BULL. AUSTRAL. MATH. SOC. 20J05
VOL. 32 (1985), 261-274.

THE KERNEL OF THE CUP PRODUCT

JONATHAN A. HILLMAN

We relate the kernel of the cup product of 1-dimensional cohomology classes for a group G acting trivially on a field R to $\mathrm{Hom}(G_2/G_3,R)$, the space of group homomorphisms of the second stage of the lower central series for G into R , by means of explicit computations with cocycles. The precise result depends on whether the characteristic of the field is 0 , an odd prime or 2 .

Let G be a finitely generated group, and R a commutative ring, considered as a trivial G-module. Then the low dimensional cohomology groups of G with coefficients in R may be computed from the standard complex of inhomogeneous cochains to be $H^0(G;R) = R$,

$$H^1(G;R) = \{f : G\to R \mid f(gh) = f(g) + f(h) \text{ for all } g,h \text{ in } G\} = \mathrm{Hom}(G,R)$$

and

$$H^2(G;R) = \{F : G^2 \to R \mid F(h,j) - F(gh,j) + F(g,hj) - F(g,h) = 0 \text{ for all } g,h,j \text{ in } G\}/B$$

where

$$B = \{\partial f : \langle g,h\rangle \mapsto f(g) + f(h) - f(gh) \text{ for all } g,h \text{ in } G \mid f : G \to R\}.$$

Received 21 February 1985.

https://doi.org/10.1017/S0004972700009941 Published online by Cambridge University Press

The cup product of two elements f_1 , f_2 in $H^1(G;R)$ is represented by the function $f_1 f_2 : \langle g,h\rangle \mapsto f_1(g)f_2(h)$ for all g,h in G . We shall show that the kernel of this cup product (of classes in degree 1) is closely related to the second stage G_2/G_3 of the lower central series of G . The connection is made essentially by dualizing the map $\eta : (G/G_2) \times (G/G_2) \to G_2/G_3$ sending $\langle gG_2, hG_2\rangle$ to the coset $ghg^{-1}h^{-1}G_3$ (for all g,h in G) . The use of this map and the statement of the principal result in the case $R = Q$ are due to Sullivan [3]. As he gave no details, and we have had occasion to use this result elsewhere, we have decided to supply an argument here, and we shall treat also the case when R is a field of positive characteristic. The odd characteristic case is similar to that for Q , but in characteristic 2 skew symmetric forms (that is $b(x,y) = -b(y,x)$) are symmetric $(b(x,y) = b(y,x))$ but no longer need be alternating $(b(x,x) = 0)$, which complicates matters.

The map η is skew symmetric and bilinear, and its image generates G_2/G_3 as an abelian group. Therefore if $\pi : G_2/G_3 \to H$ is an epimorphism, there is a corresponding monomorphism $(\pi\circ\eta)_R^* : \mathrm{Hom}(H,R) \to \Lambda^2(G/G_2,R)$, with codomain the module of skew symmetric bilinear maps. Now for any f_1, f_2 in $H^1(G;R)$ and g,h in G ,

$$f_1(g)f_2(h) + f_2(g)f_1(h) = -f_1(g)f_2(g) - f_1(h)f_2(h) - (-f_1(gh)f_2(gh))$$

so the cup product $\cup : H^1(G;R) \times H^1(G;R) \to H^2(G;R)$ is anticommutative $(f_1 \cup f_2 = -f_2 \cup f_1)$ and so gives rise to an R-homomorphism (which we shall also call cup product) from $H^1(G;R) \otimes_R H^1(G;R)/D$ to $H^2(G;R)$, where D is the submodule of the tensor product generated by

$$\{f_1 \otimes f_2 + f_2 \otimes f_1 \mid f_1, f_2 \text{ in } H^1(G;R)\} .$$

If 2 is invertible in R , the R-module $H^1(G;R) \otimes_R H^1(G;R)/D$ is just $\Lambda_2(H^1(G;R))$; if $2 = 0$ in R then it is $\mathrm{Sym}_2(H^1(G;R))$. In general, $\Lambda_2(H^1(G;R))$ is a quotient of this R-module, for any ring R .

There is a natural map $\mu : \Lambda_2(H^1(G;R)) \to \Lambda^2(G/G_2;R)$ such that $\mu(f_1 \wedge f_2)(gG_2,hG_2) = f_1(g)f_2(h) - f_2(g)f_1(h)$ for all g,h in G and f_1,f_2 in $H^1(G;R)$, which is injective if R is a field and bijective if also $H^1(G;R)$ is of finite dimension over R [1; page 20], which is certainly the case when G is finitely generated.

