THE REVIEW OF SYMBOLIC LOGIC
Volume 12, Number 2, June 2019

CAPTURING CONSEQUENCE

ALEXANDER PASEAU
Wadham College, University of Oxford

Abstract. First-order formalisations are often preferred to propositional ones because they are thought to underwrite the validity of more arguments. We compare and contrast the ability of some well-known logics—these two in particular—to formally capture valid and invalid arguments. We show that there is a precise and important sense in which first-order logic does *not* improve on propositional logic in this respect. We also prove some generalisations and related results of philosophical interest. The rest of the article investigates the results' philosophical significance. A first moral is that the correct way to state the oft-cited superiority of first-order logic vis-*à*-vis propositional logic is more nuanced than often thought. The second moral concerns semantic theory; the third logic's use as a tool for discovery. A fourth and final moral is that second-order logic's transcendence of first-order logic is greater than first-order logic's transcendence of propositional logic.

§1. Introduction. One of the main reasons to formalise is to capture implicational structure. Consider the simple valid English argument 'Felix is a cat, therefore there is a cat'. Its propositional formalisation $p \therefore q$ is not propositionally valid. In contrast, the argument's first-order formalisation $Fa \therefore \exists xFx$ is first-order valid. This example illustrates an apparently well-established moral: first-order formalisations underwrite the validity of more arguments than propositional ones. Teachers of logic often invoke this moral when introducing first-order logic to students who know only propositional logic.

This article compares and contrasts the ability of some well-known logics—propositional and first-order in particular—to capture natural language's implicational structure. Surprisingly, as we show, there is a precise and important sense in which first-order logic does *not* improve on propositional logic in this regard. Indeed, the moral is more general: propositional logic matches a wide class of logics in this respect. The correct way to state the oft-cited superiority of first-order logic vis-*à*-vis propositional logic is more nuanced, as we shall discover.[1]

Received: January 3, 2017.
2010 *Mathematics Subject Classification*: 03A05, 03B05, 03B10, 03B65.
Key words and phrases: first-order logic, propositional logic, second-order logic, compactness, countably infinite Boolean algebras, formalisation, inferential role, incompactness, validity.

[1] This article, though self-contained, is a sequel to my (2015). The quotations in §2 mostly overlap with those in §2 of my (2015), and §3 redeploys an example related to the one in §3 of my (2015). In my earlier article, I spoke of the 'inferential role' of a sentence; I now prefer 'implicational' to 'inferential', since we are interested in what follows from what rather than what can be deduced from what. The term 'inferential' carries an agential connotation—it is subjects who perform inferences—best avoided here; see Chapter 1 (esp. pp. 3–4) of Harman (1986) for a classic statement of the distinction between implication on the one hand and rational inference on the other. We'll also not be concerned with the broader sense of 'implication', which covers e.g., inductive or Gricean implications.

doi:10.1017/S1755020318000291

§2. The importance of preserving implicational structure. Although logicians are able to formalise sentences of English (including technically augmented English) with relative ease, a theoretical account of what this ability amounts to is hard to come by.[2] As Graham Priest puts it, '[formalisation] is a skill that good logicians acquire, but no one has ever spelled out the details in general' (Priest, 2006, p. 170). Nonetheless, what *is* clear is that a key constraint on (good) formalisation, if not the principal one, is that it should preserve the implicational structure of natural language, or its relevant fragment in the context at hand, as much as possible, subject to other constraints. This uncontentious aim of formalisation is our topic.

Various philosophers have stressed the importance of preserving implicational structure when formalising. Donald Davidson defends the methodology behind his account of action sentences as follows:

> ...by rewriting or rephrasing certain sentences into sentences that explicitly refer to or quantify over events, we can conveniently represent the entailment relations between the original sentences. The entailments we preanalytically recognize to hold between the original sentences become matters of quantificational logic applied to their rephrasals. (Davidson, 1967, p. 139)

Others have followed suit.[3] In discussing the success conditions of formalisation programmes, Stewart Shapiro comments:

> Although the 'translations' [formalisations] typically do not preserve anything like the grammatical form of the original natural language sentences, it is claimed that they do preserve *logical form*.... A programme is confirmed if 'intuitive consequences' among the natural language sentences are translated as model-theoretic consequences in the formal languages. (Shapiro, 1998, p. 135 n. 3)

W.V. Quine recommends that formalisation should reveal implicational structure, though no more than is required:

> *A maxim of shallow analysis* prevails: *expose no more logical structure than seems useful* for the deduction or other inquiry at hand. (Quine, 1960, p. 160)

In the following extended passage, Alex Oliver further articulates the general point:

> Philosophers are trained to put natural language sentences into logical form, i.e., to formalize them. What is this activity and what is its point? As I see it, it is an exercise in translation. One translates a sentence of English, say, into a sentence of a given logic, with the result that some of the inferential properties of the original sentence can be deduced from the logic of its translation. It is an incurably relative process, since it depends both on the choice of a logic and the particular translation into that logic. The point of the activity is to model the meanings of English sentences. The meaning of a sentence determines its inferential connections: what

[2] For recent discussions, see Baumgartner & Lampert (2008) and my (2015).

[3] Parsons (1990, pp. 7 & 10) for example explains his methodology as based on predicting correct implications between English sentences. See also Higginbotham, Pianesi, & Varzi (2000).

> follows from it (possibly in combination with others) and what it follows from. A given formalization highlights particular connections. Choosing to highlight different connections results in different translations into a given logic, perhaps even a change of logic. (Oliver, 2010, pp. 180–181)

Though they may disagree on details, these philosophers and many others all agree on one thing: respecting natural-language implications is a key constraint on formalisation. Surprisingly though, the philosophical literature has little systematic to say about this feature of formalisation. This article attempts to improve on this situation. We begin by comparing propositional logic with first-order logic.[4]

§3. Toy example. With virtually no loss of generality (see the end of §6), we take natural language to be English, understood liberally so as to include its technical parts, in particular mathematical and scientific language. English sentences here and throughout are taken to be meaningful (interpreted).[5] We generally assume that the set $Sen(E)$ of declarative English sentences is a countably infinite set, since the lexicon of English is finite and its formation rules allow for sentences of arbitrarily large finite length.[6] This assumption is as standard in philosophy as it is in linguistics. In contrast with English sentences, sentences of a formal language are uninterpreted. If $\mathcal{L}$ is a formal language and $Sen(\mathcal{L})$ its set of sentences then any function $\Phi : Sen(E) \rightarrow Sen(\mathcal{L})$ is a formalisation of English into $\mathcal{L}$; no other structure is imposed on Φ in the general case.

$\mathcal{L}$ might for instance be propositional logic with a countable infinity of atoms, PL_ω, and Φ a function from $Sen(E)$ into $Sen(\mathsf{PL}_\omega)$. Or $\mathcal{L}$ might be countable first-order logic FOL_ω and Φ a function with domain $Sen(E)$ and codomain $Sen(\mathsf{FOL}_\omega)$. As well as a countable infinity of atoms or sentence letters, PL_ω has an expressively adequate (countable) set

[4] Although I proved all the results in this article independently, I expect that some of them can be found in the algebraic logic literature or are recoverable from it. That said, I haven't been able to discover a precise formulation of any of them, be it in the encyclopedia Monk & Bonnet (1989) or elsewhere, partly because the abstract algebraic logic literature of the past few decades is principally concerned with general results about the class of logics as a whole. What is certainly without precedent is our focus on formalisation and natural language—the results' philosophical context—and the philosophical morals drawn from them. For an entry into the abstract algebraic logic literature, see Font, Jansana, & Pigozzi (2003), Lewitzka (2007) and Blok & Pigozzi (1989); Czelakowski (2001) is an extensive treatment. Barwise & Feferman (1985) remains a classic collection on abstract model theory.

[5] Sentences thus understood are often called statements.

[6] The presence of context-dependent terms allows a countably infinite language to express uncountably many propositions. For example, if there could be (in the sense of metaphysical possibility) uncountably many speakers, then 'My name is Jo' could express uncountably many propositions, one per metaphysically possible speaker who utters it. To preserve countability, we restrict attention to a particular context. For example, the argument 'I'm cold and tired, therefore I'm cold' is valid, as long as the speaker who utters the premise is the same as the speaker who utters the conclusion. This construal of logical consequence as applying to interpreted sentences in a particular context is entirely standard; see for example Quine (1982, p. 56) for an influential expression, or Rumfitt (2015, p. 33) for a more recent one. The interest of analysing the logical relations, if any, between sentences made in different contexts is limited, since naturally-occurring arguments have a common context. Alternatively, one can think of our discussion as applying to all and only meaningful English sentences uttered or written in the past, present or future by human/human-like beings, of which it is safe to suppose there are no more than a countable infinity. In a few places below, we consider the more stringent view that $Sen(E)$ is finite.

of truth-functional connectives, say $\{\neg, \wedge, \vee\}$; the usual formation rules; and its standard consequence relation $\vDash_{\mathsf{PL}_\omega}$. FOL_ω is first-order logic with a countable infinity of variables, constants, predicate and function symbols of all arities, a (countable) and expressively adequate set of truth-functional connectives, standard formation rules and its standard consequence relation $\vDash_{\mathsf{FOL}_\omega}$.[7]

The article's main theme is best introduced via a toy example. Take the fragment of English consisting of the following three sentences:

(A) There are at least three people.
(B) There are at least two people.
(C) There is at least one person.

Since the set $\{A, B, C\}$ consists of three sentences, there are $2^3 = 8$ possible choices of premise sets drawn from $\{A, B, C\}$ (including the empty set) and $3+1 = 4$ possible choices for the conclusion (including the empty conclusion). So there are $8 \times 4 = 32$ possible arguments involving A, B and C, or 31 if we ignore, as we will, the vacuous argument with empty premise set and empty conclusion. Of each of these 31 arguments, we can ask whether it is valid or invalid. A succinct description of our fragment's implicational structure is that A implies B, B implies C, and B and C together do not imply A. The full, 31-argument, implicational structure follows from this description, as well as generic facts about implication (e.g., that every sentence implies itself).

Suppose now that Φ_1 and Φ_2 are the following formalisation functions, with domain $\{A, B, C\}$ and respective codomains $Sen(\mathsf{PL}_\omega)$ and $Sen(\mathsf{FOL}_\omega)$:

$$\Phi_1(A) = p$$
$$\Phi_1(B) = q$$
$$\Phi_1(C) = r$$
$$\Phi_2(A) = \exists x \exists y \exists z (Px \wedge Py \wedge Pz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$$
$$\Phi_2(B) = \exists x \exists y (Px \wedge Py \wedge x \neq y)$$
$$\Phi_2(C) = \exists x Px$$

We may abbreviate $\Phi_2(A)$, $\Phi_2(B)$, and $\Phi_2(C)$ as $\exists_3$, $\exists_2$, and $\exists_1$ respectively. As mentioned earlier, the notions of PL_ω-validity and FOL_ω-validity are the usual ones; so, for instance, $\exists_3 \vDash_{\mathsf{FOL}_\omega} \exists_2$ and $\{p, q\} \nvDash_{\mathsf{PL}_\omega} r$. We can now compare how Φ_1 and Φ_2 respect validity for this fragment of English. For brevity, we use (as above) the symbol $\therefore$ as an inference marker, we denote validity by 1 and invalidity by 0 (for each of the three notions: validity in English, FOL_ω-validity and PL_ω-validity), we denote the empty set by $\emptyset$, and omit curly brackets around set terms. The table on the next page summarises the validity/invalidity facts.

