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On the loss of compactness in the vectorial heteroclinic connection problem

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We give an alternative proof of the theorem of Alikakos and Fusco concerning existence of heteroclinic solutions $U\colon \mathbb{R} \to \mathbb{R}^N$ to the system

$$U_{xx} = \mathrm{D}W(U),$$
$$U(\pm\infty) = a^{\pm}.$$

Here $a^{\pm}$ are local minima of a potential $W \in C^2(\mathbb{R}^N)$ with $W(a^{\pm}) = 0$. This system arises in the theory of phase transitions. Our method is variational but differs from the original artificial constraint method of Alikakos and Fusco and establishes existence by analysing the loss of compactness in minimizing sequences of the action in the appropriate functional space. Our assumptions are slightly different from those considered previously and also imply *a priori* estimates for the solution.

Keywords: Heteroclinic connection problem; loss of compactness; phase transitions; Hamiltonian system

2010 *Mathematics subject classification:* Primary 34C37; 46B50; 82B26
Secondary 37K05

1. Introduction

In this paper we consider the problem of existence of heteroclinic solutions to the Hamiltonian ordinary differential equation (ODE) system

$$\left.\begin{aligned} U_{xx} &= \mathrm{D}W(U), \quad & U &\colon \mathbb{R} \to \mathbb{R}^N, \\ U(-\infty) &= a^-, \quad & U(+\infty) &= a^+, \end{aligned}\right\} \tag{1.1}$$

where $W \in C^2(\mathbb{R}^N)$ is a potential and $a^{\pm}$ are its local minima with $W(a^{\pm}) = 0$. A typical W for $N = 2$ is shown in figures 1 and 2. Solutions to (1.1) are known as *heteroclinic connections*, being *standing waves* of the gradient diffusion system

$$u_t = u_{xx} - \mathrm{D}W(u), \quad u\colon \mathbb{R} \times (0, +\infty) \to \mathbb{R}^N. \tag{1.2}$$

System (1.1) arises in the theory of phase transitions. For details we refer the reader to [2, 8]. From the viewpoint of physics, (1.1) is the Newtonian law of motion with force $-\mathrm{D}(-W)$ induced by the potential $-W$ and with U the trajectory of a test particle that connects two maxima of $-W$. In the scalar case of $N = 1$, the existence

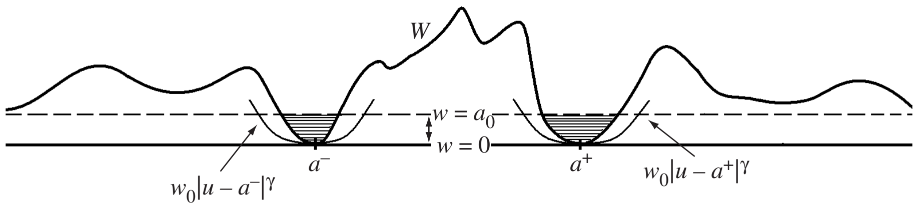

Figure 1. A typical W, which satisfies assumption (A1) and the coercivity assumption (A2′).

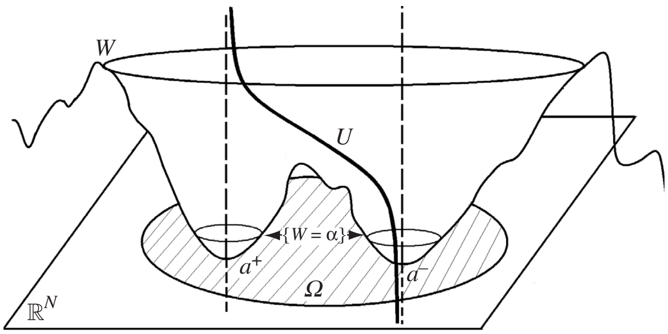

Figure 2. A typical W, the heteroclinic solution U, the localization set Ω of (A2″) and the level sets.

is textbook material by phase plane methods. For a variational approach we refer the reader to [2]. Even in this simple case the unboundedness of $\mathbb{R}$ implies that standard compactness and semi-continuity arguments fail when one tries to obtain solutions to $U_{xx} = W'(U)$ variationally as minimizers of the *action* functional

$$E(U) = \int_{\mathbb{R}} \{\tfrac{1}{2}|U_x|^2 + W(U)\}\,\mathrm{d}x. \tag{1.3}$$

However, for $N = 1$, rearrangement methods do apply [12]. When $N > 1$, solving (1.1) is much more difficult. It was first considered by Sternberg [15] as a problem arising in the study of the elliptic system $\Delta U = \mathrm{D}W(U)$. Noting the compactness problems, he uses the *Jacobi principle* to obtain solutions by studying geodesics in the Riemannian manifold $(\mathbb{R}^N \setminus \{a^{\pm}\}, \sqrt{2W}\langle\cdot,\cdot\rangle)$.

Following a different approach, Alikakos and Fusco [5] subsequently treated (1.1) using the *least action principle*. They derived their solutions as minimizers of (1.3). They introduced an artificial constraint in order to restore compactness, and applied the direct method in order to obtain solutions to (1.1) by eventually removing the constraint. The same approach has subsequently been applied by Alikakos and Katzourakis [7] to the respective travelling wave problem for (1.2), establishing the existence of a solution to the system $U_{xx} = \mathrm{D}W(U) - cU_x$ for $c \neq 0$. System (1.1) has attracted some attention in connection with the study of system $\Delta U = \mathrm{D}W(U)$, and related material appears in [1, 3, 4, 6, 10].

The problem (1.1) is non-trivial; except for the failure of the direct method for (1.3) due to the loss of compactness, an additional difficulty is that the maximum

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principle does not apply when $N > 1$. substitutes of the maximum principle for minimizers were introduced in [5, 7]. Inspired by these results, Katzourakis [11] developed related ideas that apply to general non-convex functionals. A further difficulty of (1.1) is that additional minima of W inhibit existence, and suitable assumptions on W must be imposed (see [5]).

