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OVERGROUPS OF PRIMITIVE GROUPS

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Abstract

We give a qualitative description of the set $\mathcal{O}_G(H)$ of overgroups in G of primitive subgroups H of finite alternating and symmetric groups G, and particularly of the maximal overgroups. We then show that certain weak restrictions on the lattice $\mathcal{O}_G(H)$ impose strong restrictions on H and its overgroup lattice.

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Introduction

Assume that Ω is a finite set, $S = \mathrm{Sym}(\Omega)$ is the symmetric group on Ω, G is S or the alternating group A on Ω, and H is a subgroup of G acting primitively on Ω. In this paper we study the set $\mathcal{O}_G(H)$ of overgroups of H in G. In particular, we describe (in some sense) the maximal overgroups of H. Then we use our description to establish results about the lattice $\mathcal{O}_G(H)$ when H is the intersection of a pair of maximal subgroups of G.

In [P], Cheryl Praeger also discusses the overgroups of primitive subgroups of alternating and symmetric groups, but her organization of information does not immediately lend itself to the applications we have in mind. Thus we have produced our own treatment, which we believe has some advantages over that in [P].

In [P], inclusions among primitive subgroups of S are described in terms of the notion of a 'blow-up'. In this paper, the maximal subgroups of S are described (whenever possible) as stabilizers of suitable structures on Ω. Inclusions among primitive subgroups are described in terms of the embedding of generalized Fitting subgroups, and relations among the structures preserved by the subgroups, particularly the product structures on Ω. In a sequel to this work, we introduce a partial order on product structures; together with inclusions of generalized Fitting subgroups, this

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partial order gives a qualitative description of large portions of the poset of primitive subgroups of S.

We now give some details of the work. See [A2] for the notation and terminology involving finite groups used in the paper.

We begin with a brief discussion of the relevant structures on Ω. Informally, an affine structure on Ω is an identification of Ω with the points of an affine space; see Definition 1.2 for a precise definition. The notion of a diagonal structure is less natural; see Definition 1.9 for a definition.

Informally, a regular (m, k)-product structure on Ω is an identification of Ω with the set product of k copies of a set of order m, but often it is better to view the product structure as a certain kind of chamber system on Ω. (As defined by Tits in [T], a *chamber system* on Ω is a collection of equivalence relations on Ω.) The referee has pointed out that the latter point of view is also adopted in [BPS1], where product structures are called 'Cartesian decompositions', and that Baddeley and co-workers [BP, BPS1, BPS2, BPS3] prove various results about Cartesian decompositions, and about quasiprimitive and innately transitive subgroups of S preserving Cartesian decompositions. (Quasiprimitive and innately transitive groups are classes of permutation groups more general than the class of primitive groups.) The referee also points out that it should be possible to give a shorter proof of our propositions below, using [P] and these references. However, we feel that a description (in terms of natural structures on Ω) of the poset of primitive subgroups of the symmetric groups is sufficiently fundamental to warrant a self-contained treatment that avoids an invocation of the notion of a blow-up and appeals to the broader literature on innately transitive groups.

Next we recall some facts about primitive subgroups H of S. These facts are stated in terms of the generalized Fitting subgroup $F^*(H)$ of H, so we recall that $F^*(H)$ is a certain characteristic subgroup of H which controls the structure of H. The notion of the generalized Fitting subgroup plays a crucial role in finite group theory; see [A2, Section 31] for a discussion of this topic. Set $D = F^*(H)$. The possibilities for the pair H, D and their action on Ω were determined in [AS], and are listed in Lemma 2.2. By now they are well known. In the language of Definition 2.3, there are five types of primitive groups H: affine, doubled, semisimple, diagonal, and complemented groups. When H is affine, D is the direct product of subgroups of order p for some prime p, and is regular on Ω. In the remaining cases, D is the direct product of isomorphic nonabelian simple groups, and the type of H is determined by the embedding of a point stabilizer D_ω, $\omega \in \Omega$, in D.

There are some interesting special cases: the almost simple groups, the strongly diagonal groups (see Definition 2.3), the product decomposable semisimple groups (see Definition 5.10), and the octal semisimple groups (see Definition 4.2). An almost simple group is product indecomposable if it preserves no nontrivial product structure. The general semisimple group H is product decomposable if the obvious product structure $\mathcal{F}(H)$ preserved by H, when 'composed' with the rank-two product structure preserved by a component of H (using the composition defined in Definition 1.11),

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produces a larger H-invariant product structure. An octal semisimple group H exists only when Ω has order a power of 8, and D preserves an affine structure.

There are four kinds of primitive groups maximal in G, as listed in Lemma 2.5.

(1) The affine maximals, which are the stabilizers of affine structures on Ω.
(2) The semisimple maximals, which are the stabilizers of regular product structures on Ω.
(3) The strongly diagonal maximals, which are the stabilizers of diagonal structures.
(4) The almost simple maximals.

See Section 1 for a discussion of the various structures.

The statements of our various propositions and theorems involve some specialized notation. We begin our discussion of that notation with a few words which hopefully will give the reader some flavor of the objects described by the notation. Then we direct the reader to the places in the paper where precise definitions appear. First, $\boldsymbol{\mathcal{F}}(H)$ is the set of H-invariant product structures on Ω, while $\mathcal{F}(H)$ is a certain natural member of $\boldsymbol{\mathcal{F}}(H)$. It turns out that $\mathcal{F}(H)$ is the greatest member of $\boldsymbol{\mathcal{F}}(H)$ in a certain partial order, unless H is semisimple and decomposable, where $\mathcal{F}^2(H)$ is the greatest member. One can parameterize $\boldsymbol{\mathcal{F}}(H)$ in terms of a certain collection of subgroups K of H via bijections $K \mapsto \mathcal{F}(H, K)$ or $K \mapsto \mathcal{F}^2(H, K)$, with H semisimple and product decomposable in the latter case. When H is diagonal, $\Sigma(H)$ is a certain partition of the components of H. Finally, when H is affine, $\mathcal{D}(H)$ is the set of nontrivial H-invariant direct sum decompositions of D.

A precise definition of the notation $\mathcal{F}(\mathcal{D})$ and $\Sigma(H)$ appears in Example 1.6 and Lemma 2.2, respectively. The notation $\mathcal{D}(H)$, $\mathcal{F}(H)$, $\boldsymbol{\mathcal{F}}(H)$, and $\mathcal{F}(H, K)$ is defined in Notation 2.6. Finally, the definition of $\mathcal{F}^2(H)$ and $\mathcal{F}^2(H, K)$ appears in Lemma 5.11(6).

We wish to investigate the set

$$\mathcal{O}_G(H)' = \{M \in \mathcal{O}_G(H) \mid F^*(G) \nleq M\}$$

of 'proper' overgroups of H in G and, in particular, the set $\mathcal{M}(H)$ of maximal members of $\mathcal{O}_G(H)'$. As discussed above, there are five types of primitive groups; however, for the purposes of our propositions, it is convenient to regard the almost simple primitive groups as forming a sixth type. Roughly speaking, if H is of type T_H and $M \in \mathcal{O}_G(H)'$ is of type T_M, then there are fairly strong restrictions on the pair (T_H, T_M), and often $F^*(M)$ is forced to be $F^*(H)$. Indeed, if M is not semisimple then the embedding of H in M is essentially completely controlled by the pair T_H, T_M, and the embedding of $F^*(H)$ in M, so the situation is quite satisfactory. On the other hand, when M is semisimple, things are more complicated, but at least $H \leq M$, so $\boldsymbol{\mathcal{F}}(M) \subseteq \boldsymbol{\mathcal{F}}(H)$, and the maximal semisimple overgroups of H are in one-to-one correspondence with $\boldsymbol{\mathcal{F}}(H)$, which we describe in terms of parameterizations that appear in the various propositions.

We organize this information into eleven propositions of two kinds. The first kind of result fixes the type of a primitive subgroup M, and describes the possible

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primitive subgroups of M, according to type and other restrictions. The second kind of result fixes the type of H, and describes the members of $\mathcal{O}_G(H)'$, again according to type and other restrictions. There are so many inclusions of primitive groups in semisimple groups, that it makes little sense to state a proposition in the case where M is semisimple. We continue to write D for $F^*(H)$.

PROPOSITION 1. *Assume that $M \in \mathcal{O}_G(H)'$ is almost simple. Then either*

(1) *H is almost simple; or*
(2) *$|\Omega|$ is a prime and H is affine.*

Proposition 1 is proved as part of the proof of Lemma 4.4.

PROPOSITION 2. *Assume that $M \in \mathcal{O}_G(H)'$ and that H is almost simple. Then one of the following statements holds:*

(1) *M is almost simple;*
(2) *$|\Omega| = 8$, $H \cong L_3(2)$ is octal, M is the stabilizer of one of the two H-invariant affine structures on Ω, and the two structures are conjugate under $N_S(H)$;*
(3) *H is product decomposable, M is semisimple, and M is contained in the stabilizer $N_G(\mathcal{F}^2(H))$ of $\mathcal{F}^2(H)$.*

Proposition 2 is a corollary to Proposition 5. The statement in part (2) of the proposition that H stabilizes two affine structures follows from Lemma 7.1(5).

PROPOSITION 3. *Assume that $M \in \mathcal{O}_G(H)'$ is affine and let $X = F^*(M)$. Then either*

(1) *H is affine and $F^*(M) = F^*(H)$; or*
(2) *$\Omega = 8^r$, H is octal semisimple with components $\{L_1, \ldots, L_r\}$, HX is affine with $\mathcal{D} = \{[X, L_i] \mid 1 \le i \le r\} \in \mathcal{D}(HX)$, and $\mathcal{F}(H) = \mathcal{F}(\mathcal{D})$.*

Proposition 3 follows from Lemma 4.3.

PROPOSITION 4. *Assume that $M \in \mathcal{O}_G(H)'$ and that H is affine. Then one of the following statements holds:*

(1) *M is affine and $F^*(M) = F^*(H)$;*
(2) *M is semisimple, H is imprimitive on D, and there exists $\mathcal{D} \in \mathcal{D}(H)$ such that $\mathcal{F}(\mathcal{D}) = \mathcal{F}(M)$;*
(3) *$|\Omega|$ is prime and M is almost simple.*

Proposition 4 follows from Lemma 4.1. If $|\Omega|$ is prime and (1) fails, then M is almost simple from Lemma 2.2.

The next three propositions are proved in Section 7.

PROPOSITION 5. *Assume that H is semisimple and pick a component L of H. Let $M \in \mathcal{O}_G(H)'$.*

(1) *If H is product indecomposable and not octal then M is semisimple, L is contained in a component of M, H is transitive on the components of M, and the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(L))' = \mathcal{O}_H(N_H(L)) - \{H\}$ with $\boldsymbol{\mathcal{F}}(H)$.*

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(2) *Suppose that H is octal with k components. Then M is affine or semisimple. Furthermore, the affine structures stabilized by H are conjugate under $N_S(H)$. If M is semisimple then L is contained in a component of M, H is transitive on the components of M, and the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(L))'$ with $\boldsymbol{\mathcal{F}}(H)$.*

(3) *Assume that H is product decomposable. Then M is semisimple and the map $K \mapsto \mathcal{F}^2(H, K)$ is a bijection of $\mathcal{O}_H(N_H(A_1))'$ with $\boldsymbol{\mathcal{F}}(H)$, where $A_1 \in \mathcal{O}_L(L_\omega)$ is described in Definition 5.10.*

PROPOSITION 6. *Assume that $M \in \mathcal{O}_G(H)'$ is diagonal and let $X = F^*(M)$ and k be the number of components of M. Then one of the following statements holds:*

(1) *H is diagonal, $F^*(M) = F^*(H)$, and $\Sigma(M) = \Sigma(H)$;*
(2) *H is doubled, $F^*(M) = F^*(H)$, and $k = 2|\Sigma(M)|$;*
(3) *H is complemented with $|\Sigma(M)|$ components, $F^*(M) = DC_G(D)$, and $k = 2|\Sigma(M)|$.*

PROPOSITION 7. *Assume that H is diagonal and pick $\sigma \in \Sigma(H)$. Let $M \in \mathcal{O}_G(H)'$. Then either*

(1) *M is diagonal, $F^*(M) = F^*(H)$, and $\Sigma(M) = \Sigma(H)$; or*
(2) *M is semisimple, each component of H is contained in a component of M, H is transitive on the components of M, and the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(\sigma))' = \mathcal{O}_H(N_H(\sigma)) - \{H\}$ with $\boldsymbol{\mathcal{F}}(H)$.*

The proofs of Propositions 8 and 9 appear in Section 8.

PROPOSITION 8. *Assume that $M \in \mathcal{O}_G(H)'$ is doubled. Then one of the following statements holds:*

(1) *H is doubled and $F^*(M) = F^*(H)$;*
(2) *H is complemented and $F^*(M) = DC_G(D)$.*

PROPOSITION 9. *Assume that H is doubled and pick a component L of H. Let $M \in \mathcal{O}_G(H)'$. Then one of the following statements holds:*

(1) *M is doubled and $F^*(M) = F^*(H)$;*
(2) *M is semisimple, L is contained in a component of M, H is transitive on the components of M, and the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(L))'$ with $\boldsymbol{\mathcal{F}}(H)$;*
(3) *M is diagonal, $F^*(M) = F^*(H)$, and M has $2|\Sigma(M)|$ components.*

PROPOSITION 10. *Assume that $M \in \mathcal{O}_G(H)'$ is complemented. Then H is complemented and $F^*(M) = F^*(H)$.*

Proposition 10 follows from Lemma 4.5.

PROPOSITION 11. *Assume that H is complemented and pick a component L of H. Let $M \in \mathcal{O}_G(H)'$. Then one of the following statements holds:*

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(1) *M is complemented and $F^*(M) = F^*(H)$;*
(2) *M is semisimple, L is contained in a component of M, H is transitive on the components of M, and the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(L))'$ with $\boldsymbol{\mathcal{F}}(H)$;*
(3) *M is diagonal, $F^*(M) = DC_G(D)$, H has $|\Sigma(M)|$ components, and M has $2|\Sigma(M)|$ components;*
(4) *M is doubled and $F^*(M) = DC_G(D)$.*

The proof of Proposition 11 appears in Section 8.

We are interested in the maximal overgroups of H in G. Unless $H \leq A$ and $G = S$, $\mathcal{M}(H)$ is the set of maximal overgroups of H in G. Moreover, the set $\mathcal{M}(H)$ can be retrieved from our propositions. For example, if H is affine then by Proposition 4, Lemma 2.5, and Remark 2.7, $\mathcal{M}(H) = \{N_G(D), N_G(\mathcal{F}(\mathcal{D})) \mid \mathcal{D} \in \mathcal{D}(H)\}$. Similarly, if H is semisimple and product indecomposable, but not octal or almost simple, then by Proposition 5, $\mathcal{M}(H) = \{N_G(\mathcal{F}) \mid \mathcal{F} \in \boldsymbol{\mathcal{F}}(H)\}$, and $\boldsymbol{\mathcal{F}}(H)$ is parameterized by $\mathcal{O}_H(N_H(L))'$, for L a component of H.

Overgroup lattices The question of whether each nonempty finite lattice is isomorphic to an interval in the lattice of subgroups of some finite group has been of interest for at least 25 years since the appearance of [PP]. To illustrate how our propositions can be used to investigate this question and the lattice of subgroups of S, we also include two results which give information about $\mathcal{O}_G(H)$ when H is the intersection of some (or many) pair(s) of maximal subgroups of G.

First, we (essentially) determine the pairs M_1, M_2 of distinct maximal subgroups of G such that $M_1 \cap M_2$ is primitive on Ω.

THEOREM 12. *Assume that M_1 and M_2 are distinct subgroups of G maximal subject to $F^*(G) \not\leq M_i$, such that $H = M_1 \cap M_2$ is primitive. Assume that $|\Omega|$ is not prime. Then one of the following statements holds:*

(1) *M_1, M_2, and H are almost simple;*
(2) *$M_i = N_G(\mathcal{F}_i)$ for some regular product structures $\mathcal{F}_1$ and $\mathcal{F}_2$;*
(3) *interchanging M_1 and M_2 if necessary, M_1 and H are affine with $F^*(H) = F^*(M_1)$, M_2 is semisimple, and there exists $\mathcal{D} = (D_1, \ldots, D_k) \in \mathcal{D}(H)$ such that $M_2 = N_G(\mathcal{F}(\mathcal{D}))$ and $H = N_{M_1}(\mathcal{D})$;*
(4) *H is octal semisimple and the wreath product of $L_3(2)$ by S_k, $|\Omega| = 8^k$, and M_1 and M_2 are the stabilizers of the two H-invariant affine structures on Ω.*

The proof of Theorem 12 appears in Section 9.

Next we impose a much stronger constraint on the lattice $\mathcal{O}_G(H)$.

THEOREM 13. *Assume that H is a proper primitive subgroup of G and let $\mathcal{M}$ denote the set of maximal overgroups of H in G. Assume that $|\Omega|$ is not prime and that,*

$$\textit{for each } M \in \mathcal{M}, \textit{ there exists } M' \in \mathcal{M} - \{M\} \textit{ such that } H = M \cap M'. \tag{0.1}$$

Then one of the following statements holds:

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(1) *$\mathcal{M} = \{N_G(\mathcal{F}) \mid \mathcal{F} \in \boldsymbol{\mathcal{F}}(H)\}$;*
(2) *all members of $\mathcal{O}_G(H)$ are almost simple, product indecomposable, and not octal;*
(3) *$|\Omega|$ is a prime power, H is affine, and $\mathcal{M} = \{N_G(D), N_G(\mathcal{F}(\mathcal{D})) \mid \mathcal{D} \in \mathcal{D}(H)\}$;*
(4) *$|\Omega| = 8$, $G = A$, $H \cong L_3(2)$ is octal, and $\mathcal{M}$ consists of the stabilizers of the two H-invariant affine structures on Ω;*
(5) *$|\Omega| = 8$, $G = S$, $H \cong L_3(2)$ is octal, $N_G(H) \cong PGL_2(7)$, $\mathcal{M} = \{F^*(S), N_G(H)\}$, and $\mathcal{O}_G(H) = \mathcal{M} \cup \{H, K_1, K_2, G\}$, where K_1 and K_2 are the stabilizers of the two H-invariant affine structures on Ω;*
(6) *$G = S$, $N_G(H)$ is the stabilizer in S of an affine structure, regular product structure, or diagonal structure on Ω, H is the stabilizer in A of that structure, $\mathcal{M} = \{A, N_S(H)\}$, and $\mathcal{O}_G(H) = \{H, A, N_S(H), S\}$.*

The proof of Theorem 13 appears in Section 9.

One can (with some difficulty) determine the groups H and the lattice $\mathcal{O}_G(H)$ appearing in case (2) of Theorem 13 from the lists of inclusions among almost simple primitive groups appearing in [LPS1]. The analysis of case (1) of Theorem 13 would seem to demand some machinery to analyze the relationship among regular product structures on Ω. Such machinery and a treatment of cases (1), (2), and (3) appear in a sequel to this work. In particular, Theorem 13 is extended there to give an explicit description of those primitive subgroups H of S such that the lattice $\mathcal{O}_G(H)$ satisfies condition (0.1) of Theorem 13.

A finite lattice Λ has a greatest element ∞ and least element 0, and we can view $\Lambda' = \Lambda - \{0, \infty\}$ as a graph with adjacency relation the comparability relation. Define Λ to be *disconnected* if the graph Λ' is disconnected. We observe that if H is a primitive subgroup of G such that the lattice $\mathcal{O}_G(H)$ is disconnected, then $\mathcal{O}_G(H)$ satisfies hypothesis (0.1) of Theorem 13. In particular, Theorem 13 is part of a program of J. Shareshian and the author (see [A3] and [A4]) to study interval lattices $\mathcal{O}_U(V)$ in finite groups V over $U \leq V$, and indeed to prove the following statement:

(D) There exists a class of disconnected finite lattices, such that no member of the class is an interval in the lattice of subgroups of any finite group.

In the Aschbacher–Shareshian program, the class of lattices considered consists of those disconnected lattices Λ with the property that, for each connected component C of Λ, C is the poset of nonempty proper subsets of a set of order $m(C)$, for some $m(C) > 2$.

Define a lattice Λ to be an *M-lattice* if Λ' has no edges. There is also a program in the literature to show that (D) is satisfied by some subclass of the class of M-lattices. Perhaps the major result in that program is due to Baddeley and Lucchini [BL], and reduces the problem for M-lattices to various problems involving almost simple groups. One of those problems is to show that there exists an M-lattice which is not of the form $\mathcal{O}_U(V)$ for any almost simple group U and subgroup V. The extension of Theorem 13 mentioned above establishes this fact when U is alternating or symmetric, and V is primitive.

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1. Some structures on a set

In this section we assume that Ω is a finite set and let $S = \mathrm{Sym}(\Omega)$ be the symmetric group on Ω. We recall some structures on Ω which will be important in this paper.

DEFINITION 1.1. Let a be a positive integer and write Ω^a for the set product of a copies of Ω. Given an a-ary relation $R \subseteq \Omega^a$ on Ω, define the *stabilizer* in S of R to be the subgroup $N_S(R)$ of S consisting of those $g \in S$ such that $Rg = R$. □

DEFINITION 1.2. Assume that $|\Omega| = p^e$ is a power of a prime p. An *affine structure* on Ω is a quaternary relation $R = R(\Omega, +)$ on Ω of the form

$$R = \{(a, b, c, b + c - a) \mid a, b, c \in \Omega\},$$

defined by some e-dimensional vector space structure $(\Omega, +)$ on Ω over $\mathbf{F}_p$. □

The following results are well known.