Suppose f_{1j},f_{2j} in $H^1(G;R)$ for $1 \le j \le n$ are such that

$$\sum_{1\le j\le n} f_{1j} \cup f_{2j} = 0 .$$

Then there is a map $F : G \to R$ such that for all g,h in G

$$\sum_{1\le j\le n} f_{1j}(g)f_{2j}(h) = F(g) + F(h) - F(gh) .$$

(Notice that if F' is any other such map then $F - F'$ is a homomorphism from G to R.) Then $F(gk) = F(g) + F(k) = F(kg)$ for all g in G and all k in $G(R) = \cap\{\ker \lambda \mid \lambda \text{ in } H^1(G;R) = \mathrm{Hom}(G,R)\}$. The restriction $F|G(R)$ is a homomorphism, determined on the subgroup $G_2 \subseteq G(R)$ by

$$F(ghg^{-1}h^{-1}) = F(ghg^{-1}h^{-1}hg) - F(hg) = F(gh) - F(hg)$$

$$= \sum_{1\le j\le n} (f_{1j}(h)f_{2j}(g) - f_{1j}(g)f_{2j}(h)) ,$$

and so $F|[G,G(R)] = 0$. Therefore F induces a homomorphism $\tilde{F} : G_2/G_3 \to R$ and clearly $\mu\left(\sum_{1\le j\le n} f_{1j} \wedge f_{2j}\right) = \eta_R^*(\tilde{F})$. Thus the kernel of the cup product is mapped via μ into the image of η_R^*, and so in particular if G is abelian and R is a field of characteristic different from 2, the cup product $\cup : \Lambda_2(H^1(G;R)) \to H^2(G;R)$ is injective.

We shall assume henceforth that R is a field and treat the three cases char $R = 0$, odd prime p, and 2 separately.

(1) char $R = 0$

By the argument above, μ identifies $\ker \cup$ with a subspace of $\operatorname{Im} \eta_R^*$. We claim this subspace is all of $\operatorname{Im} \eta_R^* \approx \operatorname{Hom}(G_2/G_3, R)$. Let $\theta : G_2 \to R$ be an homomorphism such that $\theta|G_3 = 0$. Then since μ is bijective, $\theta([g,h]) = \eta_R^*\theta(gG_2, hG_2) = \sum_{1\le j\le n} (f_{1j}(g)f_{2j}(h) - f_{1j}(h)f_{2j}(g))$ for some f_{1j}, f_{2j} in $H^1(G;R)$ $(1\le j\le n)$. It must be shown that the map

$: \langle g,h\rangle \to \sum_{1\le j\le n} f_{1j}(g)f_{2j}(h)$ is a coboundary, that is, that there is a map $F : G \to R$ such that for all g,h in G,

$$\sum_{1\le j\le n} f_{1j}(g)f_{2j}(h) = F(g) + F(h) - F(gh) . \qquad (*)$$

The map F may be ambiguous up to the addition of a homomorphism; in particular it will be uniquely defined on $G(R)$ (if it exists at all). As before $F(tg)$ and $F(gt)$ must equal $F(g) + F(t)$ for all g in G and t in $G(R)$, and it follows that F must be defined on G_2 by $F(ghg^{-1}h^{-1}) = F(gh) - F(hg) = -\theta([g,h])$. Hence $F|[G,G(R)] = 0$. Since $G(R)/G_2$ is easily seen to be the torsion subgroup of G/G_2, F is now determined on $G(R)$, for if g^m is in G_2 then g is in $G(R)$ so $F(g^m) = mF(g)$ and $F(g) = \frac{1}{m} F(g^m)$. If also h^n is in G_2 then $(gh)^{mn} = g^{mn}h^{mn}k$ with k in $G(R)_2$, and so is in G_2, and

$$F(gh) = \frac{1}{mn} F((gh)^{mn}) = \frac{1}{mn} (F(g^{mn}) + F(h^{mn}) + F(k)) = F(g) + F(h) .$$

Thus with this definition of F on $G(R)$ as an homomorphism to R, condition (*) is satisfied whenever g,h are in $G(R)$.