From the table, we see that Φ_1 agrees with English validity/invalidity for 26 of the 31 arguments, whereas Φ_2 agrees with it for all 31 out of the 31. It is in this sense that Φ_2 is a better formalisation than Φ_1. More precisely, for this fragment of English, (i) there is no argument for which the values in the second and fourth columns are identical but the values in the second and sixth columns are distinct; and (ii) there is at least one argument for which the values in the second and sixth columns are identical but the values in the second and fourth are distinct. The first-order formalisation Φ_2 is therefore better than the propositional formalisation Φ_1—for this tiny fragment of English and so far as respecting logical consequence is concerned.

[7] Models of FOL_ω are assumed to have nonempty domains.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

English argument	Valid?	Φ_1-formalisation	PL_ω-valid?	Φ_2-formalisation	FOL_ω-valid?
$\emptyset \therefore A$	0	$\emptyset \therefore p$	0	$\emptyset \therefore \exists_3$	0
$\emptyset \therefore B$	0	$\emptyset \therefore q$	0	$\emptyset \therefore \exists_2$	0
$\emptyset \therefore C$	0	$\emptyset \therefore r$	0	$\emptyset \therefore \exists_1$	0
$A \therefore \emptyset$	0	$p \therefore \emptyset$	0	$\exists_3 \therefore \emptyset$	0
$B \therefore \emptyset$	0	$q \therefore \emptyset$	0	$\exists_2 \therefore \emptyset$	0
$C \therefore \emptyset$	0	$r \therefore \emptyset$	0	$\exists_1 \therefore \emptyset$	0
$A, B \therefore \emptyset$	0	$p, q \therefore \emptyset$	0	$\exists_3, \exists_2 \therefore \emptyset$	0
$B, C \therefore \emptyset$	0	$q, r \therefore \emptyset$	0	$\exists_2, \exists_1 \therefore \emptyset$	0
$A, C \therefore \emptyset$	0	$p, r \therefore \emptyset$	0	$\exists_3, \exists_1 \therefore \emptyset$	0
$A, B, C \therefore \emptyset$	0	$p, q, r \therefore \emptyset$	0	$\exists_3, \exists_2, \exists_1 \therefore \emptyset$	0
$A \therefore A$	1	$p \therefore p$	1	$\exists_3 \therefore \exists_3$	1
$B \therefore A$	0	$q \therefore p$	0	$\exists_2 \therefore \exists_3$	0
$C \therefore A$	0	$r \therefore p$	0	$\exists_1 \therefore \exists_3$	0
$A, B \therefore A$	1	$p, q \therefore p$	1	$\exists_3, \exists_2 \therefore \exists_3$	1
$B, C \therefore A$	0	$q, r \therefore p$	0	$\exists_2, \exists_1 \therefore \exists_3$	0
$A, C \therefore A$	1	$p, r \therefore p$	1	$\exists_3, \exists_1 \therefore \exists_3$	1
$A, B, C \therefore A$	1	$p, q, r \therefore p$	1	$\exists_3, \exists_2, \exists_1 \therefore \exists_3$	1
$A \therefore B$	1	$p \therefore q$	0	$\exists_3 \therefore \exists_2$	1
$B \therefore B$	1	$q \therefore q$	1	$\exists_2 \therefore \exists_2$	1
$C \therefore B$	0	$r \therefore q$	0	$\exists_1 \therefore \exists_2$	0
$A, B \therefore B$	1	$p, q \therefore q$	1	$\exists_3, \exists_2 \therefore \exists_2$	1
$B, C \therefore B$	1	$q, r \therefore q$	1	$\exists_2, \exists_1 \therefore \exists_2$	1
$A, C \therefore B$	1	$p, r \therefore q$	0	$\exists_3, \exists_1 \therefore \exists_2$	1
$A, B, C \therefore B$	1	$p, q, r \therefore q$	1	$\exists_3, \exists_2, \exists_1 \therefore \exists_2$	1
$A \therefore C$	1	$p \therefore r$	0	$\exists_3 \therefore \exists_1$	1
$B \therefore C$	1	$q \therefore r$	0	$\exists_2 \therefore \exists_1$	1
$C \therefore C$	1	$r \therefore r$	1	$\exists_3 \therefore \exists_1$	1
$A, B \therefore C$	1	$p, q \therefore r$	0	$\exists_3, \exists_2 \therefore \exists_1$	1
$B, C \therefore C$	1	$q, r \therefore r$	1	$\exists_2, \exists_1 \therefore \exists_1$	1
$A, C \therefore C$	1	$p, r \therefore r$	1	$\exists_3, \exists_1 \therefore \exists_1$	1
$A, B, C \therefore C$	1	$p, q, r \therefore r$	1	$\exists_3, \exists_2, \exists_1 \therefore \exists_1$	1

Generalising to formalisation functions defined on all of English, suppose that $\Phi_1 : Sen(E) \to Sen(\mathcal{L}_1)$ and $\Phi_2 : Sen(E) \to Sen(\mathcal{L}_2)$, where each $\mathcal{L}_i$ has consequence relation $\vDash_{\mathcal{L}_i}$, for $i = 1, 2$. We say that $\Phi_2 \geq \Phi_1$ just when, for all subsets P of $Sen(E)$ and all c in $Sen(E)$ (or c is the empty conclusion),[8] if the biconditional

$$P \therefore c \text{ is valid if and only if } \Phi_1(P) \vDash_{\mathcal{L}_1} \Phi_1(c)$$

holds then so does the biconditional

$$P \therefore c \text{ is valid if and only if } \Phi_2(P) \vDash_{\mathcal{L}_2} \Phi_2(c).$$

(Throughout this article, we write $\Phi(P)$ for $\{\Phi(x) : x \in P\}$ and similarly for the setwise image of other maps.) This definition makes precise the previous paragraph's more informal description. We write $\Phi_2 \sim \Phi_1$ if both $\Phi_2 \geq \Phi_1$ and $\Phi_1 \geq \Phi_2$. It is easily checked

[8] Omitting the empty argument.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

that $\geq$ is a preorder on the class of formalisation functions;[9] it is also easily checked that it is not a linear order.

On the basis of our toy example, one might conclude that, for the fragment of English consisting of A, B and C, first-order formalisations are superior to propositional ones as far as mirroring logical consequence is concerned. Yet that conclusion would be too swift. We showed that one first-order formalisation, *viz.* Φ_2, implicationally outperforms a particular candidate propositional formalisation function, *viz.* Φ_1. Yet how do we know that Φ_2 implicationally outperforms *all* propositional formalisations (for this fragment)? Could not some other propositional formalisation match Φ_2?

It could and it does. The following formalisation $\Phi_3 : \{A, B, C\} \to Sen(\mathsf{PL}_\omega)$, does the job:

$$\Phi_3(A) = p \wedge q \wedge r$$
$$\Phi_3(B) = p \wedge q$$
$$\Phi_3(C) = p$$

Φ_3 perfectly captures the implicational structure of our fragment of English, since $\Phi_3(A) \vDash_{\mathsf{PL}_\omega} \Phi_3(B)$, $\Phi_3(B) \vDash_{\mathsf{PL}_\omega} \Phi_3(C)$ and $\{\Phi_3(B), \Phi_3(C)\} \nvDash_{\mathsf{PL}_\omega} \Phi_3(A)$. Restricting to $\{A, B, C\}$ we see that $\Phi_3 \sim \Phi_2$. In fact, for any first-order formalisation Ψ restricted to this fragment, $\Phi_3 \geq \Psi$.

To sum up: a propositional formalisation is available for this toy example that does just as well as or better than any first-order formalisation. What about the general case? What if the implicational structure we are interested in is not that of a small fragment of English, but all of it? Is it not plausible that some first-order formalisation will be superior to all available propositional ones?[10]

§4. Propositional vs first-order logic. Surprisingly, the answer is no, as we prove in this section.

We assume for simplicity that any given English argument is either valid or invalid. More precisely, consider the set of English arguments, i.e., the arguments whose premise set is a subset of $Sen(E)$ and whose conclusion is an element of $Sen(E)$ or empty.[11] In §4, we assume that every argument in this set is valid or invalid, as the case may be.

[9] That is, $\geq$ is reflexive and transitive.

[10] To forestall misunderstanding, here and throughout we are looking for a *single* propositional formalisation that does just as well as a given first-order one. In other words, we fix the formalisation of each sentence and then consider how well this formalisation respects the validity or invalidity (as the case may be) of arguments involving these sentences compared to some other formalisation. If we were allowed to pick a different formalisation function for each different argument, then trivially a propositional formalisation could not be improved upon, for an uninteresting reason. In that case, if argument $\mathcal{A}$ is valid, we can simply formalise all $\mathcal{A}$'s premises and its conclusion as p; if on the other hand $\mathcal{A}$ is invalid, formalise each of its premises as p and its conclusion as q. The point is that, as standard, we do not allow the formalisation of a given sentence to vary from argument to argument.

[11] Excluding the empty argument, a qualification henceforth understood. For presentational simplicity, we speak as if conclusions are nonempty. Strictly speaking, whenever we write say

$$\Gamma \therefore \delta \text{ where } \Gamma \subseteq Sen(\mathcal{L}_1) \text{ and } \delta \in Sen(\mathcal{L}_1),$$

this should be understood as saying *either* that Γ is empty and $\delta \in Sen(\mathcal{L}_1)$, *or* that Γ is nonempty and the conclusion is either an element of $Sen(\mathcal{L}_1)$ or is empty.

In §5, we justify this assumption, explaining in detail why it holds with little loss of generality. In §6 we compare propositional logic with other logics than first-order logic. In §7, the article's philosophical pay-off, we draw some philosophical consequences from our technical discussion. The appendices generalise the §4 results and supply some further formal details.

Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are two logics, with respective sets of sentences $Sen(\mathcal{L}_1)$, $Sen(\mathcal{L}_2)$ and respective consequence relations $\vDash_{\mathcal{L}_1}, \vDash_{\mathcal{L}_2}$. The map $j : Sen(\mathcal{L}_1) \rightarrow Sen(\mathcal{L}_2)$ is a *consequence homomorphism* just when, for all $\Gamma \subseteq Sen(\mathcal{L}_1)$ and $\delta \in Sen(\mathcal{L}_1)$,[12]

$$\Gamma \vDash_{\mathcal{L}_1} \delta \text{ if and only if } j(\Gamma) \vDash_{\mathcal{L}_2} j(\delta).$$

Equivalently, $j : Sen(\mathcal{L}_1) \rightarrow Sen(\mathcal{L}_2)$ is a consequence homomorphism when, for all $\Gamma \subseteq Sen(\mathcal{L}_1)$ and $\delta \in Sen(\mathcal{L}_2)$,

(i) if $\Gamma \vDash_{\mathcal{L}_1} \delta$ then $j(\Gamma) \vDash_{\mathcal{L}_2} j(\delta)$;
(ii) if $\Gamma \nvDash_{\mathcal{L}_1} \delta$ then $j(\Gamma) \nvDash_{\mathcal{L}_2} j(\delta)$.

A *consequence isomorphism* is a bijection $j : Sen(\mathcal{L}_1) \rightarrow Sen(\mathcal{L}_2)$ which is also a consequence homomorphism. Notice that if j is a consequence isomorphism then so is $j^{-1} : Sen(\mathcal{L}_2) \rightarrow Sen(\mathcal{L}_1)$. A *consequence embedding* is an injective consequence homomorphism.

The main fact we cite rather than prove is:

FACT. *Any two countably infinite atomless Boolean algebras are isomorphic* (*as Boolean algebras*).

