Journal of the Inst. of Math. Jussieu (2006) **5**(2), 187–227 © Cambridge University Press 187
doi:10.1017/S1474748005000241 Printed in the United Kingdom

A HAAGERUP INEQUALITY, DEFORMATION OF TRIANGLES AND AFFINE BUILDINGS

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(Received 27 October 2003; accepted 17 December 2003)

Abstract In this paper we study a Haagerup inequality in the general case of discrete groupoids. We develop two geometrical tools, pinching and tetrahedral change of faces, based on deformation of triangles, to prove it. We show how to use these tools to find all the already known results just by manipulating triangles. We use these tools for groups acting freely and by isometries on the set of vertices of any affine building and give a first reduction of this inequality to its verification on some special triangles and prove the inequality when the building is of type $\tilde{A}_{k_1} \times \cdots \times \tilde{A}_{k_n}$, where $k_i \in \{1, 2\}$, $i = 1, \dots, n$.

Keywords: property of rapid decay (RD); groups acting on buildings; groupoids; geometry of groups

AMS 2000 *Mathematics subject classification:* Primary 20E42
Secondary 46L99; 20F65

Contents

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Introduction

Let Γ be a discrete group, 1 its unit, $\mathbb{C}\Gamma$ its algebra and $\lambda : \mathbb{C}\Gamma \to B(l^2(\Gamma))$ the left regular representation. Recall that a length-function L on Γ is a positive map satisfying

(i) $L(1) = 0$,

(ii) $L(g^{-1}) = L(g)$, and

(iii) $L(gh) \leqslant L(g) + L(h)$ for all $g, h \in \Gamma$.

Γ is said to satisfy the Haagerup inequality with respect to L if there exist two positive constants C and r such that

$$\|\lambda(\phi)\| \leqslant C\|(1 + L)^r \phi\|_2, \quad \forall \phi \in \mathbb{C}\Gamma,$$

where $\|\cdot\|$ is the operator norm and $\|\cdot\|_2$ is the l^2-norm.

The Haagerup inequality was first introduced by Haagerup, who established it for finitely generated free non-Abelian groups with respect to the word length-function, his objective being to show that the metric approximation property is not enough to characterize nuclear C^*-algebras [**Haa79**]. Jolissaint then introduced the equivalent property (RD) of rapid decay [**Jol90**], which demands that the space $H_L^\infty(\Gamma)$ of rapidly decreasing functions on Γ with respect to the length-function L be contained in the reduced C^*-algebra $C_{\mathrm{r}}^*(\Gamma)$. He established property (RD) for classical hyperbolic groups and de la Harpe extended it to word hyperbolic groups *à la* Gromov [**dlH88**]. Jolissaint's property (RD) implies that both $H_L^\infty(\Gamma)$, which is in this case an algebra, and $C_{\mathrm{r}}^*(\Gamma)$, have the same K-theory [**Jol89**]. This fact plays a crucial role in the proof of Novikov's conjecture for higher signatures for Gromov hyperbolic groups [**CM90**] and in Lafforgue's work to prove the Baum–Connes conjecture for cocompact lattices in some semisimple Lie groups [**Laf98**].

In [**Jol90**], Jolissaint proved that property (RD) is stable under some constructions of groups such as taking subgroups, quotients by finite subgroups, extensions by finite groups, direct and free products, some semi-direct products and some amalgamations. On the other hand, he proved that a finitely generated amenable group satisfies the Haagerup inequality if and only if it is of polynomial growth. Such groups are virtually nilpotent. Hence, this provides us with the first examples of groups which do not satisfy the Haagerup inequality such as Grigorchuk groups of intermediate growth [**Gri85**], solvable groups of exponential growth and consequently $\mathrm{SL}_n(\mathbb{Z})$, for $n \geqslant 3$, since it

https://doi.org/10.1017/S1474748005000241 Published online by Cambridge University Press

contains such a solvable subgroup. For $\mathrm{SL}_2(\mathbb{Z})$, it satisfies the Haagerup inequality since it is a Gromov word hyperbolic group.

In fact, Jolissaint's result can be generalized to all discrete amenable groups which are not necessarily finitely generated. These groups satisfy the Haagerup inequality if and only if they are inductive limits of finitely generated virtually nilpotent groups having finite index in each other [**Tal01**, **PT76**]. This provides new examples of groups which satisfy the Haagerup inequality such as countable locally finite groups [**Har95**] and in particular the group of finite permutations of $\mathbb{N}$ and the group of rational rotations of the circle $\mathbb{Q}/\mathbb{Z}$, discrete Abelian groups of finite rank [**Rot95**] and in particular the discrete additive group $\mathbb{Q}$, etc.

In their paper [**RRS98**], Ramagge, Robertson and Steger proved that the Haagerup inequality holds for any group acting freely and by isometries on the set of vertices of an affine building of type $\tilde{A}_1 \times \tilde{A}_1$ or $\tilde{A}_2$, in particular for lattices of $\mathrm{SL}_3(\mathbb{Q}_p)$, and Lafforgue extended their result to actions with bounded isotropy groups cardinality [**Laf00**]. This provided the first examples of higher rank groups satisfying the Haagerup inequality. Lafforgue proved this inequality for cocompact lattices in $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{C})$ in [**Laf00**]. Chatterji, in [**Cha01**, **Cha03**], generalized his result to $\mathrm{SL}_3(\mathcal{H})$ and $\mathrm{SL}_3(\mathcal{O})$ and their products.

This paper is divided into two parts. In the first part, which is analytical, we formulate the Haagerup inequality in the general case of discrete local groupoids and give some properties making connection with the Haagerup inequality for groups. We then give a definition of this inequality for sets of triangles of local groupoids and develop two geometrical tools, pinching and tetrahedral change of faces, based on deformation of triangles, to prove this inequality. We show how to use these tools to find all the already known results on the Haagerup inequality just by manipulating triangles.

In the second part, which is geometrical, we use these tools for groups acting freely and by isometries on the set of vertices of any affine building and give a first reduction of this inequality to its verification on the set of what we call *reduced triangles*. We then show that these reduced triangles, in the case of buildings of type $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_2$ and their products, are contained in an apartment and prove the Haagerup inequality when the building is of type $\tilde{A}_{k_1} \times \cdots \times \tilde{A}_{k_n}$, where $k_i \in \{1, 2\}$, $i = 1, \ldots, n$. The Haagerup inequality in the case of a building of type $\tilde{A}_k \times \cdots \times \tilde{A}_k$ for $k = 1$ or 2 has been proved simultaneously and independently by Chatterji in [**Cha01**, **Cha03**].