In the present work, following Alikakos and Fusco [5], we obtain solutions to (1.1) as minimizers of (1.3). We bypass their unilateral constraint method, which is of independent interest but requires a rather delicate analysis. We establish the existence for (1.1) by analysing and then restoring by hand the loss of compactness in minimizing sequences. Our motivation comes from the theory of *concentration compactness* (see [9, 13, 14] for a related application of this principle). We note, however, that Lions's theory merely motivated the ideas used herein and we do not know if the well-known condition of 'strict inequality' applies in the present context. Our approach is conceptually different: we introduce a functional space tailored to the study of (1.1) and show that, given any minimizing sequence of (1.3), there exist uniformly decaying translates up to which compactness is restored and passage to a minimizer is available (theorem 2.1). Our main ingredients are certain energy estimates and measure bounds that relate to those of [5, 7]. Herein, however, we use a different method: we control the behaviour of the minimizing sequence by using the supremum-level sets $\{W \geqslant \alpha\}$ and compactify the sequence by suitable translations.

Our basic assumption (A1) is slightly stronger than the respective assumption of [5], but we still allow for a certain degree of degeneracy. Under this assumption we obtain the *a priori* quantitative decay estimates $(*)$ by means of energy arguments, without linearizing the equation. The assumptions (A2′), (A2″) allow Ws with several minima and possibly unbounded from below, similar to those in [5]. We believe that our proof of the Alikakos–Fusco theorem [5] provides further insights into the problem.

2. Hypotheses, set-up and the existence-compactness result

2.1. Hypotheses

We assume $W \in C^2(\mathbb{R}^N)$ with $a^{\pm}$ local minima at zero: $W(a^{\pm}) = 0$. Moreover, we have the following.

(A1) There exist $\alpha_0, w_0 > 0$ and $\gamma \geqslant 2$ such that for all $\alpha \in (0, \alpha_0]$ the sublevel sets $\{W \leqslant \alpha\}$ contain two C^2 strictly convex components $\{W \leqslant \alpha\}^{\pm}$, each enclosing $a^{\pm}$ such that $\{W = \alpha\} = \partial\{W \leqslant \alpha\}$ and

$$W(u) \geqslant w_0|u - a^{\pm}|^{\gamma}, \quad u \in \{W \leqslant \alpha_0\}^{\pm}.$$

In addition, at least one of the following two properties is satisfied:

(A2′) we have

$$\{W \leqslant \alpha_0\} = \{W \leqslant \alpha_0\}^{+} \cup \{W \leqslant \alpha_0\}^{-};$$

(A2″) there exist a convex bounded (localization) set $\Omega \subseteq \mathbb{R}^N$ and a $w_{\max} > \alpha_0$ such that $a^{\pm}$ are global minima of $W|_{\Omega}$, while

$$\Omega \subseteq \{W \leqslant w_{\max}\}, \qquad \partial\Omega \subseteq \{W = w_{\max}\}.$$

Hypothesis (A1) allows $C^{\gamma-\varepsilon}$ flatness at the minima for all $\varepsilon > 0$ (but not C^∞ flatness as in [5, 7]). The assumption (A2′) requires that $\{W \leqslant \alpha\}^\pm$ are the *only* components of the sublevel sets $\{W \leqslant \alpha\}$. We note that there is a crucial local monotonicity assumption hidden inside (A1). this monotonicity is included in the statement that the level sets coincide with the boundaries of the sublevel sets and hence 'flatness' is excluded.

Under (A2′), we immediately obtain $\liminf_{|u|\to\infty} W(u) \geqslant \alpha_0$. Assumption (A2″) permits Ws that may be unbounded from below, assuming non-negativity of W only within Ω.

Under (A2″) the existence of a local minimizer U of (1.3) with $E(U) > -\infty$ is a certain issue, but (A1) is more crucial. We shall refer to (A2′) as the *coercive* assumption and to (A2″) as the *non-coercive* assumption.

2.2. Functional set-up

We derive solutions to (1.1) as minimizers of (1.3) in an affine Sobolev space that incorporates the boundary condition $U(\pm\infty) = a^\pm$ and excludes the trivial solutions $U = a^\pm$. Let $[W^{1,p}_{\mathrm{loc}}(\mathbb{R})]^N$ denote the local Sobolev space of vector functions $U \colon \mathbb{R} \to \mathbb{R}^N$. For $\varepsilon > 0$ consider the affine function

$$U^\varepsilon_{\mathrm{aff}}(x) := \begin{cases} a^-, & x \leqslant -\varepsilon, \\ \left(\dfrac{\varepsilon - x}{2\varepsilon}\right)a^- + \left(\dfrac{\varepsilon + x}{2\varepsilon}\right)a^-, & -\varepsilon < x < \varepsilon, \\ a^+, & x \geqslant \varepsilon, \end{cases} \tag{2.1}$$

and set $U^1_{\mathrm{aff}} := U_{\mathrm{aff}}$. For $p \in (1,\infty)$, the affine L^p-space, $[L^p_{\mathrm{aff}}(\mathbb{R})]^N := [L^p(\mathbb{R})]^N + U_{\mathrm{aff}}$ is a complete metric space for the L^p distance. The function (2.1) will also serve as an *a priori* upper bound on the action (1.3) of the minimizer. For $p, q \in (1,\infty)$, we introduce the affine anisotropic Sobolev space

$$[W^{1;p,q}_{\mathrm{aff}}(\mathbb{R})]^N := \{U \in [L^p_{\mathrm{aff}}(\mathbb{R})]^N : U_x \in [L^q(\mathbb{R})]^N\}. \tag{2.2}$$

This is a complete metric space, isometric to a reflexive Banach space. The purpose of this work is to establish the following version of the Alikakos–Fusco theorem from [5].