LEMMA 1.3. *Let $V = (\Omega, +)$ be an e-dimensional vector space structure on Ω over $\mathbf{F}_p$, $R = R(V)$ the corresponding affine structure on Ω, $M = N_S(R)$ the stabilizer of R, and $0 \in \Omega$ the zero of V. Then:*

(1) *$F^*(M) = D \cong E_{p^e}$ is the group of translations $\tau_a : b \mapsto a + b$, $a \in \Omega$, on V, and D is regular on Ω;*

(2) *$M_0 \cong GL(V)$ and the map $a \mapsto \tau_a$ is an equivalence of the representation of M_0 on Ω with the representation of M_0 on D via conjugation.*

LEMMA 1.4. *Assume that p is a prime and that $E_{p^e} \cong D$ is a subgroup of S regular on Ω. Pick $\omega \in \Omega$ and define*

$$R = R(D) = \{(\omega a, \omega b, \omega c, \omega bca^{-1}) \mid a, b, c \in D\}.$$

Then R is an affine structure on Ω (independent of ω) with translation group D, and $N_S(R) = N_S(D)$.

DEFINITION 1.5. Let m, k be integers with $m \geq 5$ and $k > 1$. We recall the notion of a *regular* (m, k)*-product structure* on Ω. Informally, such a structure is a bijection $f : \Omega \to \Gamma^I$, where $I = \{1, \ldots, k\}$ and Γ is an m-set. The function f may be thought of as a family of functions $(f_i : \Omega \to \Gamma : i \in I)$ via $f(\omega) = (f_1(\omega), \ldots, f_k(\omega))$ for $\omega \in \Omega$.

Formally, a product structure is a family $\mathcal{F} = (\Omega_i : i \in I)$ of partitions Ω_i of Ω into m blocks of size m^{k-1}, such that $\mathcal{F}$ is *injective*: for each pair of distinct points ω, ω' of Ω, $\mathcal{F}(\omega) \neq \mathcal{F}(\omega')$, where $\mathcal{F}(\omega)$ is the family $(B_i : i \in I)$ of blocks defined by $\omega \in B_i \in \Omega_i$.

The family $\mathcal{F} = \mathcal{F}(f)$ defined by f has ith partition $\Omega_i = \{f_i^{-1}(\gamma) \mid \gamma \in \Gamma\}$, the fibers of f_i. An *indexing* of $\mathcal{F}$ is an indexing $\Omega_i = \{\Omega_{i,\gamma} \mid \gamma \in \Gamma\}$ of the blocks of the various partitions Ω_i. The function f defines the indexing $\Omega_{i,\gamma} = f_i^{-1}(\gamma)$, while an indexing of $\mathcal{F}$ defines a function f via $\omega \in \Omega_{i, f_i(\omega)}$. As $\mathcal{F}$ is injective, the function

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f defined by the indexing is injective, so as $|\Omega| = |\Gamma^I|$, $f : \Omega \to \Gamma^I$ is a bijection. In short, the formal definition is a 'coordinate-free' definition of product structure.

The *stabilizer* $N_S(\mathcal{F})$ in S of $\mathcal{F}$ is the subgroup consisting of those $g \in S$ such that $\mathcal{F}g = \mathcal{F}$. Write $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\Omega)$ for the set of all regular product structures on Ω. □

The referee has pointed out that the description above of a product structure as a chamber system is also used in [BP], where product structures are called *Cartesian decompositions*, and various properties of Cartesian decompositions are established.

EXAMPLE 1.6. Assume that $|\Omega| = m^k$ and let $\omega \in \Omega$ and $I = \{1, \ldots, k\}$. Assume that D is a transitive subgroup of S which is the direct product of a set $\mathcal{D} = \{D_i \mid i \in I\}$ of subgroups such that

$$D_\omega = \prod_{i \in I} D_{i,\omega},$$

with $|D_i : D_{i,\omega}| = m$ for each $i \in I$. Define $\mathcal{F} = \mathcal{F}(\mathcal{D}) = (\Omega_i \mid i \in I)$ to be the product structure on Ω such that Ω_i is the set of orbits of $D_{i'}$ on Ω, where $D_{i'} = \langle D_j \mid j \in I - \{i\}\rangle$. It is well known that $\mathcal{F}$ is a product structure, and that if G is a subgroup of S permuting $\mathcal{D}$ via conjugation then $G \leq N_S(\mathcal{F})$. □

LEMMA 1.7. *Assume the setup of Example 1.6, and assume for each $i \in I$ that $E_i \leq D_i$ with $D_i = D_{i,\omega} E_i$. Set $\mathcal{E} = \{E_i \mid i \in I\}$. Then $\mathcal{F}(\mathcal{D}) = \mathcal{F}(\mathcal{E})$.*

PROOF. As $D_i = D_{i,\omega} E_i$ for each $i \in I$, $\omega D_{i'} = \omega E_{i'}$ and Ω_i is the set of orbits of $E_{i'}$ on Ω. □

LEMMA 1.8. *Let $\mathcal{F} = (\Omega_i : i \in I)$ be a regular (m, k)-product structure on Ω, pick $\omega \in \Omega$, and set $M = N_S(\mathcal{F})$.*

(1) *$F^*(M) = D = D_1 \times \cdots \times D_k$, where, for $i \in I$, D_i acts faithfully on Ω_i as the alternating group, and $D_{i'}$ acts transitively on each block in Ω_i.*
(2) *$\mathcal{F} = \mathcal{F}(\mathcal{D})$, where $\mathcal{D} = \{D_i \mid i \in I\}$.*
(3) *M is isomorphic to the wreath product of S_m by S_k. More precisely, $D \leq \bar{D} \trianglelefteq M$ with $\bar{D} = \bar{D}_1 \times \cdots \times \bar{D}_k$, $F^*(\bar{D}_i) = D_i$, $\bar{D}_i$ acts trivially on Ω_j for $j \neq i$, and faithfully on Ω_i as the symmetric group, and there is a complement T to $\bar{D}$ in M contained in M_ω which acts faithfully as the symmetric group on $\bar{\mathcal{D}} = \{\bar{D}_i \mid i \in I\}$ via conjugation, with $N_T(\bar{D}_i)$ centralizing $\bar{D}_i$.*

PROOF. This is part of the folklore and the proof is straightforward. □

DEFINITION 1.9. Assume that $|\Omega| = c^{k-1}$ for some integer $k > 1$ and some nonabelian finite simple group L of order c, and let $\omega \in \Omega$. Assume that $D = D_1 \times \cdots \times D_k$ is a transitive subgroup of S which is the direct product of a set $\mathcal{D} = \{D_i \mid i \in I\}$ of subgroups isomorphic to L, and such that D_ω is a full diagonal subgroup F of D with respect to the direct product decomposition. (see [AS, Section 1] for a discussion of full diagonal subgroups.) Then $\mathcal{D}$ and F define a

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diagonal structure $\mathbf{d} = \operatorname{diag}(\mathcal{D}, F)$ on Ω, whose stabilizer $N_S(\mathbf{d})$ in S we decree to be the subgroup $D(N_S(D) \cap N_S(F))$. □

The following lemma is part of the folklore and easy to prove.

LEMMA 1.10. *Let* $\mathbf{d} = \operatorname{diag}(\mathcal{D}, F)$ *be a diagonal structure on* Ω, $\omega \in \Omega$, *and set* $M = N_S(\mathbf{d})$.

(1) $F^*(M) = D$ *and* $M = DM_\omega$.
(2) $M_\omega = KT$ *where* $T = C_M(F)$ *acts faithfully as the symmetric group on* $\mathcal{D}$ *via conjugation,* K *is the kernel of the action of* M_ω *on* $\mathcal{D}$, $F = F^*(K)$, *and* $K \cong \operatorname{Aut}(F)$.

We next define a notion of composition of regular product structures.

DEFINITION 1.11. Let $m, k, \hat{m}, \hat{k}$ be integers with $m, \hat{m} \geq 5$ and $k, \hat{k} > 1$. Let $I = \{1, \ldots, k\}$, $\hat{I} = \{1, \ldots, \hat{k}\}$, and let Γ be an m-set and $\hat{\Gamma}$ an $\hat{m}$-set. Let $\mathcal{F} = (\Omega_i : i \in I)$ be a regular (m, k)-product structure on Ω and $\hat{\mathcal{F}} = (\Gamma_j : j \in \hat{I})$ be a regular $(\hat{m}, \hat{k})$-product structure on Γ. Recall from Definition 1.5 that we can choose bijections $f : \Omega \to \Gamma^I$ and $\hat{f} : \Gamma \to \hat{\Gamma}^{\hat{I}}$ so that $\mathcal{F} = \mathcal{F}(f)$ and $\hat{\mathcal{F}} = \mathcal{F}(\hat{f})$. That is, $f(\omega) = (f_1(\omega), \ldots, f_k(\omega))$ and $\Omega_i = \{f_i^{-1}(\gamma) \mid \gamma \in \Gamma\}$, and similarly for $\hat{f}$. Define $\tilde{m} = \hat{m}$, $\tilde{\Gamma} = \hat{\Gamma}$, and $\tilde{I} = I \times \hat{I}$. Thus $\tilde{k} = |\tilde{I}| = k\hat{k}$. Define

$$\tilde{f} : \Omega \to \tilde{\Gamma}^{\tilde{I}}$$

by $\tilde{f}(\omega) = (\hat{f}(f_1(\omega), \ldots, \hat{f}(f_k(\omega))))$, for $\omega \in \Omega$. That is, $\tilde{f} = (\tilde{f}_{i,j} : (i, j) \in \tilde{I})$ has coordinate functions $\tilde{f}_{i,j} = \hat{f}_j \circ f_i$ for $(i, j) \in \tilde{I}$.

Visibly $\tilde{f}$ is an informal regular $(\tilde{m}, \tilde{k})$-product structure on Ω, as defined in Definition 1.5, giving rise to the formal product structure $\tilde{\mathcal{F}} = \mathcal{F}(\tilde{f}) = \{\Omega_{i,j} \mid (i, j) \in \tilde{I}\}$, where $\Omega_{i,j} = \{f_{i,j}^{-1}(\alpha) \mid \alpha \in \tilde{\Gamma}\}$. We call $\tilde{\mathcal{F}}$ a *composition* of $\hat{\mathcal{F}}$ and $\mathcal{F}$, and sometimes write $\hat{\mathcal{F}} \circ \mathcal{F}$ for such a composition.

Alternatively, as in Definition 1.5, pick indexings $\Omega_i = \{\Omega_{i,\gamma} \mid \gamma \in \Gamma\}$ and $\Gamma_j = \{\Gamma_{j,\alpha} \mid \alpha \in \hat{\Gamma}\}$, and for $(i, j) \in \tilde{I}$ define

$$\Omega_{i,j} = \{\Omega_{i,j,\alpha} \mid \alpha \in \tilde{\Gamma}\} \quad \text{where } \Omega_{i,j,\alpha} = \bigcup_{\gamma \in \Gamma_{j,\alpha}} \Omega_{i,\gamma}.$$

Then $\tilde{\mathcal{F}} = (\Omega_{i,j} \mid (i, j) \in \tilde{I})$ is a regular $(\tilde{m}, \tilde{k})$-product structure on Ω and a composition of $\hat{\mathcal{F}}$ with $\mathcal{F}$. □

LEMMA 1.12. *Let* $m, k, \hat{m}, \hat{k}$ *be integers with* $m, \hat{m} \geq 5$ *and* $k, \hat{k} > 1$. *Let* $I = \{1, \ldots, k\}$, $\hat{I} = \{1, \ldots, \hat{k}\}$, *and* $\mathcal{F} = (\Omega_i \mid i \in I)$ *be a regular* (m, k)*-product*

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structure on Ω. *Assume for each* $i \in I$ *that* $\hat{\mathcal{F}}_i = (\Delta_{i,j} \mid j \in \hat{I})$ *is a regular* $(\hat{m}, \hat{k})$*-product structure on* Ω_i. *Define* $\tilde{I} = I \times \hat{I}$, *and* $\tilde{\mathcal{F}} = (\Omega_{i,j} \mid (i, j) \in \tilde{I})$, *where*

$$\Omega_{i,j} = \{\tilde{B} \mid B \in \Delta_{i,j}\} \quad \textit{and} \quad \tilde{B} = \bigcup_{A \in B} A. \tag{1.1}$$

(1) *Then* $\tilde{\mathcal{F}}$ *is a composition* $\hat{\mathcal{F}}_1 \circ \mathcal{F}$ *of the regular product structures* $\hat{\mathcal{F}}_1$ *and* $\mathcal{F}$.
(2) *Assume that* $H \leq N_S(\mathcal{F})$ *permutes* $\{\hat{\mathcal{F}}_i \mid i \in I\}$. *Then* $H \leq N_S(\tilde{\mathcal{F}})$.

PROOF. Let $\hat{\mathcal{F}} = \hat{\mathcal{F}}_1$, $\Gamma = \Omega_1$, and $\Gamma_j = \Delta_{1,j}$ for $j \in \hat{I}$. As all regular $(\hat{m}, \hat{k})$-product structures are isomorphic, for each $i \in I$ there exists an isomorphism $\varphi_i : \hat{\mathcal{F}} \to \hat{\mathcal{F}}_i$ with $\Gamma_j \varphi_i = \Delta_{i,j}$ for $j \in \hat{I}$. Take $\varphi_1 = 1$.

For $\gamma \in \Gamma$ and $i \in I$, set $\Omega_{1,\gamma} = \gamma$ and $\Omega_{i,\gamma} = \gamma\varphi_i$. Thus $\Omega_i = \{\Omega_{i,\gamma} \mid \gamma \in \Gamma\}$ is an indexing of $\mathcal{F}$. Let $\hat{\Gamma}$ be an $\hat{m}$-set and $\Gamma_j = \{\Gamma_{j,\alpha} \mid \alpha \in \hat{\Gamma}\}$, $j \in \hat{I}$, be an indexing of $\hat{\mathcal{F}}$. For $(i, j) \in \tilde{I}$ and $\alpha \in \hat{\Gamma}$, set $\Delta_{i,j,\alpha} = \Gamma_{j,\alpha}\varphi_i$. Then

$$\Delta_{i,j} = \Gamma_j \varphi_i = \{\Gamma_{j,\alpha}\varphi_i \mid \alpha \in \hat{\Gamma}\} = \{\Delta_{i,j,\alpha} \mid \alpha \in \hat{\Gamma}\}, \tag{1.2}$$

and we can form

$$\tilde{\Delta}_{i,j,\alpha} = \bigcup_{A \in \Delta_{i,j,\alpha}} A$$

as in (1.1). Set $\Omega_{i,j,\alpha} = \tilde{\Delta}_{i,j,\alpha}$. For $A \in \Delta_{i,j,\alpha}$, $A = \gamma\varphi_i$ for some $\gamma \in \Gamma_{j,\alpha}$, so $A = \gamma\varphi_i = \Omega_{i,\gamma}$. Thus

$$\Omega_{i,j,\alpha} = \tilde{\Delta}_{i,j,\alpha} = \bigcup_{A \in \Delta_{i,j,\alpha}} A = \bigcup_{\gamma \in \Gamma_{j,\alpha}} \Omega_{i,\gamma}, \tag{1.3}$$

and by (1.1), (1.2), and (1.3),

$$\Omega_{i,j} = \{\tilde{\Delta}_{i,j,\alpha} \mid \alpha \in \hat{\Gamma}\} = \{\Omega_{i,j,\alpha} \mid \alpha \in \hat{\Gamma}\}. \tag{1.4}$$

Hence (1) follows from (1.3), (1.4), and Definition 1.11.

Assume the hypothesis of (2). As $H \leq N_S(\mathcal{F})$, for $h \in H$ and $i \in I$, we can define $ih \in I$ by $\Omega_i h = \Omega_{ih}$, to obtain a representation of H on I. As H permutes the $\hat{\mathcal{F}}_i$, for $(i, j) \in \tilde{I}$, $\Delta_{i,j} h = \Delta_{ih,j'}$ for some $j' \in \hat{I}$. Thus for $B \in \Delta_{i,j}$, $Bh \in \Delta_{ih,j'}$, so

$$\tilde{B}h = \left(\bigcup_{A \in B} A\right)h = \bigcup_{A \in B} Ah = \tilde{C},$$

where $C = Bh \in \Delta_{ih,j'}$. Thus

$$\Omega_{i,j} h = \{\tilde{B}h \mid B \in \Delta_{i,j}\} = \{\tilde{C} \mid C \in \Delta_{ih,j'}\} = \Omega_{ih,j'},$$

so indeed $H \leq N_S(\tilde{F})$. $\square$

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LEMMA 1.13. *Assume $I, m, k, D, \mathcal{D}$ satisfy the hypothesis of Example 1.6 and set $\mathcal{F} = \mathcal{F}(\mathcal{D})$. Let $\hat{m}, \hat{k}$ be integers with $\hat{m} \geq 5$ and $\hat{k} > 1$, and set $\hat{I} = \{1, \ldots, \hat{k}\}$. Assume for each $i \in I$ that $\hat{\mathcal{F}}_i = (\Sigma_{i,j} : j \in \hat{I})$ is a regular $(\hat{m}, \hat{k})$-product structure on $\Sigma_i = \omega D_i$. Define $\tilde{I} = I \times \hat{I}$, and $\tilde{\mathcal{F}} = (\Omega_{i,j} : (i, j) \in \tilde{I})$, where*

$$\Omega_{i,j} = \{\tilde{U} \mid U \in \Sigma_{i,j}\} \quad \textit{and} \quad \tilde{U} = \{ug \mid u \in U,\ g \in D_{i'}\}.$$

(1) *Then $\tilde{\mathcal{F}}$ is a composition $\hat{\mathcal{F}}_1 \circ \mathcal{F}$ of the regular product structures $\hat{\mathcal{F}}_1$ and $\mathcal{F}$.*
(2) *Assume that $D \trianglelefteq H \leq S$, D_1 acts on $\hat{\mathcal{F}}_1$, and H_ω permutes the $\hat{\mathcal{F}}_i$. Then $H \leq N_S(\tilde{\mathcal{F}})$.*

PROOF. For $i \in I$ and $\sigma \in \Sigma_i$, set $\bar{\sigma} = \sigma D_{i'} \subseteq \Omega$. Then for $U \subseteq \Sigma_i$, set $\bar{U} = \{\bar{u} \mid u \in U\}$, and for $(i, j) \in \tilde{I}$, set $\Delta_{i,j} = \{\bar{U} \mid U \in \Sigma_{i,j}\}$. Now $\sigma = \omega x$ for some $x \in D_i$, so $\bar{\sigma} = \omega x D_{i'} = \omega D_{i'} x$, and hence the map $\psi_i : \Sigma_i \to \Omega_i$ defined by $\psi_i : \sigma \mapsto \bar{\sigma}$ is a bijection. In particular as $\hat{\mathcal{F}}_i$ is a regular $(\hat{m}, \hat{k})$-product structure on Σ_i, $\mathcal{E}_i = (\Delta_{i,j} : j \in \hat{I})$ is a regular $(\hat{m}, \hat{k})$-product structure on Ω_i. Furthermore, for $U \in \Sigma_{i,j}$,

$$\tilde{U} = \{ug \mid u \in U,\ g \in D_{i'}\} = \bigcup_{u \in U} \bar{u} = \bigcup_{A \in \bar{U}} A,$$

so $\tilde{U}$ is the set $\tilde{B}$ of (1.1) in Lemma 1.12, for $B = \bar{U} \in \Delta_{i,j}$. Therefore $\Omega_{i,j} = \{\tilde{U} \mid U \in \Sigma_{i,j}\}$ is the set $\{\tilde{B} \mid B \in \Delta_{i,j}\}$ of (1.1), so (1) follows from Lemma 1.12(1).

Now assume the hypothesis of (2). As D is transitive on Ω, $H = DH_\omega$. As D_i acts on $\hat{\mathcal{F}}_i$ and ψ_i is D_i-equivariant, D_i acts on $\mathcal{E}_i$. Furthermore, $D_{i'}$ is trivial on Ω_i, so D acts on $\mathcal{E}_i$.

Next, as in the proof of Lemma 1.12, there is a representation of H on I such that $D_i^h = D_{ih}$. For $h \in H_\omega$ and $u \in \Sigma_i$, $u = \omega x$ for some $x \in D_i$, so $uh = \omega xh = \omega h x^h = \omega x^h \in \Sigma_{ih}$. By hypothesis, H_ω permutes the $\hat{\mathcal{F}}_i$, so $\Sigma_{i,j} h = \Sigma_{ih,j'}$ for some $j' \in \hat{I}$. Then for $U \in \Sigma_{i,j}$,

$$\bar{U}h = \{uD_{i'}h \mid u \in U\} = \{uhD_{(ih)'} \mid u \in U\} = \{vD_{(ih)'} \mid v \in Uh\} = \overline{Uh},$$

so $\Delta_{i,j} h = \Delta_{ih,j'}$. That is, H_ω permutes the $\mathcal{E}_i$. Then as $H = DH_\omega$, H permutes the $\mathcal{E}_i$, so (2) follows from Lemma 1.12(2). □

2. Primitive groups

In this section we make the following assumption.

HYPOTHESIS 2.1. Ω is a finite set, $S = \mathrm{Sym}(\Omega)$ is the symmetric group on Ω, $\omega \in \Omega$, G is S or the alternating group A on Ω, H is a subgroup of G primitive on Ω, and $D = F^*(H)$.