This fact is very well known and the familiar back-and-forth argument for it will not be rehearsed here; see for example Chapter 16 of Givant & Halmos (2009) for a painstaking presentation. We use *Fact* to establish the first result, probably known to Tarski as early as the 1930s.[13]

PROPOSITION 4.1. *There is a consequence isomorphism* $j : Sen(\mathsf{FOL}_\omega) \rightarrow Sen(\mathsf{PL}_\omega)$.

Proof. Let $\mathsf{FOL}_\omega/\vDash_{\mathsf{FOL}_\omega}$ be the Boolean algebra of FOL_ω-sentences quotiented by $\vDash_{\mathsf{FOL}_\omega}$-equivalence; this is known as the Lindenbaum algebra of FOL_ω. Two elements $[\gamma_1]$ and $[\gamma_2]$ of this Boolean algebra are equal if and only if $\vDash_{\mathsf{FOL}_\omega} \gamma_1 \leftrightarrow \gamma_2$. Join, meet and complement are defined in $\mathsf{FOL}_\omega/\vDash_{\mathsf{FOL}_\omega}$ as usual:

$$[\gamma_1] \vee [\gamma_2] = [\gamma_1 \vee \gamma_2]$$
$$[\gamma_1] \wedge [\gamma_2] = [\gamma_1 \wedge \gamma_2]$$
$$\overline{[\gamma]} = [\neg\gamma],$$

where we are using the symbols $\wedge, \vee$ ambiguously, as customary. Similarly define $\mathsf{PL}_\omega/\vDash_{\mathsf{PL}_\omega}$, the Boolean algebra of PL_ω quotiented by $\vDash_{\mathsf{PL}_\omega}$-equivalence.[14]

Clearly, $\mathsf{FOL}_\omega/\vDash_{\mathsf{FOL}_\omega}$ and $\mathsf{PL}_\omega/\vDash_{\mathsf{PL}_\omega}$ are countably infinite Boolean algebras, since $Sen(\mathsf{FOL}_\omega)$ and $Sen(\mathsf{PL}_\omega)$ are countably infinite, and the number of their equivalence classes when quotiented by their respective logical consequence relations is in each case infinite. Also, $\mathsf{FOL}_\omega/\vDash_{\mathsf{FOL}_\omega}$ and $\mathsf{PL}_\omega/\vDash_{\mathsf{PL}_\omega}$ are both atomless. To see this, suppose that $[\gamma]$

12 From a model-theoretic perspective, we are thus treating $\vDash$ and $\nvDash$ as relations. By definition, $j(\Gamma) = \{j(\gamma) : \gamma \in \Gamma\}$.

13 Though I have not been able to identify an exact statement in his works or anyone else's.

14 For more detail on how to do this, see e.g., Hinman (2005, pp. 74–79).

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

were an atom of $\mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}}$, for some $\gamma \in Sen(\mathsf{FOL}_\omega)$ such that $\gamma \nvDash_{\mathsf{FOL}_\omega} \bot$. Let δ be $\exists xFx$ where F does not appear in γ.[15] By an interpolation theorem for FOL_ω, $\gamma \nvDash_{\mathsf{FOL}_\omega} \delta$ and $\gamma \nvDash_{\mathsf{FOL}_\omega} \neg\delta$. Thus $[\bot] < [\gamma \wedge \delta] < [\gamma]$ in the Boolean algebra $\mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}}$. An analogous argument works for $\mathsf{PL}_\omega/{\vDash_{\mathsf{PL}_\omega}}$, using an interpolation theorem for PL_ω (let δ be p where p does not appear in γ). Note in passing that for any $\gamma \in Sen(\mathsf{FOL}_\omega)$, there are countably infinitely many sentences $\vDash_{\mathsf{FOL}_\omega}$-equivalent to it; similarly for PL_ω.

The up-to-isomorphism uniqueness of countably infinite atomless Boolean algebras, expressed in *Fact*, implies that there is a Boolean algebra isomorphism $i : \mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}} \to \mathsf{PL}_\omega/{\vDash_{\mathsf{PL}_\omega}}$. Now define $j : Sen(\mathsf{FOL}_\omega) \to Sen(\mathsf{PL}_\omega)$ so that j is bijective and respects equivalence classes, i.e., for all $\gamma_1, \gamma_2 \in \mathsf{FOL}_\omega$,

$$[j(\gamma_1)] = [j(\gamma_2)] \text{ iff } [\gamma_1] = [\gamma_2].$$

(The first identity is in $\mathsf{PL}_\omega/{\vDash_{\mathsf{PL}_\omega}}$ and the second in $\mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}}$.) Given the existence of a Boolean algebra isomorphism i, the existence of such a j is immediate, since all the equivalence classes in both $Sen(\mathsf{PL}_\omega)$ and $Sen(\mathsf{FOL}_\omega)$ are countably infinite. In fact, if $b_k : [\gamma] \to [i(\gamma)]$ is a bijection with domain the k^{th} element of $\mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}}$ for $k \in \omega$ we may define $j = \bigcup_{k\in\omega} b_k$. j can be thought of as a lift of i.

It remains to check that j is a consequence isomorphism. We first show this for finite premise sets before proceeding to the general case.

For $\gamma_1, \dots, \gamma_n, \delta \in Sen(\mathsf{FOL}_\omega)$,

$$\begin{aligned} &\{\gamma_1, \dots, \gamma_n\} \vDash_{\mathsf{FOL}_\omega} \delta \\ &\text{iff } \gamma_1 \wedge \dots \wedge \gamma_n \leq \delta \text{ in } \mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}} \\ &\text{iff } i(\gamma_1) \wedge \dots \wedge i(\gamma_n) \leq i(\delta) \text{ in } \mathsf{PL}_\omega/{\vDash_{\mathsf{PL}_\omega}} \\ &\text{iff } \{j(\gamma_1), \dots, j(\gamma_n)\} \vDash_{\mathsf{PL}_\omega} j(\delta) \end{aligned}$$

by the construction of i and j and the meaning of $\leq$ in the Boolean algebras $\mathsf{FOL}_\omega/{\vDash_{\mathsf{FOL}_\omega}}$ and $\mathsf{PL}_\omega/{\vDash_{\mathsf{PL}_\omega}}$ respectively.

Turning to the general case, suppose first that $\Gamma \subseteq Sen(\mathsf{FOL}_\omega)$ and $\delta \in Sen(\mathsf{FOL}_\omega)$ with $\Gamma \vDash_{\mathsf{FOL}_\omega} \delta$.

(1)	$\Gamma \vDash_{\mathsf{FOL}_\omega} \delta$	assumption
(2)	$\{\gamma_1, \dots, \gamma_n\} \vDash_{\mathsf{FOL}_\omega} \delta$	from (1) by compactness of $\vDash_{\mathsf{FOL}_\omega}$, for some $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$
(3)	$\{j(\gamma_1), \dots, j(\gamma_n)\} \vDash_{\mathsf{PL}_\omega} j(\delta)$	from (2) by the finite-case argument
(4)	$j(\Gamma) \vDash_{\mathsf{PL}_\omega} j(\delta)$	from (3) since $\{j(\gamma_1), \dots, j(\gamma_n)\} \subseteq j(\Gamma)$

Suppose alternatively that $\Gamma \nvDash_{\mathsf{FOL}_\omega} \delta$. In that case:

(1)	$\Gamma \nvDash_{\mathsf{FOL}_\omega} \delta$	assumption
(2)	for all $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$, $\{\gamma_1, \dots, \gamma_n\} \nvDash_{\mathsf{FOL}_\omega} \delta$	from (1)
(3)	for all $\{j(\gamma_1), \dots, j(\gamma_n)\} \subseteq j(\Gamma)$, $\{j(\gamma_1), \dots, j(\gamma_n)\} \nvDash_{\mathsf{PL}_\omega} j(\delta)$	from (2) by the finite-case argument
(4)	$j(\Gamma) \nvDash_{\mathsf{PL}_\omega} j(\delta)$	from (3) by compactness of $\vDash_{\mathsf{PL}_\omega}$

We have shown that j is a consequence isomorphism. □

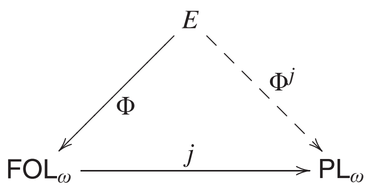
We now draw a corollary from the proposition just proved.

PROPOSITION 4.2. *Let $\Phi : Sen(E) \to Sen(\mathsf{FOL}_\omega)$ be a formalisation function. Then there is a formalisation function $\Phi^j : Sen(E) \to Sen(\mathsf{PL}_\omega)$ such that, for any $P \subseteq Sen(E)$ and $c \in Sen(E)$, $\Phi(P) \vDash_{\mathsf{FOL}_\omega} \Phi(c)$ iff $\Phi^j(P) \vDash_{\mathsf{PL}_\omega} \Phi^j(c)$.*

[15] Recall that FOL_ω contains a countable infinity of predicate symbols.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

Proof. Given $\Phi : Sen(E) \to Sen(\mathsf{FOL}_\omega)$ let $\Phi^j : Sen(E) \to Sen(\mathsf{PL}_\omega)$ be the composition of Φ with j as given in Proposition 4.1, i.e., $\Phi^j = j \circ \Phi$. Diagrammatically:



The result follows since j is a consequence isomorphism. □

As a corollary of Proposition 4.2, we obtain:

THEOREM 4.3. *Let $\Phi_1 : Sen(E) \to Sen(\mathsf{FOL}_\omega)$ be a formalisation function. Then there is a formalisation function $\Phi_2 : Sen(E) \to Sen(\mathsf{PL}_\omega)$ such that $\Phi_2 \sim \Phi_1$.*

Proof. Define Φ_2 as Φ_1^j, i.e., $\Phi_2 = j \circ \Phi_1$ where j is the above Boolean isomorphism. Since $\langle Sen(\mathsf{FOL}_\omega), \vDash_{\mathsf{FOL}_\omega}\rangle$ and $\langle Sen(\mathsf{PL}_\omega), \vDash_{\mathsf{PL}_\omega}\rangle$ are consequence-isomorphic structures, it is immediate that, for $P \subseteq Sen(E)$ and $c \in Sen(E)$, the biconditional

$$P \therefore c \text{ is valid if and only if } \Phi_1(P) \vDash_{\mathsf{FOL}_\omega} \Phi_1(c)$$

obtains just when the biconditional

$$P \therefore_E c \text{ is valid if and only if } \Phi_2(P) \vDash_{\mathsf{PL}_\omega} \Phi_2(c)$$

does. The material equivalence of these biconditionals is exactly the condition $\Phi_2 \sim \Phi_1$. □

Consequently, for any first-order formalisation some propositional formalisation respects English implication just as well. Since the converse of Theorem 4.3 is immediate, we also see that for any given propositional formalisation function there is a first-order formalisation that matches it in this same regard. To put it less precisely but more strikingly: if all we are interested in is preserving the implicational structure of English—whatever it may be—then as a class propositional formalisations are just as good as first-order ones.

Two observations are in order before moving on. First, a feature of the standard proof of *Fact* is that the isomorphism between two atomless countably infinite algebras B_1 and B_2 may be recursively constructed given a recursive enumeration of the domains of B_1 and B_2. So given a formalisation function $\Phi_1 : Sen(E) \to Sen(\mathsf{FOL}_\omega)$ and a recursive specification of $Sen(E)$ and $Sen(\mathsf{FOL}_\omega)$, can one recursively specify a formalisation function $\Phi_2 : Sen(E) \to Sen(\mathsf{PL}_\omega)$ such that $\Phi_2 \sim \Phi_1$? *No*: the consequence isomorphism j in Proposition 1 is *not* recursive. The reason is that j must map FOL_ω-tautologies to PL_ω-tautologies, since it maps a sentence of FOL_ω entailed by any other FOL_ω-sentence to a sentence of PL_ω entailed by any other PL_ω-sentence. So if j were recursive, one could apply it to an arbitrary sentence α of FOL_ω to obtain a sentence $j(\alpha)$ of PL_ω whose status as a PL_ω-tautology is decidable,[16] and thereby decide whether α is an FOL_ω-tautology. But by Church's Theorem, there is no decision procedure for being an FOL_ω-tautology, contradicting the assumption that j is recursive.