Most of the results of this paper have been obtained in my thesis [**Tal01**], and have already been announced in [**Tal02**]. To make the results more general, some definitions given here are a little bit different from the ones given in [**Tal01**, **Tal02**].

1. The Haagerup inequality and groupoids

The Haagerup inequality for groupoids acting simply transitively and by isometries on the set of vertices of a building of type either $\tilde{A}_1 \times \tilde{A}_1$ or $\tilde{A}_2$ was introduced and established in [**RRS98**]. Here, we give a definition of the Haagerup inequality for general discrete local groupoids, give some properties of this inequality and state some geometrical tools to prove it.

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1.1. Definitions and first properties

1.1.1. *Groupoids*

Recall from [**Con94**, § II.5] the following definition.

Definition 1.1. A groupoid consists of a set $\mathcal{G}$ of elements, a distinguished subset $\mathcal{G}^{(0)} \subset \mathcal{G}$ of units called the base of $\mathcal{G}$, two maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$ called the source and the range, respectively, and a partial law of composition defined from the set of composable pairs $\mathcal{G}^{(2)}$ to $\mathcal{G}$ such that

(1) $\mathcal{G}^{(2)} = \{(\alpha_1, \alpha_2) \in \mathcal{G} \times \mathcal{G},\ s(\alpha_1) = r(\alpha_2)\}$;

(2) $s(\alpha) = r(\alpha) = \alpha$, for all elements in $\mathcal{G}^{(0)}$;

(3) $\alpha s(\alpha) = r(\alpha)\alpha = \alpha$, for all elements;

(4) for any element α, there is a unique element α^{-1} called the bilateral inverse, such that $\alpha\alpha^{-1} = r(\alpha)$ and $\alpha^{-1}\alpha = s(\alpha)$;

(5) $s(\alpha_1\alpha_2) = s(\alpha_2)$, $r(\alpha_1\alpha_2) = r(\alpha_1)$, for all composable pairs;

(6) the partial law of composition is associative.

Any group is a groupoid where the set of its units is reduced to a singleton and vice versa.

If $\mathcal{G}$ is a groupoid and $u \in \mathcal{G}^{(0)}$ a unit, consider the fibres $\mathcal{G}_u = s^{-1}\{u\}$, $\mathcal{G}^u = r^{-1}\{u\}$ and the set $\mathcal{G}(u) = \mathcal{G}_u \cap \mathcal{G}^u$. $\mathcal{G}(u)$ is a group called the isotropy group at u. If two units u and v are the source and range, respectively, of an element $\alpha \in \mathcal{G}$, the map $g \mapsto \alpha g \alpha^{-1}$ is an isomorphism from $\mathcal{G}(u)$ to $\mathcal{G}(v)$.

A groupoid $\mathcal{G}$ is said to be trivial if it contains only units. It is said to be transitive if the map $(r, s) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is onto. It is said to be principal if this map is one-to-one. In a transitive groupoid all isotropy groups are isomorphic and any of the fibres $\mathcal{G}_u$ or $\mathcal{G}^u$, for $u \in \mathcal{G}^{(0)}$ generates the groupoid $\mathcal{G}$. In a principal groupoid all isotropy groups are trivial.

If $\{\mathcal{G}_i\}_{i\in I}$ is a family of groupoids then the disjoint union $\mathcal{G} = \coprod_{i\in I} \mathcal{G}_i$ is a groupoid where $\mathcal{G}^{(0)} = \coprod_{i\in I} \mathcal{G}_i^{(0)}$. Conversely, if $\mathcal{G}$ is an arbitrary groupoid, there is a canonical partition into connected components $\mathcal{G}_i$, $i \in I$, of $\mathcal{G}$ defined by the equivalence relation $\alpha \sim \beta$ if there exist $u, v \in \mathcal{G}$ such that $\beta = u\alpha v$. The groupoids $\mathcal{G}_i$ are transitive.

The groupoid given in the following definition has been introduced in [**RRS98**] and was used in [**Laf00**].

Definition 1.2. If Γ is a group acting freely on the right on a space X, it defines a groupoid $\mathcal{G}(X, \Gamma) = X \times_\Gamma X$ where the elements of $\mathcal{G}(X, \Gamma)$ are equivalence classes $[x, y]$ of elements of $X \times X$ under the relation $(x, y) \sim (xg, yg)$ for any $g \in \Gamma$. The units are the classes of the form $[x, x]$, the source and range maps are defined for all $x, y, z \in X$ by $s[x, y] = [y, y]$ and $r[x, y] = [x, x]$, respectively, and the product is defined by $[x, y][y, z] = [x, z]$.

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It is clear that $\mathcal{G}(X,\Gamma)$ is transitive and consequently all isotropy groups are isomorphic. More precisely, we have the following remark.

Remark 1.3. For any $x \in X$, the map $g \mapsto [xg, x]$ is an isomorphism between the group Γ and the isotropy group at $[x, x]$.

On the other hand, the map $x \mapsto [x, x]$ induces a bijection between the set X/Γ of orbits of Γ and the base of $\mathcal{G}(X,\Gamma)$.

In § 2, we will deal with a generalization that we will give of the groupoid $\mathcal{G}(X,\Gamma)$.

When the action of Γ on X is not necessarily free, we construct the following space on which Γ acts freely. Let $X_\Gamma = \bigcup_{x\in X}\{x\} \times \Gamma_x$, where Γ_x is the stabilizer of x. Choose for any two elements $x, y \in X$ in the same orbit an element $\alpha_{x,y} \in \Gamma$ such that

(1) $x\alpha_{x,y} = y$ for all $x, y \in X$ in the same orbit;

(2) $\alpha_{x,x} = 1$ for all $x \in X$;

(3) $\alpha_{x,y}\alpha_{y,z} = \alpha_{x,z}$ for all $x, y, z \in X$ in the same orbit.

This defines a free action of Γ on X_Γ by $(x, g)h = (xh, \alpha_{xh,x}gh)$ for all $((x, g); h) \in X_\Gamma \times \Gamma$. Let $\mathcal{G}(X_\Gamma, \Gamma, \alpha)$ be the corresponding groupoid given by Definition 1.2.

Recall from [**ADR00**, § 2.1.a] the following definition.

Definition 1.4. Let $\mathcal{G}$ be a groupoid and let X be a set together with a surjection $s' : X \to \mathcal{G}^{(0)}$. A right groupoid action of $\mathcal{G}$ on X is a map $(x, \alpha) \mapsto x\alpha$ from the fibred product $X * \mathcal{G} = \{(x, \alpha) \in X \times \mathcal{G},\ s'(x) = r(\alpha)\}$ to X satisfying the following conditions:

(1) $s'(x\alpha) = s(\alpha)$,

(2) if $(x, \alpha) \in X * \mathcal{G}$, $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then $(x\alpha)\beta = x(\alpha\beta)$.