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LEMMA 2.2. *$H = DH_\omega$ and one of the following statements holds.*

(1) *$|\Omega| = p^e$ is a power of the prime p, $D \cong E_{p^e}$ is regular on Ω, and H_ω is a complement to D in H which is irreducible on D regarded as an $\mathbf{F}_p H_\omega$-module.*

(2) *$D = D_1 \times D_2$, D_1 and D_2 are isomorphic normal subgroups of H, D_i is the direct product of k isomorphic nonabelian simple components L of H, permutated transitively by H, D_i is regular on Ω, D_ω is a full diagonal subgroup of D, and $H_\omega = N_H(D_\omega)$. Furthermore, $|\Omega| = |L|^k$.*

(3) *D is the direct product of the set $\mathcal{L}$ of components of H, H is transitive on $\mathcal{L}$, and choosing $L \in \mathcal{L}$, one of the following statements holds:*

   (i) *D_ω is the direct product of the groups F_ω, $F \in \mathcal{L}$, $L_\omega \neq 1$, and $\mathrm{Aut}_{H_\omega}(L)$ is maximal in $\mathrm{Aut}_H(L)$. Furthermore, $|\Omega| = |L : L_\omega|^{|\mathcal{L}|}$.*

   (ii) *There exists a maximal H-invariant partition $\Sigma = \Sigma(H)$ of $\mathcal{L}$ such that D_ω is the direct product of full diagonal subgroups F_σ of $D_\sigma = \langle\sigma\rangle$, $\sigma \in \Sigma$. Furthermore, $|\Omega| = |L|^{|\mathcal{L}|-|\Sigma|}$.*

   (iii) *$\mathrm{Inn}(L) \leq \mathrm{Aut}_{H_\omega}(L)$ and H_ω is a complement to D in H. Furthermore, $|\Omega| = |L|^{|\mathcal{L}|}$.*

PROOF. This is a consequence of [AS, Theorem 1]. □

DEFINITION 2.3. In case (1) of Lemma 2.2, we say that H is *affine*. In case (2), we say that H is *doubled*. In case (3i) we say that H is *semisimple*, in case (3ii) we say that H is *diagonal*, and in case (3iii) we say that H is *complemented*. We say that H is *strongly diagonal* if H is diagonal and $\Sigma(H) = \{\mathcal{L}\}$. □

In Praeger's theory [P] of overgroups in S of primitive subgroups of S, a somewhat different partition of primitive groups appears. In Praeger's terminology, affine groups are of type HA, doubled groups are of type HC, semisimple but not almost simple groups are of type PA, diagonal groups are of type CD, complemented groups are of type TW, and strongly diagonal groups are of type SD.

LEMMA 2.4. *Let $M \in \mathcal{O}_G(H)$.*

(1) *M is primitive on Ω.*

(2) *If $X \trianglelefteq M$ and $1 \neq D \cap X$ then either $D \leq X$ or H is doubled and $D_i = X \cap D$ for $i = 1$ or 2.*

PROOF. Part (1) is trivial. Part (2) follows from the fact that either D is a minimal normal subgroup of H, or H is doubled and D_1 and D_2 are the minimal normal subgroups of H. □

LEMMA 2.5. *Let M be a maximal subgroup of G primitive on Ω and $X = F^*(M)$. Then one of the following statements holds:*

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(1) *M is affine. That is, $|\Omega| = p^e$ is a power of the prime p, M is the stabilizer of an affine structure on Ω, $X \cong E_{p^e}$ is regular on Ω, and $N_S(X)_\omega$ is a complement to X in $N_S(X)$ which acts faithfully as $GL(D)$ on D regarded as an e-dimensional F_p-space.*

(2) *M is the stabilizer of a regular (m, k)-product structure on Ω, for some $m \geq 5$ and $k \geq 2$, M is semisimple, X is the direct product of k copies of A_m, $|\Omega| = m^k$, and $N_S(X)$ is the wreath product of S_m by S_k.*

(3) *M is the stabilizer of a diagonal structure on Ω, X is the direct product of k copies of some nonabelian simple group Y, M is diagonal, $|\Omega| = |Y|^{k-1}$, and $N_S(X)/X \cong S_k \times \mathrm{Out}(Y)$.*

(4) *M is almost simple and $N_S(X)$ is the stabilizer in $\mathrm{Aut}(X)$ of the equivalence class of the representation of X on Ω.*

PROOF. This follows from the O'Nan–Scott theorem, which is in turn a fairly easy consequence of Lemma 2.2; see the Appendix to [AS]. The stabilizers of the structures in (1)–(3) are almost always maximal. See Remark 2.7 for more discussion. □

NOTATION 2.6. Write $\boldsymbol{\mathcal{F}}(H)$ for the set of H-invariant regular product structures on Ω.

If H is affine, write $\mathcal{D}(H)$ for the set of systems $\mathcal{D} = \{D_1, \ldots, D_k\}$ of imprimitivity for H on D. That is, $k > 1$, $D = D_1 \times \cdots \times D_k$, and H permutes $\mathcal{D}$ transitively via conjugation. Observe that for $\mathcal{D} \in \mathcal{D}(H)$, $\mathcal{F}(\mathcal{D})$ is an H-invariant regular (d, k)-product structure on Ω by Example 1.6, where $d = |D_i|$.

In the remaining cases of Lemma 2.2, a minimal normal subgroup E of H is the direct product of its set $\mathcal{L}$ of components permutated transitively by H via conjugation. Let $\mathcal{P}(H)$ be the set of H-invariant partitions Δ of $\mathcal{L}$ such that $|\Delta| > 1$, and such that $\Sigma(H)$ is a refinement of Δ if H is diagonal. Let $\Delta \in \mathcal{P}(H)$ and, for $\delta \in \Delta$, set $E_\delta = \langle\delta\rangle$, $k = |\Delta|$, and $m = |E_\delta : E_{\delta,\omega}|$. Define $\mathcal{D}(\Delta) = \{E_\delta \mid \delta \in \Delta\}$ and set $\mathcal{F}(H, \Delta) = \mathcal{F}(\mathcal{D}(\Delta))$ the H-invariant (m, k)-product structure on Ω supplied by Example 1.6.

Observe that we have a bijection $K \mapsto \gamma_K^H$ of $\mathcal{O}_H(N_H(U))' = \mathcal{O}_H(N_H(U)) - \{H\}$ with $\mathcal{P}(H)$, where U is some fixed member of $\mathcal{L}$ and $\gamma_K = U^K$ if H is doubled, semisimple, or complemented, while $U = E_\sigma$ for some fixed $\sigma \in \Sigma(H)$ and $\gamma_K = L^K$ for $L \in \sigma$ if H is diagonal. We also write $\mathcal{F}(H, K)$ for $\mathcal{F}(H, \gamma_K^H)$.

Write $\mathcal{F}(H)$ for $\mathcal{F}(H, N_H(U))$ if H is doubled, semisimple, or complemented, and $|\mathcal{L}| > 1$. Set $\mathcal{F}(H) = \mathcal{F}(H, N_H(\sigma))$ if H is diagonal but not strongly diagonal. □

REMARK 2.7. Liebeck *et al.* [LPS1] determine when the subgroups listed in Lemma 2.5 are maximal in G. In particular, they supply an explicit list of those primitive almost simple groups which are *not* maximal.

One can also use our propositions to see that the stabilizers H listed in the first three parts of Lemma 2.5 are almost always maximal. (It should be pointed out, however, that such a treatment of the maximality of the stabilizers still depends in part on [LPS1], since the proofs of some of these propositions sometimes make

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appeals to [LPS1].) Observe that if $G = S$ and $H < L = F^*(S)$, then H is not maximal. As it is not difficult to determine when $H \leq L$, we ignore this subtlety in the following discussion. We also assume that $|\Omega|$ is not prime.

If H is almost simple, product indecomposable, and not octal, then by Lemma 8.5, $\mathcal{M}(H)$ consists of almost simple groups which are product indecomposable and not octal. On the other hand, by Proposition 1, if H is not almost simple then no member of $\mathcal{M}(H)$ is almost simple, so, in particular, maximal semisimple overgroups of H are stabilizers of product structures.

If H is affine then from Lemma 1.3, H is imprimitive on D, so $\mathcal{D}(H) = \varnothing$. Therefore H is maximal by Proposition 4.

If H is the stabilizer of a regular product structure, then the components D_i of H are alternating groups and, in particular, are product indecomposable and not octal. Furthermore, H is primitive on its components, so $\mathcal{O}_H(N_H(D_i)) = \varnothing$. Therefore, H is maximal by Proposition 5.

Finally, if H is the stabilizer of a diagonal structure, then $\Sigma(H) = \{\mathcal{L}\}$, where $\mathcal{L}$ is the set of components of H. Therefore H is maximal by Proposition 7.

3. Preliminary lemmas

LEMMA 3.1. *Assume that p is a prime, G is a nonabelian finite simple group, and $H < G$ with $|G : H| = p^a$. Then one of the following statements holds:*

(a) $G \cong A_{p^a}$ *and* $H \cong A_{p^a-1}$*;*
(b) $G \cong L_n(q)$ *for some prime power q and some prime n, H is the stabilizer of a point or a hyperplane of the projective geometry for G, and* $p^a = (q^n - 1)/(q - 1)$*;*
(c) $G \cong L_2(11)$*,* $H \cong A_5$*, and* $p^a = 11$*;*
(d) (G, H, p^a) *is* $(M_{23}, M_{22}, 23)$ *or* $(M_{11}, M_{10}, 11)$*;*
(e) $G \cong PSp_4(3) \cong U_4(2)$*, H is a maximal 2-parabolic of G which is an extension of E_{16} by A_5, and* $p^a = 3^3$.

PROOF. This is [Gu, Theorem 1]. □

LEMMA 3.2. *Assume that p is a prime, G is a nonabelian finite simple group, and $H < G$ with $|G : H| = p^a$. Let $\mathcal{Q}$ be the set of abelian complements to H in G.*

(1) $\mathcal{Q} = \varnothing$ *if and only if either*
   (i) *q is a Mersenne prime, $p = 2$, and* $G \cong L_2(q)$*; or*
   (ii) *case (e) of Lemma 3.1 holds.*
(2) *If some member of $\mathcal{Q}$ is noncyclic then case (a) of Lemma 3.1 holds.*

PROOF. Represent G by right multiplication on $\Omega = G/H$. In case (a) of Lemma 3.1, there is a subgroup $Q \cong E_{p^a}$ regular on Ω, so $Q \in \mathcal{Q}$ and the lemma holds in this case.

Suppose that case (b) of Lemma 3.1 holds. As n is prime, Zsigmondy's theorem (see [Gu, 3.2]) says that either conclusion (1i) holds, or there is a prime divisor r of $q^n - 1$

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which does not divide $q^k - 1$ for any $k < n$. In the first case the Sylow 2-subgroups of G are the complements to H in G, and are nonabelian dihedral groups, so we may assume the second case holds. Then as $p^a = (q^n - 1)/(q - 1)$, it follows that $p = r$. Let V be the natural module for $\hat{G} = GL_n(q)$ and P a subgroup of $\hat{G}$ of order p. As p does not divide $q^k - 1$ for $k < n$, P is irreducible on V and $C_{\hat{G}}(P)$ is the multiplicative group of $\mathbf{F}_{q^n}$ acting by right multiplication on V regarded as the additive group of $\mathbf{F}_{q^n}$. In particular, $C_{\hat{G}}(P)$ is cyclic, so G has a cyclic Sylow p-subgroup Q whose preimage $\hat{Q}$ in $\hat{G}$ is irreducible on V, and hence regular on the points and hyperplanes of V. Thus Q is regular on Ω so the lemma holds in this case.

In cases (c) and (d), a Sylow p-subgroup of G is of order p, and $\mathcal{Q} = \mathrm{Syl}_p(G)$, so the lemma holds.

Finally, in case (e), G acts transitively on the set $\mathcal{A}$ of abelian subgroups of G of order 3^3, and for $A \in \mathcal{A}$, $A \cong E_{3^3}$ contains a member of each of the three conjugacy classes of subgroups of G of order 3. Thus as $3 \in \pi(H)$, A is not regular on Ω, so again the lemma holds. $\square$

LEMMA 3.3. *Assume that p is a prime, let n be a positive integer and $e = \log_p((n!)_p)$ the log of the p-part of $n!$.*

(1) *If n is a power of p then $e = (n-1)/(p-1)$.*
(2) *$e \leq (n-1)/(p-1)$, with n a power of p in the case of equality.*

PROOF. Part (1) is [A1, Lemma 7.1]. Let $n = \sum_i a_i p^i$ be the p-ary expansion of n. By [A1, Lemma 7.2],

$$e = \sum_i a_i \log_p(((p^i)!)_p).$$

Thus if n is not a power of p then by induction on n,

$$e \leq \sum_i a_i (p^i - 1)/(p-1) = (n-r)/(p-1),$$

where $r = \sum_i a_i > 1$. Hence we may assume n is a power of p, where (1) completes the proof. $\square$

LEMMA 3.4. *Assume that G is an almost simple group and set $L = F^*(G)$. Assume that M is a maximal subgroup of G not contained in L and $A < L$ such that $L \cap M < A$ and*

$$L = AA' \quad \text{where } A' = \bigcap_{D \in A^M - \{A\}} D. \tag{3.1}$$

Then $v = |M : N_M(A)| = 2$ and, taking $B \in A^M - \{A\}$ and setting $U = A \cap B$, we obtain $U = L \cap M$, $M = N_G(U)$, and one of the following statements holds:

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(1) *$L \cong A_6$, $G \cong \mathrm{Aut}(L)$, $PGL_2(9)$, or M_{10}, $A \cong A_5$, $U \cong D_{10}$, and $M \cong \mathbf{Z}_2 \times F_{20}$, D_{20}, or F_{20}, respectively, where F_{20} is the Frobenius group of order* 20*;*
(2) *$L \cong M_{12}$, $G \cong \mathrm{Aut}(M_{12})$, $A \cong M_{11}$, $U \cong L_2(11)$, and $M \cong PGL_2(11)$;*
(3) *$L \cong Sp_4(q)$ for some $q > 2$ even, G is a subgroup of* $\mathrm{Aut}(L)$ *acting nontrivially on the Dynkin diagram of L, $A \cong O_4^-(q)$, and $U = N_A(T)$ is an extension of $T \cong \mathbf{Z}_{q^2+1}$ by $\mathbf{Z}_4$.*

PROOF. This lemma is essentially contained in [LPS1, proof of 4.3]. Furthermore, the referee observes that [BPS1, Lemma 5.2] can be used to give a short proof of the lemma.

Let $M_L = M \cap L$. As M is maximal in G and not contained in L, $G = ML$, $M = N_G(M_L)$, and $\mathcal{I}_L^*(M) = \{M_L\}$, where $\mathcal{I}_L^*(M)$ denotes the maximal members of the set of M-invariant proper subgroups of L. Then as $M_L < A < L$, we have $v = |M : N_M(A)| > 1$. Let $m \in M - N_M(A)$ and set $B = A^m$ and $U = A \cap B$. Thus $M_L \leq U$, and by hypothesis (3.1), $L = AB$. Therefore $|U| = |A||B|/|L| = |A|^2/|L|$.

Let $A_1 \in \mathcal{M}_L(A)$, and set $B_1 = A_1^m$ and $U_1 = A_1 \cap B_1$. Set $X = \mathrm{Aut}(L)$ and $Y = LN_X(A_1)$.

As $L = AB$ with $A \leq A_1$ and $B \leq B_1$, also $L = AB_1 = BA_1$. Hence $A_1 = A_1 \cap AB_1 = A(A_1 \cap B_1) = AU_1$, and similarly $B_1 = BU_1$. Similarly, $B_1 = B_1 \cap AB = B(B_1 \cap A) = B(U_1 \cap A)$. Then also $U_1 = U_1 \cap B_1 = U_1 \cap B(U_1 \cap A) = U_A U_B$, where $U_A = U_1 \cap A$ and $U_B = U_1 \cap B$.

Let $\alpha = |A_1 : A|$. Then as $A_1^m = B_1$ and $A^m = B$, also $\alpha = |B_1 : B|$. As $A_1 = AU_1$, then $|U_1 : U_A| = \alpha$, and similarly $|U_1 : U_B| = \alpha$.

Next $L = A_1 B_1$ is a maximal factorization, and hence is described in the main theorem of [LPS2]. Inspecting the tables of examples in that theorem for pairs (A_1, B_1) with $B_1 \cong A_1$, we conclude that one of the following statements holds:

(i) $L \cong A_6$ and $A_1 \cong A_5$;
(ii) $L \cong M_{12}$ and $A_1 \cong M_{11}$;
(iii) $L \cong Sp_4(q)$ with $q > 2$ even, and $A_1 \cong O_4^-(q)$;
(iv) $L \cong P\Omega_8^+(q)$ and $A_1 \cong \Omega_7(q)$.

Example (iii) appears in [LPS2, Table 1], using the fact that $Sp_2(q^2).2 \cong O_4^-(q)$. In example (iv), $\Omega_7(q) = Sp_6(q)$ when q is even. Using the formula $|L| = |A_1|^2/|U_1|$, we calculate $|U_1|$ and then from the subgroup structure of A_1 we conclude that $U_1 \cong D_{10}$, $L_2(11)$, an extension of $\mathbf{Z}_{q^2+1}$ by $\mathbf{Z}_4$, $G_2(q)$, in cases (i)–(iv), respectively.

Suppose that $A \neq A_1$. Then $\alpha = |A_1 : A| > 1$ and we showed above that $U_1 = U_A U_B$, with

$$|U_1 : U_C| = \alpha \quad \text{for } C \in \{A, B\}. \tag{3.2}$$

In the first two cases this is impossible, as U_1 does not have a proper subgroup U_A such that $\pi(U_A) = \pi(U_1)$. In the third case all Sylow subgroups of U_1 are cyclic, so as $U_1 = U_A U_B$, for each $p \in \pi(U_1)$, U_A or U_B contains a Sylow p-subgroup of U_1,

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and then $U_A = U_B = U_1$ by (3.2), contradicting $\alpha > 1$. Finally, (3.2) is impossible in case (iv) from the factorizations of $G_2(q)$ listed in [LPS2, Theorem B].

Therefore A and B are maximal in L and described in one of the four cases above. We next argue as in [LPS1, proof of 4.3] that case (iv) does not satisfy our hypothesis. Assume that (iv) holds. We saw that $U = U_1 \cong G_2(q)$ and $A \cong \Omega_7(q)$. Suppose first that $v > 2$. Then $A' \leq B \cap D$ where D is a third member of A^M. By symmetry, $B \cap D \cong G_2(q)$. But as $B \cap D \cong G_2(q)$, $|L|$ does not divide $|A||B \cap D|$ as $(q^2 + 1, q^6 - 1) = (q - 1, 2)$. Thus $|L|$ does not divide $|A||A'|$, and this contradicts the hypothesis in (3.1) that $L = AA'$.

Therefore $v = 2$, so U is M-invariant, so that $U = M_L$ as $\{M_L\} = \mathcal{I}_L^*(M)$. Therefore $M = N_G(U)$ and $|G : LN_G(A)| \leq |G : LN_M(A)| \leq |M : N_M(A)| = 2$. But then from [K], $N_G(U)$ acts on an $\Omega_7(q)$-subgroup of L, contradicting $\{M_L\} = \mathcal{I}_L^*(M)$.

Therefore one of cases (i)–(iii) holds. In each of these cases, $|X : Y| = 2$, so as $A^m = B \notin A^L$ for $m \in M - N_M(L)$, it follows that $v = 2$. Then as above, $U = M_L$, so $M = N_G(U)$. We proved that $A = A_1$, so A is described in (i)–(iii), and $U = U_1$ was described shortly thereafter. Finally, we can determine G, and hence also $M = N_G(U)$, by inspecting $\mathrm{Aut}(L)$, to complete the proof of the lemma. □

LEMMA 3.5. *Assume that G is a finite simple group. Then there exists $p \in \pi(G)$ such that G has cyclic Sylow p-subgroups.*

PROOF. This seems to be part of the folklore, but as I do not know of a reference, here is a sketch of a proof.

It suffices to exhibit a prime which divides $|G|$ to the first power, and it is often possible to establish this stronger statement. If G is of prime order this is trivial, so we may assume that G is nonabelian. If G is sporadic, such a prime exists by inspection of the orders of the sporadic groups (see [GLS3]).

Suppose that $G \cong A_n$. By Bertrand's postulate (see [NZ, Theorem 8.6]) there is a prime p such that $n/2 < p \leq n$. Then p divides $|G|$ to the first power, so the lemma holds in this case too.

Thus we may assume that G is of Lie type. Let G_0 be the universal group of Lie type of type G. From [GLS3, 4.10.1],

$$|G_0| = q^N \prod_i \Phi_i(q)^{n_i} \tag{3.3}$$

for suitable integers N, n_i and prime power q, where $\Phi_m(x)$ is the mth cyclotomic polynomial. Let p be a prime relatively prime to q, and m_0 the order of q in the multiplicative group of $\mathbf{F}_p$. By [GLS3, 4.10.3.a], $m_p(G_0) = n_{m_0}$. Thus it remains to pick p with $n_{m_0} = 1$.

Next by Zsigmondy's theorem (see [Gu, 3.2]), given i, there is a prime p with $m_0 = i$, unless $(q, i) = (2, 6)$ or q is a Mersenne prime and $i = 2$. Thus it suffices to exhibit an i with $n_i = 1$ and (q, i) not one of the exceptional Zsigmondy pairs. If G is of type ${}^3D_4(q)$, then the factorization (3.3) appears after [GLS3, 4.10.1], and by inspection $i = 12$ works. In the remaining case we examine the standard factorization

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$$|G_0| = q^N \prod_j (q^{d_j} - \epsilon_j)$$

appearing for example in [GLS3, Table 2.2] or [A2, Table 16.1], where the d_j are suitable positive integers and $\epsilon_j \in \{1, -1\}$. Now $\Phi_{u_j d_j}(q)$ divides $q^{d_j} - \epsilon_j$, for $u_j = 1, 2$ when $\epsilon_j = 1, -1$, respectively, and we choose $i = u_j d_j$ with d_j maximal. This choice works unless (q, i) is an exceptional Zsigmondy pair. From the factorization, $n_i = 1$, and $i = 2$ if and only if $G = L_2(q)$, while $(q, i) = (2, 6)$ if and only if G is $L_6^\epsilon(2)$, $Sp_6(2)$, or $G_2(2)$. We choose $p = q, 31, 11, 7, 7$ when G is $L_2(q)$, $L_6(2)$, $U_6(2)$, $Sp_6(2)$, $G_2(2)$, respectively. □

Recall (see [A2]) that for G a finite group and p a prime, $m_p(G)$ is the p-rank of G.