THEOREM 2.1 (existence–compactness). *Assume that W satisfies (A1) and either (A2′) or (A2″), with α_0, γ, w_0 as in (A1), (A2′), (A2″). There exists a minimizing sequence $(U_i)_1^\infty$ of the problem*

$$E(U) = \inf\{E(V) : V \in [W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N\}$$

for (1.3) with $E(U_i) \geqslant 0$. For any such $(U_i)_1^\infty$, there exist $(x_i)_1^\infty \subseteq \mathbb{R}$ and translates $\tilde{U}_i := U_i(\cdot - x_i)$ that have a subsequence converging weakly in $[W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N$ to a minimizer U which solves (1.1).

In addition, any such minimizing solution U satisfies the decay estimates

$$\left.\begin{aligned} |U(x) - a^\pm| &\leqslant (M{w_0}^{-1})^{1/\gamma}|x|^{-1/\gamma}, \\ |U_x(x)| &\leqslant (2M)^{1/2}|x|^{-1/2}, \end{aligned}\right\} \quad |x| \geqslant M\alpha_0^{-1}, \tag{$*$}$$

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as well as the bound $E(U) \leqslant M$, where

$$M = |a^+ - a^-| \max_{[a^-,a^+]} \sqrt{2W}.$$

Corollary 2.2. *Estimates (∗) imply that the solution is non-trivial. In particular, $U \not\equiv a^\pm$.*

Theorem 2.1 asserts that translation invariance of (1.1) and (1.3) causes the *only* possible loss of compactness to minimizing sequences. The space $[W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N$ plays a special role to this description. The estimates (∗) are an essential property, satisfied *uniformly* by the compactified sequence of the translates and may *not* be satisfied by the initial $(U_i)_1^\infty$. In addition, they are quantitative, in the sense that the constant depends explicitly on the potential. Moreover, they guarantee that both $U(\pm\infty) = a^\pm$ and $U_x(\pm\infty) = 0$, fully, not merely up to subsequences.

3. Proof of the main result

3.1. Control of the minimizing sequence

Let $(U_i)_1^\infty$ be any minimizing sequence of (1.3). We will tacitly identify each U_i with its precise representatives. Since

$$|U(x'') - U(x')| \leqslant (x'' - x')^{1/2}\left(\int_{x'}^{x''} |U_x|^2 \,\mathrm{d}x\right)^{1/2},$$

we have the inclusion $[W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N \subseteq [C^{1/2}(\mathbb{R})]^N$. By (2.1), we obtain

$$E(U^\varepsilon_{\mathrm{aff}}) = \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{|a^+ - a^-|^2}{8\varepsilon^2} + W\left(\left(\frac{\varepsilon - x}{2\varepsilon}\right)a^- + \left(\frac{\varepsilon + x}{2\varepsilon}\right)a^-\right)\right\} \mathrm{d}x,$$

and hence the explicit bounds

$$\frac{|a^+ - a^-|^2}{4\varepsilon} \leqslant E(U^\varepsilon_{\mathrm{aff}}) \leqslant \frac{|a^+ - a^-|^2}{4\varepsilon} + 2\varepsilon \max_{[a^-,a^+]} W. \tag{3.1}$$

We immediately get

$$\inf_{[W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N} E \leqslant \inf_{\varepsilon>0} E(U^\varepsilon_{\mathrm{aff}}) \leqslant |a^+ - a^-| \max_{[a^-,a^+]} \sqrt{2W} = M < \infty.$$

M is necessarily a *strict* upper bound, since all $U^\varepsilon_{\mathrm{aff}}$ are merely Lipschitz, while minimizing solutions to (1.1) must be smooth (the latter is a consequence of standard regularity considerations of the solutions to the Euler–Lagrange equations). Furthermore, for i large we have

$$\int_{\mathbb{R}} \tfrac{1}{2}|(U_i)_x|^2 \,\mathrm{d}x + \int_{\mathbb{R}} W(U_i)\,\mathrm{d}x \leqslant M. \tag{3.2}$$

We now derive bounds for $[L^\infty(\mathbb{R})]^N$. These are obtained in two different ways, depending on whether (A2′) or (A2″) is assumed. In the case of (A2′), the bound is a consequence of the next energy estimate. For $\alpha \in (0, \alpha_0]$ and $i = 1, 2, \dots$ we define the *control set*

$$\Lambda_i^\alpha := \{x \in \mathbb{R} \colon W(U_i(x)) > \alpha\}. \tag{3.3}$$

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Let $|\cdot|$ denote the Lebesgue measure on $\mathbb{R}$ and let M denote the constant in estimates $(*)$.

Lemma 3.1 (energy estimate I). *Assume W satisfies (A2′). Then we have*

$$M \geqslant \alpha|\Lambda_i^\alpha| + \frac{1}{2}\int_{\mathbb{R}} |(U_i)_x|^2 \,\mathrm{d}x, \tag{3.4}$$

$$\|U_i\|_{[L^\infty(\mathbb{R})]^N} \leqslant |\Lambda_i^\alpha|^{1/2}\left(\int_{\mathbb{R}} |(U_i)_x|^2 \,\mathrm{d}x\right)^{1/2} + \max_{u\in\{W\leqslant\alpha\}^\pm} |u| \tag{3.5}$$

for all $i \in \mathbb{N}$.

Proof of lemma 3.1. By (3.2) and (3.3), we have

$$\begin{aligned} M &\geqslant E(U_i)\\ &= \int_{\mathbb{R}} W(U_i)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} |(U_i)_x|^2 \,\mathrm{d}x\\ &\geqslant \int_{\Lambda_i^\alpha} W(U_i)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} |(U_i)_x|^2 \,\mathrm{d}x\\ &\geqslant \alpha|\Lambda_i^\alpha| + \frac{1}{2}\int_{\mathbb{R}} |(U_i)_x|^2 \,\mathrm{d}x. \end{aligned}$$

This proves (3.4). Now let (t', t'') be a subinterval of Λ_i^α such that the end points $U_i(t')$, $U_i(t'')$ of $U_i((t', t''))$ lie on different components of $\{W = \alpha\}$. Hence, we have

$$|U_i(t') - U_i(t'')| \leqslant |t'' - t'|^{1/2}\left(\int_{t'}^{t''} |(U_i)_x|^2 \,\mathrm{d}x\right)^{1/2} \leqslant |\Lambda_i^\alpha|^{1/2}\left(\int_{\mathbb{R}} |(U_i)_x|^2 \,\mathrm{d}x\right)^{1/2}.$$

