with zero initial function, then

$$|z(t, t_0, 0)| \le \frac{M}{\gamma} e^{(p+\gamma)\tau_0} \max_{t_0 \le s \le t} |h(s)|. \tag{3.3}$$

The following lemma is needed in the proof of our main result in this section.

LEMMA 1. *Consider the delay differential equation*

$$x'(t) + x(t - \sigma(t)) = g(t), \quad t \ge 0, \tag{3.4}$$

where

$$\sigma \in C([0, \infty), [0, \infty)) \quad \text{and} \quad g \in C([0, \infty), \mathbf{R}).$$

Suppose that

$$\lim_{t \to \infty} \sigma(t) = \tau < \pi/2 \quad \text{and} \quad \lim_{t \to \infty} g(t) = 0.$$

Then every solution of (3.4) *tends to zero as t tends to infinity.*

PROOF. Let $x(t)$ be a solution of (3.4). Choose $t_0 \ge 2\tau + 2$ such that

$$\sigma(t) \le \tau + 1 \quad \text{for } t \ge t_0$$

and

$$\frac{M}{\gamma} e^{(1+\gamma)\tau} |\tau - \sigma(t)| \le 1/2 \quad \text{for } t \ge t_0, \tag{3.5}$$

where the constants M and γ are as defined in (3.2). Let $y(t)$ be the solution of

$$y'(t) + y(t - \tau) = 0, \quad t \ge t_0,$$

with initial condition

$$y(t) = x(t), \quad t_0 - \tau \le t \le t_0.$$

Then, by noting that $\tau < \pi/2$, we see that $y(t)$ tends to zero as $t \to \infty$. Set

$$z(t) = x(t) - y(t)$$

and observe that $z(t)$ satisfies the equation

$$z'(t) + z(t-\tau) = x(t-\tau) - x(t-\sigma(t)) + g(t), \quad t \ge t_0,$$

with zero initial condition $z(t) \equiv 0$ for $t_0 - \tau \le t \le t_0$. Hence, applying (3.3) with $p = 1$ and $h(s) = x(s-\tau) - x(s-\sigma(s)) + g(s)$, we find that

$$|z(t)| \le \frac{M}{\gamma} e^{(1+\gamma)\tau} \max_{t_0 \le s \le t} |x(s-\tau) - x(s-\sigma(s)) + g(s)|. \tag{3.6}$$

By the mean value theorem, we obtain

$$\begin{aligned} |x(s-\tau) - x(s-\sigma(s))| &= |\sigma(s) - \tau||x'(\xi)| \\ &= |\sigma(s) - \tau||g(\xi) - x(\xi - \sigma(\xi))|, \end{aligned}$$

where ξ is between $s - \tau$ and $s - \sigma(s)$. Setting

$$A = \max_{t_0 \le s < \infty} \{|g(s)|\} \quad \text{and} \quad B_0 = \max_{0 \le s \le t_0} \{|x(s)|\},$$

we have

$$\begin{aligned} &\max_{t_0 \le s \le t} |x(s-\tau) - x(s-\sigma(s)) + g(s)| \\ &\quad \le \max_{t_0 \le s \le t} |x(s-\tau) - x(s-\sigma(s))| + \max_{t_0 \le s \le t} |g(s)| \\ &\quad \le \max_{t_0 \le s \le t} |\sigma(s) - \tau| \left(\max_{0 \le s \le t} |g(s)| + \max_{0 \le s \le t} |x(s)| \right) + \max_{t_0 \le s \le t} |g(s)| \\ &\quad \le \max_{t_0 \le s \le t} |\sigma(s) - \tau| \left(A + B_0 + \max_{t_0 \le s \le t} |x(s)| \right) + \max_{t_0 \le s \le t} |g(s)|. \end{aligned}$$

Hence it follows from (3.6) that

$$\begin{aligned} |x(t)| - |y(t)| \le \frac{M}{\gamma} e^{(1+\gamma)\tau} \Bigg[&\max_{t_0 \le s \le t} |\sigma(s) - \tau| \left(A + B_0 + \max_{t_0 \le s \le t} |x(s)| \right) \\ &+ \max_{t_0 \le s \le t} |g(s)| \Bigg], \end{aligned} \tag{3.7}$$

which, in view of (3.5), implies that

$$|x(t)| \leq |y(t)| + \frac{1}{2}\left(A + B_0 + \max_{t_0 \leq s \leq t} |x(s)|\right) + A\frac{M}{\gamma}e^{(1+\gamma)\tau}.$$