Obviously, any groupoid acts on the set of its units. The groupoid $\mathcal{G}(X,\Gamma)$ acts on the set X by $x[x, y] = y$ where $s' : x \mapsto [x.x]$.

When a groupoid $\mathcal{G}$ acts on a set X then, for all $u \in \mathcal{G}^{(0)}$, this action defines a group action of the isotropy group $\mathcal{G}(u)$ on the corresponding set $X_u = s'^{-1}(u)$. The action of the groupoid $\mathcal{G}$ is said to be free if the map $(x, \alpha) \mapsto (x, x\alpha)$ is one-to-one. This is equivalent to saying that the actions of the isotropy groups $\mathcal{G}(u)$ on the corresponding sets X_u are free. The action is said to be transitive if the map $(x, \alpha) \mapsto (x, x\alpha)$ is onto. It is said to be simply transitive if this map is bijective.

1.1.2. *Local groupoids*

To give a generalization of Definition 1.2 to the case of a groupoid $\mathcal{G}$, we first need to recall the following definition which is equivalent to the one given in [**vE84**, p. 282] and which generalizes the notion of groupoid.

Definition 1.5. A local groupoid consists of a set $\mathcal{G}$ of elements, a distinguished subset $\mathcal{G}^{(0)} \subset \mathcal{G}$ of units called the base of $\mathcal{G}$, two maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$ called the source and the range, respectively, and a partial law of composition defined from the set of composable pairs $\mathcal{G}^{(2)}$ to $\mathcal{G}$ such that

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(1) $\mathcal{G}^{(2)} \subset \{(\alpha_1, \alpha_2) \in \mathcal{G} \times \mathcal{G},\ s(\alpha_1) = r(\alpha_2)\}$;

(2) $s(\alpha) = r(\alpha) = \alpha$, for all elements in $\mathcal{G}^{(0)}$;

(3) for any element α, $(\alpha, s(\alpha))$ and $(r(\alpha), \alpha)$ are in $\mathcal{G}^{(2)}$ and $\alpha s(\alpha) = r(\alpha)\alpha = \alpha$;

(4) for any element α, there is a unique element α^{-1} called the bilateral inverse, such that $(\alpha, \alpha^{-1}), (\alpha^{-1}, \alpha) \in \mathcal{G}^{(2)}$ and $\alpha\alpha^{-1} = r(\alpha)$ and $\alpha^{-1}\alpha = s(\alpha)$;

(5) if $(\alpha_1, \alpha_2) \in \mathcal{G}^{(2)}$, then $(\alpha_2^{-1}, \alpha_1^{-1}) \in \mathcal{G}^{(2)}$ and $(\alpha_1\alpha_2)^{-1} = \alpha_2^{-1}\alpha_1^{-1}$;

(6) if $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3) \in \mathcal{G}^{(2)}$ and $(\alpha_1\alpha_2, \alpha_3) \in \mathcal{G}^{(2)}$, then $(\alpha_1, \alpha_2\alpha_3) \in \mathcal{G}^{(2)}$ and $\alpha_1(\alpha_2\alpha_3) = (\alpha_1\alpha_2)\alpha_3$.

A local group is by definition a local groupoid where the set of its units is reduced to a singleton.

Notice that by (5) of Definition 1.5, condition (6) is equivalent to:

(6′) if $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3) \in \mathcal{G}^{(2)}$ and $(\alpha_1, \alpha_2\alpha_3) \in \mathcal{G}^{(2)}$, then $(\alpha_1\alpha_2, \alpha_3) \in \mathcal{G}^{(2)}$ and $(\alpha_1\alpha_2)\alpha_3 = \alpha_1(\alpha_2\alpha_3)$.

In a more general case a net $\alpha_1, \dots, \alpha_n$ in a local groupoid is said to be local if the product $\alpha_1 \cdots \alpha_n$ is defined for a suitable placement of brackets. If the value of the product does not depend on the placement of brackets we say that this product is stable. We say that the product in a local groupoid is generally associative if any local net has a stable product.

Now, let $(\alpha_1, \alpha_2) \in \mathcal{G}^{(2)}$. By (4) of Definition 1.5 we have $(\alpha_1^{-1}, \alpha_1) \in \mathcal{G}^{(2)}$ and $\alpha_1^{-1}\alpha_1 = s(\alpha_1)$, which is equal to $r(\alpha_2)$ by (1). Then, we deduce by (6) that $(\alpha_1^{-1}, \alpha_1\alpha_2) \in \mathcal{G}^{(2)}$ and $\alpha_1^{-1}(\alpha_1\alpha_2) = \alpha_2$. But $(\alpha_1\alpha_2, s(\alpha_1\alpha_2)) \in \mathcal{G}^{(2)}$ by (3). By using (6) for α_1^{-1}, $\alpha_1\alpha_2$, $s(\alpha_1\alpha_2)$ we deduce that $(\alpha_2, s(\alpha_1\alpha_2)) \in \mathcal{G}^{(2)}$ and therefore $s(\alpha_2) = s(\alpha_1\alpha_2)$. We prove in the same way that $r(\alpha_1) = r(\alpha_1\alpha_2)$. This proves that any local groupoid satisfies condition (5) of Definition 1.1.

Hence, any groupoid is a local groupoid. On the other hand, a local groupoid is a groupoid if and only if the inclusion in (1) of Definition 1.5 is an equality. This is equivalent to saying in (6) of Definition 1.5 that $(\alpha_1\alpha_2, \alpha_3) \in \mathcal{G}^{(2)}$ whenever $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3) \in \mathcal{G}^{(2)}$ (take for this $\alpha_2 = s(\alpha_1) = r(\alpha_3)$).

Now let $\mathcal{G}$ be a groupoid and let $\mathcal{G}^{(0)} \subset \mathcal{G}' \subset \mathcal{G}$ with $\mathcal{G}'^{-1} = \{\alpha^{-1};\ \alpha \in \mathcal{G}'\} = \mathcal{G}'$. Then $\mathcal{G}'$ together with the restriction of the algebraic structure of the groupoid $\mathcal{G}$ is a local groupoid. Conversely, a local groupoid $\mathcal{G}'$ is said to be weakly enlargeable if there exists a groupoid $\mathcal{G}$ and a one-to-one morphism of local groupoids $\phi : \mathcal{G}' \to \mathcal{G}$. To recognize such local groupoids, we have the following useful criterion [**vE84**].