By using the fact that $U_i(t') \in \{W = \alpha\}^\pm$, we deduce

$$|U_i(t'') - U_i(t')| \geqslant |U_i(t'')| - |U_i(t')| \geqslant |U_i(t'')| - \max_{u\in\{W\leqslant\alpha\}^\pm} |u|.$$

This establishes estimate (3.5), proving lemma 3.1. □

Corollary 3.2 (L^∞ bound under (A2′)). *If W satisfies (A1), (A2′), then*

$$\sup_{i\geqslant 1} \|U_i\|_{[L^\infty(\mathbb{R})]^N} \leqslant \sqrt{\frac{2}{\alpha_0}}M + \max_{u\in\{W\leqslant\alpha_0\}^\pm} |u|. \tag{3.6}$$

Now we turn to the case of (A2″). We obtain the existence of a minimizing sequence $(U_i)_1^\infty$ of (1.3) localized inside $\bar{\Omega} \subseteq \mathbb{R}^N$ whereon $W|_\Omega \geqslant 0$.

Lemma 3.3 (L^∞ bound under (A2″)). *If W satisfies (A1), (A2″), there is a minimizing sequence $(U_i)_1^\infty$ for which $\bigcup_{i=1}^\infty U_i(\mathbb{R}) \subseteq \bar{\Omega}$ and $W(U_i) \geqslant 0$. Moreover,*

$$\sup_{i\geqslant 1} \|U_i\|_{[L^\infty(\mathbb{R})]^N} \leqslant \max_{u\in\partial\Omega} |u|. \tag{3.7}$$

Proof of lemma 3.3. We show the existence of a deformation of W to a new $\bar{W}$ such that $\bar{W} = W$ on Ω and *all* the minimizing sequences of (1.3) relative to $\bar{W}$ in

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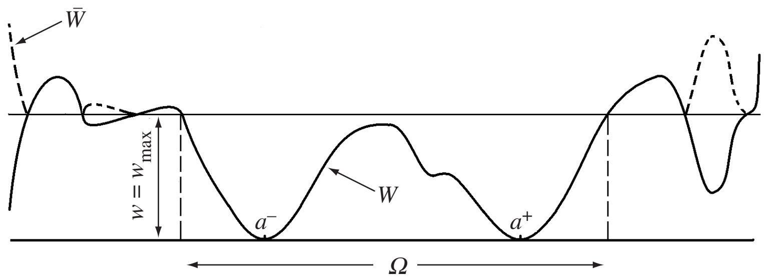

Figure 3. The deformed coercive potential $\bar{W}$, for which $w = w_{\max}$ is a lower bound outside of Ω.

$[W^{1;\gamma,2}_{\text{aff}}(\mathbb{R})]^N$ can be chosen to be localized inside Ω. By (A2″), $W \leqslant w_{\max}$ inside Ω and $W = w_{\max}$ on $\partial\Omega$. We define $\bar{W}$ by reflecting with respect to the hyperplane $\{w = w_{\max}\}$ the portions of the graph of W that lie in the half-space $\{w < w_{\max}\}$ to $\{w > w_{\max}\}$. See figure 3.

By construction, $\bar{W}(u) \geqslant w_{\max}$ for $u \in \mathbb{R}^N \setminus \Omega$. Suppose by contradiction that $\bar{W}$ has a minimizing sequence $(U_i)_1^\infty$ such that for some U_i and $a < b$ $U_i((a,b)) \subseteq \mathbb{R}^N \setminus \Omega$. This is the only case that has to be excluded, since, by the definition of $[W^{1;\gamma,2}_{\text{aff}}(\mathbb{R})]^N$, the 'tails' of each U_i approach $a^\pm \in \Omega$ asymptotically, at least along a sequence (in general, of course, there may exist countably many such intervals and we apply this argument to each of them). By replacing $U_i([a,b])$ by the straight line segment with the same end points, i.e. by defining

$$\bar{U}_i(x) := \begin{cases} U_i(x), & x \in \mathbb{R} \setminus (a,b), \\ \left(\dfrac{x-a}{b-a}\right) U_i(b) + \left(\dfrac{b-x}{b-a}\right) U_i(a), & x \in (a,b), \end{cases} \tag{3.8}$$

we obtain by the convexity of Ω that $\bar{U}_i(\mathbb{R}) \subseteq \bar{\Omega}$. By pointwise comparison,

$$\int_a^b \bar{W}(\bar{U}_i(x))\,\mathrm{d}x \leqslant \int_a^b \bar{W}(U_i(x))\,\mathrm{d}x. \tag{3.9}$$

In addition, $\bar{U}_i|_{(a,b)}$ minimizes the Dirichlet integral, since it is a straight line. Thus,

$$\frac{|\bar{U}_i(b) - \bar{U}_i(a)|^2}{b-a} = \int_a^b |(\bar{U}_i)_x|^2\,\mathrm{d}x < \int_a^b |(U_i)_x|^2\,\mathrm{d}x. \tag{3.10}$$

Formulae (3.9) and (3.10) imply that there exists a minimizing sequence of the action (1.3) with a potential $\bar{W}$ (in place of W) which lies inside $\bar{\Omega}$. Finally, $W|_\Omega = \bar{W}|_\Omega$, by construction. □

In the case when (A2″) is assumed, we fix a sequence valued inside Ω. Moreover,

$$M \geqslant \liminf_{i\to\infty} E(U_i) =: \inf\{E(V) \colon V \in [W^{1;\gamma,2}_{\text{aff}}(\mathbb{R})]^N\} \geqslant 0.$$

As the notation suggests, the right-hand side will stand for $\liminf_{i\to\infty} E(U_i)$ here and later. Now we employ (A1) to show that Λ_i^α is connected. For $\alpha \in (0, \alpha_0]$,

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$i = 1, 2, \dots,$ we set

$$\lambda_i^{\alpha-} := \inf \Lambda_i^\alpha, \qquad \lambda_i^{\alpha+} := \sup \Lambda_i^\alpha. \tag{3.11}$$

We also set

$$d_\alpha := \mathrm{dist}(\{W = \alpha\}^-, \{W = \alpha\}^+). \tag{3.12}$$

We note that d_α is the distance between the two components of the level set $\{W = \alpha\}$.