Thus

$$\max_{t_0 \leq s \leq t} |x(s)| \leq 2 \max_{t_0 \leq s \leq t} |y(s)| + A + B_0 + 2A\frac{M}{\gamma}e^{(1+\gamma)\tau}, \quad t \geq t_0,$$

and so $x(t)$ is bounded. Let B be a positive constant such that

$$|x(t)| \leq B \quad \text{for} \quad t \geq 0.$$

Then (3.7) yields

$$|x(t)| \leq |y(t)| + \frac{M}{\gamma}e^{(1+\gamma)\tau}\left((A + 2B) \max_{t_0 \leq s \leq t} |\sigma(s) - \tau| + \max_{t_0 \leq s \leq t} |g(s)|\right),$$

which clearly implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

THEOREM 2. *Consider* (1.1). *Assume that*

$$b(t) > 0, \quad \int_0^\infty b(t)dt = \infty, \tag{3.8}$$

$$\lim_{t\to\infty} \int_{t-\tau(t)}^t b(s)ds = \beta < \pi/2 \tag{3.9}$$

and

$$\lim_{t\to\infty} \frac{f(t)}{b(t)} = 0. \tag{3.10}$$

Then every solution of (1.1) *tends to zero as t tends to infinity.*

PROOF. Set

$$u = \sigma(t) \equiv \int_0^t b(s)ds, \quad t \geq 0.$$

In view of (3.8), we see that σ^{-1} exists, $\lim_{t\to\infty} u(t) = \infty$ and

$$\sigma(t - \tau(t)) = \int_0^{t-\tau(t)} b(s)ds = \int_0^t b(s)ds - \int_{t-\tau(t)}^t b(s)ds$$
$$= u - \int_{\sigma^{-1}(u)-\tau(\sigma^{-1}(u))}^{\sigma^{-1}(u)} b(s)ds.$$

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Then the transformation $z(u) = x(\sigma^{-1}(u))$ reduces (1.1) to

$$z'(u) + z\left(u - \int_{\sigma^{-1}(u)-\tau(\sigma^{-1}(u))}^{\sigma^{-1}(u)} b(s)ds\right) = \frac{f(\sigma^{-1}(u))}{b(\sigma^{-1}(u))}. \tag{3.11}$$

In view of conditions (3.9) and (3.10), the hypotheses of Lemma 1 are satisfied for (3.11) and so

$$\lim_{t\to\infty} x(t) = \lim_{u\to\infty} z(u) = 0.$$

The proof is complete.

The following result is a direct consequence of Theorem 2.

COROLLARY 2. *Consider* (1.3). *Assume that* (3.8) *and* (3.9) *hold. Then the trivial solution of* (1.3) *is asymptotically stable.*

REMARK 1. It has been shown (see [5]) that if

$$b \in C([0,\infty),[0,\infty)),$$
$$\tau \in C([0,\infty),[0,q)) \quad \text{for some number } q \geq 0, \tag{3.12}$$
$$\int_{t-\tau(t)}^{t} b(s)ds \to 0 \quad \text{as } t \to \infty \tag{3.13}$$

and

$$\int_0^\infty b(s)ds = \infty,$$

then the trivial solution of (1.3) is asymptotically stable. Here, in Corollary 2, we need the hypothesis $b(t) > 0$ for the asymptotic stability of (1.3) but remove the boundedness hypothesis (3.12) on τ and improve condition (3.13) on b.

EXAMPLE 2. Consider the delay differential equation

$$x'(t) + \frac{2}{1+t}x\left(\frac{1}{2}t\right) = \frac{1}{(1+t)^2}, \quad t \geq 0. \tag{3.14}$$

Let

$$b(t) = \frac{2}{1+t}, \quad \tau(t) = \frac{1}{2}t \quad \text{and} \quad f(t) = \frac{1}{(1+t)^2}.$$

Observe that

$$\int_0^{\infty} b(t)dt = \infty$$

and

$$\int_{t-\tau(t)}^{t} b(s)ds = \int_{\frac{1}{2}t}^{t} \frac{2}{1+s}ds$$
$$= 2\left(\ln(1+t) - \ln\left(1+\frac{1}{2}t\right)\right) \to 2\ln 2 < \frac{\pi}{2} \quad \text{as } t \to \infty.$$

Hence by Theorem 2 every solution of (3.14) tends to zero as t tends to infinity.

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