[16] As there is a decision procedure for the property of being a PL_ω-tautology, e.g., the truth table test.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

This conclusion is compatible with the fact that the isomorphism between two atomless countably infinite algebras B_1 and B_2 is recursively constructible given a recursive enumeration of their domains, as long as the enumeration of the two domains' elements presents them in a canonical way, i.e., as strings in the language of Boolean algebra. Sentences of FOL_ω however do not wear their place in FOL_ω's Lindenbaum algebra on their sleeve. For example, no recursive procedure exists for determining whether an arbitrary FOL_ω-sentence corresponds to a top or bottom element of this algebra, that is, whether it corresponds to the elements $[\exists xFx] \vee \overline{[\exists xFx]}$ or $[\exists xFx] \wedge \overline{[\exists xFx]}$ thus or similarly presented.

A second observation is that PL_ω is the smallest propositional logic for which Proposition 1 holds. This follows from the fact that PL_n (i.e., propositional logic with a finite number n of atoms) has a finite implicational structure, whereas FOL_ω contains infinite consequence chains; e.g., consider the sequence consisting of the first-order formalisations of 'There is at least one thing', 'There are at least two things', 'There are at least three things',.... The same applies to Theorem 3, since English also contains an infinite consequence chain, as the example in the previous sentence also illustrates.

§5. Natural-language validity. In §4, we took the implicational structure of English as given and, taking English as our representative natural language, assumed that every English argument is valid or invalid.[17] (For more on the equation of English with natural language, see the last paragraph of §6. To repeat, 'English' as understood here includes technical English.) Two questions may be raised about the notion of validity used thus far. First, does English really have a unique consequence relation? Second, what notion of consequence do we have in mind? We take these questions in turn.

5.1. The consequence relation? It may be clear that 'Felix is a cat, therefore there is a cat' is a valid argument, whereas 'Felix is a cat, therefore there is a dog' is not. But is *every* English argument valid or invalid? Is there really such a thing as *the* implicational structure of English, so that every English argument is valid or invalid?

The question is a deep and important one. Fortunately, we need not answer it here, since we are interested in how well various logics fare at respecting *some target implicational structure or other*. As long as there is some such clear target notion, the discussion can get off the ground and our results in §4 and their extensions in Appendix A proved. To illustrate this point, we consider three views about English validity. (The views are not intended as exclusive or exhaustive—though the first and second are in fact exclusive.)

According to the first view, every English argument is either valid or invalid (but not both). On this view, the theorems of §4 and their generalisation in Appendix A apply straightforwardly.

Naturally, different theorists will disagree about what that consequence relation is. Some argue that it's best modelled by first-order logic, others by a free logic, and still others by second-order logic; some would impose a requirement of relevance whereas others wouldn't; and so on. On this first view, all such disagreements are about what the *unique* notion of English consequence is.

According to the second view, for any given English argument A, either (i) A is determinately valid, or (ii) A is determinately invalid, or (iii) A's validity is indeterminate. In

[17] By an English argument we mean a premise set consisting of declarative English sentences and a conclusion that is also a declarative English sentence. As noted in §3, we also allow empty premise sets and conclusions.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

contrast to the first view, this second view insists that the third category (iii) is instantiated. A formalisation respects English's implicational structure if any formalisation of a valid argument is valid and any formalisation of an invalid argument is invalid. In other words, as long as the formalisation respects the determinate cases, the indeterminate cases can fall either way. Our results in §4 (and Appendix A) apply to all such logics. The reason is simple: arbitrarily assign validity values ('valid' or 'invalid') to all the neither valid nor invalid English arguments, and then run the arguments in §4 and Appendix A.[18]

According to the third and final view, there is no such thing as *the* implicational structure of English; there are only various implicational structures—note the plural—that result from looking at English through a particular theoretical lens, or that arise in a particular context, each of them as good as any other. The debate between a model-theoretic and a proof-theoretic analysis is on the first view a debate about the nature of the one true consequence relation. Other who adopt this third perspective may see each analysis as an equally compelling account of natural-language consequence. Depending on how exactly it is understood, this third view will likely be a form of logical pluralism.[19]

Relatedly, some logicians stress the relativity of validity to prevailing contextual standards. For example, Rumfitt (2011 & chap. 2 of 2015) argues that an implication such as 'This body so accelerating, therefore a force is acting on it' is valid according to one standard (that of theoretical physics) but not another (the broadest standard, typically applied by logicians). As Rumfitt sees it, there are a great many relations of consequence in natural language, and logic is concerned with the general laws obeyed by all of them.

On the third view, then, the results proved in §4 (and those in Appendix A) should be understood from within a particular theoretical or contextual perspective. Suppose you adopt theoretical perspective T_1 (alternatively, some contextually determined standard), so that your resulting notions of validity and invalidity for English arguments are validity$_{T_1}$ and invalidity$_{T_1}$. You may run the arguments in §4 to convince yourself that propositional logic is just as good as first-order logic at capturing the implicational structure of English,

[18] Suppose that Φ_1 agrees with Φ_2 on all determinate cases of validity and invalidity, but that Φ_1 formalises some indeterminately valid arguments as indeterminate, whereas Φ_2 formalises all arguments as determinately valid or invalid. One might conclude that Φ_1 is a better formalisation than Φ_2. Since virtually all standard logics viewed from within a classical metatheory are 'consequence-bivalent', meaning that for any given premise set Γ and conclusion δ either Γ implies δ or Γ does not imply δ, we ignore this possibility in what follows. (Even standard many-valued logics have this feature, since they typically possess a set of designated truth-values; an argument is valid if it preserves this set, and invalid otherwise.) The most obvious example of a logic *not* assumed by its proponents to be consequence-bivalent is intuitionistic logic. Although in intuitionistic logic *viewed from a classical metaperspective* whether a premise implies a conclusion is a determinate fact, intuitionistic logic *as understood by an intuitionist herself* cannot be assumed to be consequence-bivalent. That said, intuitionistic logic is *not* an example of a logic to which one could apply the stronger success condition just mooted. The reason is that intuitionists themselves cannot take any particular argument to be neither valid nor invalid, on pain of contradiction; nor, by the same token, can they claim that some unspecified argument or other is neither valid nor invalid. Though they refrain from asserting that every argument is determinate, intuitionists may not go so far as to assert that there are indeterminate arguments.

[19] Beall & Restall (2006) and Shapiro (2014) defend different versions of logical pluralism. A slightly different vein within analytic philosophy takes ordinary language to have no single exact logic: for a relatively early defence, see Strawson (1950) and (1952, pp. 56–57 & 230–232); for a more recent one, see Glanzberg (2015). For more on monism versus pluralism, see Griffiths & Paseau (forthcoming).

understood *à la* T_1. Another logician, looking at English implication from a distinct theoretical perspective T_2, will take the class of validities and invalidities to be distinct from yours. Yet she too may run the same arguments from her own perspective, to convince herself that propositional logic is no worse off than first-order logic as far as capturing the implicational structure of English is concerned—this structure being understood according to her own precepts, that is, *à la* T_2.

Although these three views do not exhaust all the possibilities, collectively they represent the great majority of perspectives in contemporary philosophy of logic.

5.2. The relation's nature. Having examined whether the results in §4 (and those in Appendix A) assume a unique determinate notion of English validity, we turn finally and much more briefly to whether they assume anything about its nature. The answer is that they assume *nothing* about the nature of the consequence relation, other than that its left-relata are sets of declarative English sentences and that its right-relata are declarative English sentences. A consequence relation might for instance be formal, in that it turns on the formal features of the sentences involved, consequence being understood as formal necessitation. Or it might be conceptual, in that it turns on the concepts expressed, consequence being understood as conceptual necessitation. Or it might be metaphysical, in that it turns on metaphysical possibility or impossibility, consequence being understood as metaphysical necessitation.[20] Our results apply to all these conceptions of consequence and more, as they make no substantive assumptions about its nature.[21] Even the assumption that logical consequence relates sets of declarative sentences and a declarative sentence is not strictly necessary; for example plural logicians, who see consequence as relating declarative sentences (construed plurally) and a declarative sentence may easily paraphrase our results.

This concludes our discussion of whether the technical results in §4 and their generalisation in Appendix A assume a unique notion of English consequence and also whether they assume anything about its nature. The short answer to the first question is that they don't. More precisely, our results are compatible with a great many (though not all) forms of logical monism and pluralism. The short answer to the second question is that they assume virtually nothing about its nature.

§6. Incompactness. The argument in §4 applies to first-order logic because of the latter's compactness and truth-functional completeness, as the argument's generalisation in Appendix A makes clear. The analogue of Proposition 4.1 fails for an incompact logic. Whether the analogue of Theorem 4.3 obtains or fails depends on whether the consequence relation in English is compact. This section's main concern is to substantiate the last two claims.

We first show that the analogue of Proposition 4.1 fails for any incompact logic. In fact, even the weaker claim that there is a consequence homomorphism $j : Sen(\mathcal{L}) \to Sen(\mathsf{PL}_\omega)$ fails. For suppose that $\mathcal{L}$ is an incompact logic with set of sentences $Sen(\mathcal{L})$. To say that $\mathcal{L}$ is incompact is to say that for some $\Gamma \subseteq Sen(\mathcal{L})$ and $\delta \in Sen(\mathcal{L})$,

[20] An argument such as 'The ball is red, so it is coloured' is conceptually and metaphysically valid but not formally valid; 'Jack drank some water, so Jack drank some H_2O' is metaphysically valid but conceptually and formally invalid; and 'Donna ate and drank, so Donna drank' is formally, conceptually and metaphysically valid.

[21] At the end of §4, we noted in passing that English contains infinite consequence chains. This holds on pretty much any conception of English consequence, and in any case does not affect the main claims in §4.

$$\Gamma \vDash_{\mathcal{L}} \delta \text{ but } \Gamma^{fin} \nvDash_{\mathcal{L}} \delta \text{ for all finite } \Gamma^{fin} \subseteq \Gamma.$$

If $j : Sen(\mathcal{L}) \to Sen(\mathsf{PL}_\omega)$ were a consequence homomorphism, we would have

$$j(\Gamma) \vDash_{\mathsf{PL}_\omega} j(\delta) \text{ but } j(\Gamma)^{fin} \nvDash_{\mathsf{PL}_\omega} j(\delta) \text{ for all finite } j(\Gamma)^{fin} \subseteq j(\Gamma),$$

thereby contradicting the compactness of PL_ω.

Turn next to the analogue of Theorem 4.3. We cannot here examine, still less settle, the question of whether the logical consequence relation of English is incompact. As discussed in §5, it may not even be a well-posed question; the real issue might be to determine, for each of the several possible precisifications of the notion of logical consequence, whether consequence thus precisified is compact or incompact. Talk of 'the' English consequence relation, used for brevity, should be understood throughout this article as discussed in §5. If, for instance, there is more than one such relation then our discussion applies to each of them. Maintaining neutrality on whether natural-language consequence (understood as just prescribed) is compact or incompact, we consider each alternative in turn.