Proposition 1.6 (Malcev criterion). *A local groupoid is weakly enlargeable if and only if the partial product law is generally associative.*

If $\mathcal{G}$ is a local groupoid and $u \in \mathcal{G}^{(0)}$ is a unit, we define in the same way the isotropy local group $\mathcal{G}(u)$ at u. Contrary to groupoids, it is clear from the above given example

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that there is in general no isomorphism between two isotropy local groups $\mathcal{G}(u)$ and $\mathcal{G}(v)$ though the units are the source and range, respectively, of an element $\alpha \in \mathcal{G}$.

A local groupoid $\mathcal{G}$ is said to be coherent if the image of the map (r, s) generates the principal groupoid $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ and it is said to be transitive if this image is equal to the principal groupoid.

Now let us define the notion of action of local groupoids on sets. This will generalize in a natural way the case of groupoids. We do not know whether such a notion already exists in the literature.

Definition 1.7. Let $\mathcal{G}$ be a local groupoid and let X be a set together with a surjection $s' : X \to \mathcal{G}^{(0)}$. We will call a right groupoid action of $\mathcal{G}$ on X a map $(x, \alpha) \mapsto x\alpha$ from a domain $\mathcal{D}$ subset of the fibred product $X * \mathcal{G} = \{(x, \alpha) \in X \times \mathcal{G},\ s'(x) = r(\alpha)\}$ to X satisfying the following conditions.

(1) If $\alpha \in \mathcal{G}$ such that $(\beta, \alpha) \in \mathcal{G}^{(2)}$ for all $\beta \in \mathcal{G}$ with $s(\beta) = r(\alpha)$, then $(x, \alpha) \in \mathcal{D}$ for all $x \in X$ with $s'(x) = r(\alpha)$.

(2) If $(x, \alpha) \in \mathcal{D}$, $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $(x, \alpha\beta) \in \mathcal{D}$, then $(x\alpha, \beta) \in \mathcal{D}$ and $(x\alpha)\beta = x(\alpha\beta)$.

(3) For all $(\alpha, \beta) \in \mathcal{G}^{(2)}$, there exists $x \in X$ such that $(x, \alpha^{-1}), (x, \beta) \in \mathcal{D}$.

If we take $\beta = s(\alpha)$ in (2) of Definition 1.7 we deduce that if $(x, \alpha) \in \mathcal{D}$ then $s'(x\alpha) = s(\alpha)$. If $\mathcal{G}$ is a groupoid, all elements $\alpha \in \mathcal{G}$ satisfy (1) of Definition 1.7. This means that in the case of a groupoid, $\mathcal{D} = X * \mathcal{G}$ and we again obtain Definition 1.4.

Notice that any local groupoid $\mathcal{G}$ acts on the right on its base $\mathcal{G}^{(0)}$. It also acts in a natural way on the right on itself, where $s' = r$, $\mathcal{D} = \mathcal{G}^{(2)}$ and the action is given by the partial law of composition.

As for actions of groupoids, an action of a local groupoid $\mathcal{G}$ is said to be free if the map $(x, \alpha) \mapsto (x, x\alpha)$ defined from $\mathcal{D}$ to $X \times X$ is one-to-one. It is said to be coherent if the image of this map generates the principal groupoid $X \times X$ and it is said to be transitive if this image is equal to the principal groupoid. It is said to be simply transitive if it is free and transitive.

Now let us give two generalizations of Definition 1.2. The first generalization is when the action of the group Γ on X is not necessarily free. In this case, we construct the following space on which Γ acts freely.

Let $X_\Gamma = \bigcup_{x \in X} \{x\} \times \Gamma_x$, where Γ_x is the stabilizer of x. Choose for any two elements $x, y \in X$ in the same orbit an element $\alpha_{x,y} \in \Gamma$ such that

(1) $x\alpha_{x,y} = y$ for all $x, y \in X$ in the same orbit;

(2) $\alpha_{x,x} = 1$ for all $x \in X$;

(3) $\alpha_{x,y}\alpha_{y,z} = \alpha_{x,z}$ for all $x, y, z \in X$ in the same orbit.

One can define a free action of Γ on X_Γ by $(x, g)h = (xh, \alpha_{xh,x}gh)$ for all $((x, g); h) \in X_\Gamma \times \Gamma$. Denote by $\mathcal{G}(X_\Gamma, \Gamma, \alpha)$ the corresponding groupoid defined in Definition 1.2.

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Definition 1.8. Let Γ be a group acting on the right on a space X with a not necessarily free action. We define the local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ to be equal to $\{[(x,1),(y,1)] \in \mathcal{G}(X_\Gamma,\Gamma,\alpha);\ x,y \in X\}$.

The local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ is a groupoid if and only if for all $x,y,z \in X$ with y, z in the same orbit, there exists $t \in X$ such that $\alpha_{y,z} = \alpha_{x,t}$. Indeed, if y, z are in the same orbit, $s([(x,1);(y,1)]) = r([(z,1);(u,1)])$ and $[(x,1);(y,1)] = [(x,1)\alpha_{y,z};(z,1)]$. However, they cannot be composable in $\mathcal{G}(X,\Gamma,\alpha)$ unless there exists $t \in X$ such that $(x,1)\alpha_{y,z} = (t,1)$. Notice that the local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ is transitive.

The local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ acts simply transitively on the set X by

$$x[(x,1),(y,1)] = y, \quad \text{for all } x,y \in X.$$

The second generalization of Definition 1.2 is when the group Γ is replaced by a groupoid $\mathcal{G}$.

Definition 1.9. When the action of a groupoid $\mathcal{G}$ on a set X is free, we define the local groupoid $\mathcal{G}(X,\mathcal{G}) = X \times_{\mathcal{G}} X$ such that the elements of $\mathcal{G}(X,\mathcal{G})$ are the equivalence classes $[x,y]$ of elements of $X \times X$ under the equivalence relation generated by

$$(x,y) \sim (x\alpha, y\alpha), \quad \text{for all } x,y \in X,\ \alpha \in \mathcal{G} \text{ such that } (x,\alpha),(y,\alpha) \in \mathcal{D}.$$

The units are the classes of the form $[x,x]$, the source and range maps are defined for all $x,y,z \in X$ by $s[x,y] = [y,y]$ and $r[x,y] = [x,x]$, respectively, and the product is defined by $[x,y][y,z] = [x,z]$.