The quotient $G/G(R)$ is a finitely generated torsion free abelian group, and so free of rank β, say. Choose representatives $h_1,\ldots,h_\beta$ in G for a basis of $G/G(R)$. Then any g in G can be written $g = g_0 t$ where g_0 is the unique standard representative of the coset $gG(R)$ of the form $g_0 = h_1^{w_1} \ldots h_\beta^{w_\beta}$ (with $w_1,\ldots,w_\beta$ integers) and t in $G(R)$. If F exists, then $F(g)$ must equal $F(g_0) + F(t)$, and so it

https://doi.org/10.1017/S0004972700009941 Published online by Cambridge University Press

will suffice to define F on standard elements g_0, provided that the result is consistent with (*) . If (*) is satisfied whenever g and h are standard elements, then it holds in general, for if $g = g_0 a$ and $h = h_0 b$ with g_0, h_0 standard elements and a, b in $G(R)$, then

$$\begin{aligned}
\sum_{1\le j\le n} f_{1j}(g) f_{2j}(h) \\
&= \sum_{1\le j\le n} f_{1j}(g_0) f_{2j}(h_0) \\
&= F(g_0) + F(h_0) - F(g_0 h_0) \quad \text{(by assumption)} \\
&= F(g_0) + F(h_0) - F(g_0 h_0) - F(a^{-1} h_0^{-1} a h_0) \quad (\text{since } F|[G, G(R)] = 0) \\
&= F(g_0) + F(a) + F(h_0) + F(b) - F(g_0 h_0 h_0^{-1} a h_0 b) \\
&= F(g) + F(h) - F(gh) .
\end{aligned}$$

(Notice that although $g_0 h_0$ need not itself be standard $F(g_0 h_0 t)$ will still equal $F(g_0 h_0) + F(t)$ for all t in $G(R)$, for if $g_0 h_0 = k_0 s$ with k_0 standard and s in $G(R)$, then

$F(g_0 h_0 t) = F(k_0 s t) = F(k_0) + F(st) = F(k_0) + F(s) + F(t) = F(k_0 s) + F(t)$.)

Furthermore, if condition (*) holds, then by induction on m,

$$F(g^m) = mF(g) - \frac{m(m-1)}{2} \sum_{1\le j\le n} f_{1j}(g) f_{2j}(g)$$

for all g in G and for all positive integers m, and since

$$F(h^{-1}) = -F(h) - \sum_{1\le j\le n} f_{1j}(h) f_{2j}(h)$$

it is easily seen that this formula holds for all integers m. Thus once $F(h_1), \ldots, F(h_\beta)$ are known, the condition (*) determines F uniquely on all standard elements and hence on all of G. (For instance, if $g_0 = h_1^{w_1} \dots h_\beta^{w_\beta}$, then

$$F(g_0) = F\left(h_1^{w_1} \dots h_{\beta-1}^{w_{\beta-1}}\right) + F\left(h_\beta^{w_\beta}\right) - \sum_{1\le j\le n} f_{1j}\left(h_1^{w_1} \dots h_{\beta-1}^{w_{\beta-1}}\right) f_2\left(h_\beta^{w_\beta}\right)$$

$$= \dots .$$

$$= \sum_{1\le i\le\beta} F\left(h_i^{w_i}\right) - \sum_{1\le i<j\le\beta}\ \sum_{1\le k\le n} w_i w_j p_{ki} q_{kj}$$

(where $p_{ki} = f_{1k}(h_i)$, $q_{kj} = f_{2k}(h_j)$)

$$= \sum_{1\le i\le\beta} w_i F(h_i) - \sum_{1\le i\le\beta}\ \sum_{1\le k\le n} \frac{w_i(w_i-1)}{2} p_{ki}q_{ki} - \sum_{1\le i<j\le\beta}\ \sum_{1\le k\le n} w_i w_j p_{ki} q_{kj}.)$$

We claim that $F(h_1),\dots,F(h_\beta)$ can be chosen arbitrarily. Indeed, let $F(h_i) = r_i$ in R, $1 \le i \le \beta$, and if g in G can be written as $g = g_0 t = h_1^{w_1} \dots h_\beta^{w_\beta} t$ with t in $G(R)$, define

$$F(g) = \sum_{1\le i\le\beta} w_i r_i - \sum_{1\le i\le\beta}\ \sum_{1\le k\le n} \frac{w_i(w_i-1)}{2} p_{ki}q_{ki} - \sum_{1\le i<j\le\beta}\ \sum_{1\le k\le n} w_i w_j p_{ki} q_{kj} + F(t).$$