6.1. English consequence is compact. If English consequence is compact, the arguments in §4, or more precisely their generalisation in Appendix A, may be easily amended to apply to the set of all English arguments. The only relevant features of English are: that logical equivalence is an equivalence relation on the set of declarative English sentences, $Sen(E)$; that English is truth-functionally complete;[22] that $Sen(E)$ is countable (countably infinite or finite); and that its consequence relation is compact. Given these assumptions, the argument set out in Appendix A can be applied to English, resulting in:

THEOREM 6.1. *Assume that $Sen(E)$ is countable, that logical equivalence is an equivalence relation on $Sen(E)$, that English is truth-functionally complete, and that the consequence relation in English is compact. Then there is a propositional formalisation $\Phi : Sen(E) \to Sen(\mathsf{PL}_\omega)$ that perfectly mirrors the implicational structure of English.*

A minor modification is needed if some English arguments are neither valid nor invalid. In that case, there is a formalisation $\Phi : Sen(E) \to Sen(\mathsf{PL}_\omega)$ that is best-possible in the following sense: if $P \therefore c$ is valid then so is $\Phi(P) \vDash_{\mathsf{PL}_\omega} \Phi(c)$, and if $P \therefore c$ is invalid then $\Phi(P) \nvDash_{\mathsf{PL}_\omega} \Phi(c)$. As mentioned in §5, the argument for this amended conclusion is a simple amendment of the given argument.

6.2. English consequence is incompact. Alternatively, suppose that English logical consequence is incompact, so that there exists at least one valid argument

$$\{E_1, \ldots, E_n, \ldots\} \therefore E_\infty$$

where $E_i \in Sen(E)$ for $i \in \{1, 2, \ldots, \infty\}$, such that

$$\{E_{n_1}, \ldots, E_{n_k}\} \therefore E_\infty$$

is not valid for any finite $k, n_1, \ldots, n_k$. In that case, Theorem 4.3's analogue fails for any incompact logic. An argument many philosophers[23] believe witnesses the incompactness of English is:

[22] It is perhaps moot whether the sentential connectives 'it is not the case that' and 'and' are truth-functional in ordinary English. But English here includes the technical vocabulary of mathematics and so is truth-functionally complete, virtually by stipulation.

[23] See e.g., Oliver & Smiley (2013, p. 238), or Yi (2006, p. 262) for a similar example.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

$$E_1 = \text{'There is at least one thing.'}$$
$$E_2 = \text{'There are at least two things.'}$$
$$\dots$$
$$E_n = \text{'There are at least } n \text{ things.'}$$
$$\dots$$
$$\therefore$$
$$E_\infty = \text{'There are infinitely many things.'}$$

We illustrate the failure of Theorem 4.3 under the assumption that English consequence is incompact for the case of countably infinite second-order logic SOL_ω with standard (full) semantics, and more briefly for the infinitary logics $\mathcal{L}_{\omega_1\omega}$ and $\mathsf{PL}_\omega^{\omega_1}$.[24] The language of SOL_ω has a countable infinity of: first-order variables, second-order predicate and function variables of all arities, nonlogical predicates and function symbols of all arities (including arity 0, i.e., constants). In interpreting SOL_ω, second-order n-place predicate variables range over *all* n-tuples from the domain of interpretation, and similarly for functional variables. $\mathcal{L}_{\omega_1\omega}$ extends FOL_ω by allowing countably infinite conjunctions and disjunctions of well-formed formulas, and extends FOL_ω's semantics in the obvious way. $\mathsf{PL}_\omega^{\omega_1}$ extends PL_ω by allowing countably infinite conjunctions and disjunctions of well-formed formulas, and extends PL_ω's semantics in the obvious way. SOL_ω, $\mathcal{L}_{\omega_1\omega}$ and $\mathsf{PL}_\omega^{\omega_1}$ thus understood are all incompact.

Now let γ_1 be the second-order (indeed first-order) sentence $\exists x(x = x)$, γ_2 the sentence $\exists x \exists y(x \neq y)$, and so on for all finite n, and let γ_∞ be the second-order sentence

$$\exists R(R \text{ is functional } \wedge R \text{ is injective } \wedge \neg R \text{ is surjective})$$

where 'R is functional' abbreviates $\forall x \exists ! y Rxy$, '$R$ is injective' abbreviates $\forall x \forall y \forall z((Rxz \wedge Ryz) \rightarrow x = y)$ and 'R is surjective' abbreviates $\forall y \exists x Rxy$.

Suppose next that $\Phi_1 : Sen(E) \rightarrow Sen(\mathsf{SOL}_\omega)$ is a formalisation function such that $\Phi(E_i) = \gamma_i$ for $i \in \{1, 2, \dots, n, \dots, \infty\}$. Φ_1 thus respects the validity of the English argument

$$\{E_1, \dots, E_n, \dots\} \therefore E_\infty$$

and the invalidity of all the English arguments

$$\{E_{n_1}, \dots, E_{n_k}\} \therefore E_\infty$$

for any finite $k, n_1, \dots, n_k$. If there were a formalisation function $\Phi_2 : Sen(E) \rightarrow Sen(\mathsf{PL}_\omega)$ such that $\Phi_2 \sim \Phi_1$, we would have $\Phi_2(\{E_1, \dots, E_n, \dots\}) \vDash_{\mathsf{PL}_\omega} \Phi_2(E_\infty)$ and $\Phi_2(\{E_{n_1}, \dots, E_{n_k}\}) \nvDash_{\mathsf{PL}_\omega} \Phi_2(E_\infty)$ for all finite subsets $\{E_{n_1}, \dots, E_{n_k}\}$ of $\{E_1, \dots, E_n, \dots\}$, contradicting PL_ω's compactness. Thus the analogue of Theorem 4.3, that if $\Phi_1 : Sen(E) \rightarrow Sen(\mathsf{SOL}_\omega)$ then there is a $\Phi_2 : Sen(E) \rightarrow Sen(\mathsf{PL}_\omega)$ such that $\Phi_2 \sim \Phi_1$, fails, on the assumption that English consequence is incompact. Under the assumption of incompactness, English is implicationally too 'capacious' for PL_ω.

Similarly for $\mathcal{L}_{\omega_1\omega}$: formalise E_i as γ_i for $i \in \{1, 2, \dots, n, \dots\}$ and E_∞ as their conjunction, which exists in $\mathcal{L}_{\omega_1\omega}$ since this logic's set of sentences is closed under countable conjunction. It is easy to see that the resulting $\mathcal{L}_{\omega_1\omega}$-formalisation perfectly captures the implicational structure of the English fragment under consideration. Indeed, abstractly

[24] There are other well-known incompact logics of course, e.g., first-order logic supplemented with the generalised quantifier 'Most Φ are Ψ', or various plural logics, or other higher-order logics.

conceived, the fragment's consequence structure is that of an 'inverted ω-sequence plus limit', i.e., the structure

$$\begin{array}{c} E_1 \\ E_2 \\ \dots \\ E_n \\ \dots \\ E_\infty \end{array}$$

in which if one sentence is below another, then the first is strictly logically stronger than the second, and the sentences $E_1, \dots, E_n, \dots$ tend to E_∞, in the sense that no sentence logically implies all the E_n for finite n but not E_∞.[25] Similarly for $\mathsf{PL}_\omega^{\omega_1}$, formalising E_1 as p, E_2 as $p \wedge q, \dots$, and E_∞ as their infinitary conjunction.

We note in passing that $\mathcal{L}_{\omega_1\omega}$ and SOL_ω are not consequence-isomorphic. This is because $\mathcal{L}_{\omega_1\omega}$ but not SOL_ω has the following property: for any countable subset Γ of the set of sentences of the language, there is a sentence δ that is logically equivalent to Γ, i.e., $\Gamma \vDash \delta$ and $\{\delta\} \vDash \gamma$ for all $\gamma \in \Gamma$. (In terms of models: Γ and δ share the same models.) That this property is preserved under consequence-isomorphism is immediate. It is equally evident that $\mathcal{L}_{\omega_1\omega}$ and $\mathsf{PL}_\omega^{\omega_1}$ have the property, since given Γ we may take δ to be the conjunction of Γ's countably many elements. SOL_ω however lacks the property, as Appendix C demonstrates.

Clearly, any logic closed under countable conjunction can play the role of $\mathcal{L}_{\omega_1\omega}$ in the argument just given, including $\mathsf{PL}_\omega^{\omega_1}$. The analogue of the Schröder–Bernstein theorem in this context would be that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are consequence isomorphic iff there is a consequence embedding from $\mathcal{L}_1$ to $\mathcal{L}_2$ as well as a consequence embedding from $\mathcal{L}_2$ to $\mathcal{L}_1$. Without knowing whether such a result holds, one *cannot* conclude from the fact that $\mathcal{L}_{\omega_1\omega}$ and SOL_ω (or $\mathsf{PL}_\omega^{\omega_1}$ and SOL_ω) are not consequence-isomorphic that at least one of them does not consequence-embed into the other.[26]

The argument for the failure of Theorem 4.3's analogue when FOL_ω is replaced by SOL_ω or $\mathcal{L}_{\omega_1\omega}$ or $\mathsf{PL}_\omega^{\omega_1}$ applies to any incompact logic whose incompactness is witnessed by a countably infinite premise set, in particular any countable and incompact logic. For suppose $\{\delta_1, \dots, \delta_n, \dots\} \vDash_{\mathcal{L}} \delta_\infty$ but $\{\delta_{n_1}, \dots, \delta_{n_k}\} \nvDash_{\mathcal{L}} \delta_\infty$ for all finite $\{\delta_{n_1}, \dots, \delta_{n_k}\} \subseteq \{\delta_1, \dots, \delta_n, \dots\}$, where $\delta_1, \dots, \delta_n, \dots, \delta_\infty \in Sen(\mathcal{L})$. Then no propositional formalisation can match the formalisation function $\Phi_1 : Sen(E) \rightarrow Sen(\mathcal{L})$ given by $\Phi(E_i) = \delta_i$ for $i \in \{1, 2, \dots, n, \dots, \infty\}$. The argument however does *not* generalise to incompact logics whose incompactness is witnessed only by uncountably infinite premise sets. Such logics are hardly ever used as codomains of formalisation maps, so perhaps little generality is lost here. The further claims that second-order logic, $\mathcal{L}_{\omega_1\omega}$ and $\mathsf{PL}_\omega^{\omega_1}$ *perfectly* capture the implicational structure of the fragment of English consisting of arguments drawn from $\{E_1, \dots, E_n, \dots, E_\infty\}$ depend on the particular example chosen—that is, on the fact that the English argument at the start of §6.2 has the structure of an 'inverted ω-sequence plus limit'. The argument also generalises if we replace PL_ω with the propositional logic PL_κ for any cardinal κ (i.e., propositional logic with κ atoms), which is compact. Moreover, it generalises to other codomains than PL_κ, e.g., FOL_κ for infinite κ. The upshot is that

[25] An alternative characterisation of the notion of 'tending to' for formal languages is that the conclusion and the premise set have the same set of models.

[26] See Griffiths & Paseau (forthcoming) for more details.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

if natural-language consequence (or its appropriate precisification) is incompact then an incompact logic such as SOL_ω, $\mathcal{L}_{\omega_1\omega}$ or $\mathsf{PL}_\omega^{\omega_1}$ implicationally matches it better than propositional or first-order logic.