LEMMA 3.6. *Assume that Ω is a finite set, $G \leq \mathrm{Sym}(\Omega)$, and H is a subgroup of G regular on Ω. Let p be a prime, $P \in \mathrm{Syl}_p(H)$, and $P \leq Q \in \mathrm{Syl}_p(G)$.*

(1) *There exists $\omega \in \Omega$ such that $Q_\omega \in \mathrm{Syl}_p(G_\omega)$, and for each such ω, P is a complement to Q_ω in Q.*
(2) *If Q is abelian then $Q = P \times Q_\omega$ and $m_p(G) = m_p(H) + m_p(G_\omega)$.*

PROOF. The first remark in (1) follows from Sylow's theorem. Next $|G| = |\Omega||G_\omega|$, so

$$|Q| = |G|_p = |\Omega|_p |G_\omega|_p = |P||Q_\omega|, \tag{3.4}$$

as H is regular on Ω, $P \in \mathrm{Syl}_p(H)$, and $Q_\omega \in \mathrm{Syl}_p(G_\omega)$. As P is semiregular on Ω, $P \cap Q_\omega = 1$, and (1) follows from this observation together with (3.4). Then (1) implies (2). □

LEMMA 3.7. *Assume that G is almost simple and set $L = F^*(G)$. Assume that $1 \neq H < L$ and $|G : L|$ is prime. Let $\mathcal{M}$ be the set of maximal overgroups of H in G and assume that,*

$$\textit{for each } M \in \mathcal{M}, \textit{ there exists } M' \in \mathcal{M} - \{M\} \textit{ with } H = M \cap M'. \tag{3.5}$$

Then:

(1) *L and $N_G(H)$ are in $\mathcal{M}$;*
(2) *$H = N_L(H)$;*
(3) *$|N_G(H) : H| = |G : L|$;*
(4) *for each $M \in \mathcal{M} - \{N_G(H)\}$, $H = M \cap N_G(H)$.*

PROOF. As $|G : L|$ is prime and $H \leq L$, $L \in \mathcal{M}$. Thus by (3.5), there exists $K \in \mathcal{M} - \{L\}$ with $H = L \cap K$. As $L \trianglelefteq G$, $H \trianglelefteq K$, so $K = N_G(H)$ by maximality of K. Thus (1) holds, and $H = L \cap K = N_L(H)$, establishing (2). As $|G : L|$ is prime and $K \in \mathcal{M} - \{L\}$, $|G : L| = |K : K \cap L| = |K : H|$, so (3) holds. Finally, if $M \in \mathcal{M} - \{K\}$ then $M \cap K = N_M(H) = H$ or K as $H \leq M \cap K$ and $|K : H|$ is prime. But as M and K are distinct maximal subgroups of G, $K \not\leq M$, so $M \cap K = H$, establishing (4). □

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LEMMA 3.8. *Assume that G is almost simple and set $L = F^*(G)$. Assume that $|G : L|$ is prime and H is a maximal subgroup of L such that $N_G(H) \not\leq L$. Let $\mathcal{M}$ be the set of maximal overgroups of H in G.*

(1) $\mathcal{M} = \{L, N_G(H)\}$.
(2) $\mathcal{O}_G(H) = \{H, L, N_G(H), G\}$.

PROOF. As $|G : L|$ is prime and $H \leq L$, $L \in \mathcal{M}$. As $|G : L|$ is prime and $K = N_G(H) \not\leq L$, $G = LK$ and $|K : K \cap L| = |G : L|$. Let $J \in \mathcal{O}_G(H) - \{G, L\}$. Then as H is maximal in L, $H = J \cap L \trianglelefteq J$, so $J \leq K$. In particular, $H = K \cap L$, so $|K : H|$ is prime, and hence $J = H$ or K. The lemma follows. □

4. Overgroups of primitive groups

In this section we assume Hypothesis 2.1. Recall the definition of the various types of primitive groups from Definition 2.3. Recall also that for M a suitable primitive subgroup of S, $\mathcal{F}(M)$ is the M-invariant product structure on Ω defined in Notation 2.6. Similarly, for H affine, $\mathcal{D}(H)$ is the set of H-invariant direct sum decompositions of $F^*(H)$ defined in Notation 2.6. Given a suitable collection $\mathcal{L}$ of subgroups of S, the product structure $\mathcal{F}(\mathcal{L})$ is defined in Example 1.6.

LEMMA 4.1. *Assume that H is affine and $M \in \mathcal{O}_G(H)$. Then H preserves the affine structure $R = R(D)$ defined in Lemma 1.4, $D \leq F^*(M)$, and one of the following statements holds:*

(1) *M is affine with $F^*(M) = D$, so $M \leq N_G(R)$.*
(2) *H is imprimitive on D and there exists $\mathcal{D} = \{D_1, \ldots, D_k\} \in \mathcal{D}(H)$ such that M is semisimple and $\mathcal{F}(M) = \mathcal{F}(\mathcal{D})$. Moreover, $d = |D_1| \geq 5$, and $D_i = D \cap X_i$ is a complement to $X_{i,\omega}$ in X_i, where $\{X_1, \ldots, X_k\}$ are the components of M. Furthermore, if $M = N_G(\mathcal{F}(M))$ then $N_M(D)$ is the stabilizer in $N_G(D)$ of $\mathcal{D}$.*
(3) *$|\Omega| = p$ is prime.*

PROOF. By Lemma 1.4, H preserves R.

Let $X = F^*(M)$. By Lemma 2.4, M is primitive on Ω, so X is transitive on Ω. In particular, $p \in \pi(X)$, so $1 \neq C_X(D)$. But as D is regular on Ω, $D = C_S(D)$, so $1 \neq D \cap X$, and hence $D \leq X$ by Lemma 2.4(2). As D is regular on Ω, D is a complement to X_ω in X.

Next M is described in Lemma 2.2, so as $|\Omega| = p^e$ is a power of a prime, we conclude (using Lemma 1.4 in case (3i) of Lemma 2.2) that one of the following statements holds:

(i) M is affine;
(ii) M is semisimple and stabilizes the (d, k)-product structure $\mathcal{F} = \mathcal{F}(M) = \mathcal{F}(\mathcal{L})$, where $\mathcal{L} = (X_i : 1 \leq i \leq k)$ is the set of components of M, $d = p^f$ for some divisor f of e, and $k = e/f > 1$;
(iii) M is almost simple.

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In case (i), $|X| = |\Omega| = |D|$, so $X = D$ and hence (1) holds. Thus we may assume that case (ii) or (iii) holds. Furthermore, we may assume (3) does not hold, so $e > 1$.

Suppose that case (iii) holds. Recall that D is an abelian complement to X_ω in X. Thus as $e > 1$, Lemma 3.2 says that case (a) of Lemma 3.1 holds. This is a contradiction as $F^*(S) \not\leq M$.

Thus we may assume that case (ii) holds. Let D_i be the projection of D on X_i. Then $D_i \leq C_S(D) = D$, so $D = \prod_i D_i$ and $\mathcal{D} = \{D_1, \ldots, D_k\} \in \mathcal{D}(H)$. As $X = X_\omega D$ and $X_{i,\omega}$ is the projection of X_ω on X_i, it follows that $X_i = X_{i\omega} D_i$. Then as D is regular on Ω, D_i is a complement to $X_{i,\omega}$ in X_i. Hence, $\mathcal{F}(\mathcal{D}) = \mathcal{F}(M)$ by Lemma 1.7. Therefore, (2) holds in this case, and the proof of the lemma is complete. □

DEFINITION 4.2. Define a semisimple group H to be *octal* if for each component L of H, and for each $\omega \in \Omega$, $L \cong \mathrm{Aut}_H(L) \cong L_3(2)$ and $|\omega L| = 8$.

LEMMA 4.3. *Let $M \in \mathcal{O}_G(H)$. Then either*

(1) $D \leq F^*(M)$*; or*

(2) *H is semisimple and octal with r components $\{E_i \mid 1 \leq i \leq r\}$, $E_i \cong L_3(2)$, $|\Omega| = 8^r$, and M is affine with*

$$X = F^*(M) = \prod_{i=1}^{r} X_i,$$

where $X_i = [X, E_i] \cong E_8$. In particular, HX is affine, $\mathcal{D} = \{X_1, \ldots, X_r\} \in \mathcal{D}(HX)$, and $\mathcal{F}(H) = \mathcal{F}(\mathcal{D})$.

PROOF. Assume that M is a counterexample, and let X be a minimal normal subgroup of M. By Lemma 4.1, H is not affine. By Lemma 2.4(2), either $D \cap X = 1$ or H is doubled and we may assume that $D_1 \cap X = 1$. Set $E = D_1$ if H is doubled and $E = D$ otherwise. Thus $E \cap X = E \cap F^*(M) = 1$.

Suppose first that M is affine. Then $n = |\Omega| = |X| = p^e$, so from Lemma 2.2, H is semisimple, and hence D is the direct product of r simple groups E_i permuted transitively by H and $|E_i : E_{i,\omega}| = p^f$, with $rf = e$. By Lemma 3.2, one of the following statements holds:

(i) for each i there exists an abelian complement Q_i to $E_{i,\omega}$ in E_i;

(ii) $p = 2$ and $E_i \cong L_2(q)$, where $q = 2^f - 1$ is a Mersenne prime;

(iii) $p = 3$, $f = 3$, and $E_i \cong PSp_4(3)$.

Assume first that (i) holds, and let $Q = Q_1 \cdots Q_r$. Then Q is abelian and regular on Ω, so $C_S(Q) = Q$. This is a contradiction as $C_X(Q) \neq 1 = D \cap X$.

Next assume that (ii) or (iii) holds. As D is transitive on Ω, D is not contained in the complement M_ω to X in M, so $H^1(H, X) \neq 0$. On the other hand, $H \cong K = HX \cap M_\omega$, and HX is primitive on Ω by Lemma 2.4, so K is irreducible on X. Therefore H is irreducible on X, so by [AS, Theorem 3],

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$$X = \bigoplus_{i=1}^{r} X_i,$$

where $X_i = [X, E_i]$. Let $d = \dim([X, E_i])$. Then $rd = e = rf$, so $d = f$. But $|PSp_4(3)|$ does not divide $|L_3(3)|$, so (iii) does not hold. Furthermore, in (ii), if $f > 3$ and $P \in \mathrm{Syl}_q(E_i)$ then $N_{GL_f(2)}(P)$ is P extended by a group of order f, whereas $|N_{E_i}(P)| = q(q-1)/2$, which is a contradiction. Thus, in case (ii), $f = 3$ and $E_i \cong L_3(2)$, so that (2) holds.

Thus we may assume that M is not affine. If M is almost simple then (1) holds by the Schreier property (see [GLS3, 7.1.1]). Therefore, as M appears in case (2) or (3) of Lemma 2.2, X is the direct product of k simple components X_i, permuted transitively by M, and one of the following statements holds:

(a) M is semisimple and $n = m^k$, where $|X_i : X_{i,\omega}| = m$;
(b) M is doubled or complemented, and $n = y^k$, where $|X_i| = y$;
(c) M is diagonal and $n = y^{k-s}$, where $|X_i| = y$ and $s = |\Sigma(M)|$.

Next $F^*(M) \leq U \trianglelefteq M$ with $U/F^*(M)$ solvable and $M/U \leq S_k$. Therefore, as E is the direct product of simple components and $E \cap F^*(M) = 1$, we conclude that $E \cap U = 1$, so E is isomorphic to a subgroup of S_k. However, E is transitive on Ω, so $|E|$ is divisible by $n = m^k$, y^k, or y^{k-s}, in (a), (b), or (c), respectively. Pick p to be a prime divisor of m or y in the respective case. We may pick p to be odd, except possibly in case (a). Thus $a = \log_p(|E|_p) \geq k$ in (a) and (b), while $a \geq k - s$ in (c). In particular, $a \geq k$ if $p = 2$. But by Lemma 3.3, $l = \log_p((k!)_p) \leq (k-1)/(p-1)$, while if $p > 2$ then $(k-1)/(p-1) \leq (k-1)/2 < k - s$. Thus in any event, $l < a$, in contradiction to E being isomorphic to a subgroup of S_k. □

LEMMA 4.4. *Let $M \in \mathcal{O}_G(H)$ be almost simple. Then one of the following statements holds:*

(1) *$M = S$ or $F^*(S)$;*
(2) *H is almost simple;*
(3) *H is affine and Ω is prime.*

PROOF. Assume that the lemma is false and pick a pair $\mathcal{P} = (M, H)$ which is a counterexample to the lemma, and with $|M : H|$ minimal subject to this constraint. By Lemma 4.1, H is not affine, and as $\mathcal{P}$ is a counterexample, H is not almost simple. Therefore H is doubled, complemented, or semisimple. Let $X = F^*(M)$. As HX satisfies the hypothesis of the lemma, $M = HX$ by minimality of $\mathcal{P}$. By Lemma 4.3, $D \leq X$.

Let $K = M_\omega$ and $J = K \cap X$. As D is transitive on Ω, $M = KD$. Then as $D \leq X$, also $X = JD$. By Lemma 2.4, M is primitive on Ω, so K is maximal in M.

Suppose that H is not maximal in M, and let $I \in \mathcal{O}_M(H)$ with H maximal in I. By Lemma 2.4, I is primitive on Ω. By minimality of $\mathcal{P}$, I is not almost simple. If $|\Omega|$ is a prime, then H is affine or almost simple by Lemma 2.2, contrary to an earlier

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observation. Thus $M = S$ or $F^*(S)$ by minimality of $\mathcal{P}$, contrary to the choice of $\mathcal{P}$ as a counterexample.

Therefore H and K are maximal in M, so $M = KH$ is a maximal factorization of M with the following properties:

(a) $X = (X \cap K)(X \cap H)$ is also a factorization;
(b) H is primitive on Ω so $K \cap H = H_\omega$ is maximal in H;
(c) $D = F^*(H)$ is the direct product of $r' > 1$ isomorphic nonabelian simple groups.

To complete the proof of the lemma, we inspect the list of maximal factorizations of the almost simple groups in [LPS2], and verify that no pair on the lists satisfies (a)–(c).

We begin with the case where X is a classical group. Thus we appeal to [LPS2, Theorem A], and inspect [LPS2, Tables 1–4]. We first search for pairs satisfying (c). In Table 1, the only examples occur with $X = Sp_{2m}(q)$, q even. The last four rows of that subtable contain a $(*)$, so they do not satisfy (a). The remaining pairs (A, B) satisfying (c) are $(Sp_2(q^2).2, O_4^+(q))$ and $(O_{2m}^-(q), Sp_m(q) \text{ wr } \mathbf{Z}_2)$ with m even. Both pairs fail to satisfy (b): see the discussion in [LPS2, 3.2.1.d p. 48, and 3.2.4.b p. 50, respectively].

In Table 2, the only pair satisfying (c) is $(Sz(q), O_4^+(q))$ in $PSp_4(q)$. However, here $A \cap B$ is a dihedral group of order $2(q-1)$, which is not maximal in H, so (b) fails.

There are no examples of pairs in Table 3 satisfying (c). In Table 4 there are two examples: $(\Omega_7(2), (L_2(4) \times L_2(4)).2^2)$ in $\Omega_8^+(2)$, and an example for $\Omega_8^+(4)$ which fails (a) as a $(*)$ appears in the corresponding row. Consider the first example. Here (see the discussion in [LPS2, 3.6.1.c]) X is $\Omega_8^+(2)$, and if we write V for the orthogonal space defining X, then, up to conjugation in $\mathrm{Aut}(X)$, A is the stabilizer of a nonsingular vector $v \in V$, and B is the stabilizer of an extension field structure over $\mathbf{F}_4$, and isomorphic to $O_4^+(4)$ extended by a field automorphism. Now $A \cap B = C_B(v) \cong \mathbf{Z}_2 \times O_3(4)$ is not maximal in B, giving a contradiction. This completes the analysis in the case when X is a classical group.

When X is an exceptional group of Lie type, we see from [LPS2, Theorem B and Table 5] that there are no examples of pairs satisfying (c). Similarly, from [LPS2, Theorem C and Table 6], when X is sporadic there are no examples satisfying (c).

Finally, assume that X is an alternating group A_n with $n = 7$ or $n > 8$. We appeal to [LPS2, Theorem D]. We first consider the generic examples in that theorem, where A is the stabilizer of a k-subset Γ in the n-set Σ permuted by M, with $1 \le k \le 5$, and B is k-homogeneous on Σ. As $\mathcal{P}$ does not satisfy conclusion (1) of our lemma, $k > 1$. As B is k-homogenous, either B is 2-transitive or $k = 2$ and B is of odd order. In either case B does not satisfy (c), so A satisfies (c). Thus $k = 5$ and $n = 10$. But as $B < M$ is 5-homogeneous, $n = 12$ or 24 and B is M_{12} or M_{24}, giving a contradiction.

This leaves the exceptional cases in Theorem D, and as n is not 6 or 8, we have $n = 10$. The only example satisfying (c) has A equal to $L_2(8)$ or ${}^2G_2(3)$, and $D \cong A_5 \times A_5$. From the discussion on [LPS2, p. 124], $A \cap B \cong A_4$, so $A \cap B$ is not maximal in B, and hence (b) is not satisfied. Thus the proof is complete at last. $\square$

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LEMMA 4.5. *If $M \in \mathcal{O}_G(H)$ is complemented then H is complemented and $F^*(M) = F^*(H)$.*

PROOF. By Lemma 4.3, $D \leq F^*(M) = X$. As M is complemented, X is regular on Ω, so as D is transitive on Ω, D is also regular on Ω. In particular, $|D| = |\Omega| = |X|$, so $D = X$. As M is complemented, $|\Omega|$ is not a prime power. But from Lemma 2.2, as D is regular on Ω, either H is affine and $|\Omega|$ is a prime power, or H is complemented. Therefore H is complemented. □

LEMMA 4.6. *Assume that H is complemented and let $D_2 = C_G(D)$.*

(1) *The map $\varphi : g \mapsto \omega g$ is an equivalence of the representation ρ of D by right multiplication on D with the representation of D on Ω.*
(2) *Define $\lambda : D \to \mathrm{Sym}(D)$ by $x\lambda : g \mapsto x^{-1}g$, and let $\psi = \lambda\varphi^*$, where $\varphi^* : \mathrm{Sym}(D) \to S$ is defined by $\varphi^* : \beta \mapsto \varphi^{-1}\beta\varphi$. Then $\psi : D \to S$ is a permutation representation with $D\psi = D_2$.*
(3) *Let $X = DD_2$ and $F = \{g \cdot g\psi \mid g \in D\}$. Then $F = X_\omega$ is an H_ω-invariant full diagonal subgroup of X.*
(4) *$HC_G(D) = XH \in \mathcal{O}_G(H)$ is doubled.*

PROOF. Part (1) is a restatement of the fact in Lemma 2.2 that D is regular on Ω.

Visibly λ and ψ are permutation representations with $D\psi$ a subgroup of D_2 regular on Ω. As D is transitive on Ω, D_2 is semiregular on Ω, so as $D\psi$ is a transitive subgroup of D_2, (2) holds.

For $g \in D$, let $g\alpha = g \cdot g\psi$. Observe that $1(g\rho \cdot g\lambda) = g^{-1}g = 1$, so $g\rho \cdot g\lambda$ fixes 1, and hence $g\alpha = g\rho\varphi^* \cdot g\lambda\varphi^* \in X_\omega$. As D is transitive on Ω, $|X_\omega| = |X|/|\Omega| = |F|$, so $F = X_\omega$. As H_ω acts on D, it also acts on $X = DC_G(D)$, and then on $X_\omega = F$, completing the proof of (3).

As H acts on $C_G(D)$, $HC_G(D) = HX$ is a subgroup of G, so $XH \in \mathcal{O}_G(H)$. Then XH is primitive on Ω by Lemma 2.4, and from (3), HX is doubled. □

In the next lemma, $\mathrm{diag}(\mathcal{B}, F)$ is defined in Definition 1.9, while $\mathcal{P}(H)$ and $\mathcal{F}(H.\Delta)$ are defined in Notation 2.6.

LEMMA 4.7. *Assume that H is doubled with minimal normal subgroups D_1 and D_2, and let $\mathcal{L}$ be the set of components of D_1. Let $k = |\mathcal{L}|$ and $m = |L|$ for $L \in \mathcal{L}$.*

(1) *If $k > 1$, then for each $\Delta \in \mathcal{P}(H)$, $\mathcal{F}(H, \Delta)$ is a regular $(m^{k/d}, d)$-product structure preserved by H, where $d = |\Delta|$. In particular, $\mathcal{F}(H)$ is an (m, k)-structure.*
(2) *If $k = 1$, set $\mathcal{B} = \{D_1, D_2\}$ and $F = D_\omega$. Then $\mathbf{d}(H) = \mathrm{diag}(\mathcal{B}, F)$ is an H-invariant diagonal structure on Ω.*

PROOF. Part (1) follows from Notation 2.6. On the other hand, if $k = 1$ then (2) follows from the definition of a diagonal structure in Definition 1.9. □

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LEMMA 4.8. *Assume that $M \in \mathcal{O}_G(H)$.*

(1) *If M is doubled or diagonal then H is not semisimple.*
(2) *If M is doubled then H is not diagonal.*
(3) *If M and H are doubled then $F^*(H) = F^*(M)$.*

PROOF. Assume the hypothesis of one of (1)–(3), and let $X = F^*(M)$. By Lemma 4.3, $D \leq X$, so $D_\omega \leq X_\omega$.

First assume the hypothesis of (3). Then $|X| = |\Omega|^2 = |D|$, so $D = X$ and hence (3) holds.

Now assume that M, H is a counterexample to (1) or (2). If H is semisimple let $\mathcal{L}$ be the set of components of H, while if H is diagonal let $\mathcal{L} = \{D_\sigma \mid \sigma \in \Sigma(H)\}$ (in the notation of Lemma 2.2). If M is doubled let J be one of the minimal normal subgroups of M, while if M is diagonal, then for $\gamma \in \Sigma(M)$, let J_γ be the product of $|\gamma| - 1$ of the components of X_γ, and set $J = \langle J_\gamma : \gamma \in \Sigma(M)\rangle$.