Then F maps G to R and the coboundary of F represents the cup product $\sum_{1\le j\le n} f_{1j} \cup f_{2j}$. For, as remarked above, it suffices to check that condition (*) is satisfied when $g_0 = h_1^{w_1} \dots h_\beta^{w_\beta}$ and $h_0 = h_1^{z_1} \dots h_\beta^{z_\beta}$ are standard elements. The standard representative g_0h_0 is then $h_1^{w_1+z_1} \dots h^{w_\beta+z_\beta}$, and in fact

$$g_0h_0 = h_1^{w_1} \dots h_\beta^{w_\beta}.h_1^{z_1} \dots h_\beta^{z_\beta} = h_1^{w_1+z_1} \dots h_\beta^{w_\beta+z_\beta}\left[\left(h_2^{w_2} \dots h_\beta^{w_\beta}\right)^{-1}, h_1^{-z_1}\right] .$$

$$\dots \left[h_\beta^{-w_\beta}, h_\beta^{-z_{\beta-1}}\right] k$$

for some k in G_3, and so

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$$F(g_0h_0) = F\left(h_1^{w_1+z_1}\dots h_\beta^{w_\beta+z_\beta}\right) + F\left(\left[\left(h_2^{w_1}\dots h_\beta^{w_\beta}\right)^{-1}, h_1^{-z_1}\right]\right) + \dots + F\left(\left[h_\beta^{-w_\beta}, h_{\beta-1}^{-z_{\beta-1}}\right]\right).$$

Therefore

$$F(g_0) + F(h_0) - F(g_0h_0)$$

$$= \sum_{1\le i\le\beta} w_i r_i - \sum_{1\le i\le\beta}\sum_{1\le k\le n} \frac{w_i(w_i-1)}{2} p_{ki}q_{ki}$$

$$- \sum_{1\le i<j\le\beta}\sum_{1\le k\le n} w_iw_jp_{ki}q_{kj} + \sum_{1\le i\le\beta} z_ir_i - \sum_{1\le i\le\beta}\sum_{1\le k\le n} \frac{z_i(z_i-1)}{2} p_{ki}q_{ki}$$

$$- \sum_{1\le i<j\le\beta}\sum_{1\le k\le n} z_iz_jp_{ki}q_{kj} - \sum_{1\le i\le\beta}(w_i+z_i)r_i +$$

$$+ \sum_{1\le i\le\beta}\sum_{1\le k\le n} \frac{(w_i+z_i)(w_i+z_i-1)}{2} p_{ki}q_{kj} + \sum_{1\le i<j\le\beta}\sum_{1\le k\le n}(w_i+z_i)(w_j+z_j)p_{ki}q_{kj}$$

$$- \sum_{1\le k\le n}\left(f_{1k}\left(h_1^{-z_1}\right)f_{2k}\left(\left(h_2^{w_2}\dots h_\beta^{w_\beta}\right)^{-1}\right) - f_{1k}\left(\left(h_2^{w_2}\dots h_\beta^{w_\beta}\right)^{-1}\right)f_{2k}\left(h_1^{-z_1}\right)\right)$$

$$- \dots - \sum_{1\le k\le n}\left(f_{1k}\left(h_{\beta-1}^{-z_{\beta-1}}\right)f_{2k}\left(h_\beta^{-w_\beta}\right) - f_{1k}\left(h_\beta^{-w_\beta}\right)f_{2k}\left(h_{\beta-1}^{-z_{\beta-1}}\right)\right)$$

$$= \sum_{1\le i\le\beta}\sum_{1\le k\le n} w_iz_ip_{ki}q_{ki} + \sum_{1\le i<j\le\beta}\sum_{1\le k\le n}(w_iz_j+w_jz_i)p_{ki}q_{kj}$$

$$- \sum_{1\le k\le n}\sum_{1\le i<j\le\beta} z_iw_j(p_{ki}q_{kj}-p_{kj}q_{ki})$$

$$= \sum_{1\le k\le n}\left(\sum_{1\le i\le\beta} w_iz_ip_{ki}q_{ki} + \sum_{1\le i<j\le\beta}(w_iz_jp_{ki}q_{kj}+z_iw_jp_{kj}q_{ki})\right)$$

$$= \sum_{1\le k\le n}\sum_{1\le i,j\le\beta} w_iz_jp_{ki}q_{kj} = \sum_{1\le k\le n}\left(\sum_{1\le i\le\beta} w_ip_{ki}\right)\left(\sum_{1\le j\le\beta} z_jp_{kj}\right)$$

$$= \sum_{1\le k\le n} f_{1k}(g_0)f_{2k}(h_0)\ .$$

Thus the second claim, and hence the first, is proven and so

$$\mu(\ker \cup : \Lambda_2 H^1(G;R) \to H^2(G;R)) = \eta^*(\mathrm{Hom}(G_2/G_3,R)) ,$$

for R any field of characteristic 0 .