Lemma 3.4 (control on the $\lambda^{\alpha\pm}$ times). *Assume W satisfies (A1) and either (A2′) or (A2″). Then, for $\alpha \in (0, \alpha_0]$, if $(U_i)_1^\infty$ is the minimizing sequence constructed previously, then the respective sets Λ_i^α are intervals, and hence*

$$\Lambda_i^\alpha = (\lambda_i^{\alpha-}, \lambda_i^{\alpha+}).$$

Proof of lemma 3.4. The claim follows by a direct application of [7, replacement lemma 12, p. 1381] by choosing as μ the Lebesgue measure on $\mathbb{R}$. In order to make the presentation self-contained, we also provide an alternative proof that bypasses this maximum principle type of result of [7]. We note that the result follows by the replacement lemma of [5] as well, but this is not entirely direct since here we use convex level sets and not balls.

We fix a term U_i of the minimizing sequence and its respective Λ_i^α and we drop the subscript i. Since $\Lambda^\alpha = \{W(U) > \alpha\}$ is open, there exist countably many open intervals such that

$$\Lambda^\alpha = \bigcup_{p=0}^{\infty} (x_{2p}^\alpha, x_{2p+1}^\alpha). \tag{3.13}$$

Since $U \in [C^0(\mathbb{R})]^N$, each image $U((x_{2p}^\alpha, x_{2p+1}^\alpha))$ is connected, with end points on $\{W(U) = \alpha\}$ and

$$U(\Lambda^\alpha) = \bigcup_{p=0}^{\infty} U((x_{2p}^\alpha, x_{2p+1}^\alpha)). \tag{3.14}$$

Claim 3.5. *For all $p \in \mathbb{N}$, the image $U((x_{2p}^\alpha, x_{2p+1}^\alpha))$ has end points on different components $\{W(U) = \alpha\}^\pm$ of $\{W(U) = \alpha\}$.*

Indeed, suppose by contradiction that for some p both $U(x_{2p}^\alpha)$ and $U(x_{2p+1}^\alpha)$ are on $\{W(U) = \alpha\}^+$. The deformation of lemma 3.3, together with the strictness of assumption (A1), contradicts the minimality of U. The same holds if the end points are on $\{W(U) = \alpha\}^-$. The claim follows.

Claim 3.6. *The set Λ^α consists of finitely many intervals of odd number.*

By claim 3.5, for each p, $U((x_{2p}^\alpha, x_{2p+1}^\alpha))$ has end points on different components $\{W(U) = \alpha\}$. Hence, in view (3.12) we have

$$d_\alpha \leqslant |U(x_{2p+1}^\alpha) - U(x_{2p}^\alpha)| \leqslant \int_{x_{2p}^\alpha}^{x_{2p+1}^\alpha} |U_x|$$

and hence, for each $q \in \mathbb{N}$, by (3.13),

$$q d_\alpha \leqslant \sum_{p=0}^{q} \int_{x_{2p}^\alpha}^{x_{2p+1}^\alpha} |U_x| \leqslant \int_{\Lambda^\alpha} |U_x| \leqslant |\Lambda^\alpha|^{1/2} \biggl(\int_{\mathbb{R}} |U_x|^2 \biggr)^{1/2}.$$

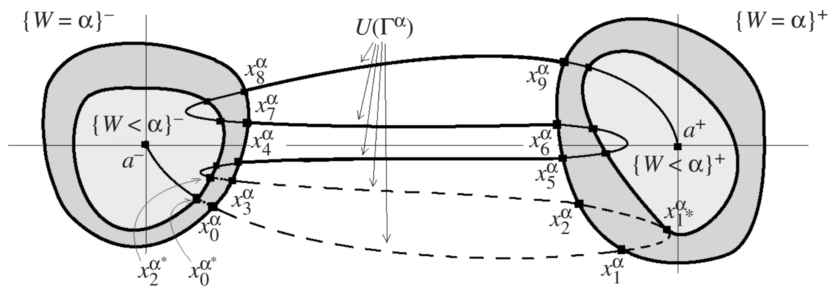

Figure 4. Illustration with $p_\alpha = 4$. By minimality, the dashed line with end points $U(x_0^{\alpha *})$, $U(x_2^{\alpha *})$ cannot exist. For brevity we denote the points $U(x_p^\alpha)$ by x_p^α.

Hence, by lemma 3.1, we have

$$q \leqslant \frac{1}{d_\alpha}\left(\frac{M}{\alpha}\right)^{1/2} M^{1/2},$$

which implies that there exists a $p_\alpha \in \mathbb{N}$ no greater than the integer part of $M/\sqrt{\alpha}d_\alpha$ such that

$$\Lambda^\alpha = \bigcup_{p=0}^{p_\alpha} (x_{2p}^\alpha, x_{2p+1}^\alpha).$$

Since

$$\mathbb{R} \setminus \Lambda^\alpha = (-\infty, x_0^\alpha] \cup [x_1^\alpha, x_2^\alpha] \cup \cdots \cup [x_{2p^\alpha-1}^\alpha, x_{2p^\alpha}^\alpha] \cup [x_{2p_\alpha+1}^\alpha, +\infty)$$

and $\mathbb{R} \setminus \Lambda^\alpha = \{W(U) \leqslant \alpha\}$, U exits $\{W(U) \leqslant \alpha\}^-$ for the first time at $x = x_0^\alpha$ and stays inside $\{W(U) \leqslant \alpha\}^+$ after $x = x_{2p_\alpha+1}^\alpha$ (figure 4). Since

$$\begin{aligned} U(x_0^\alpha) &\in \{W = \alpha\}^-, \\ U(x_1^\alpha), U(x_2^\alpha) &\in \{W = \alpha\}^+, \\ U(x_3^\alpha), U(x_4^\alpha) &\in \{W = \alpha\}^-, \\ &\vdots \end{aligned}$$

in view of (3.14), the number of intervals has to be odd, otherwise U stays inside $\{W \leqslant \alpha\}^-$ for an infinite time and this contradicts the conjecture that (at least along a sequence) $U(x)$ converges to a^+ as $x \to \infty$.