If $x,y \in X$ with $s'(x) \neq s'(y)$, the class $[x,y]$ contains only one element (x,y). If moreover $y',z \in X$ with $s'(y') \neq s'(z)$ and y, y' are in the same orbit but different, then $([x,y],[y',z]) \notin \mathcal{G}(X,\mathcal{G})^{(2)}$ though $s[x,y] = r[y',z]$. Hence, $\mathcal{G}(X,\mathcal{G})$ is not a groupoid whenever $\mathcal{G}$ is neither a group nor trivial. On the other hand, $\mathcal{G}(X,\mathcal{G})$ is a local group if and only if the free action of the groupoid $\mathcal{G}$ on X is transitive.

As for $\mathcal{G}(X,\Gamma)$, it is clear that the local groupoid $\mathcal{G}(X,\mathcal{G})$ is transitive.

Notice that, for any $u \in \mathcal{G}^{(0)}$, the inclusion of $X_u \times X_u$ in $X \times X$ induces a one-to-one morphism from the groupoid $\mathcal{G}(X_u,\mathcal{G}(u))$ to the local groupoid $\mathcal{G}(X,\mathcal{G})$. Now, for two units $u,v \in \mathcal{G}^{(0)}$, both the images in $\mathcal{G}(X,\mathcal{G})$ of the groupoids $\mathcal{G}(X_u,\mathcal{G}(u))$ and $\mathcal{G}(X_v,\mathcal{G}(v))$ are either the same if u and v are in the same connected component of $\mathcal{G}$ or disjoint if not.

Let $\{\mathcal{G}_i\}_{i\in I}$ be the canonical partition of the groupoid $\mathcal{G}$ into connected components. If $\{u_i\}_{i\in I}$ is a family of units of $\mathcal{G}$ such that u_i is in $\mathcal{G}_i$ for all $i \in I$, there is a bijection between the disjoint union $\coprod_{i\in I} X_{u_i}/\mathcal{G}(u_i)$ and the base of $\mathcal{G}(X,\mathcal{G})$.

To get a generalization of Remark 1.3, the natural way to do it is to consider, for any family $\{x_u\}_{u\in\mathcal{G}^{(0)}}$ of elements of X such that $s'(x_u) = u$ for all $u \in \mathcal{G}^{(0)}$, the morphism from the groupoid $\mathcal{G}$ to the local groupoid $\mathcal{G}(X,\mathcal{G})$ defined by $\alpha \mapsto [x_{r(\alpha)}\alpha, x_{s(\alpha)}]$. But this morphism is one-to-one if and only if the family $\{x_u\}_{u\in\mathcal{G}^{(0)}}$ is such that for all non-units $\alpha \in \mathcal{G}$, $x_{r(\alpha)}\alpha \neq x_{s(\alpha)}$. This condition is too strong. To avoid it, we introduce the following definition.

Definition 1.10. When the action of a groupoid $\mathcal{G}$ on a set X is free, we define the local groupoid $\mathcal{G}'(X,\mathcal{G}) = \mathcal{G}(X,\mathcal{G}) \times \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$, where the units are the elements of the form $([x,x],u,u)$, the source and range maps are defined by $s([x,y],u,v) = ([y,y],v,v)$ and $r([x,y],u,v) = ([x,x],u,u)$, respectively, and the product is defined by $([x,y],u,v)([y,z],v,w) = ([x,z],u,w)$, for all $x,y,z \in X$, $u,v \in \mathcal{G}^{(0)}$.

The local groupoid $\mathcal{G}'(X,\mathcal{G})$ is nothing but the direct product of the local groupoid $\mathcal{G}(X,\mathcal{G})$ by the principal groupoid $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. Notice that when $\mathcal{G}$ is a group, the base $\mathcal{G}^{(0)}$ is reduced to a singleton and then both $\mathcal{G}(X,\mathcal{G})$ and $\mathcal{G}'(X,\mathcal{G})$ are isomorphic. In general, we have the following remark which generalizes Remark 1.3.

Remark 1.11. Any family $\{x_u\}_{u\in\mathcal{G}^{(0)}}$ of elements of X such that $s'(x_u) = u$, for all $u \in \mathcal{G}^{(0)}$, defines a one-to-one morphism from the groupoid $\mathcal{G}$ to the local groupoid $\mathcal{G}'(X,\mathcal{G})$ by $\alpha \mapsto ([x_{r(\alpha)}\alpha, x_{s(\alpha)}], r(\alpha), s(\alpha))$.

Local groupoids $\mathcal{G}(X,\mathcal{G})$ and $\mathcal{G}'(X,\mathcal{G})$ act transitively on the set X by $x[x,y] = y$ and $x([x,y],u,v) = y$, respectively, for all $x,y \in X$. However, only the action of $\mathcal{G}(X,\mathcal{G})$ is simply transitive.

The only local groupoids which are not groupoids and will appear in this paper are those defined in Definitions 1.8–1.10.

1.1.3. *The Haagerup inequality*

Our aim in this paper is to study the Haagerup inequality for groups and groupoids. The generalization to local groupoids has been given only because of the examples that appear naturally as tools in Definitions 1.8–1.10. Thus most of remarks that we will make in this paragraph will be restricted to groupoids, although some of them can be extended to local groupoids.

Let $\mathcal{G}$ be a discrete local groupoid. For $\mathbb{K} = \mathbb{R}_+$, $\mathbb{R}$ or $\mathbb{C}$, denote by $\mathbb{K}\mathcal{G}$ the set of functions of finite support in $\mathcal{G}$ having values in $\mathbb{K}$. The convolution product and the involution on $\mathbb{K}\mathcal{G}$ are defined by

$$\phi_1 * \phi_2(\gamma) = \sum_{\alpha\beta=\gamma} \phi_1(\alpha)\phi_2(\beta) \quad \text{and} \quad \phi^*(\gamma) = \overline{\phi(\gamma^{-1})}, \quad \text{for } \gamma \in \mathcal{G} \text{ and } \phi_1,\phi_2,\phi \in \mathbb{K}\mathcal{G},$$

where the sum is taken on composable pairs (α,β). This convolution extends to $l^2(\mathcal{G})$ and defines a representation λ of $\mathbb{C}\mathcal{G}$ on $B(l^2(\mathcal{G}))$ by

$$\lambda(\phi) : \psi \mapsto \phi * \psi, \quad \forall \phi \in \mathbb{C}\mathcal{G},$$

called the left regular representation.

Notice that if the $\mathcal{G}_i$, $i \in I$, are the connected components of a groupoid $\mathcal{G}$, the operator $\lambda(\phi)$ is diagonal on the decomposition

$$l^2(\mathcal{G}) = \bigoplus_{i\in I} l^2(\mathcal{G}_i).$$

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A length-function on a local groupoid $\mathcal{G}$ is a map $L : \mathcal{G} \to \mathbb{R}_+$ such that

(1) $L(\alpha) = 0$, for all $\alpha \in \mathcal{G}^{(0)}$;

(2) $L(\alpha^{-1}) = L(\alpha)$, for all $\alpha \in \mathcal{G}$;

(3) $L(\alpha_1\alpha_2) \leqslant L(\alpha_1) + L(\alpha_2)$, for all composable pair (α_1, α_2).