(2) char $R = p$, p an odd prime

The map μ is again bijective, and so identifies $\ker \cup$ with a subspace of $\mathrm{Im}\ \eta_R^*$. If $F : G \to R$ is a function for which there exist homomorphisms f_{1j}, f_{2j} in $\mathrm{Hom}(G,R)$ $(1 \le j < n)$ such that for all g,h in G

$$\sum_{1\le i\le n} f_{1i}(g)f_{2i}(h) = F(g) + F(h) - F(gh) ,$$

then, as before, $F|G(R)$ is a homomorphism and $F|[G,G(R)] = 0$. Furthermore, $F(g^p) = pF(g) - \frac{p(p-1)}{2} \sum_{1\le i\le n} f_{1i}(g)f_{2i}(g) = 0$ for all g in G . (Notice this uses the fact that p is odd.) Therefore $F|X^p(G) = 0$, where $X^p(G)$ is the verbal subgroup generated by all p^{th} powers, and so F gives rise to a homomorphism $\tilde{F} : G_2/[G,G(R)].(X^p(G) \cap G_2) \to R$ whose image under η_R^* is $\mu\left(\sum_{1\le i\le n} f_{1i} \wedge f_{2i}\right)$. We claim that $\mu(\ker \cup)$ is the subspace $\eta_R^*\ \mathrm{Hom}(G_2/[G,G(R)].(X^p(G) \cap G_2),R)$ of $\mathrm{Im}\ \eta_R^*$. It is easily seen that $G(R) = G_2.X^p(G)$, and $[G,G(R)] = G_3.[G,X^p(G)] \subset G_3.(X^p(G) \cap G_2)$, so this subspace is isomorphic to

$$\mathrm{Hom}(G_2/G_3.(X^p(G) \cap G_2),R) = \mathrm{Hom}(G_2.X^p(G)/G_3.X^p(G),R) .$$

(In general however this is smaller than $\mathrm{Hom}(G_2/G_3,R)$.) The argument proceeds as in case (1), by showing that if $\theta : G_2.X^p(G) \to R$ is a homomorphism such that $\theta|G_3.X^p(G) = 0$, then

$$\theta([g,h]) = \sum_{1\le i\le n} (f_{1i}(g)f_{2i}(h) - f_{1i}(h)f_{2i}(g)) \quad \text{for suitable } f_{1i}, f_{2i} ,$$

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and then constructing a function $F : G \to R$ with coboundary

$F(g) + F(h) - F(gh) = \sum_{1 \le i \le n} f_{1i}(g) f_{2i}(h)$ for all g, h in G. The map is uniquely definable on $G(R)$, and is an homomorphism there, and is extendable to all of G on choosing a $\mathbb{Z}/p\mathbb{Z}$-basis for the finitely generated $\mathbb{Z}/p\mathbb{Z}$-vector space $G/G(R)$, using the formulae of case (1) read modulo p (and with the r_i now being arbitrarily chosen elements of R).

(3) char $R = 2$

The map μ is no longer appropriate; consider instead the natural map

$$\sigma : \mathrm{Sym}_2(H^1(G;R)) \to \mathrm{Sym}^2(G/G_2, R) = \Lambda^2(G/G_2, R)$$

given by

$$\sigma(f_1 \odot f_2)(gG_2, hG_2) = f_1(g) f_2(h) + f_1(h) f_2(g) \quad \text{for all } g, h \text{ in } G .$$

