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Asymptotic problems for a kinetic model of two-phase flow

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We consider asymptotic problems for coupled equations modelling interactions between particles and a viscous fluid. The particles are driven by a Vlasov-like equation, involving the velocity of the fluid. We obtain, as certain parameters tend to 0, hydrodynamic equations for the macroscopic density and the velocity.

1. Introduction

In this work we are interested in asymptotic problems for a simple kinetic model for two-phase flows where a dispersed phase interacts with a fluid. Such equations arise in the description of various combustion phenomena, e.g. diesel engines. The dispersed phase is a spray of droplets having the same radius $r > 0$. Let ρ_l be the mass density of the material contained in these droplets. We describe the evolution of this phase by the distribution function $f(t, x, v)$ of particles occupying, at time t, the position x with velocity v. These particles move in a viscous gas characterized by its mass density ρ_g, its kinematic viscosity $\nu_g > 0$ (assumed to be constant) and its velocity u. Let us collect the assumptions we make in order to derive the model. (We refer to [5, 7, 8, 16] and K. Hamdache (personal communication) for detailed discussions on these hypothesis and further information on the model. A comprehensive presentation of the problem, as well as a deep discussion on the relevant scalings, can be found in [4].)

(i) We restrict the model to a situation where the flow is directed towards the x direction: the problem is mono-dimensional, so that $x \in \mathbb{R}$, $v \in \mathbb{R}$. With this in mind, let us introduce the following macroscopic quantities

$$\rho_p(t, x) = \int_{\mathbb{R}} f(t, x, v)\,\mathrm{d}v, \qquad \rho_p V_p(t, x) = \int_{\mathbb{R}} v f(t, x, v)\,\mathrm{d}v.$$

Hence $\rho_l \rho_p$ is the mass density of the cloud of droplets, while $\rho_l \rho_p V_p$ is its momentum.

(ii) We neglect interactions between particles. Indeed, the size of the droplets is constant, thus coalescence or break-up processes are prohibited. Moreover, we assume that the spray is dilute enough to neglect inter-particle collisions. In turn, we can assume that the gas density $\rho_g > 0$ is constant, since the volume fraction occupied by the particles is small compared to that of the gas.

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(iii) We neglect gravity effects. Hence the force on the particles is given by the following Stokes drag force,

$$F = \frac{9\mu_g}{2\rho_l r^2}(u(t,x) - v), \tag{1.1}$$

which describes the friction of the viscous fluid on the droplets. In (1.1), we have used the dynamic viscosity $\mu_g = \rho_g \nu_g$ and u is the bulk velocity of the fluid. Therefore, the spray equation reads as

$$\partial_t f + v\partial_x f + \partial_v(Ff) = 0. \tag{1.2}$$

(iv) Finally, the evolution of the gas is governed by the following viscous Burgers equation for the velocity $u(t,x)$:

$$\rho_g(\partial_t u + \partial_x u^2 - \nu_g \partial_x^2 u) = E. \tag{1.3}$$

In (1.3), the force E describes the exchange of impulse between the gas and the particles, hence it is related to the drag force as follows:

$$E(t,x) = \frac{9\mu_g}{2\rho_l r^2}\rho_l \rho_p(t,x)(V_p(t,x) - u(t,x)). \tag{1.4}$$

Equations (1.1), (1.2), coupled to (1.3), (1.4), have been considered in [7], where existence–uniqueness results, considering regular data, are obtained, with an extensive study of travelling waves for this system. More complicated fluid equation is dealt with by Hamdache [8], with a complete discussion on existence and time asymptotic behaviour of the solutions. Other models of interactions between fluid and particles are studied in [9,14]. We also mention interesting works of Jabin [10,11] and Jabin and Perthame [12], who treat questions of singular perturbations similar to ours for related models. Such problems also arise in astrophysics and plasmas physics, as studied by Nieto *et al.* [13], and it is certainly worth referring to the situation of granular media recently introduced by Benedetto *et al.* [1–3].

Here we investigate hydrodynamic limits for the simple model (1.1)–(1.4), supplemented by initial conditions

$$u_{|t=0} = u_0, \qquad f_{|t=0} = f_0 \geqslant 0.$$

Introducing a reference time T, two parameters without dimension appear in the equations: $9T\nu_g/2r^2 = T/\tau$ and ρ_g/ρ_l. Therefore, writing the equations in an adimensionalized form, we are interested in the behaviour of solutions $(f_\varepsilon, u_\varepsilon)$ of the following systems,

$$\left.\begin{aligned} &\partial_t f_\varepsilon + v\partial_x f_\varepsilon + \frac{1}{\varepsilon}\partial_v((u_\varepsilon - v)f_\varepsilon) = 0,\\ &\partial_t u_\varepsilon + \partial_x u_\varepsilon^2 - \nu\partial_x^2 u_\varepsilon = \frac{1}{\varepsilon}\rho_\varepsilon(V_\varepsilon - u_\varepsilon), \end{aligned}\right\} \tag{1.5}$$

where $1/\varepsilon = T/\tau$, $\rho_g/\rho_l = 1$ and

$$\left.\begin{aligned} &\partial_t f_\varepsilon + v\partial_x f_\varepsilon + \frac{1}{\varepsilon}\partial_v((u_\varepsilon - v)f_\varepsilon) = 0,\\ &\partial_t u_\varepsilon + \partial_x u_\varepsilon^2 - \nu\partial_x^2 u_\varepsilon = \rho_\varepsilon(V_\varepsilon - u_\varepsilon), \end{aligned}\right\} \tag{1.6}$$

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when $1/\varepsilon = \rho_g/\rho_l$, $T/\tau = 1$. We recall that in both (1.5) and (1.6) we denote

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}} f_\varepsilon(t,x,v)\,\mathrm{d}v, \qquad \rho_\varepsilon V_\varepsilon(t,x) = \int_{\mathbb{R}} v f_\varepsilon(t,x,v)\,\mathrm{d}v.$$

Following K. Hamdache (personal communication), the first asymptotic will be referred to as the hydrodymamic limit, while the second will be the stratified limit. Our main results are stated as follows.