In either case, J is regular on Ω and we can write $X = J \times J'$, where J' is the product of the components of M not contained in J. Let $\pi : X \to J$ be the projection map with respect to this direct sum decomposition. Let $\alpha = \pi_{|X_\omega} : X_\omega \to J$, and observe that α is an injection as J' is semiregular on Ω. Thus, $\alpha : L_\omega \to J$ is injective for each $L \in \mathcal{L}$. Therefore as L is a minimal normal subgroup of $N_H(L) \cap N_H(J)$, and as $L_\omega \neq 1$, it follows that $\pi : L \to J$ is injective. Then as each nontrivial normal subgroup of D intersects some $L \in \mathcal{L}$ nontrivial, it follows that $\pi : D \to J$ is an injection. But now $|\Omega| < |D| \leq |J| = |\Omega|$, giving a contradiction. □

5. Semisimple overgroups of primitive groups

In this section we make the following assumption.

HYPOTHESIS 5.1. Hypothesis 2.1 is satisfied and H is not affine. Furthermore, $M \in \mathcal{O}_G(H)$ and M is semisimple but not almost simple.

The referee pointed out that [BPS2] contains results similar to those in this section, and indicates that [BPS2] can be used to prove many of the lemmas in the section.

NOTATION 5.2. Let $X = F^*(M)$, $I = \{1, \ldots, k\}$, and $\bar{I} = \{1, \ldots, r\}$. Pick $\omega \in \Omega$.

Let $\mathcal{X} = \{X_i \mid i \in I\}$ be the set of components of M. For $\gamma \subseteq I$, set $\gamma' = I - \gamma$, $X_\gamma = \langle X_i : i \in \gamma\rangle$, $\Gamma_\gamma = \omega X_\gamma$, and $\pi_\gamma : X \to X_\gamma$ the projection map with respect to the direct sum decomposition $X = X_\gamma \times X_{\gamma'}$.

As M is semisimple but not almost simple, $k > 1$ and M preserves the (m, k)-product structure $\mathcal{F} = \mathcal{F}(M) = (\Omega_i : i \in I)$ defined in Notation 2.6, with X_i transitive on Ω_i and $X_{i'}$ the kernel of the action of X on Ω_i. This allows us to identify Ω with $\prod_{i \in I} \Gamma_i$, as in the discussion in Definition 1.5.

Let E be a minimal normal subgroup of H and $\mathcal{E} = \{E_i \mid i \in \bar{I}\}$ the set of components of E. By Lemma 4.3, $E \leq X$. For $i \in \bar{I}$ let

$$\beta_i = \{j \in I \mid E_i\pi_j \neq 1\},$$

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and for $i \in I$ let

$$S_i = \{j \in \bar{I} \mid E_j\pi_i \neq 1\}.$$

Represent H on I and $\bar{I}$ so that the maps $i \mapsto X_i$ and $j \mapsto E_j$ are equivalences of permutation representations. As H is transitive on $\mathcal{E}$ and acts on $\mathcal{X}$, H is transitive on $\bar{I}$ and

$$b = b(M, H) = |\beta_i|$$

is independent of $i \in \bar{I}$. For $\gamma \subseteq I$ define

$$Q_\gamma = \{e \in E \mid e\pi_\gamma \in X_{\gamma,\omega}\}. \qquad \square$$

See Notation 2.6 for the definition of $\mathcal{F}(H, K)$ for suitable $K \leq H$.

LEMMA 5.3. (1) *For each $\gamma \subseteq I$, $E = Q_\gamma Q_{\gamma'}$.*
(2) *For each proper nonempty subset γ of I, $E_\omega < Q_\gamma < E$, and $E\pi_\gamma$ is transitive on Γ_γ.*
(3) *If $i, j \in I$ are distinct, then $E = Q_i Q_j$ and $Q_j \notin Q_i^E$.*
(4) *H is transitive on $\mathcal{X}$ and I.*

PROOF. Let $\gamma \subseteq I$ and $x \in X_\gamma$. As E is transitive on Ω, $x = ev$ for some $e \in E$ and $v \in X_\omega$. As M is semisimple, $X_\omega = X_{\gamma,\omega} \times X_{\gamma',\omega}$, so $v\pi_\alpha \in X_{\alpha,\omega}$ for $\alpha \in \{\gamma, \gamma'\}$. Thus $E_\omega \leq Q_\gamma$ and

$$e\pi_{\gamma'} = (xv^{-1})\pi_{\gamma'} = (v^{-1})\pi_{\gamma'} \in X_{\gamma',\omega},$$

so $e \in Q_{\gamma'}$. By symmetry, for $x' \in X_{\gamma'}$, $x' = e'v'$ with $e' \in Q_\gamma$ and $v' \in X_\omega$. Next for each $g \in E$, $g = xx'$ with $x \in X_\gamma$ and $x' \in X_{\gamma'}$, so $g = eve'v'$ with $e \in Q_{\gamma'}$ and $ve'v' \in Q_\gamma$. This establishes (1).

As $X = X_\omega E$, $X_\gamma = X\pi_\gamma = X_\omega\pi_\gamma E\pi_\gamma = X_{\gamma,\omega}E\pi_\gamma$, so $E\pi_\gamma$ is transitive on Γ_γ, and hence if $\gamma \neq \varnothing$ then $E\pi_\gamma \nleq X_{\gamma,\omega}$, so that $Q_\gamma \neq E$. Similarly, by (1), $E\pi_{\gamma'} \leq Q_\gamma\pi_{\gamma'}X_{\gamma',\omega}$ and if $\gamma' \neq \varnothing$ then $X_{\gamma',\omega} \neq X_{\gamma'} = E\pi_{\gamma'}X_{\gamma',\omega}$, so $Q_\gamma \neq E_\omega$. Therefore (2) holds.

Let γ be an orbit of H on I. Then H_ω acts on Q_γ, and if $\gamma \neq I$ then by (2), $H_\omega < H_\omega Q_\gamma < H$, in contradiction to H being primitive on Ω. Thus (4) holds.

Finally, suppose that $i, j \in I$ are distinct. Then $j \in i'$, so $Q_{i'} \leq Q_j$, and hence $E = Q_i Q_j$ by (1). Therefore Q_j is transitive on E/Q_i, and by (2), $|E/Q_i| > 1$, so that Q_j fixes no point of E/Q_i. Hence $Q_j \notin Q_i^E$, establishing (3). $\square$

LEMMA 5.4. (1) *$s = |S_i|$ is independent of $i \in I$.*
(2) *$ks = rb$.*
(3) *$b = 1$ if and only if each component of E is contained in a component of X.*
(4) *$|\Omega| = m^k$.*

PROOF. Part (1) follows from Lemma 5.3(4). Then (2) follows from counting the order of $\{(i, j) \in I \times \bar{I} \mid E_j\pi_i \neq 1\}$ in two ways. Part (3) is trivial, and (4) follows from Lemma 2.2. $\square$

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LEMMA 5.5. *Assume that $b = 1$. Then the following statements hold.*

(1) $r = ks$.
(2) *For $i \in I$, $S_i = \{j \in \bar{I} \mid E_j \leq X_i\}$.*
(3) *$\mathcal{S} = \{S_i \mid i \in I\}$ is an H-invariant partition of $\bar{I}$ into k blocks of size s.*
(4) *For $i \in I$, let $E(i) = \langle E_j \mid j \in S_i\rangle$ and H_i be the stabilizer in H of i. Then $E(i) = E\pi_i$ is transitive on Γ_i, H_i is transitive on S_i, and H_i is primitive on Γ_i.*
(5) *Either H is semisimple and $s = 1$, or $X_i \cong A_m$ is the alternating group on Γ_i.*
(6) $\mathcal{F} = \mathcal{F}(H, H_1)$.
(7) *If H is doubled or semisimple, then $\mathrm{Aut}_H(X_1)$ is doubled or semisimple on Γ_1, respectively.*
(8) *If H is diagonal then $\Sigma(H)$ corresponds to an H-invariant partition Σ of $\bar{I}$ under the equivalence of Notation 5.2, the partition Σ is a refinement of $\mathcal{S}$, and $\mathrm{Aut}_H(X_1)$ is diagonal on Γ_1.*
(9) *If H is complemented then $\mathrm{Aut}_H(X_1)$ is complemented or doubled on Γ_1.*

PROOF. Part (1) follows from Lemma 5.4(2), while (2) follows from Lemma 5.4(3). Then (2) implies (3).

Let $i \in I$. By (2), $E(i) = E\pi_i$, so $E(i)$ is transitive on Γ_i by Lemma 5.3(2). By (3) and Lemma 5.3(4), H_i is transitive on S_i and for $j \in S_i$, $N_H(E_j) \leq H_i$.

If H is semisimple then by Lemma 2.2, E_ω is the direct product of the groups $E_{j,\omega}$, $j \in \bar{I}$, and $\mathrm{Aut}_{H_\omega}(E_j)$ is maximal in $\mathrm{Aut}_H(E_j)$. Thus as $N_H(E_j) \leq H_i$, as H_i is transitive on S_i, and as $E(i)$ is transitive on Γ_i, H_i is primitive and semisimple on Γ_i. Therefore in this case, (4) and (7) hold, while (5) follows from Lemma 4.4 and (6) follows from Notation 2.6.

If H is doubled then by Lemma 2.2, $D = E \times \tilde{E}$ with D_ω a full diagonal subgroup of D. As $E(i)$ is transitive on Γ_i and semiregular on Ω, $E(i)$ is regular on Γ_i. From part (1) of the next lemma (whose proof does not depend upon this lemma), $b(M, \tilde{E}) = 1$, so by symmetry between E and $\tilde{E}$, $\tilde{E}\pi_i = \tilde{E} \cap X_i = \tilde{E}(i)$ is regular on Γ_i, and it the direct product of $\tilde{s}$ components. By (1), $\tilde{s} = s$. Thus Y_ω is a full diagonal subgroup of $Y = E(i)\tilde{E}(i)$. Hence Y is primitive on Γ_i, and as $\mathrm{Aut}_H(X_i)$ has two distinct minimal normal subgroups, the group is doubled. Thus (4) and (7) hold in this case, and, as above, (5) and (6) follow from Lemma 4.4 and Notation 2.6.

Assume that H is diagonal. Then the first statement in (8) follows from Lemma 2.2, which also says that for $\sigma \in \Sigma$, the global stabilizer $H(\sigma)$ in H of σ is primitive on σ. Let $1 \in S_i \cap \sigma$. As $H(\sigma)$ is primitive on σ, either $\sigma \subseteq S_i$ or $\{1\} = S_i \cap \sigma$. In the first case (8) holds, and (5) and (6) follow from Lemma 4.4 and Notation 2.6. Suppose that the second case holds. Then $E(i)$ is regular on Γ_i, so $m = |\Gamma_i| = |E(i)| = e^s$, where $e = |E_1|$. Thus appealing to Lemmas 2.2, 5.4(4), and (1),

$$e^{r-|\Sigma|} = |\Omega| = m^k = e^{ks} = e^r,$$

giving a contradiction.

Finally, assume that H is complemented. Then E is regular on Ω, so $E(i)$ is regular on Γ_i. If $H_{i,\omega}$ acts on some nontrivial proper subgroup F of $E(i)$, then

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$H_\omega < \langle F^{H_\omega}\rangle H_\omega < H$, contradicting the maximality of H_ω in H. Therefore no such F exists, so $H_{i,\omega}$ is maximal in H_i, and hence H_i is primitive on Γ_i. As $E(i)$ is regular on Γ_i, $\mathrm{Aut}_H(X_i)$ is complemented or doubled. Thus (4) and (9) hold in this case, and (5) and (6) follow as usual. □

LEMMA 5.6. *Assume that H is doubled or diagonal.*

(1) $b = 1$.
(2) $X = F^*(M')$ *where* $M' = N_G(X) = N_G(\mathcal{F})$, *and* $\mathcal{F} = \mathcal{F}(H, H_1)$, *where* $H_1 = N_H(X_1)$.

PROOF. First assume that $b > 1$ and let $Y = X_{1'}$ and $\Gamma = \Gamma_{1'}$. Then $E \cap X_1 = 1$ by Lemma 5.4(3), so $\pi = \pi_{1'} : E \to Y$ is an injection. Let $F = E\pi$ and for $i \in \bar{I}$, let $F_i = E_i\pi$. By Lemma 5.3(2) and the injectivity of π, $E_\omega\pi < Q_{1'}\pi = P = F_\omega < F$, so there exists $a \in P - E_\omega\pi$. Let $a = a_1 \cdots a_u$ with $a_i \in F_i^{\#}$, and $A = F_1 \cdots F_u$.

Suppose first that H is doubled. Then $D = E \times \tilde{E}$ and D_ω is a full diagonal subgroup of D. Then $B = D\pi = F\tilde{E}\pi$ and $R = D_\omega\pi \le B_\omega$, so R acts on $P = B_\omega \cap F$. Furthermore, $\mathrm{Aut}_R(A) = A$, so as $a_i \in F_i^{\#}$ and F_i is simple, $A = [a, R] \le P$. Then as $A \trianglelefteq F$ and F is transitive on Γ by Lemma 5.3(2), it follows that A fixes Γ pointwise, in contradiction to Y being faithful on Γ.

Hence H is diagonal. Then F_i is contained in a block α_i of the partition of $\mathcal{E}$ in Lemma 2.2(3ii), and there is a full diagonal subgroup U_i of $F_{\alpha_i} = \langle K : K \in \alpha_i\rangle$ contained in F_ω. Then as above, $A = [a, U_1 \cdots U_u] \le P$, and we obtain the same contradiction. This completes the proof of (1).

By (1) and Lemma 5.5(5), X_1 acts as the alternating group on Γ_1. Thus $M' = N_G(X) = N_G(\mathcal{F})$ by Lemma 1.8. Then (2) follows from Lemma 5.5(6). □

LEMMA 5.7. *Let $\gamma \subseteq I$ and $\mu \subseteq \bar{I}$. For $\eta \subseteq \bar{I}$, let $E_\eta = \langle E_i : i \in \eta\rangle$ and $\eta' = \bar{I} - \eta$. Let $\sigma : E \to E_\mu$ be the projection map with respect to the direct sum decomposition $E = E_\mu \times E_{\mu'}$.*

(1) *If $Q_\gamma\sigma = E_\mu$ then $E_{\mu'}\pi_\gamma$ is transitive on Γ_γ and $E_\mu\pi_\gamma$ is semiregular on Γ_γ.*
(2) *Assume that H is semisimple or complemented, $\mu = \{i\}$ for some $i \in \bar{I}$, and γ is $N_{H_\omega}(E_i)$-invariant. Then for each $\alpha \in \{\gamma, \gamma'\}$, $Q_\alpha\sigma \in \{E_{i,\omega}, E_i\}$, and $Q_\alpha\sigma = E_i$ for some $\alpha \in \{\gamma, \gamma'\}$.*

PROOF. Let $W = E_\mu$ and $P = Q_\gamma\sigma$. Suppose that $P = W$. Then for each $w \in W$, there exists $u \in Q_\gamma$ such that $w = u\sigma$, or equivalently there exists $v \in E_{\mu'}$ such that $u = wv$. Thus

$$\omega(w\pi_\gamma) = \omega(u\pi_\gamma v^{-1}\pi_\gamma) = \omega(v^{-1}\pi_\gamma) \in \omega E_{\mu'}\pi_\gamma.$$

Therefore $\omega E\pi_\gamma = \omega E_{\mu'}\pi_\gamma$, so by Lemma 5.3(2), $V = E_{\mu'}\pi_\gamma$ is transitive on $\Gamma = \Gamma_\gamma$. Then as $W\pi_\gamma$ centralizes V, it follows that $W\pi_\gamma$ is semiregular on Γ, so (1) holds.

Now assume the hypothesis of (2). Suppose first that

$$W_\omega < P < W. \tag{5.1}$$

As γ is $N_{H_\omega}(W)$-invariant, so are Q_γ and P. However, if H is semisimple then $\mathrm{Aut}_{H_\omega}(W)$ is maximal in $\mathrm{Aut}_H(W)$, contrary to (5.1). Similarly, if H is complemented then $\mathrm{Inn}(W) \leq \mathrm{Aut}_{H_\omega}(W)$, again contrary to (5.1). Thus (5.1) fails, which establishes the first statement in (2). Furthermore, $W_\omega \leq Q_\gamma \cap Q_{\gamma'}$, so $W_\omega = W_\omega\sigma \leq P \cap R$, where $R = Q_{\gamma'}\sigma$. Thus if the second statement fails, then $P = R = W_\omega$ by the first statement, whereas $W = PR$ by Lemma 5.3(1). This contradiction completes the proof of (2). □

LEMMA 5.8. *Assume that H is semisimple, let $L = E_1$, $\beta = \beta_1$, take $1 \in \beta$, let A be the stabilizer in L of $\omega \in \Gamma_1$ under the representation $\pi_1 : L \to X_1 \leq \mathrm{Sym}(\Gamma_1)$, let $\sigma : E \to L$ be the projection map with respect to the direct sum decomposition $E = L \times E_{1'}$, and set $c = |L : L_\omega|$ and $d = |L : A|$.*

(1) $|\Omega| = c^r$.
(2) *$N_H(L)$ is transitive on β.*
(3) $b \leq 2$.
(4) *Assume that $b = 2$. Then $c = d^2$, $A = Q_1\sigma$, $|N_{H_\omega}(L) : N_{H_\omega}(A)| = 2$, and one of the following statements holds:*
    (i) *$L \cong A_6$, $A \cong A_5$, $L_\omega \cong D_{10}$, and $d = 6$;*
    (ii) *$L \cong M_{12}$, $A \cong M_{11}$, $L_\omega \cong L_2(11)$, and $d = 12$;*
    (iii) *$L \cong Sp_4(q)$ for some $q > 2$ even, $A \cong O_4^-(q)$, L_ω is an extension of $\mathbf{Z}_{q^2+1}$ by $\mathbf{Z}_4$, and $d = q^2(q^2-1)/2$.*

PROOF. Part (1) follows from Lemma 2.2(3i). Assume that (2) fails and let γ be an orbit of $K = N_H(L)$ on β. As $\varnothing \neq \gamma \subset \beta$, π_α is an injection on L for $\alpha \in \{\gamma, \gamma'\}$, so $1 \neq L_\omega\pi_\alpha \leq Q_\alpha$. In particular, $L\pi_\alpha$ is not semiregular on Γ_α, so as α is K-invariant, Lemma 5.7 supplies a contradiction, completing the proof of (2).

Suppose that $b > 1$. By Lemma 5.3(1), $L = PP'$, where $P = Q_1\sigma$ and $P' = Q_{1'}\sigma$. By (2), K is transitive on β, and as E is transitive on Ω, $K = EK_\omega$, so $J = K_\omega$ is transitive on β. Therefore $P^J = \{Q_i\sigma \mid i \in \beta\}$. Also for $j \in I - \beta$, $L \leq Q_j$, so $Q_j\sigma = L$. Thus as

$$Q_{1'} = \bigcap_{i \in 1'} Q_i,$$

it follows that

$$P' \leq \bigcap_{1 \neq i \in \beta} Q_i\sigma. \tag{5.2}$$

As in the previous paragraph, $L\pi_\alpha$ is not semiregular on Γ_α for $\varnothing \neq \alpha \subset \beta$, so by Lemma 5.7(1), $P < L$. Then as $L = PP'$, (5.2) says that $Q_i\sigma \neq P$ for $1 \neq i \in \beta$, so as $P^J = \{Q_i\sigma \mid i \in \beta\}$, it follows that

$$\text{The map } i \mapsto Q_i\sigma \text{ is a bijection of } \beta \text{ with } P^J. \tag{5.3}$$

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Furthermore, if $L_\omega = P$ then as $L_\omega \trianglelefteq J$, $P^J = \{P\}$, contradicting (5.3) and $b > 1$. Thus $L_\omega < P$. On the other hand, by Lemma 2.2(3i), $\mathrm{Aut}_J(L) = \mathrm{Aut}_{H_\omega}(L)$ is maximal in $\mathrm{Aut}_H(L) = \mathrm{Aut}_K(L)$. Thus $(\mathrm{Aut}_K(L), \mathrm{Aut}_J(L), L, P, b)$ satisfies the hypothesis in Lemma 3.4 on (G, M, L, A, v), so (4) follows from that lemma, once we show that P is the group A defined in this lemma. But Q_1 is the stabilizer in E of $\omega \in \Gamma_1$ under the representation $\pi_1 : E \to \mathrm{Sym}(\Gamma_1)$, so $A = L \cap Q_1$ and $L_\omega \leq Q_1$. As Q_1 acts on L as its projection P, $\langle L_\omega^P \rangle = \langle L_\omega^{Q_1} \rangle \leq L \cap Q_1 = A \leq P$. Finally, from the list of groups in Lemma 3.4, $P = \langle L_\omega^P \rangle$, so $P = A$, completing the proof. □

LEMMA 5.9. *Assume that H is semisimple and $b = 2$. Define d as in Lemma 5.8.*

(1) *$m = d^s$ and $|\Omega| = d^{2r} = d^{ks}$.*
(2) *$N_H(X_1)$ is transitive on S_1.*
(3) *$N_H(X_1)$ is primitive on Γ_1 with $\mathrm{Aut}_H(X_1)$ semisimple and $F^*(\mathrm{Aut}_H(X_1)) = \mathrm{Aut}_E(X_1)$ is the direct product of the s copies $\mathrm{Aut}_{E_i}(X_1)$, $i \in S_1$, of E_1.*
(4) *Either*
   (i) *X_1 is the alternating group A_m on Γ_1; or*
   (ii) *$s = 1$, $m = d$, and $\mathcal{B} = \{\beta_i \mid i \in \bar{I}\}$ is a system of imprimitivity for H on I. In addition, either:*
      (a) *$X_1 \cong E_1$; or*
      (b) *$E_1 \cong Sp_4(q)$, $d = q^2(q^2 - 1)/2$, $q = q_0^e$ for some integer $e > 1$, $X_1 \cong Sp_{4e}(q_0)$, and $X_{1,\omega} \cong O^-_{4e}(q_0)$.*

PROOF. Adopt the notation from Lemma 5.8; in particular, $c = d^2$. For $i \in \bar{I}$, let $F_i = E_i\pi_1$ and set $F = \langle F_i : i \in S_1 \rangle$. Then $E\pi_1 = F$, so F is transitive on $\Gamma = \Gamma_1$ by Lemma 5.3(2).