The map σ is neither injective nor surjective (unless $H^1(G;R) = 0$). However the image of σ is easily seen to be the subspace of even symmetric bilinear maps (bilinear maps $b : (G/G_2) \times (G/G_2) \to R$ such that $b(gG_2, gG_2) = 0$ for all g in G), which clearly contains $\mathrm{Im}\, \eta_R^*$. There is a $\mathbb{Z}/2\mathbb{Z}$-linear map $\Delta : H^1(G;R) \to \mathrm{Sym}_2(H^1(G;R))$ such that $\Delta(f) = f \odot f$ for all f in $H^1(G;R)$, and clearly $\sigma \circ \Delta = 0$. If $\sigma(\sum_{1 \le i \le n} f_i \odot g_i) = 0$, we may suppose the f_i linearly independent, so there are x_i in G with $f_i(x_j) = \delta_{ij}$, and then $g_i(y) = \sum_{1 \le j \le n} f_j(y) g_j(x_i)$ for all y in G and $i \le n$. Therefore $g_i(x_j) = g_j(x_i)$ for $i, j \le n$, so $\sum_{1 \le i \le n} f_i \odot g_i = \sum_{1 \le i, j \le n} f_i \odot f_j g_j(x_i) = \sum_{1 \le k \le n} f_k \odot f_k g_k(x_k)$. Thus $\ker \sigma$ is the R-span of $\mathrm{Im}\Delta$, and is exactly $\mathrm{Im}\Delta$ if R is perfect. If there are f_{1j}, f_{2j} in $H^1(G,R)$ and a function $F : G \to R$ such that for all g, h in G

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$$\sum_{1\leq i\leq n} f_{1i}(g)f_{2i}(h) = F(g) + F(h) - F(gh) ,$$

then as before $F|G(R)$ is a homomorphism and $F|[G,G(R)] = 0$. Furthermore, $G(R) = G_2.X^2(G)$ and $F|X^4(G) = 0$, so F gives rise to a homomorphism $\tilde{F} : G_2/[G,X^2(G)].(X^4(G) \cap G_2) \to R$ whose image under η_R^* is $\sigma\left(\sum_{1\leq i\leq n} f_{1i} \odot f_{2i}\right)$. We claim that $\sigma(\ker \cup)$ is the subspace $\eta_R^* \operatorname{Hom}(G_2/[G,X^2(G)].X^4(G) \cap G_2,R)$ of $\operatorname{Im} \eta_R^*$, which is clearly isomorphic to $\operatorname{Hom}(G_2X^4(G)/[G,X^2(G)].X^4(G),R)$. As before, given an homomorphism $\theta : G_2X^4(G) \to R$ such that $\theta|[G,X^2(G)].X^4(G) = 0$ there are homomorphisms $f_{1i},f_{2i} : G \to R$ (for $1\leq i\leq n$) such that for all g,h in G

$$\theta([g,h]) = \sum_{1\leq i\leq n} (f_{1i}(g)f_{2i}(h) + f_{1i}(h)f_{2i}(g)) .$$

The next step is somewhat different, as σ is not injective. It must be shown that $\sum_{1\leq i\leq n} f_{1i} \odot f_{2i}$ is in the kernel of the cup product modulo the kernel of σ , that is that there is a function $F : G \to R$, homomorphisms $h_1,\ldots,h_q$ in $\operatorname{Hom}(G,R)$ and elements $s_1,\ldots,s_q$ in R such that for all g,h in G ,

$$\sum_{1\leq i\leq n} f_{1i}(g)f_{2i}(h) = F(g) + F(h) + F(gh) + \sum_{1\leq \ell\leq q} s_\ell h_\ell(g)h_\ell(h) .$$

The map F is uniquely definable on $G_2X^4(G)$ by θ , and is a homomorphism there. The quotient group $G/G_2X^4(G)$ is a finite abelian group of exponent 4 ; choose elements $x_1,\ldots,x_\gamma$ of G which represent a minimal generating set for this quotient, and such that $x_1,\ldots,x_\alpha$ represent elements of order 4 and $x_{\alpha+1},\ldots,x_\gamma$ represent elements of order 2 in $G/G_2X^4(G)$. $F(x_i)$, $h(x_i)$ can be chosen arbitrarily, provided that if $\alpha+1 \leq j \leq \beta$, (so x_j^2

is in $G_2X^4(G)$) then $\sum_{1\le \ell\le q} x_\ell (h_\ell(x_j))^2 = F(x_j^2) + \sum_{1\le i\le n} f_{1i}(x_j) f_{2i}(x_j)$.