THEOREM 1.1. *Let u_ε^0 and f_ε^0 satisfy*

$$\left.\begin{aligned} u_\varepsilon^0 \rightharpoonup u_0 \quad & \textit{weakly in } L^2(\mathbb{R}), \\ f_\varepsilon^0 \rightharpoonup f^0 \quad & \textit{weakly* in } \mathcal{M}^1(\mathbb{R}\times\mathbb{R}), \\ f_\varepsilon^0 \geqslant 0, \quad \int_{\mathbb{R}}\int_{\mathbb{R}} & (1+v^2) f_\varepsilon^0 \,\mathrm{d}v\mathrm{d}x = C_0 < \infty. \end{aligned}\right\} \tag{1.7}$$

Then, up to a subsequence, solutions of (1.5) satisfy

$$\begin{aligned} \rho_\varepsilon \to \rho \quad & \textit{strongly in } C^0([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R})) \textit{ and weakly* in } L^\infty(\mathbb{R};\mathcal{M}^1(\mathbb{R})), \\ u_\varepsilon \rightharpoonup u \quad & \textit{weakly in } L^2(0,T;H^1(\mathbb{R})), \end{aligned}$$

where (ρ, u) satisfy

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t((1+\rho)u) + \partial_x((1+\rho)uu) - \nu\partial_x^2 u = 0 \end{aligned}$$

and initial data

$$\rho_{|t=0} = \int_{\mathbb{R}} f^0\,\mathrm{d}v, \qquad \{(1+\rho)u\}_{|t=0} = u^0 + \int_{\mathbb{R}} v\,\mathrm{d}f^0(v).$$

(*This trace in time makes sense since it will be shown that $(1+\rho_\varepsilon)u_\varepsilon$ lies in a compact set of $C^0([0,T];H^{-1}(I))$ for any bounded set I.*)

THEOREM 1.2. *Let u_ε^0 and f_ε^0 satisfy (1.7). Then, up to a subsequence, solutions of (1.6) satisfy*

$$\begin{aligned} f_\varepsilon \rightharpoonup \rho(t,x)\delta_{v=u(t,x)} \quad & \textit{weakly* in } L^\infty(\mathbb{R}^+;\mathcal{M}^1(\mathbb{R}\times\mathbb{R})), \\ \rho_\varepsilon \to \rho \quad & \textit{strongly in } C^0([0,T];H^{-1}_{\mathrm{loc}}(\mathbb{R})) \textit{ and weakly* in } L^\infty(\mathbb{R};\mathcal{M}^1(\mathbb{R})), \\ u_\varepsilon \to u \quad & \textit{strongly in } L^2(0,T;C^0_{\mathrm{loc}}(\mathbb{R})) \textit{ and in } C^0([0,T];H^{-1}_{\mathrm{loc}}(\mathbb{R})) \\ & \textit{and weakly in } L^2(0,T;H^1(\mathbb{R})), \end{aligned}$$

where (ρ, u) satisfy

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t u + \partial_x u^2 - \nu\partial_x^2 u = 0 \end{aligned}$$

and initial data

$$\rho_{|t=0} = \int_{\mathbb{R}} \mathrm{d}f_0(v), \qquad u_{|t=0} = u^0.$$

2. Hydrodynamic limit

This section is devoted to the proof of theorem 1.1. The proof falls into two steps: first, we derive some estimates, which provide compactness properties that we then use in the next step to pass to the limit.

2.1. *A priori* estimates

First of all, a simple integration of the kinetic equation leads to the following natural conservation relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\int_{\mathbb{R}} f_\varepsilon \,\mathrm{d}v\mathrm{d}x = 0. \tag{2.1}$$

Next, multiply the kinetic equation by $\frac{1}{2}v^2$ and integrate. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}v^2 f_\varepsilon \,\mathrm{d}v\mathrm{d}x - \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}} v(u_\varepsilon - v) f_\varepsilon \,\mathrm{d}v\mathrm{d}x = 0,$$

while integrating the fluid equation multiplied by u_ε gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \tfrac{1}{2}u_\varepsilon^2 \,\mathrm{d}x + \nu\int_{\mathbb{R}} |\partial_x u_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}} \rho_\varepsilon u_\varepsilon(u_\varepsilon - V_\varepsilon)\,\mathrm{d}x = 0.$$

We note that

$$\int_{\mathbb{R}} \rho_\varepsilon u_\varepsilon(u_\varepsilon - V_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} f_\varepsilon u_\varepsilon(u_\varepsilon - v)\,\mathrm{d}v\mathrm{d}x.$$

Therefore, by adding the two previous relations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg\{\int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}v^2 f_\varepsilon \,\mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}} \tfrac{1}{2}u_\varepsilon^2 \,\mathrm{d}x\bigg\} + \nu\int_{\mathbb{R}} |\partial_x u_\varepsilon|^2 \,\mathrm{d}x + 1/\varepsilon\int_{\mathbb{R}}\int_{\mathbb{R}} f_\varepsilon (u_\varepsilon - v)^2 \,\mathrm{d}v\mathrm{d}x = 0.$$

We can now estimate the momentum by the mass and energy as follows:

$$\int_{\mathbb{R}} |\rho_\varepsilon V_\varepsilon|\,\mathrm{d}x = \int_{\mathbb{R}}\bigg|\int_{\mathbb{R}} v f_\varepsilon \,\mathrm{d}v\bigg|\,\mathrm{d}x \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}(1+v^2) f_\varepsilon \,\mathrm{d}v\mathrm{d}x.$$

We can collect this information in the following statement.

PROPOSITION 2.1. *Let u_ε^0 and f_ε^0 satisfy (1.7). Then*

(i) $(1+v^2)f_\varepsilon$ *is bounded in* $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))$,

(ii) ρ_ε *is bounded in* $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}))$,

(iii) $J_\varepsilon = \rho_\varepsilon V_\varepsilon$ *is bounded in* $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}))$,

(iv) u_ε *is bounded in* $L^\infty(\mathbb{R}^+; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$,

(v) $(1/\varepsilon)(v-u_\varepsilon)^2 f_\varepsilon$ *is bounded in* $L^1(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})$.