Claim 3.7. *All subsets* $U((x_1^\alpha, x_2^\alpha)), U((x_3^\alpha, x_4^\alpha)), \dots, U((x_{2p_\alpha-1}^\alpha, x_{2p_\alpha}^\alpha))$ *of the image* $U(\mathbb{R} \setminus \Lambda^\alpha)$ *lie inside the interior* $\{W < \alpha\}$ *and cannot touch the boundary* $\{W = \alpha\}$ *(figure 4).*

Fix a $q \in \{1, \dots, p_\alpha\}$ and assume by contradiction that there exists $[a, b] \subseteq (x_{2q-1}^\alpha, x_{2q}^\alpha)$ such that $U([a, b])$ lies on the boundary $\{W = \alpha\}$. Then, by replacing $U([a, b])$ by the straight line segment with the same end points (as in lemma 3.1), we obtain a contradiction.

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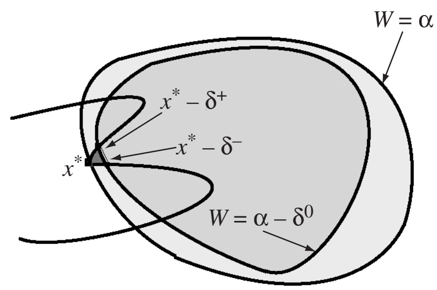

Figure 5. The idea of the proof of claim 3.7.

Hence, if $U((x^\alpha_{2q-1}, x^\alpha_{2q}))$ touches the boundary $\{W = \alpha\}$, this happens at isolated points (otherwise it is inside $\{W < \alpha\}$). See figure 5.

Fix such a point and call it x^*. By continuity and by assumption (A1), there exist $\delta^0, \delta^\pm > 0$ such that $U((x^* - \delta^-, x^* + \delta^+))$ lies outside $\{W < \alpha - \delta^0\}$.

By replacing $U((x^* - \delta^-, x^* + \delta^+))$ by the straight line segment with the same end points (as in lemma 3.1), we obtain a contradiction. By arguing for all such points x^*, we see that $U((x^\alpha_{2q-1}, x^\alpha_{2q}))$ lies inside $\{W < \alpha\}$, as desired.

Claim 3.8. *$p_\alpha = 0$, i.e. Λ^α has only one connected component; hence, $x^\alpha_1 = x^\alpha_{2p_\alpha+1}$.*

We argue by contradiction. Suppose that $p \in \{1, \dots, p_\alpha\}$ and consider the set

$$A := \left\{ \beta \in (0, \alpha) \;\middle|\; U((x^\alpha_{2p-1}, x^\alpha_{2p})) \bigcap_{p=1}^{p_\alpha} \{W < \beta\} \neq \emptyset \right\}. \tag{3.15}$$

Since $U((x^\alpha_{2p-1}, x^\alpha_{2p}))$ lies strictly inside the sublevel set, we have that $A \neq \emptyset$. We set

$$\alpha_* := \inf A.$$

Since there are finitely many components of $U((x^\alpha_{2p-1}, x^\alpha_{2p}))$, their distance from the minimum of W is bounded away frow zero, and hence $0 < \alpha_* < \alpha$. By definition of α_*, there exists at least one of the components $U((x^\alpha_{2p-1}, x^\alpha_{2p}))$, say for $p = 1$, that touches only the boundary of $\{W = \alpha_*\} = \partial\{W < \alpha_*\}$ and does not intersect $\{W < \alpha_*\}$. Moreover, it cannot touch the boundary at more than one point. Hence,

$$\Lambda^{\alpha_*} = (x^{\alpha_*}_0, x^{\alpha_*}_1) \cup (x^{\alpha_*}_0, x^{\alpha_*}_1) \cup \cdots,$$

and consequently $U((x^{\alpha_*}_0, x^{\alpha_*}_2))$ is contained in $\{W \geqslant \alpha_*\}$ and only $U(x^{\alpha_*}_1)$ is on $\{W = \alpha_*\}^+$, having both its end points $U(x^{\alpha_*}_0)$, $U(x^{\alpha_*}_2)$ on $\{W = \alpha_*\}^-$. By arguing as in lemma 3.1 for $U|_{(x^{\alpha_*}_0, x^{\alpha_*}_2)}$, we obtain a contradiction to the minimality of the action of U. Hence, $p_\alpha = 0$.

By proving claims 3.5–3.8, we see that lemma 3.4 is established. □

The following sharpens (3.4), under the additional information that Λ^α_i is connected.

Lemma 3.9 (energy estimate II). *For all $\alpha \in (0, \alpha_0]$ and $i \geqslant 1$, we have*

$$M \geqslant E(U_i) \geqslant \frac{d^2_\alpha}{2(\lambda^{\alpha+}_i - \lambda^{\alpha-}_i)} + \alpha(\lambda^{\alpha+}_i - \lambda^{\alpha-}_i). \tag{3.16}$$

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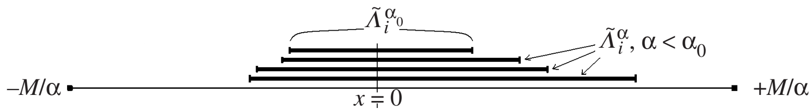

Figure 6. The control sets of $\tilde{U}_i$ are symmetric for $\alpha = \alpha_0$.
$\alpha < \alpha_0$ may not exist, but $0 \in \tilde{\Lambda}_i^\alpha$.