For any element g of G may be written uniquely as
$g = g_0 t = x_1^{w_1} \dots x_\beta^{w_\beta} t$ with t in $G_2X^4(G)$, and $0 \le w_j \le 4$ if $1 \le j \le \alpha$ and $w_j = 0$ or 1 if $\alpha + 1 \le j \le \beta$. Then $F(g)$ must be defined to be

$$F(g_0) + F(t) = \sum_{1\le j\le \gamma} F\left(x_j^{w_j}\right)$$

$$+ \sum_{1\le j<k\le \gamma} w_j w_k \left(\sum_{1\le i\le n} f_{1i}(x_j) f_{2i}(x_k) + \sum_{1\le \ell\le q} s_\ell h_\ell(x_k) h_\ell(x_k) \right) + F(t) .$$

Furthermore $F(x^w)$ depends only on the value of w modulo 4 , and $F(x^2)$ must be defined as $\sum_{1\le k\le n} f_{1k}(x) f_{2k}(x) + \sum_{1\le \ell\le q} s_\ell h_\ell(x)^2$ and $F(x^3) = F(x.x^2)$ as $F(x) + F(x^2)$. It remains to check the consistency of the coboundary formula. For ease of reading, let $p_{ki}, q_{ki}, r_i, h_{\ell i}$ denote $f_{1k}(x_i)$, $f_{2k}(x_i), F(x_i)$ and $h\ (x_i)$ respectively, and let t_{ij} denote $\sum_{1\le k\le n} f_{1k}(x_i) f_{2k}(x_j) + \sum_{1\le \ell\le q} s_\ell h_\ell(x_i) h_\ell(x_j)$. It must be shown that for all

$$F(g) + F(h) + F(gh) = \sum_{1\le k\le n} f_{1k}(g) f_{2k}(h) + \sum_{1\le \ell\le q} s_\ell h_\ell(g) h_\ell(h) .$$

As before, it suffices to assume that $g = g_0 = x_1^{w_1} \dots x_\gamma^{w_\gamma}$ and $h = h_0 = x_1^{z_1} \dots x_\gamma^{z_\gamma}$ are standard elements, with $0 \le w_j , z_j \le 4$ for $1 \le j \le \alpha$ and $w_j, z_j = 0$ or 1 if $\alpha + 1 \le j \le \gamma$. (Notice that the homomorphisms h_ℓ vanish on $X^2(G)$ and hence on $G_2X^4(G)$.) Furthermore

$$g_0 h_0 = x_1^{w_1+z_1} \dots x_\gamma^{w_\gamma+z_\gamma} \left[\left(x_2^{w_2} \dots x_\gamma^{w_\gamma} \right)^{-1}, x_1^{-z_1} \right] \dots \left[x_\gamma^{-w_\gamma}, x_{\gamma-1}^{-z_{\gamma-1}} \right] k$$

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for some k in G_3. However this will not be the standard expression for g_0h_0 if some exponent $w_j + z_j$ is greater than or equal to 4 (if $j \le \alpha$) or is greater than or equal to 2 (if $j \ge \alpha + 1$). Write $w_j + z_j = u_j + 4v_j$ with $u_j, v_j \ge 0$ and $u_j < 4$ if $1 \le j \le \alpha$, and $w_j + z_j = u_j + 2v_j$ with $u_j = 0$ or 1 and $v_j \ge 0$ if $\alpha + 1 \le j \le \gamma$.

Then

$$g_0h_0 = x_1^{w_1} \cdots x_\gamma^{u_\gamma} x_1^{4\;1} \cdots x_\alpha^{4v} x_{\alpha+1}^{2v_{\alpha+1}} \cdots x_\gamma^{2v_\gamma}\left[\left(x_2^{w_2} \cdots x_\gamma^{w_\gamma}\right)^{-1}, x_1^{-z_1}\right]$$

$$\cdots \left[x_\gamma^{-w_\gamma}, x_{\gamma-1}^{-z_{\gamma-1}}\right]kk',$$

for some k in G_3 and k' in $[G, X^2(G)].X^4(G)$. Therefore

$$F(g_0) + F(h_0) + F(g_0h_0) + \sum_{1\le i\le\gamma} F\left(x_i^{w_i}\right) + F\left(x_i^{z_i}\right) + F\left(x_i^{u_i}\right) + \sum_{\alpha+1\le i\le\gamma} F\left(x_i^{2v_i}\right)$$

$$= \sum_{1\le i<j\le\gamma}\left((w_iw_j+z_iz_j+u_iu_j)t_{ij}+z_iw_j\left(\sum_{1\le k\le n} p_{ki}q_{kj}+p_{kj}q_{ki}\right)\right)$$