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2.2. Moment equations and limit

Having disposed of this preliminary, we can deal with a subsequence such that

$$\left.\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\mathbb{R})),\\ f_\varepsilon &\rightharpoonup f \quad \text{weakly}* \text{ in } L^\infty(\mathbb{R}^+;\mathcal{M}^1(\mathbb{R}\times\mathbb{R})),\\ \rho_\varepsilon &\rightharpoonup \rho \quad \text{weakly}* \text{ in } L^\infty(\mathbb{R}^+;\mathcal{M}^1(\mathbb{R})),\\ J_\varepsilon = \rho_\varepsilon V_\varepsilon &\rightharpoonup J \quad \text{weakly}* \text{ in } L^\infty(\mathbb{R}^+;\mathcal{M}^1(\mathbb{R})). \end{aligned}\right\} \tag{2.2}$$

Here, $\mathcal{M}^1(\Omega)$ stands for the set of bounded measures on the domain Ω. Then, integrating the kinetic equation with respect to v, we obtain the following conservation relation,

$$\partial_t\rho_\varepsilon + \partial_x J_\varepsilon = 0, \tag{2.3}$$

which gives, at least in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R})$,

$$\partial_t\rho + \partial_x J = 0$$

as ε goes to 0. Furthermore, equation (2.3) combined with the estimate (iii) on J_ε, says that $\partial_t\rho_\varepsilon$ belongs to a bounded set in $L^\infty(\mathbb{R}^+;W^{-1,1}(\mathbb{R}))$. Let $0<R<\infty$ and $I=(-R,+R)$. Since the embeddings $W_0^{1,\infty}(I)\subset H_0^1(I)\subset_{\text{comp}} C^0(\bar{I})$ hold, we have

$$L^1(I)\subset\mathcal{M}^1(I)=(C^0(\bar{I}))'\subset_{\text{comp}} H^{-1}(I)=(H_0^1(I))'\subset W^{-1,1}(I)=(W_0^{1,\infty}(I))'.$$

Therefore, one deduces that

$$\rho_\varepsilon\to\rho \quad \text{in } C^0([0,T];H^{-1}_{\text{loc}}(\mathbb{R})), \tag{2.4}$$

by applying a classical compactness theorem (see, for instance, corollary 4 in [15]). Hence the product $\rho_\varepsilon u_\varepsilon$ passes to the limit

$$\rho_\varepsilon u_\varepsilon \rightharpoonup \rho u \quad \text{(at least) in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}). \tag{2.5}$$

Let us come back to the fluid equation. We have

$$\rho_\varepsilon(V_\varepsilon-u_\varepsilon)=\varepsilon(\partial_t u_\varepsilon+\partial_x u_\varepsilon^2-\nu\partial_x^2 u_\varepsilon),$$

which becomes, as $\varepsilon\to 0$,

$$\rho_\varepsilon(V_\varepsilon-u_\varepsilon)\rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}). \tag{2.6}$$

Actually, by applying the Cauchy–Schwarz inequality and estimate (v) in proposition 2.1, $\rho_\varepsilon(V_\varepsilon-u_\varepsilon)$ tends to 0 in $L^1((0,T)\times\mathbb{R})$ with a rate $\sqrt{\varepsilon}$. Then, by combining this last relation to (2.5), we can now identify the weak limit J, since

$$J_\varepsilon=\rho_\varepsilon V_\varepsilon=\rho_\varepsilon(V_\varepsilon-u_\varepsilon)+\rho_\varepsilon u_\varepsilon\rightharpoonup J=0+\rho u=\rho u \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}). \tag{2.7}$$

Next, an equation for J_ε is obtained by multiplying the kinetic equation and integrating with respect to v. We have

$$\begin{aligned}\partial_t J_\varepsilon+\partial_x p_\varepsilon-\frac{1}{\varepsilon}\int_{\mathbb{R}}(u_\varepsilon-v)f_\varepsilon\,\mathrm{d}v=0&\\ &=\partial_t J_\varepsilon+\partial_x p_\varepsilon-\frac{1}{\varepsilon}\rho_\varepsilon(u_\varepsilon-V_\varepsilon)\\ &=\partial_t(u_\varepsilon+J_\varepsilon)+\partial_x(u_\varepsilon^2+p_\varepsilon)-\nu\partial_x^2u_\varepsilon,\end{aligned} \tag{2.8}$$

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where

$$p_\varepsilon = \int_{\mathbb{R}} v^2 f_\varepsilon \, \mathrm{d}v$$

and we have made use of the fluid equation. We can pass to the limit without difficulties in the first and the last terms of (2.8) by using (2.2) and (2.7). Furthermore, equation (2.8) provides a bound on $\partial_t(u_\varepsilon + J_\varepsilon)$ in

$$L^\infty(\mathbb{R}^+; W^{-1,1}(\mathbb{R})) + L^2(\mathbb{R}^+; H^{-1}(\mathbb{R})) \subset L^2((0,T); W^{-1,1}_{\mathrm{loc}}(\mathbb{R})),$$

while $u_\varepsilon + J_\varepsilon$ is bounded in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R})) + L^\infty(\mathbb{R}^+; L^1(\mathbb{R})) \subset L^\infty(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}))$. By using the compactness theorem of [15], one deduces that $u_\varepsilon + J_\varepsilon$ belongs to a compact set of $C^0([0,T]; H^{-1}(I))$ for any bounded set I. Hence the product with u_ε passes to the limit and we are led to

$$\left.\begin{aligned} u_\varepsilon + J_\varepsilon \to u + \rho u \qquad & \text{in } C^0([0,T]; H^{-1}_{\mathrm{loc}}(\mathbb{R})) \quad \text{(by (2.2) and (2.7))}, \\ (u_\varepsilon + J_\varepsilon)u_\varepsilon \rightharpoonup (u + \rho u)u \quad & \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}). \end{aligned}\right\} \tag{2.9}$$