Next $N_H(E_1)$ is transitive on β by Lemma 5.8(2), so $N_H(X_1)$ is transitive on S_1, establishing (2). From Lemma 5.8(4) and the proof of Lemma 3.4, $F_{i,\omega}$ is maximal in F_i. Thus as the projection of F_ω on F_i acts on $F_{i,\omega}$, it follows that

(5) F_ω is the product of the groups $F_{i,\omega}$, $i \in S_1$, and $F_{i,\omega}$ is maximal in F_i.

By definition, $|F_i : F_{i,\omega}| = d$. Thus as F is transitive on Γ, it follows from (5) that $m = |F : F_\omega| = d^s$, so (1) follows from Lemmas 5.8(1) and 5.4(2). Also (3) follows from (2) and (5), and the transitivity of F on Γ. Then by Lemma 4.4, either (4i) holds or $s = 1$, and we may assume the latter. Then $m = d$ by (1), and visibly $\mathcal{B}$ is a system of imprimitivity for H on I. Without loss, $1 \in S_1$. Then from the main theorem of [LPS1], either $N_{\mathrm{Sym}(\Gamma)}(F_1)$ is the unique maximal overgroup of F_1 in $\mathrm{Sym}(\Gamma)$, so that (4i) or (4iia) holds, or (4iib) holds. This completes the proof of (4) and the lemma. □

DEFINITION 5.10. Define a semisimple group H to be *product decomposable* if for L a component of H, $c = |L : L_\omega|$, and $c = d^2$, one of the following statements holds:

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(i) $L \cong A_6$, $L_\omega \cong D_{10}$, and $d = 6$;
(ii) $L \cong M_{12}$, $L_\omega \cong L_2(11)$, and $d = 12$;
(iii) $L \cong Sp_4(q)$ for some $q > 2$ even, L_ω is an extension of $\mathbf{Z}_{q^2+1}$ by $\mathbf{Z}_4$ with $A \in \mathcal{O}_L(L_\omega)$ isomorphic to $O_4^-(q)$, and $d = q^2(q^2 - 1)/2$.

Define a semisimple group H to be *product indecomposable* if H is not product decomposable.

Observe that $\mathcal{O}_L(L_\omega)$ contains two members A_i, $i = 1, 2$, of index d in L, as described in Lemma 5.8(4). Furthermore, as H is semisimple, $\mathrm{Aut}_{H_\omega}(L)$ is maximal in $\mathrm{Aut}_H(L)$ by Lemma 2.2, so $N_{H_\omega}(L)$ is transitive on $\{A_1, A_2\}$, and hence $|N_{H_\omega}(L) : N_{H_\omega}(A_1)| = 2$. $\square$

Observe from Lemma 5.8 that if H is almost simple then H is product decomposable precisely when H preserves a nontrivial product structure on Ω. We will see in the next lemma that the general semisimple group H is product decomposable if $\mathcal{F}(H)$ can be composed with the rank-two product structure preserved by a component of H (using the composition defined in Definition 1.11) to produce a larger H-invariant product structure $\mathcal{F}^2(H)$.

LEMMA 5.11. *Assume that H is semisimple and product decomposable. Let $\hat{I} = \{1, 2\}$ and $\tilde{I} = \hat{I} \times \bar{I}$. Set $\Sigma_i = \omega E_i$.*

(1) *H preserves the regular (c, r)-product structure $\bar{\mathcal{F}} = \mathcal{F}(H)$ on Ω.*
(2) *For $i \in \bar{I}$, $E_i N_{H_\omega}(E_i)$ preserves a unique product structure $\hat{\mathcal{F}}_i$ on Σ_i. Indeed, $\hat{\mathcal{F}}_i = (\Sigma_{i,1}, \Sigma_{i,2})$ is the $(d, 2)$-structure defined by $\Sigma_{i,j} = \Sigma_{i,j}(\omega) A_{i,3-j}$ and $\Sigma_{i,j}(\omega) = \omega A_{i,j}$, where $\{A_{i,1}, A_{i,2}\}$ are the maximal overgroups of $E_{i,\omega}$ of index d in E_i described in Lemma 5.8(4).*
(3) *H preserves the $(d, 2k)$-product structure $\tilde{\mathcal{F}} = (\Omega_{i,j} : (i, j) \in \tilde{I})$ on Ω, defined by $\Omega_{i,j} = \{\Sigma_{i,j}(\omega) D_{i'} g \mid g \in A_{i,3-j}\}$. Furthermore, $\tilde{\mathcal{F}} = \hat{\mathcal{F}}_1 \circ \bar{\mathcal{F}}$.*
(4) *Let $\tilde{M} = N_G(\tilde{\mathcal{F}})$, $\tilde{X} = F^*(\tilde{M})$, and for $(i, j) \in \tilde{I}$ let $X_{i,j}$ be the component of $\tilde{M}$ fixing all blocks in $\Omega_{u,v}$ for all $(u, v) \in \tilde{I} - \{(i, j)\}$, $Y_i = N_{H_\omega}(A_{i,1}) \cap N_{H_\omega}(A_{i,2})$, and $K_i = Y_i E$. Then $E_i \le X_{i,1} \times X_{i,2}$ and $K_i = N_H(X_{i,j})$.*
(5) *Suppose that $b = 2$, for $i \in \bar{I}$ let $\beta_i = \{i_1, i_2\}$, and for $(i, j) \in \tilde{I}$ let $P_{i,j}$ be the projection of E on $X_{i,j}$ and $E_{i,j} = E_i \pi_{i_j}$. Then we can choose notation so that $P_{i,j} = E_{i,j}$.*
(6) *Identify the H-sets $\tilde{I}$ and $\{X_{i,j} \mid (i, j) \in \tilde{I}\}$ as in Notation 5.2. For $\gamma \subseteq \tilde{I}$, set $E_\gamma = \langle P_{i,j} : (i, j) \in \gamma \rangle$ and $\gamma' = \tilde{I} - \gamma$. For $K \in \mathcal{O}_H(K_1)$, let $\gamma_K = (1, 1)K$. Then $\mathcal{P}_K = \gamma_K^H$ is a partition of $\tilde{I}$. Set $\mathcal{E}_K = \{E_\gamma \mid \gamma \in \mathcal{P}_K\}$ and $\mathcal{F}^2(H, K) = \mathcal{F}(\mathcal{E}_K)$. Then $\mathcal{F}^2(H, K)$ is an (m_K, r_K)-product structure on Ω, where $r_K = |H : K|$ and $m_K = d^{|K:K_1|}$, and the map $\phi : K \mapsto \mathcal{F}^2(H, K)$ is a bijection of $\mathcal{O}_H(K_1)' = \mathcal{O}_H(K_1) - \{H\}$ with $\boldsymbol{\mathcal{F}}(H)$. Further, $\mathcal{F}^2(H) = \mathcal{F}^2(H, K_1) = \tilde{\mathcal{F}}$ and $b(N_G(\mathcal{F}^2(H, K)), H) = 1$ if and only if $N_H(E_1) \le K$.*

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PROOF. Part (1) is a consequence of Notation 2.6.

From Definition 5.10, $N_{H_\omega}(E_i)$ acts transitively on $\{A_{i,1}, A_{i,2}\}$ via conjugation. Thus $N_{H_\omega}(E_i) = Y_i\langle t_i\rangle$, where t_i fixes ω and interchanges $A_{i,1}$ and $A_{i,2}$. Thus $\omega A_{i,j}t_i = \omega A_{i,j}^{t_i} = \omega A_{i,3-j}$, so t_i interchanges $\Sigma_{i,1}$ and $\Sigma_{i,2}$. As $E_i = A_{i,1}A_{i,2}$, E_i permutes the blocks in $\Sigma_{i,j}$ for $j \in \hat{I}$. As Y_i fixes ω and acts on $A_{i,j}$, Y_i fixes the blocks. Thus $J_i = E_iY_i$ permutes the blocks, so as t_i interchanges $\Sigma_{1,1}$ and $\Sigma_{i,2}$, $\hat{J}_i = E_iN_{H_\omega}(E_i) = J_i\langle t_i\rangle$ preserves $\hat{\mathcal{F}}_i$. Moreover, this argument shows that J_i is the subgroup of index 2 in $\hat{J}_i$ which permutes $\Sigma_{i,j}$ for $j \in \hat{I}$. As the factorization $E_i = A_{i,1}A_{i,2}$ is determined up to conjugation in E_i and the ordering of the factors, the product structure $\hat{\mathcal{F}}_i$ is unique, completing the proof of (2). Notice also that $N_H(E_i) = EN_{H_\omega}(E_i) = E\hat{J}_i$, so $|N_H(E_i) : K_i| = |E\hat{J}_i : EY_i| = |\hat{J}_i : J_i| = 2$.

From the previous paragraph, the hypothesis of Lemma 1.13 is satisfied, so by that lemma $\tilde{\mathcal{F}}$ is the composition of the structures $\bar{\mathcal{F}}$ and $\hat{\mathcal{F}}_1$, and (3) holds.

From the description of $\tilde{\mathcal{F}}$ in (3), E_i acts on each block of $\Omega_{u,v}$ for $(u, v) \in \tilde{I} - \{(i, j)\}$, so $E_i \le X_{i,1} \times X_{i,2}$. Thus as H permutes $\mathcal{E}$ and the components $X_{i,j}$, $N_H(X_{i,j}) \le N_H(E_i)$. Then $K_i = EJ_i = N_H(X_{i,j})$, as J_i is the stabilizer in $N_{H_\omega}(E_i)$ of $\Sigma_{i,j}$, establishing (4).

Assume the hypothesis of (5) and let $\pi_{i,j} : \tilde{X} \to X_{i,j}$ be the projection map. As $E_i \le X_{i,1} \times X_{i,2}$, $P_{i,j} = E\pi_{i,j} = E_i\pi_{i,j} \cong E_i$. Then $P_{i,j}$ is characterized by the property that it is trivial on all blocks of $\tilde{\mathcal{F}}$ except for those in $\Omega_{i,j}$, and is isomorphic to E_i. As $E_{i,j}$ centralizes $E_{i'}$ and $E_{i,j,\omega} \ne 1$ acts on $\Sigma_{u,v}(\omega)$ for each $(u, v) \in \tilde{I}$ with $u \ne i$, $E_{i,j} = [E_{i,j}, E_{i,j,\omega}]$ fixes all fibers not in $\Sigma_{i,l}$ for some $l \in \hat{I}$. Furthermore, from Example 1.6, the orbits of $E_{i,1}$ and $E_{i,2}$ on Σ_i form a product structure on Σ_i, so from (2) this structure is $\hat{\mathcal{F}}$. Thus we may choose notation such that $E_{i,j} \cong E_i$ is trivial on all blocks of $\tilde{\mathcal{F}}$ except those in $\Sigma_{i,j}$, so that $E_{i,j} = P_{i,j}$, completing the proof of (5).

Let $K \in \mathcal{O}_H(K_1)$. By (4), $K_1 = N_H(X_{1,j})$, so K_1 is the stabilizer of $(1, j) \in \tilde{I}$ in H. Thus as $K_1 \le K$, $\mathcal{P}_K$ is a partition of $\tilde{I}$. Hence by Example 1.6, $\mathcal{F}^2(H, K)$ is an H-invariant (m_K, r_K)-product structure on Ω. Visibly the map ϕ is injective. Thus it remains to show that ϕ is surjective. Hence we may assume that M is the stabilizer in G of $\mathcal{F}$, and it suffices to show that $\mathcal{F} = \mathcal{F}^2(H, K)$, where $K = N_H(X_1)$. Pick notation so that E_1 projects nontrivially on X_1.

First, $b = 1$ if and only if $E_1 \le X_1$ if and only if $N_H(E_1) \le K$ by Lemma 5.8(2). In that event, since $K_1 \le N_H(E_1)$, then $K_1 \le K$, so indeed $K \in \mathcal{O}_H(K_1)$. Furthermore, we saw earlier that $|N_H(E_1) : K_1| = 2$, so $\gamma_K = \{(i, j) \mid E_i \le X_1\}$, and hence $E_{\gamma_K} = \langle E_i : E_i \le X_1\rangle$. Thus $\omega X_1 = \omega E_{\gamma_K}$, so indeed $\mathcal{F}^2(H, K) = \mathcal{F}$ by Lemma 1.7. Thus we may assume that $b = 2$.

By (5), $E_{1,1} = E_1 \cap X_{1,1}$ and $K_1 = N_H(X_{1,1})$, so $E_{1,1}$ is K_1-invariant. Thus $K_1 \le K$, so again $K \in \mathcal{O}_H(K_1)$. By Lemma 5.9(2), K is transitive on S_1, so $\gamma_K = \{(i, j(i)) \mid i \in S_1\}$ for some function $j : S_1 \to \hat{I}$ such that $\Gamma_1 = \omega F$, where $F = \langle E_{i,j(i)} : i \in S_1\rangle$. Then $\Gamma_1 = \omega F = \omega E_{\gamma_K}$ as $E_{i,j(i)} = P_{i,j}$ by (5). Thus $\mathcal{F} = \mathcal{F}^2(H, K)$, completing the proof. □

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LEMMA 5.12. *Assume that H is semisimple and product indecomposable. Then:*

(1) $b = b(M, H) = 1$ *for each* $M \in \mathcal{O}_G(H)$ *which is semisimple;*
(2) *the map* $K \mapsto \mathcal{F}(H, K)$ *is a bijection of* $\mathcal{O}_H(N_H(E_1))'$ *with* $\boldsymbol{\mathcal{F}}(H)$.

PROOF. As H is indecomposable, (1) follows from parts (3) and (4) of Lemma 5.8. As defined in Notation 2.6, $\mathcal{F}(H, K)$ is an H-invariant product structure. If $\mathcal{F}'$ is an H-invariant product structure, then $H \leq M' = N_G(\mathcal{F}')$, and M' is semisimple. Then by (1) and Lemma 5.5(6), $\mathcal{F}' = \mathcal{F}(H, K)$, where $K = N_H(X_1')$ and X_1' is the component of $F^*(M')$ containing E_1. By Lemma 5.3(4), $K \neq H$, so the map ϕ of (2) is surjective. Visibly ϕ is injective, completing the proof. □

LEMMA 5.13. *Assume that H is complemented and maximal in M. Then $b = 1$.*

PROOF. Let $\rho : M \to \mathrm{Sym}(M/H)$ be the representation of M on M/H via right multiplication. Suppose that $K = \ker(\rho) \neq 1$. Then as X is the unique minimal normal subgroup of M, $X \leq K \leq H \leq N_M(E)$. By Lemma 4.3, $E \leq X$, so $E \trianglelefteq X$. Then as $C_H(E) = 1$, $E = X$, contradicting $E_\omega = 1 \neq X_\omega$.

Therefore ρ is faithful. As H is maximal in M, M is primitive on M/H. Thus, applying Lemma 2.2 to ρ and recalling that $X = E(M)$ is the unique minimal normal subgroup of M, we conclude that M is semisimple, complemented, or diagonal on M/H. As $1 \neq E \leq H \cap M$, M is not complemented. Suppose that M is semisimple on M/H. Then $H \cap X$ is the direct product of the groups $H \cap X_i$, $i \in I$. Let $j \in S_i$. Then $1 \neq [H \cap X_i, E_j]$, so $E_j \leq [H \cap X_i, E_j] \leq X_i$, so $b = 1$ and the lemma holds. Thus we may assume that M is diagonal on M/H. Hence there is an H-invariant partition Δ of I with $H \cap X$ the product of full diagonal subgroups F_δ of X_δ, $\delta \in \Delta$. Therefore $\mathcal{E} = \{F_\delta \mid \delta \in \Delta\}$, $E_1 \cong X_1$, $s = 1$, and $b = |\delta|$ for $\delta \in \Delta$. By Lemma 3.5 we may pick $p \in \pi(X_1)$ with $m_p(X_1) = 1$. Then by Lemma 3.6,

$$k = m_p(X) = m_p(E) + m_p(X_\omega) = r + km_p(X_{1,\omega}),$$

contradicting $k = rb$ and $b > 1$. □

6. Diagonal overgroups of primitive groups

In this section we make the following assumption.

HYPOTHESIS 6.1. Hypothesis 2.1 is satisfied and H is not affine. Furthermore, $M \in \mathcal{O}_G(H)$ and M is diagonal.

NOTATION 6.2. Let $X = F^*(M)$, $I = \{1, \ldots, k\}$, and $\bar{I} = \{1, \ldots, r\}$.

Let $\mathcal{X} = \{X_i \mid i \in I\}$ be the set of components of M. For $\gamma \subseteq I$, set $\gamma' = I - \gamma$, $X_\gamma = \langle X_i : i \in \gamma \rangle$, $\Gamma_\gamma = \omega X_\gamma$, and $\pi_\gamma : X \to X_\gamma$ the projection map with respect to the direct sum decomposition $X = X_\gamma \times X_{\gamma'}$. Set $c = |X_i|$.

Let E be a minimal normal subgroup of H and $\mathcal{E} = \{E_i \mid i \in \bar{I}\}$ the set of components of E. Represent M on I and H on $\bar{I}$ so that the maps $i \mapsto X_i$ and

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$j \mapsto E_j$ are equivalences of permutation representations. As H is transitive on $\mathcal{E}$, H is transitive on $\bar{I}$.

As M is diagonal, there exists a maximal M-invariant partition $\Sigma = \Sigma(M)$ of I such that X_ω is the direct product of full diagonal subgroups F_σ, $\sigma \in \Sigma$, of X_σ. Recall that M is *strongly diagonal* if $\Sigma = \{I\}$. In that event write $\mathbf{d}(M)$ for the diagonal structure $\mathrm{diag}(\mathcal{X}, X_\omega)$ defined by M as in Definition 1.9. Let $k_M = |\Sigma|$ and $m_M = c^{|\sigma|-1}$ for $\sigma \in \Sigma$. □

LEMMA 6.3. *Assume that M is not strongly diagonal and let $\mathcal{D} = \{X_\sigma \mid \sigma \in \Sigma\}$.*

(1) *$\mathcal{F}(M) = \mathcal{F}(\mathcal{D})$ is an M-invariant regular (m_M, k_M)-product structure on Ω.*

(2) *Let $M' = N_G(\mathcal{F}(M))$ and let Y_σ be the component of M' acting faithfully on the σ-factor Γ_σ of the product structure. Then $\mathrm{Aut}_M(Y_\sigma)$ is strongly diagonal on Γ_σ.*

PROOF. Part (1) follows from Example 1.6. By construction $\mathrm{Aut}_X(Y_\sigma) = X_\sigma$ is the direct product of $|\sigma| > 1$ copies of X_1 with $X_{\sigma,\omega} = F_\sigma$ a full diagonal subgroup, and $N_H(\sigma)$ is primitive on σ, so (2) holds. □

LEMMA 6.4. *Assume that M is strongly diagonal.*

(1) *$X_\omega \cong X_1$ is an M_ω-invariant full diagonal subgroup of X.*

(2) *$|\Omega| = c^{k-1}$.*

PROOF. This follows from Lemma 2.2. □

LEMMA 6.5. *Assume that H is doubled. Then $k = 2r$, $r = k_M$, $|\sigma| = 2$ for $\sigma \in \Sigma$, $X = D$, and:*

(1) *if M is strongly diagonal then $r = 1$ and $\mathbf{d}(M) = \mathbf{d}(H)$ as defined in Lemma 4.7(2);*

(2) *if M is not strongly diagonal then $\mathcal{F}(M) = \mathcal{F}(H)$ as defined in Notation 2.6.*

PROOF. First assume that M is strongly diagonal. As H is doubled, $|D_\omega| = |E| = |\Omega|$. Thus $|D_\omega| = c^{k-1}$ by Lemma 6.4(2). But by Lemma 4.3, $D_\omega \leq X_\omega$ and by Lemma 6.4(1), $X_\omega \cong X_1$ is simple of order c. Therefore $k = 2$ and $D_\omega = X_\omega$ is simple. Therefore E is simple, so $r = 1$ and $X = D$. Now Lemma 4.7(2) completes the proof in this case.

So assume that M is not strongly diagonal, form the product structure $\mathcal{F} = \mathcal{F}(M)$ of Lemma 6.3, and adopt the notation of Lemma 6.3(2). By Lemma 6.3, $H \leq M \leq M'$, so by Lemma 5.6(1), $b' = b(M', H) = 1$. Then by Lemma 5.5(7), $\mathrm{Aut}_H(Y_\sigma)$ is doubled on Γ_σ with $D \cap Y_\sigma$ the direct product $(E \cap Y_\sigma) \times (\tilde{E} \cap Y_\sigma)$, where $\tilde{E}$ is the second minimal normal subgroup of H. Now by Lemma 5.6(2), $\mathcal{F} = \mathcal{F}(H, H_\sigma)$, where H_σ is the stabilizer in H of σ. Moreover, $D \cap Y_\sigma \leq X \cap Y_\sigma = X_\sigma$, $\mathrm{Aut}_M(Y_\sigma)$ is strongly diagonal on Γ_σ by Lemma 6.3(2), and we saw that $\mathrm{Aut}_H(Y_\sigma)$ is doubled on Γ_σ, so by (1), $|\sigma| = 2$ and $X_\sigma = D \cap X_\sigma$. Thus $k = |\sigma| k_M = 2r$ and $X = D$, and Lemma 4.7(1) completes the proof. □

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LEMMA 6.6. *If H is diagonal then $D = X$ and $\Sigma(M) = \Sigma(H)$.*

PROOF. Assume that H is diagonal, let $\Delta = \Sigma(H)$ be of order k_H, and set $e = |E_1|$. Then by Lemma 2.2(3ii),

$$e^{r-k_H} = |\Omega| = c^{k-k_M}, \tag{6.1}$$

so e and c have the same set Π of prime divisors. By Lemma 3.5, there exists $p \in \Pi$ such that X_1 has cyclic Sylow p-subgroups. Thus $m_p(X) = k$.