$$= \sum_{1\le i<j\le\gamma}\left(w_iz_j\sum_{1\le k\le n} p_{ki}q_{kj}+w_jz_i\sum_{1\le k\le n} p_{kj}q_{ki}+(w_iz_j+w_jz_i)\left(\sum_{1\le \ell\le q} s_\ell h_{\ell i}h_{\ell j}\right)\right)$$

$$= \sum_{1\le k\le n}\left(\sum_{1\le i,j\le\gamma} w_ip_{ki}z_jq_{kj} + \sum_{1\le i\le\gamma} w_ip_{ku}z_iz_{ki}\right)$$

$$+ \sum_{1\le \ell\le q} s_\ell\left(\sum_{1\le i,j\le\gamma} w_ih_{\ell i}z_jh_{\ell j} + \sum_{1\le i\le\gamma} w_ih_{\ell i}z_ih_{\ell i}\right)$$

$$= \sum_{1\le k\le n} f_{1k}(g_0)f_{2k}(h_0) + \sum_{1\le \ell\le q} s_\ell h_\ell(g_0)h_\ell(h_0) + \sum_{1\le k\le n}\ \sum_{1\le \ell\le\gamma} w_iz_ip_{ki}q_{ki}$$

$$+ \sum_{1\le \ell\le q} s_\ell \sum_{1\le i\le\gamma} w_iz_ih_{\ell i}^2 .$$

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It may be checked that if $1 \leq i \leq \alpha$,

$$F\begin{pmatrix} w_i \\ x_i \end{pmatrix} + F\begin{pmatrix} z_i \\ x_i \end{pmatrix} + F\begin{pmatrix} u_i \\ x_i \end{pmatrix} = \sum_{1 \leq k \leq n} w_i z_i p_{ki} q_{ki} + \sum_{1 \leq \ell \leq q} s_\ell w_i z_i h^2_{\ell i} ,$$

and if $\alpha + 1 \leq i \leq \gamma$ then

$$F\begin{pmatrix} w_i \\ x_i \end{pmatrix} + F\begin{pmatrix} z_i \\ x_i \end{pmatrix} + F\begin{pmatrix} u_i \\ x_i \end{pmatrix} + F\begin{pmatrix} 2v_i \\ x_i \end{pmatrix} = \sum_{1 \leq k \leq n} w_i z_i p_{ki} q_{ki} + \sum_{1 \leq \ell \leq q} s_\ell w_i z_i h^2_{\ell i} .$$

Thus F can be defined on all of G consistently with the coboundary formula, and so

$$\sigma(\ker \cup : \mathrm{Sym}_2(H^1(G;R)) \to H^2(G;R)) = \eta_R^* \operatorname{Hom}(G_2/[G, X^2(G)].(X^4(G) \cap G_2), R) .$$

The kernel of $\sigma|\ker \cup$ is the intersection $(\ker \sigma) \cap (\ker \cup)$ and if R is a perfect field,

$$(\ker \sigma) \cap (\ker \cup) = \{h \text{ in } H^1(G;R) | h \cup h = 0\} .$$

In particular if $R = \mathbb{Z}/2\mathbb{Z}$ this subspace is the kernel of the Bockstein map associated with the coefficient sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and so is the image in $H^1(G, \mathbb{Z}/2\mathbb{Z})$ of $H^1(G, \mathbb{Z}/4\mathbb{Z})$ via the reduction modulo 2 map [2; page 280].

REMARK. For p an odd prime, the group G presented by

$$\{x,y,z | x^p = x^p = z^{p^2} = 1 , [x,y] = z^p , [x,z] = [y,z] = 1\}$$

satisfies $G_2 \subset X^p(G)$ and $G_2/G_3 = \mathbb{Z}/p\mathbb{Z}$ so

$$0 = \operatorname{Hom}(G_2/G_3.(X^p(G) \cap G_2), \mathbb{Z}/p\mathbb{Z}) \neq \operatorname{Hom}(G_2/G_3, \mathbb{Z}/p\mathbb{Z}) .$$

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Thus the cup product may be injective even if $\mathrm{Hom}(G_2/G_3,R)$ is nonzero.

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Department of Mathematics,
Faculty of Science,
Australian National University,
G.P.O. Box 4,
CANBERRA, A.C.T. 2600.
AUSTRALIA.

https://doi.org/10.1017/S0004972700009941 Published online by Cambridge University Press