Then we rewrite the pressure term as follows:

$$u_\varepsilon^2 + p_\varepsilon = (u_\varepsilon + J_\varepsilon)u_\varepsilon + (p_\varepsilon - J_\varepsilon u_\varepsilon).$$

Let $q_\varepsilon = \int_{\mathbb{R}} (v - u_\varepsilon) v f_\varepsilon \, \mathrm{d}v$ designate the last term. We get

$$\begin{aligned} \int_0^\infty \left\{ \int_{\mathbb{R}} |q_\varepsilon| \, \mathrm{d}x \right\}^2 \mathrm{d}t & \leqslant \int_0^\infty \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} |v - u_\varepsilon| \sqrt{f_\varepsilon} |v| \sqrt{f_\varepsilon} \, \mathrm{d}v \mathrm{d}x \right\}^2 \mathrm{d}t \\ & \leqslant \int_0^\infty \left\{ \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |v - u_\varepsilon|^2 f_\varepsilon \, \mathrm{d}v \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} v^2 f_\varepsilon \, \mathrm{d}v \mathrm{d}x \right)^{1/2} \right\}^2 \mathrm{d}t \\ & \leqslant \|v^2 f_\varepsilon\|_{L^\infty(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))} \| |v - u_\varepsilon|^2 f_\varepsilon \|_{L^1(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})} \leqslant C\varepsilon. \end{aligned}$$

Hence we obtain

$$u_\varepsilon^2 + p_\varepsilon \rightharpoonup (u + \rho u)u \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$$

by using (2.9), so that (2.8) becomes

$$\partial_t((1 + \rho)u) + \partial_x((1 + \rho)u^2) - \nu \partial_x^2 u = 0$$

as $\varepsilon \to 0$. Note that the initial data also pass to the limit thanks to the convergences (2.4) and (2.9). This achieves the proof of theorem 1.1. □

One may wonder what happens at the kinetic level. In view of (1.5), we could believe that the sequence f_ε tends to the Dirac mass $\rho \delta_{v=u(t,x)}$. However, the convergence of u_ε seems to be not strong enough to obtain such a result, while we can show that

$$f_\varepsilon - \rho_\varepsilon \delta_{v=u_\varepsilon} \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}).$$

Indeed, for a regular test function $\varphi(t,x,v) = \zeta(t,x)\psi(v)$ with compact support, we have

$$\langle f_\varepsilon - \rho_\varepsilon \delta_{v=u_\varepsilon}, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \zeta(t,x)(\psi(v) - \psi(u_\varepsilon)) f_\varepsilon(t,x,v) \, \mathrm{d}v \mathrm{d}x \mathrm{d}t.$$

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Let $\delta > 0$ be fixed and let $\eta > 0$ to be determined. We split this integral, considering separately the domain $|v - u_\varepsilon| \leqslant \eta$ and $|v - u_\varepsilon| > \eta$. By choosing η small enough and using Heine's theorem, the difference $|\psi(v) - \psi(u_\varepsilon)|$ can be arbitrarily small as $|v - u_\varepsilon| \leqslant \eta$, so that

$$\left|\int_{|v-u_\varepsilon|\leqslant\eta} \cdots \, \mathrm{d}v\mathrm{d}x\mathrm{d}t\right| \leqslant \|\zeta\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}}\int_{\mathbb{R}} f_\varepsilon \, \mathrm{d}v\mathrm{d}x\mathrm{d}t \sup_{|v-u_\varepsilon|\leqslant\eta} |\psi(v) - \psi(u_\varepsilon)| \leqslant \delta.$$

On the other hand, the energy estimate gives

$$\left|\int_{|v-u_\varepsilon|\geqslant\eta} \cdots \, \mathrm{d}v\mathrm{d}x\mathrm{d}t\right| \leqslant \frac{2}{\eta^2}\|\zeta\|_{L^\infty}\|\psi\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}}\int_{\mathbb{R}} |v - u_\varepsilon|^2 f_\varepsilon \, \mathrm{d}t\mathrm{d}x\mathrm{d}v \leqslant \frac{\varepsilon C}{\eta^2},$$

where C depends on ζ, ψ and C_0. It follows that

$$|\langle f_\varepsilon - \rho_\varepsilon \delta_{v=u_\varepsilon}, \varphi\rangle| \leqslant 2\delta$$

for ε small enough. However, it is not clear at all that $\rho_\varepsilon\delta_{v=u_\varepsilon}$ is close to $\rho\delta_{v=u}$.

Let us end this section by remarking that convergence (2.4) also means that ρ_ε converges to ρ in $C^0([0,T];\mathcal{M}^1(\mathbb{R})\text{-weak}*)$. Indeed, a continuous function φ vanishing at ∞ can be approached uniformly on $\mathbb{R}$ by $\varphi_n \in C_0^\infty(\mathbb{R})$. Hence we can write

$$\langle \rho_\varepsilon(t), \varphi\rangle_{\mathcal{M}^1,C^0} = \langle \rho_\varepsilon(t), \varphi_n\rangle_{H^{-1},H_0^1} + \langle \rho_\varepsilon(t), \varphi - \varphi_n\rangle_{\mathcal{M}^1,C^0},$$

where the last term can be given arbitrarily small by choosing n large enough. The same remark also applies to (2.9).

3. Stratified limit

Let us now deal with the stratified limit (1.6), which will appear slightly simpler since we can obtain some strong compactness for the velocity field.

Here, the energy estimate becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}v^2 f_\varepsilon \, \mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}} \tfrac{1}{2}u_\varepsilon^2 \, \mathrm{d}x\right\} + \nu\int_{\mathbb{R}} |\partial_x u_\varepsilon|^2 \, \mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} f_\varepsilon(u_\varepsilon - v)^2 \, \mathrm{d}v\mathrm{d}x = 0, \tag{3.1}$$

while we keep the mass conservation (2.1). One deduces the following claim.