The only feature of English we relied on throughout the article is that the set $Sen(E)$ of English sentences is countably infinite. Any natural language with a finite lexicon and whose formation rules allow for sentences of arbitrarily large finite length shares this feature with English. Some of the results in this section were conditional on English's truth-functional completeness and its compactness or incompactness, as the case may be. As the reader may verify, the results in §4 and in Appendix A go through easily on the assumption that $|Sen(E)|$ is finite; the arguments only become simpler on this assumption. Some related results also go through on other assumptions; for example, one may formulate an analogue of Appendix A's Theorem A.3 for PL_κ in which $\kappa > \omega$ if English is construed as a compact language of cardinality κ. But results along these lines seem like academic exercises, since English is in fact a countable language. Presumably, all languages humans have ever grasped or will ever grasp are also countable. Of course, mathematicians investigate infinitary languages; but the language in which all of mathematics is ultimately done is countable: we use a countable language—e.g., the language of ZFC—to investigate infinitary languages. This is why we have taken $Sen(E)$ to be countable. (And note once more that $Sen(E)$ includes not just the sentences of ordinary English but of its technical adjuncts as well, including mathematical language.)

§7. Philosophical significance. Several philosophical morals may be drawn from our discussion. I extract four, the last of which is conditional. Collectively, the morals lead to a better understanding of how first-order logic improves on propositional logic and how these two logics relate to others, including second-order logic in particular.

7.1. The platitude. First, the seeming platitude that first-order logic better respects the implicational structure of English than propositional logic is simply not true. Theorem 4.3 in §4 puts paid to that thought. Many philosophers and logicians emphasise the importance of preserving implicational structure in formalisations. See for example, among many others, the quotation by Stewart Shapiro in §2 which suggests that this is all there is to good formalisation, or Davidson's reply to Cargile (Davidson, 1980, pp. 137–146), which comes close to the same suggestion. If the suggestion were true—if there were no more to formalisation than respecting implicational structure—it would be impossible to justify the thought that first-order logic improves on propositional logic as a formalisation tool.

Our discussion does not, however, preclude first-order logic from implicationally outperforming propositional logic *when other constraints are in place*. There are other criteria of good formalisation (sometimes called criteria of adequate formalisation) besides capturing implicational structure. Another such might be *respecting grammatical form*. To gloss this very roughly, a formalisation Φ respects the grammatical form of an English sentence s if the syntax of the formal sentence $\Phi(s)$ in some sense reflects the syntax of the sentence s.[27]

Whatever exactly they might be, as soon as other constraints on formalisation are in play, first-order formalisations may implicationally outperform propositional ones. Recall

[27] How exactly to cash out this thought is tricky. It turns on whether by the grammatical form of an English sentence we understand its surface grammatical form or something more theoretical, such as the syntactic form attributed to it by the best theory(ies) of English syntax.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

our opening example of an argument: ‘Felix is a cat, therefore there is a cat’. This English argument is evidently valid. A grammatical parsing of the argument might be:

$$\underbrace{\text{Felix}}_{\text{Noun}}\ \underbrace{\text{is a cat}}_{\text{Predicate}},\ \underbrace{\text{therefore}}_{\text{Inference marker}}\ \underbrace{\text{there}}_{\text{Quantifier}}\ \underbrace{\text{is a cat}}_{\text{Predicate}}.$$

yielding the first-order formalisation

$$\underbrace{a}_{\text{Constant}}\ \underbrace{F}_{\text{Predicate}}\ \therefore\ \underbrace{\exists x}_{\text{Quantifier}}\ \underbrace{Fx}_{\text{Predicate}}$$

or $Fa \therefore \exists x Fx$ as it is more conventionally written. A first-order formalisation grammatically constrained in this manner respects the original English argument’s validity. Contrast a propositional formalisation. A propositional parsing of the argument closest to its surface grammar is:

$$\underbrace{\text{Felix is a cat}}_{\text{Sentence 1}}\ \underbrace{\text{therefore}}_{\text{Inference marker}}\ \underbrace{\text{there is a cat}}_{\text{Sentence 2}}.$$

resulting in the propositional formalisation

$$p \therefore q,$$

which is not valid. This analysis unfolds an extremely familiar train of thought, appreciated by all but the most callow student of logic. Yet the key here is to see it in light of the above results. *It is only because formalisations are usually constrained to respect grammatical form that first-order formalisations mirror the implicational structure of English more faithfully than propositional ones.* Although this conclusion may have the ring of logically informed common sense, it had not prior to this article been *proved*. Philosophers who imagine that formalisation is much less constrained in philosophy than in linguistics should take note. The conclusion thereby raises a challenge for ‘first-orderists’, who take first-order logic to be their foundational logic of choice. If grammatical constraints are important, why stop at FOL rather than the kind of extensions, mooted by linguists, that cleave much more closely to the syntax of natural language?

On the subject of grammar, an aspect of the usual grammatical routines (such as the one in the previous paragraph) we employ when formalising into first-order logic is worth highlighting. These routines cannot be divorced from implicational concerns, for two reasons. First, in deploying the routines we are often guided by pre-theoretic judgments about what follows from what in the case at hand. We might well revise grammatically irreproachable formalisations were they to disagree with entrenched pretheoretic judgments. A paradigmatic example is the Davidsonian programme of formalising action sentences, mentioned in §2. The programme consciously rejects grammatical appearances for the sake of an implicationally accurate formalisation. Similarly for the older Russellian programme which denies, for example, that the formalisation of negative existentials such as ‘Pegasus does not exist’ is of subject-predicate form (Russell, 1905). Likewise, if most mathematicians were convinced that a mathematical argument is valid (invalid), it would be a constraint on the argument’s formalisation in a sufficiently expressive logic that it be valid (invalid). In such cases, grammar is secondary to preserving implicational structure.

The second reason is that grammatical formalisation routines have themselves been devised with an eye towards implicational structure. For example, the first-order formalisation of ‘Horses are mammals’ as $\forall x(Hx \rightarrow Mx)$ (to adapt an example of Davidson’s) is no mere matter of respecting surface grammatical form. For what in the surface grammar corresponds to the quantifier ∀? to the variable x? to the material conditional $\rightarrow$?[28] Past experience with handling sentences such as 'Horses are mammals'—in particular past experience with their implicational features—heavily influences our grammatical routines. To sum up these two points: our formalisation routines are heavily influenced by implicational considerations, both in particular applications and in their general contours.

The correct way to understand the platitude that first-order logic implicationally outperforms propositional logic is thus that it does so *if*—and only *if*—other constraints on formalisation are in play. One of the ambitions behind this article and related work,[29] is to try to better understand how one might assess the different desiderata in formalisation. A reasonable first step is to try to isolate their respective contributions. What we have seen is that merely preserving implicational structure does not allow a line to be drawn between first-order and propositional logic. A question to be pursued in future work is when exactly the grammatical constraint kicks in: To what extent can a propositional formalisation respect surface grammar and still retain implicational parity with first-order logic?

7.2. Semantic theory. Suppose one uses first-order logic (or any compact and truth-functionally complete logic—see Appendix A) to give an account of the truth-conditions of English sentences. Schematically, such an account might proceed as follows: use a formalisation function Φ to map English sentences to those of FOL_ω; interpret FOL_ω-sentences by means of the interpretation function $\mathcal{I}$; and map the resulting interpretations to sets of possible worlds via the function *Ext*. Thus if s is an English sentence, $\Phi(s)$ is an FOL_ω-sentence, $\mathcal{I}(\Phi(s))$ is its interpretation,[30] and $Ext(\mathcal{I}(\Phi(s)))$ is a set of possible worlds, i.e., a subset or subclass of W, where W is the set or class of worlds. Can a propositional logic play the role of first-order logic in such accounts, without loss of theoretical power?[31]

The §4 results show that, extensionally speaking, propositional logic can match first-order logic. For if we let $j \circ \Phi$ be the formalisation from $Sen(E)$ to $Sen(\mathsf{PL}_\omega)$, where j is a consequence isomorphism from $Sen(\mathsf{FOL}_\omega)$ to $Sen(\mathsf{PL}_\omega)$ as in §4, and use $\mathcal{I} \circ j^{-1}$ to interpret the sentences of $Sen(\mathsf{PL}_\omega)$, the resulting account of English sentences' truth-conditions is the same as the original one, since $\mathcal{I} \circ j^{-1}(j \circ \Phi(s)) = \mathcal{I}(\Phi(s))$. Typically, $Ext \circ \mathcal{I}$ will be constrained to respect Boolean operations; e.g., the conjunction of two formal sentences is mapped by $Ext \circ \mathcal{I}$ to the intersection of their conjuncts' $Ext \circ \mathcal{I}$-images. In that case, the sets of worlds the first-order and propositional accounts associate with English sentences consist of all the intersections, unions and W-complementations of the $Ext \circ \mathcal{I}$-images of $Sen(\mathsf{PL}_\omega)$'s countably many atoms. From this extensional point of view, a first-order logic-based semantics is no better than a propositional one.

One might impose further constraints on a semantic theory to discriminate between first-order and propositional accounts. One constraint might be grammatical or syntactic: the image under Φ of an English sentence should in some sense be syntactically akin to the

[28] One could argue that this formalisation reflects the *theoretical* grammatical form. But theoretical construals of grammatical form are even more infused with implicational-role considerations.

[29] See my (2015).

[30] We retain neutrality on what this interpretation is.

[31] We are not claiming that all semantic accounts take this form; only that some do. For a brief and accessible introduction to possible world semantics, see Chapter 12 of Heim & Kratzer (1998).

sentence itself. This syntactic/grammatical desideratum would allow us to discriminate between a first-order and a propositional account, along the lines discussed in §7.1. Another constraint might be that $Ext \circ \mathcal{I}$ should *recursively* assign sets of worlds to sentences of the formal language modelling English; or that $\mathcal{I}$ should *recursively* assign meanings to such sentences. Since the consequence isomorphism $j : Sen(\mathsf{FOL}_\omega) \to Sen(\mathsf{PL}_\omega)$ is nonrecursive, the recursiveness of the function $Ext \circ \mathcal{I}$ from $Sen(\mathsf{FOL}_\omega)$ to subsets/classes of W is compatible with the nonrecursiveness of the function $Ext \circ \mathcal{I} \circ j^{-1}$ from $Sen(\mathsf{PL}_\omega)$ to the same codomain. Thus a FOL_ω-based formalisation might respect the recursiveness constraint whereas a PL_ω-based one fails it.[32] If philosophical semanticists wish to drive a wedge between semantic theories based on first-order formalisations and those based on propositional ones, they must develop arguments along these lines or others for the same conclusion.

7.3. Logic as a tool for discovery and explanation. Our third moral relates our discussion thus far with the role of logic as a tool for the *discovery* of consequence relations. When formalising, we often care not just about preservation in the sense of setting down (however we get them) some formulas standing in the right formal relations. We also care about elucidation or discovery, meaning that we care about deploying formal representations that enable us to learn more about implicational structure. Taking a logic in use as a tool for discovery, it seems that first-order formalisations give us 'more bang for our buck', that is, they allow us to elucidate/discover implicational structure better than propositional ones.

It seems to me that this observation is along the right lines. But the observation is best understood in light of the foregoing results. The §4 discussion affords us a more precise understanding of first-order logic's superiority as a tool for the *discovery* of implicational structure, as opposed to its *reflection* once discovered. We saw at the end of §4 that no consequence isomorphism from $Sen(\mathsf{PL}_\omega)$ to $Sen(\mathsf{FOL}_\omega)$ can be recursive. So the mere existence of such an isomorphism does not amount to our knowing one. In fact, we *know* that such an isomorphism cannot be recursively specified. The God's eye view in §4 is distinct from our perspective as cognitively limited formalisers. Our result thus shows *why* first-order logic's virtues should be understood in epistemic terms.