Proof of lemma 3.9. Proceeding as in lemma 3.1, we recall (3.2), to obtain

$$M \geqslant E(U_i) \geqslant \alpha(\lambda_i^{\alpha+} - \lambda_i^{\alpha-}) + \frac{1}{2}\int_{\lambda_i^{\alpha-}}^{\lambda_i^{\alpha+}} |(U_i)_x|^2 \, \mathrm{d}x,$$

where we have also used lemma 3.4. In addition,

$$d_\alpha \leqslant |U_i(\lambda_i^{\alpha-}) - U_i(\lambda_i^{\alpha+})| \leqslant (\lambda_i^{\alpha+} - \lambda_i^{\alpha-})^{1/2}\left(\int_{\lambda_i^{\alpha}}^{\lambda_i^{\alpha+}} |(U_i)_x|^2 \, \mathrm{d}x\right)^{1/2}.$$

The lemma follows. □

Corollary 3.10 (uniform bounds on $|\Lambda_i^\alpha|$). *For $i = 1, 2, \ldots, \alpha \in [0, \alpha_0]$, we have*

$$\frac{d_\alpha^2}{2M} \leqslant |\Lambda_i^\alpha| = \lambda_i^{\alpha+} - \lambda_i^{\alpha-} \leqslant \frac{M}{\alpha}. \tag{3.17}$$

3.2. Restoration of compactness

The bounds (3.17) provide information that allows us to control the behaviour of each U_i by 'tracking' the Λ_i^αs. In the terminology of [1], translation invariance of (1.3) and (1.1) allows us to 'fix a centre' for the U_is and align the minimizing sequence, preventing the terms from escaping to $\pm\infty$. For $i = 1, 2, \ldots,$ we set

$$x_i := \tfrac{1}{2}(\lambda_i^{\alpha_0+} + \lambda_i^{\alpha_0-}), \tag{3.18}$$

which is the centre of the control set $\Lambda_i^{\alpha_0} = (\lambda_i^{\alpha_0-}, \lambda_i^{\alpha_0+})$. We define the *translates* of the minimizing sequence $(U_i)_1^\infty$ by

$$\tilde{U}_i := U_i(\cdot - x_i), \quad i = 1, 2, \ldots. \tag{3.19}$$

The control sets $\tilde{\Lambda}_i^{\alpha_0} = (\tilde{\lambda}_i^{\alpha_0-}, \tilde{\lambda}_i^{\alpha_0+})$ for these translates are centred at $x = 0$, being symmetric (figure 6). The control sets $\tilde{\Lambda}_i^\alpha$ of $\tilde{U}_i$ and Λ_i^α of U_i are related by

$$(\tilde{\lambda}_i^{\alpha-}, \tilde{\lambda}_i^{\alpha+}) = \tilde{\Lambda}_i^\alpha = (\lambda_i^{\alpha-} - x_i, \lambda_i^{\alpha+} - x_i). \tag{3.20}$$

The translates $(\tilde{U}_i)_1^\infty$ defined by (3.18), (3.19) will be referred to as the *compactified sequence* relative to the initial $(U_i)_1^\infty$. The sequence $(\tilde{U}_i)_1^\infty$ will turn out to be weakly precompact in $[W_{\mathrm{aff}}^{1;\gamma,2}(\mathbb{R})]^N$, converging to a solution of (1.1).

Corollary 3.11 (uniform bounds for the compactified sequence).
For $i = 1, 2, \ldots$ and $\alpha \in (0, \alpha_0]$, (3.17) can be rewritten in view of (3.18)–(3.20) as

$$\frac{d_\alpha^2}{2M} \leqslant |\tilde{\Lambda}_i^\alpha| = \tilde{\lambda}_i^{\alpha+} - \tilde{\lambda}_i^{\alpha-} \leqslant \frac{M}{\alpha}. \tag{3.21}$$

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In particular, since $0 \in \tilde{\Lambda}_i^{\alpha}$ *for* $\alpha \in (0, \alpha_0]$ *and* $i = 1, 2, \dots,$ *we have*

$$\max\{|\tilde{\lambda}_i^{\alpha+}|, |\tilde{\lambda}_i^{\alpha-}|\} \leqslant \frac{M}{\alpha}. \tag{3.22}$$

3.3. Bounds and decay estimates for the compactified sequence

The $[L^2(\mathbb{R})]^N$ bound on the derivatives $(\tilde{U}_i)_x$ is immediate by the kinetic energy term of (1.3). The more interesting uniform $[L^{\gamma}_{\text{aff}}(\mathbb{R})]^N$ bound is a consequence of our assumption (A1) on the non-convex potential term.

Lemma 3.12 (estimates for the compactified sequence). *Let* $(\tilde{U}_i)_1^{\infty}$ *be given by definitions (3.18) and (3.19). If* W *satisfies (A1) and either (A2′) or (A2″), then* $(\tilde{U}_i)_1^{\infty}$ *lies in a ball of* $[W^{1;\gamma,2}_{\text{aff}}(\mathbb{R})]^N \cap [L^{\infty}(\mathbb{R})]^N$ *centred at* U_{aff}*. Moreover,*

$$\sup_{i\geqslant 1}\|\tilde{U}_i - U_{\text{aff}}\|_{[L^{\gamma}(\mathbb{R})]^N} \leqslant M^{1/\gamma}\left\{\frac{1}{w_0} + \frac{2}{\alpha_0}\left\{\sup_{i\geqslant 1}\|\tilde{U}_i\|_{[L^{\infty}(\mathbb{R})]^N}\right\}^{\gamma}\right\}^{1/\gamma}, \tag{3.23}$$

$$\sup_{i\geqslant 1}\|\tilde{U}_i\|_{[L^{\infty}(\mathbb{R})]^N} \leqslant \begin{cases} \sqrt{\dfrac{2}{\alpha_0}}M + \max\limits_{u\in\{W\leqslant\alpha_0\}^{\pm}}|u| & \textit{under (A2′)},\\ \max_{u\in\partial\Omega}|u| & \textit{under (A2″)}, \end{cases} \tag{3.24}$$

and

$$\sup_{i\geqslant 1}\|(\tilde{U}_i)_x\|_{[L^2(\mathbb{R})]^N} \leqslant \sqrt{2M}. \tag{3.25}$$