PROPOSITION 3.1. *Let u_ε^0 and f_ε^0 fulfil (1.7). Then*

(i) *f_ε is bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))$,*

(ii) *ρ_ε is bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}))$,*

(iii) *u_ε is bounded in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$,*

(iv) *$(v - u_\varepsilon)^2 f_\varepsilon$ is bounded in $L^1(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})$,*

(v) *$|V_\varepsilon - u_\varepsilon|\rho_\varepsilon$ is bounded in $L^2(\mathbb{R}^+; L^1(\mathbb{R}))$,*

(vi) *vf_ε and $u_\varepsilon f_\varepsilon$ are bounded in $L^2(0,T; L^1(\mathbb{R}\times\mathbb{R}))$.*

Proof. Estimates (i)–(iv) read directly on (2.1) and (3.1). Next, since $\rho_\varepsilon|V_\varepsilon - u_\varepsilon| = |\rho_\varepsilon(V_\varepsilon - u_\varepsilon)|$, we get

$$\begin{aligned}
\int_0^\infty \bigg\{ \int_{\mathbb{R}} \rho_\varepsilon |V_\varepsilon - u_\varepsilon| \, \mathrm{d}x \bigg\}^2 \mathrm{d}t
&= \int_0^\infty \bigg\{ \int_{\mathbb{R}} \bigg| \int_{\mathbb{R}} (v - u_\varepsilon) f_\varepsilon \, \mathrm{d}v \bigg| \, \mathrm{d}x \bigg\}^2 \mathrm{d}t \\
&\leqslant \int_0^\infty \bigg\{ \int_{\mathbb{R}} \int_{\mathbb{R}} |v - u_\varepsilon| \sqrt{f_\varepsilon} \sqrt{f_\varepsilon} \, \mathrm{d}v \mathrm{d}x \bigg\}^2 \mathrm{d}t \\
&\leqslant \int_0^\infty \bigg\{ \bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} |v - u_\varepsilon|^2 f_\varepsilon \, \mathrm{d}v \mathrm{d}x \bigg)^{1/2} \bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} f_\varepsilon \, \mathrm{d}v \mathrm{d}x \bigg)^{1/2} \bigg\}^2 \mathrm{d}t \\
&\leqslant \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))} \| |v - u_\varepsilon|^2 f_\varepsilon \|_{L^1(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})} \leqslant C, \qquad (3.2)
\end{aligned}$$

which proves (v). Now, by using the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ (with constant C_S), we obtain

$$\begin{aligned}
\int_0^T \bigg\{ \int_{\mathbb{R}} \rho_\varepsilon |u_\varepsilon| \, \mathrm{d}x \bigg\}^2 \mathrm{d}t &= \int_0^T \bigg\{ \int_{\mathbb{R}} \int_{\mathbb{R}} |u_\varepsilon| f_\varepsilon \, \mathrm{d}v \mathrm{d}x \bigg\}^2 \mathrm{d}t \\
&\leqslant \int_0^T \|u_\varepsilon(t, \cdot)\|^2_{L^\infty(\mathbb{R})} \bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} f_\varepsilon \, \mathrm{d}v \mathrm{d}x \bigg)^2 \mathrm{d}t \\
&\leqslant \|f_\varepsilon\|^2_{L^\infty(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))} \int_0^T \|u_\varepsilon(t, \cdot)\|^2_{L^\infty(\mathbb{R})} \, \mathrm{d}t \\
&\leqslant C_S \|f_\varepsilon\|^2_{L^\infty(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))} \|u_\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}))} \leqslant C. \qquad (3.3)
\end{aligned}$$

This shows that $\rho_\varepsilon u_\varepsilon$ is bounded in $L^2(0, T; L^1(\mathbb{R}))$. We then combine (3.2) and (3.3) to deduce that

$$\begin{aligned}
\int_0^T \bigg\{ \int_{\mathbb{R}} \rho_\varepsilon |V_\varepsilon| \, \mathrm{d}x \bigg\}^2 \mathrm{d}t
&= \int_0^T \bigg\{ \int_{\mathbb{R}} \bigg| \int_{\mathbb{R}} v f_\varepsilon \, \mathrm{d}v \bigg| \, \mathrm{d}x \bigg\}^2 \mathrm{d}t \\
&\leqslant \int_0^T \bigg\{ \int_{\mathbb{R}} \int_{\mathbb{R}} (|v - u_\varepsilon| + |u_\varepsilon|) f_\varepsilon \, \mathrm{d}v \mathrm{d}x \bigg\}^2 \mathrm{d}t \\
&\leqslant 2(\| |v - u_\varepsilon| f_\varepsilon \|^2_{L^2(\mathbb{R}^+; L^1(\mathbb{R}\times\mathbb{R}))} + \| |u_\varepsilon| f_\varepsilon \|^2_{L^2(0,T; L^1(\mathbb{R}\times\mathbb{R}))}) \leqslant C
\end{aligned}$$

holds. □

In view of the energy estimate, one may wonder that the kinetic energy cannot be bounded; however, this is actually the case.

COROLLARY 3.2. *$v^2 f_\varepsilon$ is bounded in $L^1((0, T) \times \mathbb{R} \times \mathbb{R})$.*

Proof. Following what we did in (3.3), we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\rho_\varepsilon|u_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t &\leqslant \int_0^T \|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\biggl(\int_{\mathbb{R}}\int_{\mathbb{R}} f_\varepsilon\,\mathrm{d}v\mathrm{d}x\biggr)\,\mathrm{d}t\\ &\leqslant \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^1(\mathbb{R}\times\mathbb{R}))}\int_0^T \|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\,\mathrm{d}t\\ &\leqslant C_0C_S\|u_\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}))}\leqslant C.\end{aligned}$$

It follows that

$$\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} v^2 f_\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\leqslant 2\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}((v-u_\varepsilon)^2+u_\varepsilon^2)f_\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\leqslant C,$$

using (iv). □

Furthermore, proposition 3.1, combined with the fluid equation, also leads to the following compactness property.