Suppose that M is strongly diagonal. Then $X_\omega \cong X_1$, so $m_p(X_1) = 1$. But E_ω is the direct product of k_H copies of E_1, and $E_\omega \leq X_\omega$, so

$$1 = m_p(X_\omega) \geq m_p(E_\omega) = k_H m_p(E_1).$$

It follows that $k_H = 1$ and $m_p(E_1) = 1$. Thus H is strongly diagonal.

Next the product J of $r-1$ components of E is a regular normal subgroup of E, so by Lemma 3.6(2),

$$k = m_p(X) = m_p(J) + m_p(X_\omega) = (r-1)m_p(E_1) + 1 = r.$$

Thus $k = r$, so it follows from (6.1) that $c = e$. Then $|X| = c^k = e^r = |E|$, so $X = E = D$ and as both H and M are strongly diagonal, also $\Sigma(H) = \Sigma$. Thus the lemma holds in this case.

So assume $k_M > 1$, form the product structure $\mathcal{F} = \mathcal{F}(M)$ of Lemma 6.3, and adopt the notation of Lemma 6.3(2). Let $I_H = \bar{I}$, $I_M = I$, and for $U \in \{H, M\}$ and $\sigma \in \Sigma$, define $S_{U,\sigma} = \{j \in I_U \mid F^*(U)_j \leq X_\sigma\}$ and the partition $\mathcal{S}_U = \{S_{U,\sigma} \mid \sigma \in \Sigma\}$ of I_U as in Notation 5.2 and Lemma 5.5(3). By construction in Lemma 6.3, $S_{M,\sigma} = \{\sigma\}$, so $\mathcal{S}_M = \Sigma$. From Lemma 5.6(1), $b' = b(M', H) = 1$, and then from Lemma 5.5(8), $\mathrm{Aut}_H(Y_\sigma)$ is diagonal on Γ_σ, with $\Delta = \Sigma(H)$ a refinement of $\mathcal{S}_H$. As usual, $\mathrm{Aut}_H(Y_\sigma) \leq \mathrm{Aut}_M(Y_\sigma)$, which is strongly diagonal on Γ_σ by Lemma 6.3(2). Thus from the case treated above, $X_\sigma = D \cap Y_\sigma$ and $\mathrm{Aut}_H(Y_\sigma)$ is strongly diagonal. Therefore $X = D$ and $N_H(Y_\sigma)$ is primitive on σ, so $\Sigma = \Delta$ as Δ is a refinement of $\mathcal{S}_H$. Thus the lemma holds in this case too. □

In the next lemma, for $K \in \{H, M\}$, $\mathcal{P}(K)$ is the set of partitions of the components of a minimal normal subgroup of K defined in Notation 2.6.

LEMMA 6.7. *Assume that H is complemented. Then $k = 2r$, $r = k_M$, $E \trianglelefteq X = EC_G(E)$, and $\mathcal{P}(H) = \mathcal{P}(M)$.*

PROOF. Assume otherwise. For $i \in \bar{I}$ let

$$\beta_i = \{j \in I \mid E_i\pi_j \neq 1\},$$

and for $i \in I$ let $S_i = \{j \in \bar{I} \mid E_j\pi_i \neq 1\}$. As H is transitive on $\bar{I}$, $b = |\beta_i|$ is independent of $i \in \bar{I}$. For $\alpha \subseteq \bar{I}$, let $E_\alpha = \langle E_i : i \in \alpha \rangle$.

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Let $e = |E_i|$. Then

(1) $e^r = |\Omega| = c^{k-k_M}$,

so the set Π of primes dividing e and c are the same. By Lemma 3.5, there exists $p \in \Pi$ with $m_p(X_1) = 1$. Thus as $\pi_i : E_{S_i} \to X_i$ is an injection for each $i \in I$, it follows that

(2) for all $i \in I$, $|S_i| \leq 1$,

and

(3) for all $j \in \bar{I}$, $m_p(E_j) = 1$.

It follows from (2) that

(4) $\mathcal{B} = \{\beta_j \mid j \in \bar{I}\}$ is a partition of $I' = \{i \in I \mid |S_i| = 1\}$, so $|I'| = rb$; and
(5) $r = k - k_M$.

Namely by (3), $m_p(E) = r$, while by Lemma 3.6(2),

$$k = m_p(X) = m_p(E) + m_p(X_\omega) = r + k_M,$$

establishing (5). Then by (1) and (5), $c = e$, so as $\pi_i : E_1 \to X_i$ is injective for $i \in \beta_1$,

(6) $E_1 \cong X_1$ and $c = e$.

Let $u = |\sigma|$ for $\sigma \in \Sigma$; thus $k = k_M u$. We next show that

(7) either $b = 1$, or $b = u = 2$, $k = 2r$, $r = k_M$, and $I = I'$.

Namely by (4) and (5), $rb \leq k = r + k_M$, so $k_M \geq r(b-1)$. Furthermore, $k = k_M u$, so $r = k_M(u-1)$ by (5). Thus $k_M \geq k_M(u-1)(b-1)$, so as $u > 1$, it follows that either $b = 1$ or $b = u = 2$ and all inequalities are equalities. That is, (7) holds.

(8) Either
 (i) H is transitive on I, or
 (ii) $u = b = 2$, $k = 2r$, $r = k_M$, $I = I'$, H has orbits γ and γ' on I of length r, and E is a full diagonal subgroup of $X_\gamma \times X_{\gamma'}$.

For assume that $\gamma \neq I$ is an orbit of H on I. Then X_γ and $X_{\gamma'}$ are normal in HX, so HX has at least two minimal normal subgroups. But HX is primitive by Lemma 2.4, so from Lemma 2.2, HX is doubled and X_α is regular on Ω for $\alpha \in \{\gamma, \gamma'\}$. It follows from (1) that $k_M = |\gamma|$ and $k = 2k_M$, so $u = 2$. Next as E is a minimal normal subgroup of H, either $E \cap X_\gamma = 1$ or $E \leq X_\gamma$. But as $|E| = |X_\gamma|$ by (1), in the latter case $E = X_\gamma$. Then $\mathcal{P}(M) = \mathcal{P}(H)$, the product structure in Notation 2.6, and $X = EC_G(X)$ by Lemma 4.6, contrary to the choice of M as a counterexample. Hence $b \neq 1$, so (ii) holds by (7).

(9) $u = b = 2$, $k = 2r$, $r = k_M$, and $I = I'$.

If not, then $b = 1$ and H is transitive on I by (7) and (8). As $b = 1$, we may take $E_1 \le X_1$, so $E_1 = X_1$ by (6). Then as H is transitive on I, $E = X$ and $r = k$, contrary to (1).

(10) $M = HX$ and H is transitive on I.

For $H \le HX = \tilde{M}$, and by Lemma 2.4, $\tilde{M}$ is primitive on Ω. As M is diagonal, X_ω is the product of full diagonal subgroups of the groups X_σ, $\sigma \in \Sigma$. If H is transitive on I then X is the unique minimal normal subgroup of $\tilde{M}$, so $\tilde{M}$ is diagonal, and hence as $b \ne 1$, $M = \tilde{M}$ by induction on $|M|$. On the other hand, if M is not transitive on I then $\tilde{M}$ has at least two minimal normal subgroups, so $\tilde{M}$ is doubled. Then applying Lemma 8.1 (whose proof does not depend upon this lemma) to $\tilde{M}$, we conclude that $D \trianglelefteq X$, again contradicting $b \ne 1$.

For $i \in I$, let σ_i be the block in Σ containing i. Define a graph $\mathcal{G}$ on Σ by σ adjacent to μ if $\sigma \cap \beta_i \ne \varnothing \ne \mu \cap \beta_i$ for some $i \in \bar{I}$. Let Δ be a connected component of $\mathcal{G}$, $J = \{i \in I \mid \sigma_i \in \Delta\}$ and $\bar{J} = \{i \in \bar{I} \mid \beta_i \subseteq J\}$. For $i \in \bar{I}$, let $K_i = O^2(H_i)$.

(11) $|J| = 2|\bar{J}|$, $K_i = K$ is independent of $i \in \bar{J}$, and K fixes J and $\bar{J}$ pointwise.

Observe that Δ is a partition of J. Hence if $i \in \bar{I}$ with $\beta_i \cap J \ne \varnothing$, then $\beta_i \cap \sigma \ne \varnothing$ for some $\sigma \in \Delta$. Then as Δ is a connected component of $\mathcal{G}$, for each $\mu \in \Sigma$ with $\beta_i \cap \mu \ne \varnothing$, $\mu \in \Delta$. Thus $\beta_i \subseteq J$, so $i \in \bar{J}$. Hence by (4) and (9), $\{\beta_i \mid i \in \bar{J}\}$ is a partition of J. Next $I = I'$ by (9), so the map $i \mapsto \beta_i$ is a bijection of $\bar{I}$ with $\mathcal{B}$. Thus as $b = 2$, $|J| = 2|\bar{J}|$. Let $i \in \bar{J}$ and $j \in \beta_i$. As $b = 2$, K_i fixes β_i pointwise, and then as $u = 2$, K_i fixes σ_j pointwise for $j \in \beta_i$. Then as Δ is connected, K_i fixes J and $\bar{J}$ pointwise. Hence for $j \in \bar{J}$, $K_i \le O^2(H_j) = K_j$, so as H is transitive on $\bar{I}$, $K_i = K_j$. This completes the proof of (11).

(12) $J \ne I$.

Suppose that $\bar{J} = \bar{I}$. Then by (11), K fixes $\bar{I}$ pointwise, so $K^\infty \le H_{\bar{I}}^\infty = D$. Now $Y = N_{H_\omega}(E_1)^\infty \le H_1^\infty = K^\infty \le D$. Thus $Y = 1$ as D is regular on Ω, which is a contradiction as $\mathrm{Aut}_Y(E_1) = \mathrm{Inn}(E_1)$ by Lemma 2.2. Therefore $\bar{J} \ne \bar{I}$, so as $|I| = 2|\bar{I}|$ by (9) and $|J| = 2|\bar{J}|$ by (11), also $J \ne I$.

We now obtain a contradiction, establishing the lemma. Namely H permutes the connected components of $\mathcal{G}$, so as $M = HX$ by (10), M is transitive on those components, and hence $\Gamma = J^M$ is a partition of I. Then as $J \ne I$ by (12), $\Gamma \in \mathcal{P}(M)$. Therefore, from Notation 2.6 and Lemma 5.5(8), $N_M(J)$ acts primitively as a diagonal group M' on $\Omega' = \omega M'$. Similarly, by Lemma 5.5(9), $N_H(J)$ acts primitively as a complemented or doubled group H' on Ω'. In the first case, proceeding by induction on $|\Omega|$, $b = 1$, contrary to (9). Therefore $N_H(J)$ is doubled on Ω'. Then from Lemma 6.5, $X = EC_X(E)$, contradicting $b \ne 1$. □

7. The proofs of Propositions 5, 6, and 7

In this section we assume Hypothesis 2.1. Refer to Example 1.6 and Notation 2.6 for the definitions of $\mathcal{F}(\mathcal{L})$, $\mathcal{F}(H)$, and $\mathcal{F}(H, K)$.

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LEMMA 7.1. *Assume that H is octal semisimple with components $\mathcal{L} = \{L_1, \ldots, L_k\}$.*

(1) *If $k > 1$ then $N_S(D) \leq N_S(\mathcal{F}(\mathcal{L}))$ and $\mathcal{F}(\mathcal{L}) = \mathcal{F}(H)$.*

(2) *$N_S(D) = DUT$, where $U = \langle u_1, \ldots, u_k\rangle \cong E_{2^k}$, u_i induces an outer automorphism on L_i and centralizes L_j for $j \neq i$, T acts faithfully as $\mathrm{Sym}(\mathcal{L})$ on $\mathcal{L}$, and on U as the $\mathbf{F}_2T$-permutation module, and $N_S(D)_\omega = D_\omega UT$.*

(3) *Let $\mathcal{H}$ be the set of primitive semisimple subgroups $\tilde{H}$ of S such that $D = F^*(\tilde{H})$. Then U acts regularly on the set $\mathcal{A}$ of affine structures invariant under some member of $\mathcal{H}$.*

(4) *For $R \in \mathcal{A}$, $N_S(R) \cap N_S(D) = DT_R$ for some complement T_R to DU in $N_S(D)$ fixing ω.*

(5) *Let $\mathcal{A}(H)$ be the set of H-invariant affine structures on Ω. If $\mathcal{A}(H) \neq \varnothing$, then $|\mathcal{A}(H)| = |C_{UD/D}(H)|$ and $N_S(H)$ is transitive on $\mathcal{A}(H)$.*

PROOF. First (see [AS, 1.1]), $A = \mathrm{Aut}(D) = DUT$, where UT and its action on D are described in (2). The representation of A on $\Omega' = A/D_\omega UT$ via right multiplication embeds A in $\mathrm{Sym}(\Omega')$ in such a way that $D_\omega UT$ is the stabilizer of $\omega' = D_\omega UT \in \Omega'$, and so that A preserves the product structure $\mathcal{F}(\mathcal{L})$ on Ω' when $k > 1$. Then identifying Ω with Ω', we conclude that (2) holds, and $N_S(D) = A$ with $A \leq N_S(\mathcal{F}(\mathcal{L}))$ when $k > 1$. By definition of $\mathcal{F}(H)$ in Notation 2.6, $\mathcal{F} = \mathcal{F}(L) = \mathcal{F}(H)$ when $k > 1$, completing the proof of (1).

When $k > 1$, let $M = N_S(\mathcal{F})$ and $Y = F^*(M)$. Then using Lemma 1.8, $Y = Y_1 \times \cdots \times Y_k$, with Y_i acting faithfully as the alternating group on the ith set Ω_i of partitions in $\mathcal{F}$, and with Y_i trivial on Ω_j for $j \neq i$. When $k = 1$, let $Y = Y_1 = F^*(S)$, $M = S$, and $\Omega = \Omega_1$. Observe that UT is a complement to Y in M with $\langle u_i\rangle Y_i \cong S_8$ and $[u_i, Y_j] = 1$ for $i \neq j$.

In Y_i there exists $X_i \cong E_8$ regular on Ω_i with L_i acting as $GL(X_i)$ on X_i. Then $X = X_1 \times \cdots X_k \cong E_{2^{3k}}$ is regular on Ω, so D stabilizes the affine structure $R = R(X)$ of Lemma 1.4. That is, $R \in \mathcal{A}$. Let $\tilde{M} = N_S(R)$, so that $\tilde{M} = \tilde{M}_\omega X$ with $\tilde{M}_\omega$ acting faithfully as $GL(X)$ on X by Lemma 1.3.

Observe that $N_{\tilde{M}}(D) = D\tilde{T}$, where $\tilde{T}$ acts faithfully as $\mathrm{Sym}(\mathcal{L})$ on $\mathcal{L}$ and fixes ω. Thus (4) holds.

The representation of D on X is determined up to quasiequivalence, so if $g \in S$ with $D^g \leq \tilde{M}$, then there exists $c \in \tilde{M}$ with $L_i^{gc} X = L_i X$ for each i. As $H^1(L_i, X) \cong \mathbf{Z}_2$, there are two conjugacy classes $L_i^{X_i}$ and $K_i^{X_i}$ of complements to X_i in $X_i L_i$. Furthermore, we may choose $K_i \leq \tilde{M}_\omega$. Therefore, as L_i has no fixed points on Ω, $L_i^{gc} \in L_i^{X_i}$, so $D^g \in D^{\tilde{M}}$. Thus $\tilde{M}$ is transitive on $D^S \cap \tilde{M}$, so $N_S(D)$ is transitive on $\mathcal{A}$. Then as $D\tilde{T} = N_{\tilde{M}}(D)$, (3) follows.

Let $B = N_S(D)$ and $\bar{B} = B/D$. By (3), $\bar{U}$ is regular on $\mathcal{A}$. Assume that $\mathcal{A}(H) \neq \varnothing$; then conjugating in U, we may assume that $H \leq \tilde{M}$. Then as $\bar{U}$ is regular on $\mathcal{A}$, it follows that $C_{\bar{U}}(\bar{H})$ is regular on $\mathcal{A}(H)$, so as $C_{\bar{U}}(\bar{H}) = \overline{N_{UD}(H)}$, $N_{UD}(H)$ is transitive on $\mathcal{A}(H)$, establishing (5). $\square$

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We next prove Proposition 5. Assume the hypothesis of Proposition 5. By Lemma 4.5, M is not complemented, while by Lemma 4.8, M is not doubled or diagonal. Therefore M is affine or semisimple. Set $X = F^*(M)$.

Suppose that M is affine. Then by Lemma 4.3, H is octal semisimple. By Lemma 7.1(5), $N_S(H)$ is transitive on $\mathcal{A}(H)$. Thus conclusion (2) of Proposition 5 holds in this case, so we may assume that M is semisimple, and adopt Notation 5.2.

If H is product decomposable, then conclusion (3) of Proposition 5 holds by Lemma 5.11(6). Hence we may assume that H is product indecomposable. Then by Lemma 5.12(1), $b = b(M, H) = 1$, so each component of H is contained in a component of M. By Lemma 5.3(4), H is transitive on the components of M. Finally, by Lemma 5.12(2), the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(L))'$ with $\boldsymbol{\mathcal{F}}(H)$. This completes the proof of Proposition 5.

We now prove Proposition 6. Assume the hypothesis of that proposition. By Proposition 4, H is not affine. Thus Hypothesis 6.1 is satisfied, so we can adopt Notation 6.2. By Lemma 4.8, H is not semisimple. If H is doubled then conclusion (2) of Proposition 6 holds by Lemma 6.5, so we may assume that H is not doubled. Similarly, if H is diagonal then conclusion (1) of Proposition 6 holds by Lemma 6.6, so we may assume that H is complemented. Then Lemma 6.7 says that conclusion (3) of Proposition 6 holds, completing the proof of the proposition.

Finally, we prove Proposition 7, so we assume the hypothesis of Proposition 7. By Proposition 3, M is not affine. By Lemma 4.5, M is not complemented, and by Lemma 4.8, M is not doubled. If M is diagonal then conclusion (1) of Proposition 7 holds by Proposition 6. Thus we may assume that M is semisimple, and adopt Notation 5.2. By Lemma 5.6, $b = b(M, H) = 1$, so each component of H is contained in a component of M. By Lemma 5.3(4), H is transitive on the components of M. Finally, by Lemma 5.5(8) and Lemma 5.6(2), the map $K \mapsto \mathcal{F}(H, K)$ is a bijection of $\mathcal{O}_H(N_H(\sigma))'$ with $\boldsymbol{\mathcal{F}}(H)$. This completes the proof of Proposition 7.

8. The proofs of Propositions 8, 9, and 11

In this section we assume that Hypothesis 2.1 is satisfied.

LEMMA 8.1. *Assume that $M \in \mathcal{O}_G(H)$, H is complemented, and M is doubled. Then $F^*(M) = DC_G(D)$.*

PROOF. Let $X = F^*(M)$. Replacing M by HX, we may assume that $M = HX$, so that also $M = H_\omega X$. As M is doubled, M has two minimal normal subgroups X_1 and X_2, $X = X_1 \times X_2$, and X_ω is a full diagonal subgroup of X with respect to this direct sum decomposition. Let $I = \{1, \ldots, k\}$ and $\mathcal{L} = \{L_i \mid i \in I\}$ the set of components of X_1. Let $\pi_i : X \to X_i$ be the projection map, $\sigma_i = \pi_{i|X_\omega} \to X_i$ the restriction of π_i to X_ω, and $\alpha = \sigma_1^{-1}\sigma_2 : X_1 \to X_2$. Then α is an H_ω-equivariant isomorphism.

By Lemma 4.3, $D \leq X$, and as H is complemented, D is regular on Ω and the unique minimal normal subgroup of H. Thus if $D \cap X_i \neq 1$ for some i, then $D \leq X_i$, and then as $|D| = |\Omega| = |X_i|$, we have $D = X_i$. Then the lemma follows

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from Lemma 4.6 in this case, so we may assume that $D \cap X_i = 1$ for $i = 1, 2$. Then as $|D| = |X_i|$, D is also a full diagonal subgroup of X, so writing $\rho_i : D \to X_i$ for the projection of D on X_i, as above, we obtain an H_ω-equivariant isomorphism $\beta = \rho_1^{-1}\rho_2 : X_1 \to X_2$. Let $D_i = L_i\rho_1^{-1}$. As $\rho_1 : D \to X_1$ is an isomorphism, $\mathcal{D} = \{D_i \mid i \in I\}$ is the set of components of H.

We now argue as in the last few paragraphs of the proof of Lemma 6.7 to obtain a contradiction. As in Sections 5 and 6, we represent M on I so that the map $i \mapsto L_i$ is an equivalence of that representation with the representation of M on $\mathcal{L}$ via conjugation. As ρ_i is an H-equivariant isomorphism and H_ω is transitive on $\mathcal{D}$, H_ω is also transitive on $\mathcal{L}$, and hence also on I.