A second point is that in using a logic $\mathcal{L}$ to discover facts about English consequence one thing you want to know, at the outset if possible, is whether $\mathcal{L}$ is implicationally capacious enough to model any fragment of English. To take a very simple example, if PL_1 is the fragment of propositional logic with just one sentence letter, p, then PL_1 only has four sentences up to logical equivalence: p, $\neg p$, $p \vee \neg p$ and $p \wedge \neg p$. Clearly, PL_1 is incapable of capturing all but a very limited set of English arguments: the sparseness of PL_1's implicational structure is an inherent limitation on its ability to model consequence. For this reason, PL_1 could never be an adequate tool for the *discovery* of all but very simple small finite fragments of English's implicational structure. English simply won't fit into PL_1, so to speak. The general point is that you can use a logic $\mathcal{L}$ as a tool for discovery only if you are confident that English (or its relevant fragment) will fit into $\mathcal{L}$. Such discovery, of course, happens in stages. It is not as if we somehow intuit consequence facts prior to logical analysis and thereby infer which logics' implicational structures the structure of English fits into. Rather, the process is gradual. We use formali-

[32] What exactly it is for a function with codomain the set/class of worlds to be recursive would have to be spelt out if this idea is to be substantiated.

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

sation procedures tested on arguments about whose validity/invalidity there is widespread agreement to discover further implicational facts, which in turn inform our formalisation procedures. At any stage in this process we can ask whether, given what we know so far, some logics (or sub-logics) are ruled out as too limited to offer an implicationally perfect model of English consequence. The results in §4 and Appendix A speak to exactly this question.

A final epistemic point: suppose you believe that the validity of an English argument can be *explained* by its first-order formalisation's validity. It does not follow that the validity of an English argument can be explained just as well by its propositional formalisation's validity. For all we have said, explanatoriness (or some degree thereof) might be lost by transforming first-order formalisations into consequence-isomorphic propositional ones, perhaps because proof-theoretic considerations are relevant to explanation. For example, perhaps the validity of 'Felix is a cat therefore there is a cat' is better explained by the validity of its first-order formalisation $Fa \therefore \exists x Fx$ than by that of $p \therefore p \vee q$. Suppose for instance that the first-order argument's validity is best explained by appeal to the rule of existential generalisation. Then this might form part of a better explanation of the original English argument's validity than an explanation based in part on disjunction introduction. $Sen(\mathsf{PL}_\omega)$'s ability to mimic $Sen(\mathsf{FOL}_\omega)$'s implicational structure is thus compatible with a standpoint—requiring further elaboration of course—that sees first-order formalisations as ultimately offering better explanations of the implicational data than propositional ones.

7.4. Second-order logic's transcendence. Our fourth and final moral is conditional. Suppose the English consequence relation is incompact (or if you like, following §5: the English consequence relation seen through a particular theoretical lens, or according to some contextual standards, is incompact). As we saw in §6, under this assumption a second-order formalisation can implicationally outperform all propositional formalisations even when no other constraints are in place. Proponents of second-order logic laud its ability to satisfactorily formalise a range of arguments of apparently logical character that first-order logic cannot.[33] The satisfactoriness of a formalisation here refers to its ability to respect English validity, widely regarded as the key constraint on formalisation, as remarked in §2. We have shown that, under the assumption of English's incompactness, second-order logic implicationally outstrips both first-order and propositional logic *even when no other constraints are in place*. Thus the sense in which second-order formalisations implicationally outperform first-order ones is stronger than the sense in which first-order formalisations implicationally outperform propositional ones. To put it another way, second-order logic's implicational structure is more capacious than first-order logic's: the former embeds the latter (since first-order logic is a sublogic of second-order logic) but not the other way round. We are familiar with the fact that second-order languages' greater expressive richness is intimately connected to their underlying logic's incompactness.[34] Much less familiar, until now, is the fact that second-order logic is implicationally more capacious than first-order logic. This difference is also closely connected to the former being incompact and the latter compact.

[33] A classic statement is by Boolos (1975/1998, p. 49): '. . . another reason for regarding second-order logic as logic is that there are notions of a palpably logical character (*ancestral*, *identity*), which can be defined in second-order logic (but not first-) and which figure critically in inferences whose validity second-order logic (but not first-) can represent'.

[34] Shapiro (1991) has an extended discussion.

We see then that, implicationally speaking, the gulf between second-order logic and first-order logic is wider than that between first-order and propositional logic. It would perhaps be an overstatement to call the line of thought just articuled a new argument for second-order logic, even granted the assumption that English is incompact. But it is certainly a striking feature of second-order logic's strength.[35]

Appendices. Appendix A generalises §4's results. Appendix B raises a natural technical question. Appendix C proves a result used in §6.

Appendix A*: *Generalisation of the §4 results. To generalise §4's Theorem 4.3, we relax the requirement of countability on FOL_ω, and show that it depends only on the compactness of first-order logic plus the fact that it contains the Boolean connectives.

Towards this generalisation, note first a consequence of *Fact* from §4. Though regularly cited in the literature and easy enough to prove, this consequence is seldom proved, so we provide its proof here for completeness. Recall that an embedding of Boolean algebras is an injective homomorphism between two Boolean algebras. Recall also that if B is a Boolean algebra then the Boolean algebra generated by a set $X \subseteq dom(B)$ is the intersection of all the subalgebras of B with domain a (proper or improper) superset of X. It is easily verified that the subalgebra generated by X is countable if X is. (The word 'countable' in this article always means finite or countably infinite.)

PROPOSITION A.1. *Any Boolean algebra B generated by a countable (i.e., finite or countably infinite) set X may be embedded into a/the countably infinite atomless Boolean algebra.*

Sketch Proof. As is easy to check, any finite Boolean algebra (of size 2^N for some N) may be embedded into a/the countably infinite atomless Boolean algebra.

Suppose then that B is the Boolean algebra generated by the countably infinite set X; B is countably infinite because X is countable, as earlier observed. Next consider the first-order language of Boolean algebras with the usual functional symbols for join, meet and complement and augmented with constants c_b for each $b \in B$. Let T be the union of the set of axioms of atomless Boolean algebras, which are all first-order expressible, together with the sets

$$\{c_{b_1} \neq c_{b_2} : b_1, b_2 \in B \text{ s.t. } b_1 \neq b_2\}$$
$$\{c_{b_1} \wedge c_{b_2} = c_{b_3} : b_1, b_2, b_3 \in B \text{ s.t. } b_1 \wedge b_2 = b_3\}$$
$$\{c_{b_1} \vee c_{b_2} = c_{b_3} : b_1, b_2, b_3 \in B \text{ s.t. } b_1 \vee b_2 = b_3\}$$
$$\{c_{b_1} = \overline{c_{b_2}} : b_1, b_2 \in B \text{ s.t. } b_1 = \overline{b_2}\}.$$

($\neq$, $\wedge$ and $\vee$ are being used ambiguously, both as function symbols of the formal language and as operations on B.) Any finite subset of T is satisfiable, the reason being as earlier that any finite Boolean algebra may be embedded into a/the countably infinite atomless Boolean algebra. So by the compactness theorem for first-order logic, T is satisfiable, in a necessarily infinite domain since B is infinite. By the Löwenheim-Skolem Theorem

[35] I am grateful to Michael Baumgartner for asking the question that prompted this article. Thanks also to audiences at the MCMP, the Cambridge Logic seminar, the Oxford Philosophy of Mathematics seminar, the Philosophy department at MIT and the universities of Glasgow and Hambur, the IUSS in Pavia, and Unilog 2018 in Vichy for useful feedback. Particular thanks are owed to Bruno Whittle, Felix Weitkamper, Matthias Jenny, Owen Griffiths, Peter Smith and Rob Leek.

for first-order logic and the fact that T is expressed in a countable language, T has a countably infinite model $\mathcal{M}$, which must be a countably infinite atomless Boolean algebra. The mapping $b \mapsto c_b^{\mathcal{M}}$ is an embedding of B into $\mathcal{M}$. □

Next, call a logic truth-functionally complete if it can express ¬ and ∧; for notational simplicity, we may assume that such a logic in fact contains ¬, ∧ and ∨. If $\mathcal{L}$ is a truth-functionally complete logic and $X \subseteq Sen(\mathcal{L})$, define X^{tf}, the truth-functional completion of X, as the intersection of all subsets Y of $Sen(\mathcal{L})$ with the properties (i) $X \subseteq Y$, (ii) if $\phi \in Y$ then $\neg\phi \in Y$, and (iii) if $\phi, \psi \in Y$ then $\phi \vee \psi, \phi \wedge \psi \in Y$. (The notion of a truth-functional completion, though it does not feature in the statement of the next result, is needed for a subtlety in its proof.) As ever, $Sen(E)$ is the set of sentences of English, including its technical fragment, which is countably infinite. In what follows, the cardinality of $Sen(\mathcal{L})$ is unconstrained.

The relevant analogue of §4's Proposition 4.1 is now:

PROPOSITION A.2. *Let $\mathcal{L}$ be a compact and truth-functionally complete logic, and $\Phi : Sen(E) \to Sen(\mathcal{L})$ a formalisation function. Write $\Phi(Sen(E))$ for the image of $Sen(E)$ under Φ. There is then a consequence embedding $j : \Phi(Sen(E)) \to Sen(\mathsf{PL}_\omega)$.*

Sketch Proof. We mimic §4's proof of Proposition 4.1 as much as possible. Consider the Boolean algebra $\mathsf{L}/\vDash_{\mathsf{L}}$ with the usual join, meet and complement operations. Let S be the subalgebra generated by the set $\{[\Phi(s)] : s \in Sen(E)\}$. Since $Sen(E)$ is countable, so is $\{[\Phi(s)] : s \in Sen(E)\}$ and thus S is countable too. By Proposition A.1, there is a Boolean algebra embedding $i : S \to \mathsf{PL}_\omega/\vDash_{\mathsf{PL}_\omega}$. As in the proof of Theorem 4.3, we lift i to an embedding $\underline{j} : (\Phi(Sen(E)))^{\text{tf}} \to Sen(\mathsf{PL}_\omega)$; we call this embedding $\underline{j}$ because we will reserve the symbol j for a restriction of $\underline{j}$. Observe that the embedding $\underline{j}$ must be defined on $(\Phi(Sen(E)))^{\text{tf}}$ rather than $(\Phi(Sen(E)))$, since there may be elements in the domain of S that are not the image of any element of $\{[\Phi(s)] : s \in Sen(E)\}$. Observe further that $\underline{j}$ can always be chosen to be an injection. Unlike in the proof of Theorem 4.3, however, $\underline{j}$ cannot always be guaranteed to be a bijection. The reason is that some equivalence classes of the domain of the Boolean algebra $\mathsf{L}/\vDash_{\mathsf{L}}$ may be finite, and in such cases are injectible but not bijectible into the respective equivalence classes of $\mathsf{PL}_\omega/\vDash_{\mathsf{PL}_\omega}$. The proof that $\underline{j}$ is a consequence embedding is the exact analogue of the Theorem 4.3 argument, here exploiting the compactness of $\vDash_{\mathsf{L}}$. Finally, let $j : \Phi(Sen(E)) \to Sen(\mathsf{PL}_\omega)$ be the restriction of the map $\underline{j} : (\Phi(Sen(E)))^{\text{tf}} \to Sen(\mathsf{PL}_\omega)$ to the domain $\Phi(Sen(E))$. □

As in §4, we have, as a corollary of Proposition A.2:

THEOREM A.3. *Let $\mathcal{L}$ be a compact and truth-functionally complete logic, and $\Phi_1 : Sen(E) \to Sen(\mathcal{L})$ a formalisation function. Then there is a formalisation function $\Phi_2 : Sen(E) \to Sen(\mathsf{PL}_\omega)$ such that $\Phi_2 \sim \Phi_1$.*

Proof. Analogous to the proof of Theorem 4.3. Define Φ_2 as Φ_1^j, i.e., $\Phi_2 = j \circ \Phi_1$ where j is the consequence embedding in Proposition A.2. Since $\langle \Phi(Sen(E)), \vDash_{\mathsf{L}} \rangle$ and $\langle j(\Phi(Sen(E))), \vDash_{\mathsf{PL}_\omega} \rangle$ are consequence-isomorphic structures, the result follows. □

Thus for any formalisation function Φ into a compact and truth-functionally complete logic $\mathcal{L}$ there is a propositional formalisation that matches it.