COROLLARY 3.3. *u_ε lies in a compact set of $L^2(0,T;C^0(I))\cap C^0([0,T];H^{-1}(I))$ for any bounded interval I.*

Proof. We rewrite the fluid equation as

$$\partial_t u_\varepsilon=\rho_\varepsilon(V_\varepsilon-u_\varepsilon)-\partial_x u_\varepsilon^2+\nu\partial_x^2 u_\varepsilon, \tag{3.4}$$

where one sees that the right-hand side is bounded in

$$L^2(0,T;L^1(\mathbb{R}))+L^2(0,T;H^{-1}(\mathbb{R}))+L^2(0,T;H^{-1}(\mathbb{R})).$$

The second estimate comes from the injection $H^1\subset L^\infty$, thus u_ε^2 belongs (at least) to L^2. Applying corollary 4 of [15], with the compact embeddings $H^1(I)\subset C^0(I)$ and $L^2(I)\subset H^{-1}(I)$, respectively, leads to the expected compactness. □

LEMMA 3.4. *Up to a subsequence, u_ε converges to u strongly in $L^2((0,T);C^0_{\mathrm{loc}}(\mathbb{R}))$ and f_ε tends to $\rho(t,x)\delta_{v=u(t,x)}$ weakly$*$ in $L^\infty(\mathbb{R}^+;\mathcal{M}^1(\mathbb{R}\times\mathbb{R}))$, where $\rho_\varepsilon(t,x)$ converges to $\rho(t,x)$ strongly in $C^0([0,T];H^{-1}_{\mathrm{loc}}(\mathbb{R}))$.*

Proof. The convergence of u_ε is a consequence of the previous corollary. In view of the bound (i) in proposition 3.1, we can also assume that

$$f_\varepsilon\rightharpoonup f\quad\text{in } L^\infty(\mathbb{R}^+;\mathcal{M}^1(\mathbb{R}\times\mathbb{R}))\text{ weak}*.$$

Then the following convergence holds,

$$u_\varepsilon f_\varepsilon\rightharpoonup uf\quad\text{in }\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})\text{ (at least)}, \tag{3.5}$$

since, for any test function φ (with support in the ball $B_\infty(0,T)$), one has

$$\begin{aligned}&|\langle u_\varepsilon f_\varepsilon-uf,\varphi\rangle_{\mathcal{D}',\mathcal{D}}|\\ &\quad=\biggl|\int_0^\infty\langle f_\varepsilon(t,\cdot),(u_\varepsilon-u)\varphi\rangle_{\mathcal{M}^1,C^0}\,\mathrm{d}t+\int_0^\infty\langle (f_\varepsilon-f)(t,\cdot),u\varphi\rangle_{\mathcal{M}^1,C^0}\,\mathrm{d}t\biggr|\\ &\quad\leqslant\|f_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^1(\mathbb{R}\times\mathbb{R}))}\|\varphi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R})}\sqrt{T}\|u_\varepsilon-u\|_{L^2(0,T;L^\infty((-T;T)))}\\ &\qquad+\biggl|\int_0^\infty\langle (f_\varepsilon-f)(t,\cdot),u\varphi\rangle_{\mathcal{M}^1,C^0}\,\mathrm{d}t\biggr|.\end{aligned}$$

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The first term goes to 0 by using the strong convergence of u_ε, while the second term vanishes by the weak$*$ convergence of f_ε. This permits us to identify the limit f, since

$$\varepsilon(\partial_t f_\varepsilon + v\partial_x f_\varepsilon) + \partial_v((u_\varepsilon - v)f_\varepsilon) = 0$$

gives, as $\varepsilon \to 0$,

$$\partial_v((u-v)f) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}).$$

It follows that $(v-u)f = C(t,x)$ does not depend on v. Since this distribution is actually a finite measure with respect to the variable $v \in \mathbb{R}$ (see proposition 2 (iv)), one deduces that $(v-u)f = 0 \in L^\infty(\mathbb{R}^+; \mathcal{M}^1(\mathbb{R}\times\mathbb{R}))$ and $f = \rho(t,x)\delta_{v=u(t,x)}$. Furthermore, by corollary 3.2, moments of f_ε also pass to the limit

$$\left.\begin{aligned} \rho_\varepsilon(t,x) &= \int_{\mathbb{R}} f_\varepsilon(t,x,v)\,\mathrm{d}v \\ &\rightharpoonup \int_{\mathbb{R}} \mathrm{d}f(t,x,v) \\ &= \rho(t,x) \quad \text{in } L^\infty(\mathbb{R}^+; \mathcal{M}^1(\mathbb{R})) \text{ weak}*, \\ \rho_\varepsilon V_\varepsilon(t,x) &= \int_{\mathbb{R}} v f_\varepsilon(t,x,v)\,\mathrm{d}v \\ &\rightharpoonup \int_{\mathbb{R}} v\,\mathrm{d}f(t,x,v) \\ &= \rho(t,x)u(t,x) \quad \text{in } L^2(\mathbb{R}^+; \mathcal{M}^1(\mathbb{R})) \text{ weak}*. \end{aligned}\right\} \tag{3.6}$$

It remains to show that the first convergence in (3.6) holds strongly. Indeed, one has

$$\partial_t \rho_\varepsilon = -\partial_x j_\varepsilon, \tag{3.7}$$

which implies that $\partial_t \rho_\varepsilon$ is bounded in $L^2(\mathbb{R}^+; W^{-1,1}(\mathbb{R}))$. As in the previous section, by [15], one gets the strong compactness of ρ_ε in $C^0([0,T]; H^{-1}(I))$ for any bounded interval I.