Let $\gamma = \alpha\beta^{-1} : X_1 \to X_1$. Then $\gamma \in \mathrm{Aut}(X_1)$ is an H_ω-equivariant automorphism of X_1, which induces $\tau \in \mathrm{Sym}(I)$ commuting with the action of H_ω on I. As H_ω is transitive on I, the set Σ of orbits of $T = \langle\tau\rangle$ on I is an H_ω-invariant regular partition of I. Let J be an orbit of T on I, $X_J = \langle L_j L_j\alpha : j \in J\rangle$, $D_J = \langle D_j : j \in J\rangle$, $H_J = N_{H_\omega}(J)X_J$, and $\Omega_J = \omega X_J$. For $j \in J$, $D_j \leq L_j L_j\beta$ and $L_j\beta = L_j\gamma^{-1}\alpha = L_{j\tau^{-1}}\alpha \leq X_J$, so $D_J \leq X_J$. Arguing as in the proof of Lemma 6.7, D_J is a minimal normal subgroup of H_J regular on Ω_J, so H_J acts primitively on Ω_J as a complemented or doubled group. Then $M_J = X_J H_J$ acts primitively on Ω_J as a doubled group with minimal normal subgroups $X_{J,i} = X_i \cap X_J$, $i = 1, 2$. If H_J is doubled, then by Lemma 4.8(3), $D_J = X_{J,i}$ for $i = 1$ or 2, contradicting $D \cap X_i = 1$. Thus H_J is complemented, so proceeding by induction on $|\Omega|$, we may assume that $\Omega_J = \Omega$. Thus T is transitive on I, so as $\tau \in C_{\mathrm{Sym}(I)}(H_\omega)$, H_ω acts as T on I. Therefore $\mathrm{Aut}_{H_\omega^\infty}(D)$ is contained in the kernel of the action of $\mathrm{Aut}(D)$ on $\mathcal{D}$, so $H_\omega^\infty \leq D_\omega = 1$. Thus H_ω is solvable, whereas H is complemented, so $\mathrm{Inn}(D_1) \leq \mathrm{Aut}_{H_\omega}(D_1)$, giving a contradiction. □

We now give proofs of Propositions 8, 9, and 11.

Assume the hypothesis of Proposition 8. By Propositions 4, 5, and 7, H is not affine, semisimple, or diagonal, so H is doubled or complemented. In the first case conclusion (1) of Proposition 8 holds by Lemma 4.8(3), while in the second case conclusion (2) of Proposition 8 holds by Lemma 8.1. This establishes Proposition 8.

Assume the hypothesis of Proposition 9. By Proposition 3, M is not affine, and by Lemma 4.5, M is not complemented. If M is diagonal or doubled, then the proposition follows from Propositions 6 or 8, respectively. Therefore we may assume that M is semisimple, and adopt Notation 5.2. By Lemma 5.6(1), each component L of H is contained in a component of M, while by Lemma 5.3(4), H is transitive on the components of M. Then conclusion (2) of Proposition 9 holds by Lemma 5.6(2). This completes the proof of Proposition 9.

Assume the hypothesis of Proposition 11. Thus H is complemented and $M \in \mathcal{O}_G(H)$. By Proposition 3, M is not affine. If M is diagonal, doubled, or complemented, then conclusion (3), (4), or (1) of Proposition 11 holds by Proposition 6, 8, or 10, respectively. Thus we may assume that M is semisimple, and adopt the notation of Section 5. It remains to show that the map $\phi : K \mapsto \mathcal{F}(H, K)$

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(see Notation 2.6) is a bijection of $\mathcal{O}_H(N_H(L))'$ with $\boldsymbol{\mathcal{F}}(H)$, where L is some choice of component of H, that L is contained in a component of M, and that H is transitive on the components of M. By Lemma 5.3(4), H is transitive on the components of M. Recall from Lemma 5.4(3) that L is contained in a component of M if and only if the parameter $b = b(M, H)$ of Notation 5.2 is 1. From Notation 2.6, $\mathcal{F}(H, K) \in \boldsymbol{\mathcal{F}}(H)$ and ϕ is injective, so to show that ϕ is a bijection we must show that ϕ is surjective. That is, we must show that $M = \mathcal{F}(H, K)$ for some $K \in \mathcal{O}_H(N_H(L))'$. But if $b = 1$ then $\mathcal{F}(M) = \mathcal{F}(H, H_1)$ by Lemma 5.5(6), where X_1 is the component of X containing L and $H_1 = N_H(X_1)$. Thus it remains to show that $b = 1$.

If H is maximal in M then $b = 1$ by Lemma 5.13. Thus we may assume that $H \leq M' < M$ with H maximal in M'. Let $X' = F^*(M')$. By induction on $n = |M : H|$, the pair (M', H) satisfies one of the conclusions of Proposition 11, so, in particular, L is contained in a component L' of X'. Thus if L' is contained in a component of M, then $b = 1$ and the proof is complete. If M' is complemented, then as $|M : M'| < |M : H|$, L' is contained in a component of M by induction on n. If M' is diagonal or doubled, then $b(M, M') = 1$ by Lemma 5.6(1), so L' is contained in a component of M. Thus we may assume that M' is semisimple, and that $b(M, M') \neq 1$. Therefore by parts (3) and (4) of Lemma 5.8, M' is octal. But as $L \leq L'$, $b(M', H) = 1$, so applying Lemma 5.5(5) to the pair (M', H), we conclude that L' is the alternating group on $\omega L'$, in contradiction to M' being octal. Thus the proof of Proposition 11 is at last complete.

See Notation 2.6 and Example 1.6 for the definition of $\mathcal{D}(H)$ and $\mathcal{F}(\mathcal{D})$ appearing in the next theorem.

THEOREM 8.2. *Assume that M_1 and M_2 are distinct subgroups of G maximal subject to $F^*(G) \not\leq M_i$, such that $H = M_1 \cap M_2$ is primitive, and M_1 is not semisimple.*

(1) *$|\Omega| = p^{fk}$ for some prime p and integers f, k such that $p^f \geq 5$ and $k \geq 1$.*
(2) *M_1 is affine.*
(3) *One of the following statements holds:*
  (i) *$|\Omega| = p$, H is affine with $F^*(H) = F^*(M_1)$, M_2 is almost simple, and $H = N_{M_2}(D)$;*
  (ii) *$k > 1$, H is affine with $F^*(H) = F^*(M_1)$, M_2 is semisimple, and there exists $\mathcal{D} = (D_1, \ldots, D_k) \in \mathcal{D}(H)$ such that $M_2 = N_G(\mathcal{F}(\mathcal{D}))$ and $H = N_{M_1}(\mathcal{D})$;*
  (iii) *H is octal semisimple with k components, $p^f = 8$, H is the wreath product of $L_3(2)$ by S_k, and M_1 and M_2 are the stabilizers of the two H-invariant affine structures on Ω.*

PROOF. Let $Y = F^*(M_1)$. As M_1 is not semisimple, M_1 is affine or strongly diagonal by Lemma 2.5. Assume the latter and let k be the number of components of M_1. As H is primitive and $|\Sigma(M_1)| = 1$, Proposition 6 implies that one of the following statements holds:

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(i) H is strongly diagonal and $Y = D$;
(ii) H is doubled, $k = 2$, and $Y = D$;
(iii) H is complemented, $k = 2$, and D is a component of Y.

Conclusion (iii) is impossible since if H is complemented then H is not almost simple. Thus (i) or (ii) holds, and in particular $Y = D$. In case (i), as $M_2 \in \mathcal{M}(H)$ and $|\Sigma(H)| = 1$, Proposition 7 says that M_2 is strongly diagonal with $F^*(M_2) = D$. But then $M_1 = N_G(D) = M_2$, contradicting $M_1 \neq M_2$. Therefore case (ii) holds. Hence Proposition 9 says that M_2 is doubled or diagonal, and $D = F^*(M_2)$, or M_2 is semisimple. But in the first two cases, again $M_1 = N_G(D) = M_2$, for the same contradiction. Thus M_2 is semisimple, so by Proposition 9, a component D_1 of H is contained in a component of M_2, and H is transitive on the components of M_2. Then as $D_1 \trianglelefteq H$, M_2 is almost simple, contrary to Proposition 1.

Therefore M_1 is affine, so conclusions (1) and (2) of Theorem 8.2 hold, except possibly $p^k < 5$. As H is primitive, Lemma 4.3 says that either

(a) $D \leq Y$; or
(b) H is octal semisimple, $D = L_1 \times \cdots \times L_k$, $L_i \cong L_3(2)$, $p^f = 8$, HY is affine, $\mathcal{D} = \{Y_1, \ldots, Y_k\} \in \mathcal{D}(HY)$, and $\mathcal{F}(H) = \mathcal{F}(\mathcal{D})$, where $Y_i = [Y, L_i] \cong E_8$.

Suppose that case (a) holds. Then D is solvable, so H is affine and $D = Y$. By Proposition 4, one of the following statements holds:

(I) M_2 is affine and $D = F^*(M_2)$;
(II) H is imprimitive on Y, and there exists $\mathcal{D} = \{D_1, \ldots, D_k\} \in \mathcal{D}(H)$ with $M_2 = N_G(\mathcal{F}(\mathcal{D}))$;
(III) $|\Omega|$ is prime and M_2 is almost simple.

In case (I) we have our usual contradiction, while in cases (II) and (III) conclusions (3ii) and (3i) of Theorem 8.2 hold, respectively.

Finally, assume that case (b) holds. As $M_1 \neq M_2$, Proposition 5 says that either

(A) M_2 is affine and the stabilizer of an affine structure on Ω preserved by H; or
(B) M_2 is the stabilizer of $\mathcal{F}(H, K)$ for some $K \in \mathcal{O}_H(N_H(L_1))'$.

Suppose that case (A) holds. By Lemma 7.1(4), $N_{M_1}(D) = DT_1$, where T_1 acts faithfully as $\mathrm{Sym}(\mathcal{L})$ on $\mathcal{L} = \{L_1, \ldots, L_k\}$. Thus $H = DT_H$, where $T_H \leq T_1$ is transitive on $\mathcal{L}$. By Lemma 7.1(2), U is the permutation module for T_1 and T_H, so by Lemma 7.1(5), $|\mathcal{A}(H)| = |C_{UD/D}(H)| = 2$. Thus $M_2 = M_1^u$, where $u \in N_U(H)$ Similarly $|C_{UD/D}(DT_1)| = 2$, so as $T_H \leq T_1$, $C_{UD/D}(H) = C_{UD/D}(DT_1)$, and hence $DT_1 \leq M_1 \cap M_2 = H$, so $T_H = T_1 \cong S_k$. But now conclusion (3iii) of Proposition 12 holds.

So assume that case (B) holds, let $Y_{1,K} = \langle Y_1^K \rangle$, and $\mathcal{D}_K = Y_{1,K}^H$. Then $\mathcal{D}_K \in \mathcal{D}(HY)$ and, by Notation 2.6, $\omega D_{\gamma_K^h} = \omega Y_{1,K}^h$, so by Lemma 1.7, $\mathcal{F}(\mathcal{D}_K) = \mathcal{F}(H, K)$. Thus $HY \leq M_1 \cap M_2 = H$, giving a contradiction. This completes the proof of the theorem. $\square$

See Lemma 5.11(6) for the definition of $\mathcal{F}^2(H)$ appearing in the next lemma.

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LEMMA 8.3. *Assume that H is almost simple and product decomposable.*

(1) *If $M \in \mathcal{O}_G(H)'$ with M almost simple, then $F^*(M) = F^*(H)$.*
(2) *$N_G(\mathcal{F}^2(H))$ is the unique maximal member of $\mathcal{O}_G(H)'$.*

PROOF. Let $d^2 = |D : D_\omega| = |\Omega|$, and $A_i \in \mathcal{O}_D(D_\omega)$, $i = 1, 2$, with $|A_i : D_\omega| = |D : A_i| = d$ as in Definition 5.10. Set $\Lambda = \mathrm{Aut}(D)$ and $\Sigma = N_\Lambda(D_\omega)$. From Lemma 3.4, $\Lambda = \mathrm{Inn}(D)\Sigma$, $|\Sigma : N_\Sigma(A_1)| = 2$ with $\Sigma = H_\omega N_\Sigma(A_1)$, and $A_1^h = A_2$ for $h \in H_\omega - N_\Sigma(A_1)$. It follows that $N_S(D) \cong \Lambda$ and from Lemma 5.11(6), $N_S(D) \leq N_S(\mathcal{F}^2(H))$. Thus if (1) holds then each almost simple member of $\mathcal{O}_G(H)'$ is contained in $\tilde{M} = N_G(\mathcal{F}^2(H))$. On the other hand, by Proposition 2, each member of $\mathcal{O}_G(H)'$ which is not almost simple is contained in $\tilde{M}$. Thus (1) implies (2), so it remains to prove (1). Therefore we may assume that $M \in \mathcal{O}_G(H)'$ with M almost simple and $F^*(M) \neq D$, and it remains to derive a contradiction. Choose H, M so that H is maximal subject to these constraints, and M is minimal subject to this further constraint. From the maximality of H, $H = N_M(D)$.

Let $H \leq M' \leq M$ with H maximal in M'. By Proposition 2, M' is semisimple, and hence, as $M' \leq M$, M' is almost simple by Lemma 4.4. Therefore as $H = N_M(D)$, $M = M'$ by minimality of M. That is, H is maximal in M.

By Lemma 2.4, M is primitive on Ω, so we have shown that:

(a) H is a maximal subgroup of the primitive almost simple subgroup M of S on Ω;
(b) H is almost simple, primitive, and product indecomposable on Ω of order d^2.

To complete the proof we inspect the list of possible inclusions in part (B) of the main theorem of [LPS1] for pairs (H, M) satisfying (a) and (b). In particular, by (b) and Definition 5.10, $F^*(H) \cong A_6$, M_{12}, or $Sp_4(q)$, $q > 2$ even, and $n = |\Omega| = d^2$, where $d = 6$, 12, or $q^2(q^2 - 1)/2$, respectively. As there are no such pairs, we obtain a contradiction, which completes the proof. □

THEOREM 8.4. *Assume that M_1 and M_2 are distinct subgroups of G maximal subject to $F^*(G) \not\leq M_i$, such that $H = M_1 \cap M_2$ is primitive, M_1 is almost simple, and M_2 is semisimple. Then the following statements hold:*

(1) *M_2 is also almost simple.*
(2) *Either H is almost simple, or $|\Omega|$ is prime and H is affine.*

PROOF. If $|\Omega|$ is prime, then as M_2 is semisimple, we conclude from Lemma 2.2 that (1) holds, and then (2) follows from Lemma 4.4.

Thus we may assume that $|\Omega|$ is not prime. Therefore H is almost simple by Lemma 4.4, so (2) holds. Next by Proposition 2, either (1) holds or $|\Omega| = 8$ and $H \cong L_3(2)$ is octal. But now $\mathcal{M}(H)$ consists of the stabilizers of the two affine structures on Ω if $G = F^*(S)$, or that pair of subgroups together with $N_S(H) \cong PGL_2(7)$ if $G = S$. But in either case the hypothesis of the theorem is not satisfied. Thus the proof of Theorem 8.4 is complete. □

LEMMA 8.5. *Assume that H is almost simple, product indecomposable, and not octal. Then each member of $\mathcal{O}_S(H)$ is almost simple, product indecomposable, and not octal.*

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PROOF. Let $M \in \mathcal{O}_S(H)'$. By Proposition 2, M is almost simple, so it remains to show that M is product indecomposable and not octal. Proceeding by induction on $|M : H|$, we may assume that M is octal or product decomposable, and that H is maximal in M. If M is octal then $|\Omega| = 8$ and $M \cong L_3(2)$, so all proper subgroups of M are solvable, contradicting $H < M$. Therefore M is product decomposable. Hence by Definition 5.10, $|\Omega| = d^2$, where d is 6, 12, or $q^2(q^2-1)/2$, and M appears in case (i), (ii), or (iii) of Definition 5.10, respectively. As M is product decomposable but H is not, $F^*(M) \neq D$. Thus the pair D, $F^*(M)$ appears in one of [LPS1, Tables III–VI]. Inspecting these tables for examples with $|\Omega| = d^2$ and $F^*(M)$ listed in Definition 5.10, we obtain a contradiction. □

9. The proofs of Theorems 12 and 13

In this section we assume Hypothesis 2.1.

We begin with a proof of Theorem 12, so assume the hypothesis of that theorem. Suppose first that M_1 is not semisimple. Then by Theorem 8.2, M_1 is affine and conclusion (ii) or (iii) of Theorem 8.2(3) holds. But now conclusion (3) or (4) of Theorem 12 holds. Therefore we may assume that M_1 and M_2 are semisimple.

Suppose next that M_1 is almost simple. Then by Theorem 8.4, M_2 and H are almost simple, so conclusion (1) of Theorem 12 holds. Hence we may assume that neither M_1 nor M_2 is almost simple. Then by maximality of M_i and Lemma 2.5, $M_i = N_G(\mathcal{F}(M_i))$, so conclusion (2) of Theorem 12 holds, completing the proof of Theorem 12.

The remainder of the section is devoted to a proof of Theorem 13. In particular, we assume the hypothesis and notation of Theorem 13. For example, $\mathcal{M}$ denotes the set of maximal overgroups of H in G. In addition, assume that H is a counterexample to Theorem 13. Set $L = F^*(G)$, so that L is the alternating group on Ω. We begin a short series of reductions.

LEMMA 9.1. *Suppose that $G = S$ and $H \leq L$.*

(1) *L and $N_G(H)$ are in $\mathcal{M}$.*
(2) *For each $M \in \mathcal{M} - \{N_G(H)\}$, $M \cap N_G(H) = H$.*
(3) *H and $N_G(H)$ are almost simple.*

PROOF. Parts (1) and (2) follow from Lemma 3.7. If $N_G(H)$ is the stabilizer in S of an affine structure, regular product structure, or diagonal structure, then by (1) and (2), H is the stabilizer in L of that structure, and hence is maximal in L by Remark 2.7. But then by Lemma 3.8, case (6) of Theorem 13 holds, contrary to the choice of H as a counterexample. Hence $N_G(H)$ is almost simple by Lemma 2.5, so H is also almost simple, establishing (3). □

LEMMA 9.2.

(1) *H is not almost simple.*
(2) *If $G = S$ then $H \not\leq L$.*
(3) *$\mathcal{M} = \mathcal{M}(H)$.*

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PROOF. Suppose first that H is almost simple. As H is a counterexample to Theorem 13, case (2) of that theorem does not hold, so either H is octal or product decomposable, or some member of $\mathcal{O}_G(H)$ is not almost simple, product indecomposable, and not octal. The latter contradicts Lemma 8.5, so the former holds. Hence by Proposition 2, one of the following statements holds:

(i) $H \cong L_3(2)$ is octal, and either
   (a) $G = L$, $\mathcal{M} = \{M_1, M_2\}$ with M_1 and M_2 affine, and $\mathcal{O}_G(H) = \{H, M_1, M_2, G\}$; or
   (b) $G = S$, $\mathcal{M} = \{L, N_S(H)\}$, and $\mathcal{O}_G(S) = \{H, M_1, M_2, L, N_G(H), G\}$.

(ii) H is product decomposable.

Conclusion (i) contradicts the choice of H as a counterexample. If conclusion (ii) holds, then by Lemma 8.3, $\mathcal{M}(H) = \{N_G(\mathcal{F}^2(H))\}$. Therefore by hypothesis (0.1) of Theorem 13, $\mathcal{M} \neq \mathcal{M}(H)$, so $G = S$ and $H \leq L$. Therefore $H = N_L(\mathcal{F}^2(H))$ by Lemma 9.1, which is not the case. Hence (1) is established.

Observe that (1) and Lemma 9.1(3) imply (2), while (2) implies (3). □

LEMMA 9.3. *No member of $\mathcal{M}$ is almost simple.*

PROOF. Assume that $M \in \mathcal{M}$ is almost simple. Then by Lemma 9.2(3) and Theorem 12, H is almost simple, contrary to Lemma 9.2(1). □

LEMMA 9.4. *There exists $M \in \mathcal{M}$ such that M is not semisimple.*

PROOF. Assume otherwise and let $M \in \mathcal{M}$. By Lemma 9.3, M is not almost simple, so by Lemma 2.5 and maximality of M, $M = N_G(\mathcal{F}(M))$. As $H \leq M$, $\mathcal{F}(M) \in \boldsymbol{\mathcal{F}}(H)$, so conclusion (1) of Theorem 13 holds, contrary to the choice of H as a counterexample. □

Let Δ be the set of pairs (M_1, M_2) of distinct $M_i \in \mathcal{M}$, $i = 1, 2$, such that $H = M_1 \cap M_2$ and M_1 is not semisimple. By Lemma 9.4 and hypothesis (0.1) of Theorem 13, $\Delta \neq \varnothing$.

LEMMA 9.5. *Let $(M_1, M_2) \in \Delta$. Then either*

(1) *H is affine, $M_1 = N_G(D)$, and $M_2 = N_G(\mathcal{D})$ for some $\mathcal{D} \in \mathcal{D}(H)$; or*

(2) *$H \cong L_3(2)$ wr $\mathrm{S_k}$ is octal semisimple, and M_1 and M_2 are the stabilizers of the two H-invariant affine structures on Ω.*

PROOF. As $(M_1, M_2) \in \Delta$, H, M_1, M_2 satisfy one of the conclusions of Theorem 12. Then as M_1 is not semisimple, the lemma follows by inspection of the possibilities listed in Theorem 12. □

Suppose that $(M_1, M_2) \in \Delta$ satisfies Lemma 9.5(2). Then M_i is perfect, so $M_i \leq L$. Thus $G = L$ by maximality of M_i. By Proposition 5, and as H is primitive on the set of components of H, $\mathcal{M}(H) = \{M_1, M_2, M_3\}$, where $M_3 = N_G(\mathcal{F}(H))$. By hypothesis (0.1) in Theorem 13, $M_i \cap M_3 = H$ for $i = 1$ or 2, contrary to Theorem 12.

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We have shown that for each $(M_1, M_2) \in \Delta$, (M_1, M_2) satisfies Lemma 9.5(1). In particular, H is affine and $M_1 = N_G(D)$, so all members of $\mathcal{M} - \{N_G(D)\}$ are semisimple. Then it follows from Proposition 4 that case (3) of Theorem 13 holds, contrary to the choice of H as a counterexample to Theorem 13. This contradiction completes the proof of Theorem 13.

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