For any positive integer n, let $B_L(n) = \{\alpha \in \mathcal{G},\ L(\alpha) \leqslant n\}$.

Of course, this definition is a generalization of definition of a length-function on a group. When $\mathcal{G}$ is a groupoid we have the following lemma.

Lemma 1.12. *Let $\mathcal{G}$ be a transitive groupoid, u a unit of $\mathcal{G}$ and Γ the isotropy group at u. Any length-function on Γ has an extension to a length-function on $\mathcal{G}$ such that for any two isotropy groups, there is an isomorphism preserving lengths.*

Proof. Let L be a length-function on Γ. Since the groupoid $\mathcal{G}$ is transitive, one can choose for any unit $v \in \mathcal{G}^{(0)} \setminus \{u\}$, an element in $\mathcal{G}$, denoted by $[uv]$, such that $r([uv]) = u$ and $s([uv]) = v$. Denote by $[vu]$ the inverse of $[uv]$ and by $[uu]$ the unit u.

$$\text{For any } \alpha \in \mathcal{G},\ \text{the element } \tilde{\alpha} = [ur(\alpha)]\alpha[s(\alpha)u] \in \Gamma. \text{ Set } \tilde{L}(\alpha) = L(\tilde{\alpha}).$$

The map $\tilde{L}$ extends L and is a length-function on $\mathcal{G}$, and for any $v \in \mathcal{G}^{(0)}$, the isomorphism $\alpha \mapsto [vu]\alpha[uv]$ between the group Γ and the isotropy group at v preserves the length-function $\tilde{L}$. □

Recall that a pseudo-metric space is a space X together with a mapping $d : X \times X \to \mathbb{R}_+$ satisfying (i) $d(x, x) = 0$ for all $x \in X$, (ii) $d(y, x) = d(x, y)$ for all $x, y \in X$, (iii) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Proposition 1.13.

(i) *When a group Γ acts by isometries on a pseudo-metric space (X, d), any $x \in X$ defines a length-function $L_x : g \mapsto d(x, gx)$ on Γ.*

(ii) *Moreover, if the action is free, we can define a length-function on $\mathcal{G}(X, \Gamma)$ by $L : [x, y] \mapsto d(x, y)$.*

(iii) *However, if the action is not free we define a length-function L on $\mathcal{G}(X_\Gamma, \Gamma, \alpha)$ and its restriction to $\mathcal{G}(X, \Gamma, \alpha)$ by $L : [(x, g), (y, h)] \mapsto d(x, y)$.*

Notice that if the action of Γ on (X, d) is by isometries but not free, we define a pseudo-distance d on X_Γ by $d((x, g), (y, h)) = d(x, y)$ and Γ acts freely and by isometries on X_Γ. Thus, the length-function defined by Proposition 1.13 (iii) for the triple (Γ, X, d) coincides with the one defined by Proposition 1.13 (ii) for the triple (Γ, X_Γ, d).

Remark 1.14. Notice that the map $g \mapsto [xg, x]$ given in Remark 1.3 is an isometry between (Γ, L_x) and $(\mathcal{G}(X, \Gamma)([x, x]), L)$.

In general, we will say that the action of a local groupoid $\mathcal{G}$ on a pseudo-metric space (X,d) is by isometries if $d(x\alpha, y\alpha) = d(x,y)$ for all $x, y \in X$, $\alpha \in \mathcal{G}$ such that $(x,\alpha),(y,\alpha) \in \mathcal{D}$. Notice that the actions of local groupoids $\mathcal{G}(X,\Gamma,\alpha)$, $\mathcal{G}(X,\mathcal{G})$ and $\mathcal{G}'(X,\mathcal{G})$ defined in Definitions 1.8, 1.9 and 1.10, respectively, are all by isometries on X whenever the actions of Γ and $\mathcal{G}$ are by isometries on X.

We have the following proposition which generalizes Proposition 1.13 and Remark 1.14 to the case of groupoids.

Proposition 1.15.

(1) *When a groupoid $\mathcal{G}$ acts by isometries on a pseudo-metric space (X,d), any family $\{x_u\}_{u\in\mathcal{G}^{(0)}}$ of elements of X such that $s'(x_u) = u$ for all $u \in \mathcal{G}^{(0)}$ defines a length-function on $\mathcal{G}$ by $L_{\{x_u\}}(\alpha) = d(x_{r(\alpha)}\alpha, x_{s(\alpha)})$ for all $\alpha \in \mathcal{G}$.*

(2) *If, moreover, the action is free, we can define a length-function on both local groupoids $\mathcal{G}(X,\mathcal{G})$ and $\mathcal{G}'(X,\mathcal{G})$ by $L : [x,y] \mapsto d(x,y)$ and $L' : ([x,y],u,v) \mapsto d(x,y)$, respectively.*

(3) *The morphism defined in Remark 1.11 is an isometric embedding of $(\mathcal{G}, L_{\{x_u\}})$ into $(\mathcal{G}'(X,\mathcal{G}), L')$.*

The proofs of both Proposition 1.13 and Proposition 1.15 are straightforward and are omitted. Let us now give the definition of the Haagerup inequality for general discrete local groupoids.

Definition 1.16. The discrete local groupoid $\mathcal{G}$ is said to satisfy the Haagerup inequality if there exists a length-function L on $\mathcal{G}$ and two positive constants C, r such that $\forall\phi \in \mathbb{C}\mathcal{G}$, $\|\lambda(\phi)\| \leqslant C\|\phi(1+L)^r\|_2$, where $\|\cdot\|$ is the operator norm on $B(l^2(\mathcal{G}))$ and $\|\cdot\|_2$ is the l^2-norm on $\mathbb{C}\mathcal{G}$.

It is obvious that if a local groupoid satisfies the Haagerup inequality with respect to a length-function, any local subgroupoid satisfies this inequality with respect to the restriction of this length-function. Notice on the other hand that if the groupoids $\mathcal{G}_i$, $i \in I$, are the connected components of a groupoid $\mathcal{G}$, Definition 1.16 is equivalent to saying that all the $\mathcal{G}_i$ satisfy the Haagerup inequality for the same constants C, r. In fact we can say more.