Now, letting $\varepsilon \to 0$ in (3.7) yields

$$\partial_t \rho + \partial_x(\rho u) = 0$$

by (3.6). On the other hand, we remark that

$$\rho_\varepsilon(V_\varepsilon - u_\varepsilon) \rightharpoonup \rho u - \rho u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}),$$

still by (3.6) combined with (3.5). Thus, as ε goes to 0 in (3.4), we are led to

$$\partial_t u + \partial_x u^2 - \nu \partial_x^2 u = 0,$$

using the convergence given by corollary 3.3. This ends the proof of theorem 1.2. □

4. Existence of solutions for the coupled kinetic/fluid problem

This section is devoted to the existence of solutions as $\varepsilon > 0$ is fixed. For the sake of simplicity, we set from now on $\varepsilon = 1$ and we are interested in solving the following

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coupled equations:

$$\left.\begin{aligned} \partial_t f + v\partial_x f + \partial_v((u-v)f) = 0, \\ \partial_t u + \partial_x u^2 - \nu\partial_x^2 u = \rho(V-u), \\ u_{|t=0} = u_0, \qquad f_{|t=0} = f_0. \end{aligned}\right\} \tag{4.1}$$

When the initial data (u_0, f_0) are regular, we can prove global existence and uniqueness of a regular solution (u, f), by combining a tedious analysis of the characteristic curves associated to the field $b(t, x, v) = (v, u(t, x) - v)$ and an application of the classical Banach fixed point. Precisely, according to [7], for $u_0 \in C^2(\mathbb{R})$ and $f_0 \in C_0^1(\mathbb{R} \times \mathbb{R})$, $f_0 \geqslant 0$, one proves the existence–uniqueness of a solution

$$u \in C^0([0,T]; C^2(\mathbb{R})) \quad \text{and} \quad f \in C^0([0,T]; C_0^1(\mathbb{R} \times \mathbb{R}), \qquad f \geqslant 0.$$

Of course, this solution satisfies the following energy equality:

$$\begin{aligned} \int_{\mathbb{R}} \tfrac{1}{2} u(T,x)^2 \,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2} v^2 f(T,x,v)\,\mathrm{d}v\mathrm{d}x \\ + \nu \int_0^T \int_{\mathbb{R}} |\partial_x u|^2 \,\mathrm{d}x\mathrm{d}s + \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} (v-u)^2 f \,\mathrm{d}v\mathrm{d}x\mathrm{d}s \\ = \int_{\mathbb{R}} \tfrac{1}{2} u_0^2 \,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2} v^2 f_0 \,\mathrm{d}v\mathrm{d}x. \end{aligned} \tag{4.2}$$

This result is clearly sufficient to perform the asymptotic analysis of the previous sections, dealing with the sequences $(u_\varepsilon, f_\varepsilon)$ associated to these regular data and taking into account the introduction of the relevant scaling. However, it is also interesting to derive some existence results for less regular data. This is the aim of this section.

First of all, we can consider initial regular data (u_0^n, f_0^n) converging to some (u_0, f_0) in $L^2(\mathbb{R})$ and $L^1(\mathbb{R} \times \mathbb{R})$, respectively, where $v^2 f_0$ is also integrable. Then, by using the bounds (uniform with respect to n) given by (4.2), we can easily pass to the limit in the equation. This strategy provides the existence of a solution $(u, f) \in L^2(0, T; H^1(\mathbb{R})) \times L^\infty(0, T; \mathcal{M}^1(\mathbb{R} \times \mathbb{R}))$, which satisfies (4.2), the equals sign being replaced with '$\leqslant$'. Furthermore, if one assumes that f_0^n is bounded in $L^\infty(\mathbb{R} \times \mathbb{R})$, the maximum principle gives an additional L^∞ bound on the approximated sequence f^n (namely, $\|f^n(t)\|_{L^\infty} \leqslant \mathrm{e}^T \|f_0^n\|_{L^\infty}$). In turn, the solution f obtained by regularization of the initial data belongs to $L^\infty((0, T) \times \mathbb{R} \times \mathbb{R})$. Then, applying general results of [6] on transport equations, one proves that this solution f lies in $C^0[0, T]; L^p_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}))$ for $1 \leqslant p < \infty$. Let us give below a similar result, obtained by a fixed-point approach. Note finally that such a result is close to those of [8] or [11] for multi-dimensional Vlasov–Stokes equations and we are able to prove the existence of non-regular solutions, but uniqueness is far from clear in this context.

Proposition 4.1. *Let u_0 and $f_0 \geqslant 0$, $f_0 \in L^1 \cap L^2(\mathbb{R} \times \mathbb{R})$, satisfy*

$$\int_{\mathbb{R}} u_0^2 \,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} (1+v^2) f_0 \,\mathrm{d}v\mathrm{d}x \leqslant C_0 < \infty.$$

Then there exists a solution of (4.1) with $u \in C^0([0,T];L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R}))$ *and* $f \in C^0([0,T];L^1(\mathbb{R}\times\mathbb{R}))$. *Moreover, the following energy inequality holds:*

$$\int_{\mathbb{R}} \tfrac{1}{2}u(T,x)^2\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}v^2 f(T,x,v)\,\mathrm{d}v\mathrm{d}x + \nu\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\,\mathrm{d}x\mathrm{d}s + \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}(v-u)^2 f\,\mathrm{d}v\mathrm{d}x\mathrm{d}s \leqslant \int_{\mathbb{R}}\tfrac{1}{2}u_0^2\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}}\tfrac{1}{2}v^2 f_0\,\mathrm{d}v\mathrm{d}x.$$

Proof. We can follow closely the arguments used in Hamdache's paper [8] by introducing a suitable regularized problem and then letting the regularization parameter go to 0. Instead, we use here a direct approach, which takes advantage of the particularities of the mono-dimensional framework. Let $0 < T < \infty$ and set

$$\mathbb{F} = C^0([0,T];L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R})),$$

endowed with the norm

$$\|u\|_{\mathbb{F}}^2 = \int_0^T\int_{\mathbb{R}}(u^2 + |\partial_x u|^2)\,\mathrm{d}x.$$

We consider the map

$$\mathcal{T} : \mathbb{F} \to \mathbb{F}, \qquad u^* \mapsto u = \mathcal{T}(u^*)$$

defined by the following scheme.

(a) Solve the kinetic equation

$$\partial_t f + v\partial_x f + \partial_v((u^* - v)f) = 0,$$

with initial data f_0.

(b) Solve the fluid equation

$$\partial_t u + \partial_x u^2 - \nu\partial_x^2 u + \rho u = \rho V,$$

with initial data u_0.