Proof of lemma 3.12. Formula (3.25) follows from translation invariance, while formula (3.24) follows by (3.6), (3.7) and translation invariance. Thus, we only need to prove (3.23). As

$$M \geqslant \int_{\mathbb{R}} W(U_i)\,\mathrm{d}x = \int_{\mathbb{R}} W(\tilde{U}_i)\,\mathrm{d}x \geqslant \int_{-\infty}^{-M/\alpha} W(\tilde{U}_i)\,\mathrm{d}x + \int_{+M/\alpha}^{+\infty} W(\tilde{U}_i)\,\mathrm{d}x,$$

using (3.22), we obtain $W(\tilde{U}_i(x)) \leqslant \alpha$ for $i = 1, 2, \dots$ when $|x| \geqslant M\alpha^{-1}$. Thus, for such x we are in the domain of validity of (A1). For $\alpha = \alpha_0$, we get

$$w_0\left(\int_{-\infty}^{-M/\alpha_0}|\tilde{U}_i - a^-|^{\gamma}\,\mathrm{d}x + \int_{+M/\alpha_0}^{+\infty}|\tilde{U}_i - a^+|^{\gamma}\,\mathrm{d}x\right) \leqslant M.$$

By restricting to smaller $\alpha \leqslant \alpha_1 (< \alpha_0)$, we may assume that $(-M\alpha_0^{-1}, +M\alpha_0^{-1}) \supseteq (-1, 1)$. Hence, $U_{\text{aff}} = a^{\pm}$ for $|x| \geqslant M\alpha_0^{-1}$. To conclude, we employ (3.24) to get

$$\int_{-M/\alpha_0}^{+M/\alpha_0}|\tilde{U}_i - U_{\text{aff}}|^{\gamma}\,\mathrm{d}x \leqslant \frac{2M}{\alpha_0}\{\|\tilde{U}_i\|_{[L^{\infty}(\mathbb{R})]^N}\}^{\gamma}.$$

Putting these estimates together, we see that (3.23) is established. □

Lemma 3.13 (uniform decay estimate). *If* W *satisfies (A1), then the compactified sequence* $(\tilde{U}_i)_1^{\infty}$ *satisfies* $|\tilde{U}_i(x) - a^{\pm}| \leqslant (Mw_0^{-1})^{1/\gamma}|x|^{-1/\gamma}$ *for* $|x| \geqslant M\alpha_0^{-1}$.

Proof of lemma 3.13. We saw in lemma 3.12 that (3.22) implies $W(\tilde{U}_i(x)) \leqslant \alpha$ for $i = 1, 2, \ldots$ when $|x| \geqslant M\alpha^{-1}$. By (A1), we have

$$w_0|\tilde{U}_i(x) - a^{\pm}|^{\gamma} \leqslant W(\tilde{U}_i(x))$$

for all such $x \in \mathbb{R}$. Therefore,

$$|\tilde{U}_i(x) - a^{\pm}|^{\gamma} \leqslant \frac{\alpha}{w_0}$$

for all $|x| \geqslant M\alpha^{-1}$ and all $\alpha \leqslant \alpha_0$. We fix an $x \in \mathbb{R}$ for which $|x| \geqslant M\alpha_0^{-1}$ and choose

$$\alpha(x) := \frac{|x|}{M}.$$

This is a legitimate choice since $|x| = M\alpha(x)^{-1} \geqslant M\alpha_0^{-1}$. We thus obtain that

$$|\tilde{U}_i(x) - a^{\pm}|^{\gamma} \leqslant \frac{\alpha(x)}{w_0} \leqslant \frac{M}{w_0|x|}$$

and, by letting x vary, the estimate follows. □

COROLLARY 3.14 (*a priori* decay estimates). *Assume W satisfies (A1). Then, if a solution U to (1.1) exists, it must satisfy estimates $(*)$ of theorem 2.1.*

*Proof of corollary 3.14.*We recall from [5] the equipartition property $|U_x|^2 = 2W(U)$ satisfied by solutions of (1.1). Equipartition implies $|U_x|^2 = 2W(U) \leqslant 2\alpha$ for $|x| \geqslant M\alpha^{-1}$ and $\alpha \leqslant \alpha_0$. The rest of the proof closely follows that of lemma 3.13. □

3.4. Passage to a minimizing solution

We conclude by proving existence of minimizers. By (3.23)–(3.25), the sequence of translates $(\tilde{U}_i)_1^{\infty}$ converges along a subsequence to some U weakly in $[W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N$. By again denoting the subsequence by $(\tilde{U}_i)_1^{\infty}$, we have that $\tilde{U}_i - U - \rightharpoonup 0$ in $[L^{\gamma}(\mathbb{R})]^N$ and $(\tilde{U}_i - U)_x - \rightharpoonup 0$ in $[L^2(\mathbb{R})]^N$ as $i \to \infty$. Up to a further subsequence, we have $\tilde{U}_i \to U$ in $[L^2_{\mathrm{loc}}(\mathbb{R})]^N$ and almost everywhere on $\mathbb{R}$ as $i \to \infty$. By the weak lower semi-continuity of the L^2 norm and the Fatou lemma, we obtain

$$E(U) \leqslant \liminf_{i\to\infty} E(\tilde{U}_i).$$

By (3.1), we also get $0 \leqslant E(U) \leqslant M$. Thus, U is a local minimizer of the functional (1.3) in $[W^{1;\gamma,2}_{\mathrm{aff}}(\mathbb{R})]^N$. Hence, U solves (1.1) classically and satisfies the estimates $(*)$. The proof of theorem 2.1 is complete.

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