Proposition 1.17. *A discrete groupoid $\mathcal{G}$ satisfies the Haagerup inequality with respect to a length-function L if and only if all its isotropy groups satisfy the Haagerup inequality with respect to the restrictions of the length-function L and for the same constants C, r.*

As a direct consequence of Proposition 1.17, we have the following corollary.

Corollary 1.18. *Any principal groupoid satisfies the Haagerup inequality with respect to any length-function on it.*

The proof of Proposition 1.17 will be given in § 1.2. By mixing results of Proposition 1.17 and Lemma 1.12 we deduce the following proposition.

Proposition 1.19. *Let $\mathcal{G}$ be a transitive groupoid, u a unit of $\mathcal{G}$ and Γ the isotropy group at u. The groupoid $\mathcal{G}$ satisfies the Haagerup inequality if and only if the group Γ satisfies the Haagerup inequality.*

This result is in general not true for any length-function on $\mathcal{G}$. Indeed, let $\{u, v\}$ be a set with two elements and let $\mathcal{G}$ be the set of elements of the form (mx, ny) where $m, n \in \mathbb{Z}$ and $x, y \in \{u, v\}$. Let $\mathcal{G}^{(0)}$ be the set of elements of the form $(0u, 0u)$, $(0v, 0v)$, which we shall denote by u, v, respectively. Define the range and source maps by $r(mx, ny) = x$, $s(mx, ny) = y$, respectively, and the partial law of composition by

$$(mx, ny)(py, qz) = \begin{cases} (mx, qz) & \text{if } x = z \neq y, \\ ((m+n+p)x, qz) & \text{if } x = y \neq z, \\ (mx, (n+p+q)z) & \text{if } x \neq y = z, \\ ((m+n)x, (p+q)z) & \text{if } x = y = z. \end{cases}$$

Now define the length-function L on the groupoid $\mathcal{G}$ by

$$L(mx, ny) = \begin{cases} |m+n| & \text{if } x = y = u, \\ \log(1 + |m+n|) & \text{if } x = y = v, \\ |m| + \log(1 + |n|) & \text{if } x = u,\ y = v, \\ \log(1 + |m|) + |n| & \text{if } x = v,\ y = u. \end{cases}$$

The isotropy group at u then satisfies the Haagerup inequality with respect to the restriction of the length-function L, but $(\mathcal{G}, L)$ does not satisfy the Haagerup inequality since the isotropy group at v is amenable and of exponential growth with respect to the restriction of the length-function L.

Let us give the following useful proposition, which will be proved in § 1.2.

Proposition 1.20. *A direct product of two local groupoids $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ satisfies the Haagerup inequality with respect to the length-function $L = L_1 + L_2$ if and only if the local groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the Haagerup inequality with respect to the length-functions L_1 and L_2, respectively.*

The following two propositions motivate Definition 1.16 for local groupoids.

Proposition 1.21. *Let $\mathcal{G}$ be a groupoid acting freely and by isometries on a pseudo-metric space X and let L and $L_{\{x_u\}}$ be the length-functions defined in Proposition 1.15. If the local groupoid $\mathcal{G}(X, \mathcal{G})$ satisfies the Haagerup inequality with respect to L, the pair $(\mathcal{G}, L_{\{x_u\}})$ satisfies the Haagerup inequality for any family $\{x_u\}_{u \in \mathcal{G}^{(0)}}$ of elements of X such that $s'(x_u) = u$ for all $u \in \mathcal{G}^{(0)}$.*

Proof. When $\mathcal{G} = \Gamma$ is a group, this is nothing but Proposition 2.1 of [**Laf00**] extracted from [**RRS98**]. By Remarks 1.3 and 1.14, the proof becomes obvious in this situation.

For the general case, we need the following lemma, which we deduce from Propositions 1.18 and 1.20.

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Lemma 1.22. *If $(\mathcal{G}(X,\mathcal{G}),L)$ satisfies the Haagerup inequality, then $(\mathcal{G}'(X,\mathcal{G}),L')$ satisfies the Haagerup inequality.*

Using Lemma 1.22, Remark 1.11 and Proposition 1.15, Proposition 1.21 follows. □

Proposition 1.23. *Let Γ be a group acting by isometries on a pseudo-metric space X with a bounded order of stabilizers. Let L and L_x be the length-functions defined in Proposition 1.13. If the local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ satisfies the Haagerup inequality with respect to L, the pair (Γ,L_x) satisfies the Haagerup inequality for any $x\in X$.*

Proof. The proof follows from the following lemma, which will be proved in § 1.3.1.

Lemma 1.24. *If the local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ satisfies the Haagerup inequality with respect to L, then the groupoid $\mathcal{G}(X_\Gamma,\Gamma,\alpha)$ satisfies the Haagerup inequality with respect to L.*

Using Lemma 1.24, Remark 1.3, Proposition 1.13 and Remark 1.14, Proposition 1.23 follows. □

Propositions 1.21 and 1.23 give us an improvement of Theorem 1.9 and Corollary 1.10 of [**RRS98**]. Notice that the proof of Theorem 1.9 of [**RRS98**] still goes through if we replace groupoids by local groupoids. Since our local groupoids act simply transitively, we obtain the following improvement.

Theorem 1.25. *Let (X,d) be the set of vertices of an affine building of type $\tilde{A}_1\times\tilde{A}_1$ or $\tilde{A}_2$ together with the metric given by the 1-skeleton of the building.*

(1) *Any groupoid acting freely and by isometries on (X,d) satisfies the Haagerup inequality.*

(2) *Any group acting by isometries on (X,d) with a bounded order of stabilizers satisfies the Haagerup inequality.*

Part (2) has already been obtained in [**Laf00**]. A generalization of the above theorem will be given in § 2.

1.2. The Haagerup inequality and triangles

Let $\mathcal{G}$ be a local groupoid and let $\mathcal{G}_0^{(3)}$ be the set of its triangles $(\alpha_1,\alpha_2,\alpha_3)$. These are the composable triplets of $\mathcal{G}$ such that their products $\alpha_1\alpha_2\alpha_3$ are units. We will call such a triangle degenerate if its three sides α_i, $i=1,2,3$, are units.

Given two functions $\phi_1,\phi_2\in\mathbb{C}\mathcal{G}$ and a set $\mathcal{T}\subset\mathcal{G}_0^{(3)}$ of triangles of $\mathcal{G}$, we define a non-associative partial convolution product on $\mathcal{T}$ by

$$\phi_1 *_{\mathcal{T}} \phi_2 : \gamma \mapsto \sum_{(\alpha,\beta,\gamma^{-1})\in\mathcal{T}} \phi_1(\alpha)\phi_2(\beta),$$

with the convention that the empty sum is zero. It is clear that $\phi_1 *_{\mathcal{G}_0^{(3)}} \phi_2 = \phi_1 * \phi_2$.