Note that the kinetic equation in (a) can be rewritten

$$\partial_t f + b\cdot\nabla_{x,v} f - f = 0,$$

where $b(t,x,v) = (v, u^*(t,x) - v)$ lies (at least) in

$$L^1(0,T;H^1((-K,+K)\times(-K,+K))), \quad 0 < K < \infty,$$

with $\mathrm{div}_{x,v}\, b = -1 \in L^\infty((0,T)\times\mathbb{R}\times\mathbb{R})$, and $|b|/(1+|x|+|v|)$ is bounded, which appeals to the general results in [6]. In particular, for $u^* \in \mathbb{F}$, there exists a unique non-negative solution $f \in C^0([0,T]; L^2 \cap L^1(\mathbb{R}\times\mathbb{R}))$, as the initial data satisfy $f_0 \in L^2 \cap L^1(\mathbb{R}\times\mathbb{R})$, $f_0 \geqslant 0$ (see corollaries II-1 and II-2 of [6]). Let $M > 0$ to be specified later. Assume that

$$\int_0^T \|u^*\|^2_{L^\infty(\mathbb{R})}\,\mathrm{d}t \leqslant C_S\|u^*\|^2_{\mathbb{F}} \leqslant C_S M,$$

using the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ (we denote by C_S the constant corresponding to this injection). Then one checks that

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}v^2 f\,\mathrm{d}v\mathrm{d}x &= \int_{\mathbb{R}}\int_{\mathbb{R}} \tfrac{1}{2}v^2 f_0\,\mathrm{d}v\mathrm{d}x + \int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}(u^*-v)vf\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\leqslant \tfrac{1}{2}C_0 + \frac{1}{2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}(u^*)^2 f\,\mathrm{d}v\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} v^2 f\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\leqslant \tfrac{1}{2}C_0 + \frac{1}{2}\int_0^T\left(\|u^*\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,\mathrm{d}v\mathrm{d}x\right)\mathrm{d}t \\
&\leqslant \tfrac{1}{2}C_0 + \frac{1}{2}\int_0^T\left(\|u^*\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\,\mathrm{d}v\mathrm{d}x\right)\mathrm{d}t \\
&\leqslant \tfrac{1}{2}C_0(1+MC_S),
\end{aligned}$$

using the positivity of f and the mass conservation. This estimate yields

$$\int_{\mathbb{R}}\rho|V|\,\mathrm{d}x \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}}|v|f\,\mathrm{d}v\mathrm{d}x \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}}\tfrac{1}{2}(1+v^2)f\,\mathrm{d}v\mathrm{d}x \leqslant C_0(1+\tfrac{1}{2}C_S M) = K.$$

Now we turn to the fluid equation (b). Let $C_1 = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R})} < C_0$ and $M > 2C_0/\nu$. We shall show that the set

$$\mathcal{C} = \{u \in \mathbb{F},\ \|u\|^2_{\mathbb{F}} \leqslant M\}$$

is left invariant by $\mathcal{T}$, provided T is small enough. Indeed, we get (since $\rho \geqslant 0$)

$$\begin{aligned}
\int_{\mathbb{R}}\tfrac{1}{2}u^2\,\mathrm{d}x + \nu\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\,\mathrm{d}x\mathrm{d}t &\leqslant \int_{\mathbb{R}}\tfrac{1}{2}u_0^2\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}}\rho V u\,\mathrm{d}x\mathrm{d}t \\
&\leqslant C_1 + K\int_0^T\|u\|_{L^\infty(\mathbb{R})}\,\mathrm{d}t \\
&\leqslant C_1 + KC_S\int_0^T\|u\|_{H^1}\,\mathrm{d}t.
\end{aligned}$$

Then classical tricks lead to

$$\int_{\mathbb{R}}\tfrac{1}{2}u^2\,\mathrm{d}x + \tfrac{1}{2}\nu\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\,\mathrm{d}x\mathrm{d}t \leqslant C_1 + C_2T + \nu\int_0^T\int_{\mathbb{R}}\tfrac{1}{2}u^2\,\mathrm{d}x\mathrm{d}t,$$

where C_2 depends on C_0, C_S, M and ν. It follows that

$$\int_{\mathbb{R}}\tfrac{1}{2}u^2\,\mathrm{d}x \leqslant (C_1 + C_2T)\mathrm{e}^{\nu T},$$

$$\tfrac{1}{2}\nu\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\,\mathrm{d}x\mathrm{d}t \leqslant C_1 + C_2T + \nu T(C_1 + C_2T)\mathrm{e}^{\nu T},$$

hence we get

$$\|u\|_{\mathbb{F}}^2 \leqslant 2C_1/\nu + \phi(T),$$

which remains less than or equal to M for T small enough, since $\phi(T) \to 0$ as T goes to 0.

Having defined the invariant set $\mathcal{C}$, we go on to prove the continuity of $\mathcal{T}$ in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R})$. Let u_k^* be a sequence in $\mathcal{C}$ with $u_k^* \to u^*$ in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R})$. Then f_k, associated to u_k^*, converges to f, associated to u^*, in $C^0([0,T];L^1(\mathbb{R}\times\mathbb{R}))$ (see theorems II-4 and II-3 of [6]). One deduces that ρ_k and $\rho_k V_k$ converge in $C^0([0,T];L^1(\mathbb{R}))$. In turn, this finally gives the convergence of u^k to u in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R})$. Now we show that $\mathcal{T}$ is also compact, since for u_k^* in $\mathcal{C}$, $u_k = \mathcal{T}(u_k^*) \in \mathcal{C}$, with $\partial_t u_k$ bounded in $L^1(0,T;H^{-1}_{\mathrm{loc}}(\mathbb{R}))$. It follows that u_k belongs to a compact set in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R})$, by applying results in [15].

Schauder's theorem provides the existence of a fixed point $u = \mathcal{T}(u)$ that actually defines a solution (u, f) of (4.1) in the time-interval $[0,T]$. Furthermore, the energy inequality holds. Hence we can now reproduce our arguments on $[T, 2T]$ and so on, obtaining in this way existence on any time-interval. □

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