We note that the class of compact and truth-functionally logics is very large. It includes in particular any truth-functionally complete logic with a sound and complete deductive procedure, e.g., modal logics such as S5 or S4, since compactness is of course an immediate

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

consequence of soundness and completeness.[36] Unlike the first-order case, however, some propositional formalisations may be better than *all* formalisations into $\mathcal{L}$, depending on $\mathcal{L}$. If for instance $|Sen(\mathcal{L})|$ is a small finite number then any $\mathcal{L}$-formalisation will be implicationally outperformed by some propositional formalisation. In contrast to Theorem 4.3, the converse of Theorem A.3 therefore fails: given $\Phi_1 : Sen(E) \to Sen(\mathsf{PL}_\omega)$, there may be no $\Phi_2 : Sen(E) \to Sen(\mathcal{L})$ that improves on Φ_1.

Appendix B*: *ω-universal logic. Our article raises a natural technical question: What is the size of the smallest ω-universal logic? An ω-universal logic is a logic into which any logic with a countable sentence set consequence-embeds. As we now show, an upper bound for the smallest ω-universal logic is $2^{2^{\aleph_0}}$.

To see this, let $\mathcal{L}$ be a logic such that $|Sen(\mathcal{L})| = \aleph_0$. Since $|\mathbb{P}(Sen(\mathcal{L}))| = 2^{\aleph_0}$, there are $2^{\aleph_0}.\aleph_0 = 2^{\aleph_0}$ ordered pairs $\langle S, s\rangle$, with $S \subseteq Sen(\mathcal{L})$ and $s \in Sen(\mathcal{L})$. It follows that there are $2^{2^{\aleph_0}}$ possible consequence relations one might put on the language of $\mathcal{L}$, assuming no restrictions. We may without loss of generality think of the sentences of $\mathcal{L}$ as elements of ω, i.e., $Sen(\mathcal{L}) = \omega$. Enumerate the possible relations between $\mathbb{P}(\omega)$ and ω under the identification just given as $\{\vDash_\xi : \xi < 2^{2^{\aleph_0}}\}$. More strictly, the left-relata of $\vDash_\xi$ are subsets of $\omega \times \{\xi\}$ and its right-relata are elements of $\omega \times \{\xi\}$. Now define an ω-universal logic $\mathcal{L}^{\mathcal{U}}$ as follows:

$$\text{Sen}\,(\mathcal{L}^{\mathcal{U}}) = \bigcup\{\omega \times \{\xi\} : \xi < 2^{2^{\aleph_0}}\}; \text{ and for } \Gamma \subseteq Sen(\mathcal{L}^{\mathcal{U}}) \text{ and } \phi \in Sen(\mathcal{L}^{\mathcal{U}}):$$

$$\Gamma \vDash_{\mathcal{L}^{\mathcal{U}}} \phi \text{ iff } \exists \xi < 2^{2^{\aleph_0}} \text{ s.t. } \Gamma \subseteq \omega \times \{\xi\}, \phi \in \omega \times \{\xi\}, \text{ and } \Gamma \vDash_\xi \phi.$$

Informally, $\mathcal{L}^{\mathcal{U}}$ is obtained by pasting all the $2^{2^{\aleph_0}}$-many possible consequence structures for a countable logic into a single logic. No further consequence facts obtain other than those for the individual logics thus collated. By construction, any countable logic embeds into $\mathcal{L}^{\mathcal{U}}$. The size of $\mathcal{L}^{\mathcal{U}}$, i.e., the cardinality of its sentence set, is $2^{2^{\aleph_0}}$. An obvious question is whether this crude upper bound can be improved upon.

Appendix C*: *More on second-order logic. The argument in §6 that $\mathcal{L}_{\omega_1\omega}$ and SOL_ω are not consequence-isomorphic assumed that some countably infinite set of SOL_ω-sentences is not equivalent to any single SOL_ω-sentence. We now prove this fact.

Let α be the least cardinal with the property that SOL_ω cannot define the notion of being at least as large as α.[37] Such an α exists for SOL_ω, and must be the aleph of a countable ordinal, since $Sen(\mathsf{SOL}_\omega)$ is countably infinite (and α is obviously infinite).[38] To show that α is a limit cardinal (i.e., the aleph of a limit ordinal), suppose for contradiction that $\alpha = \aleph_{\beta+1}$, and that $\phi_{\geq\aleph_\beta}(X)$ with free second-order variable X is a formula defining the notion $\geq \aleph_\beta$. (If a sentence ψ defines the notion of $\geq \aleph_\beta$ then take $\phi_{\geq\aleph_\beta}(X)$ as $\psi \wedge X = X$.) In that case,

$$\phi_{\aleph_{\beta+1}}(X) =_{df} \exists Y(\phi_{\geq\aleph_\beta}(Y) \wedge Y < X)$$

[36] By definition/stipulation, a proof has finitely many premises.

[37] A sentence ϕ defines the notion of being $\geq \alpha$ iff the models of ϕ are all and only those of domain size $\geq \alpha$. In the argument to follow, we use the well-known facts that $X < Y$ is definable in second-order logic (there is an injection from X to Y but not vice-versa), as is $X \leq Y$ (there is an injection from X to Y).

[38] We assume a background set theory strong enough to prove this claim. A detailed account of the second-order definability of various cardinals may be found in Väänänen (2012).

https://doi.org/10.1017/S1755020318000291 Published online by Cambridge University Press

would be a formula defining the notion of being at least as large as $\alpha = \aleph_{\beta+1}$, and $\exists X \phi_{\aleph_{\beta+1}}(X)$ a sentence with the same property.

If we let $(\beta_i)_{i<\omega}$ be a cofinal sequence in the limit ordinal λ, where $\alpha = \aleph_\lambda$ as just shown, and let

$$\Gamma = \{\phi_{\geq \aleph_{\beta_i}} : i < \omega\},$$

then no SOL_ω-sentence (or formula) δ can be equivalent to Γ. For such a sentence δ would have to be equivalent to a sentence $\phi_{\geq \aleph_\lambda}$ defining the notion of being greater or equal to α, and by assumption no such $\phi_{\geq \aleph_\lambda}$ exists.

BIBLIOGRAPHY

[1] Barwise, J. & Feferman, S. (editors) (1985). *Model-Theoretic Logics*. New York: Springer-Verlag.

[2] Baumgartner, M. & Lampert, T. (2008). Adequate Formalization. *Synthese*, **164**, 93–115.

[3] Beall, J. C. & Restall, G. (2006). *Logical Pluralism*. Oxford: Oxford University Press.

[4] Blok, W. & Pigozzi, D. (1989). *Algebraizable Logics*. Memoirs of the American Mathematical Society, Vol. 77. Providence, RI: American Mathematical Society.

[5] Boolos, G. (1975). On second-order logic. *The Journal of Philosophy*, **72**. Reprinted in his *Logic, Logic, and Logic*. Cambridge, MA: MIT Press, 1998, pp. 37–53.

[6] Czelakowski, J. (2001). *Protoalgebraic Logics*. Dordrecht: Kluwer.

[7] Davidson, D. (1967). The logical form of action sentences. In Rescher, N., editor. *The Logic of Decision and Action*. University of Pittsburgh Press. Reprinted with criticism, comment and defence in his *Essays on Actions & Events*. Oxford: Oxford University Press, pp. 105–148.

[8] Font, J. M., Jansana, R., & Pigozzi, D. (2003). A survey of abstract algebraic logic. *Studia Logica*, **74**, 13–97.

[9] Givant, S. & Halmos, P. (2009). *Introduction to Boolean Algebras*. New York: Springer.

[10] Glanzberg, M. (2015). Logical consequence and natural language. In Caret, C. R. and Hjortland, O. T., editors. *Foundations of Logical Consequence*. New York: Oxford University Press, pp. 71–120.

[11] Griffiths, O. & Paseau, A. C. (forthcoming). *One True Logic*. Oxford University Press.

[12] Harman, G. (1986). *Change in View*. Cambridge, MA: MIT Press.

[13] Heim, I. & Kratzer, A. (1998). *Semantics in Generative Grammar*. Malden, MA: Blackwell.

[14] Higginbotham, J., Pianesi, F., & Varzi, A. (editors) (2000). *Speaking of Events*. New York: Oxford University Press.

[15] Hinman, P. G. (2005). *Fundamentals of Mathematical Logic*. Wellesley, MA: A.K. Peters.

[16] Lewitzka, S. (2007). A topological approach to universal logic: Model-theoretical abstract logics. In Beziau, J.-Y., editor. *Logica Universalis, Second Edition*. Basel: Birkhäuser, pp. 35–61.

[17] Monk, J. D. & Bonnet, R. (editors) (1989). *Handbook of Boolean Algebras*, Vol. 3. Amsterdam: North-Holland.

[18] Oliver, A. D. (2010). The matter of form: Logic's beginnings. In Lear, J. and Oliver, A. D., editors. *The Force of Argument*: *Essays in Honor of Timothy Smiley*. New York: Routledge, pp. 165–185.

[19] Oliver, A. D. & Smiley, T. (2013). *Plural Logic*. Oxford: Oxford University Press.

[20] Parsons, T. (1990). *Events in the Semantics of English*. Cambridge, MA: MIT Press.

[21] Paseau, A. C. (2015). A measure of inferential-role preservation. *Synthese*, doi: 10.1007/s11229-015-0705-5.

[22] Priest, G. (2006). *Doubt Truth to be a Liar*. Oxford: Oxford University Press.

[23] Quine, W. V. (1960). *Word and Object*. Cambridge, MA: MIT Press.

[24] Quine, W. V. (1982). *Methods of Logic* (fourth edition). Cambridge, MA: Harvard University Press.

[25] Rumfitt, I. (2011). What is logic? In Novák, Z. and Simonyi, A., editors. *Truth, Reference and Realism*. Budapest: Central European University Press, pp. 125–75.

[26] Rumfitt, I. (2015). *The Boundary Stones of Thought*. Oxford: Oxford University Press.

[27] Russell, B. (1905). On denoting. *Mind*, **14**, 479–493.

[28] Shapiro, S. (1991). *Foundations without Foundationalism*. New York: Clarendon Press.

[29] Shapiro, S. (1998). Logical consequence: Models and modality. In Schirn, M., editor. *The Philosophy of Mathematics Today*. Oxford: Oxford University Press, pp. 131–156.

[30] Shapiro, S. (2014). *Varieties of Logic*. New York: Oxford University Press.

[31] Strawson, P. F. (1950). On referring. *Mind*, **59**, 320–344.

[32] Strawson, P. F. (1952). *Introduction to Logical Theory*. New York: Wiley.

[33] Väänänen, J. (2012). Second order logic or set theory. *Bulletin of Symbolic Logic*, **18**, 91–121.

[34] Yi, B.-U. (2006). The logic and meaning of plurals. Part II. *Journal of Philosophical Logic*, **35**, 239–288.

WADHAM COLLEGE
UNIVERSITY OF OXFORD
OXFORD OX1 3PN, UK
E-mail: alexander.paseau@philosophy.ox.ac.uk