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Proposition 1.26. *Let L be a length-function on $\mathcal{G}$ and let $\mathcal{T}$ be a set of triangles of $\mathcal{G}$. The following two inequalities are equivalent.*

(1) *There exists a polynomial P for which, $\forall n \in \mathbb{N}$, $\forall \phi_1, \phi_2 \in \mathbb{C}\mathcal{G}$ with the support of ϕ_1 in $B_L(n)$, we have $\|\phi_1 *_{\mathcal{T}} \phi_2\|_2 \leqslant P(n)\|\phi_1\|_2\|\phi_2\|_2$.*

(2) *There exists a polynomial P for which, $\forall n \in \mathbb{N}$, $\forall \phi_1, \phi_2, \phi_3 \in \mathbb{R}_+\mathcal{G}$ with the support of ϕ_1 in $B_L(n)$, we have*

$$\sum_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{T}} \phi_1(\alpha_1)\phi_2(\alpha_2)\phi_3(\alpha_3) \leqslant P(n)\|\phi_1\|_2\|\phi_2\|_2\|\phi_3\|_2.$$

Proof. For all $\phi_1, \phi_2, \phi_3 \in \mathbb{C}\mathcal{G}$, we have

$$\sum_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{T}} \phi_1(\alpha_1)\phi_2(\alpha_2)\phi_3(\alpha_3) = \sum_{\alpha_3\in\mathcal{G}} \phi_1 *_{\mathcal{T}} \phi_2(\alpha_3^{-1})\phi_3(\alpha_3) = \langle \phi_1 *_{\mathcal{T}} \phi_2, \check{\phi}_3\rangle,$$

where $\langle\cdot\,,\cdot\rangle$ is the inner product on $l^2(\mathcal{G})$ and $\check{\phi}_3 : \alpha \mapsto \phi_3(\alpha^{-1})$.

It follows from this and the Cauchy–Schwartz inequality that (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (1) follows from setting $\check{\phi}_3 = \phi_1 *_{\mathcal{T}} \phi_2$ and using the fact that the absolute value of the sum is less than or equal to the sum of the absolute values. □

Definition 1.27. If one of the two inequalities of Proposition 1.26 holds, we say that $(\mathcal{T}, L)$ satisfies the Haagerup inequality.

Notice that the local groupoid $\mathcal{G}$ satisfies the Haagerup inequality if and only if the set $\mathcal{G}_0^{(3)}$ of all its triangles satisfies it. On the other hand, we have the following straightforward result.

Proposition 1.28. *The set of degenerate triangles of $\mathcal{G}$ satisfies the Haagerup inequality for any length-function on $\mathcal{G}$.*

In a more general case, we have the following lemma, which will be used later.

Lemma 1.29. *Let X_i, $i \in \mathbb{Z}/3\mathbb{Z}$, be three discrete sets and let $f_i \in l^2(X_i \times X_{i+1})$ for $i \in \mathbb{Z}/3\mathbb{Z}$. Then*

$$\left|\sum_{x_i\in X_i,\, i\in\mathbb{Z}/3\mathbb{Z}} f_1(x_1,x_2)f_2(x_2,x_3)f(x_3,x_1)\right| \leqslant \|f_1\|_2\|f_2\|_2\|f_3\|_2.$$

To prove this lemma we only have to apply the Cauchy–Schwartz inequality three times.

Our strategy to prove the Haagerup inequality for a given local groupoid $\mathcal{G}$ with respect to a length-function L consists of applying successive deformations to the set $\mathcal{G}_0^{(3)}$ of triangles of $\mathcal{G}$, which are 'compatible' with this inequality, to reduce the proof of the inequality to the set of degenerate triangles. But before presenting these deformations let us give the proofs of Propositions 1.17 and 1.20.

Proof of Proposition 1.17. The direct sense is obvious. For the opposite, denote by Γ_u the isotropy group at u, for any $u \in \mathcal{G}^{(0)}$. Denote by P the polynomial of a Haagerup inequality satisfied by all Γ_u. Since $\mathcal{G}$ is transitive, choose for any $u, v \in \mathcal{G}^{(0)}$, an element in $\mathcal{G}$, denoted by $[uv]$, such that $r([uv]) = u$, $s([uv]) = v$, its length is minimal and such that $[vu]$ is the inverse of $[uv]$. Let n be a non-negative integer and let $\phi_1, \phi_2, \phi_3 \in \mathbb{R}_+\mathcal{G}$ such that the support of ϕ_1 is in $B_L(n)$.

For all $u \in \mathcal{G}^{(0)}$, $\alpha \in \Gamma_u$, set

$$\phi_{1,u}(\alpha) = \sqrt{\sum_{v \in \mathcal{G}^{(0)}} \phi_1^2(\alpha[uv])},$$

$$\phi_{2,u}(\alpha) = \sqrt{\sum_{v,w \in \mathcal{G}^{(0)}} \phi_2^2([vu]\alpha[uw])},$$

$$\phi_{3,u}(\alpha) = \sqrt{\sum_{w \in \mathcal{G}^{(0)}} \phi_3^2([wu]\alpha)}.$$

Notice that $\|\phi_2\|_2 = \|\phi_{2,u}\|_2$ and that

$$\|\phi_i\|_2 = \sqrt{\sum_u \|\phi_{i,u}\|_2^2} \quad \text{for } i = 1, 3.$$

Notice, on the other hand, that the support of $\phi_{1,u}$ is in $B_L(2n)$. We have by using Lemma 1.29 and the Haagerup inequality for the Γ_u,

$$\begin{aligned}
\sum_{(\alpha,\beta,\gamma)\in\mathcal{G}_0^{(3)}} \phi_1(\alpha)\phi_2(\beta)\phi_3(\gamma) &= \sum_{u\in\mathcal{G}^{(0)}} \sum_{\substack{\alpha\beta\gamma=u \\ \alpha,\beta,\gamma\in\Gamma_u}} \sum_{v,w\in\mathcal{G}^{(0)}} \phi_1(\alpha[uv])\phi_2([vu]\beta[uw])\phi_3([wu]\gamma) \\
&\leqslant \sum_{u\in\mathcal{G}^{(0)}} \sum_{\substack{\alpha\beta\gamma=u \\ \alpha,\beta,\gamma\in\Gamma_u}} \phi_{1,u}(\alpha)\phi_{2,u}(\beta)\phi_{3,u}(\gamma) \\
&\leqslant \sum_{u\in\mathcal{G}^{(0)}} P(2n)\|\phi_{1,u}\|_2\|\phi_{2,u}\|_2\|\phi_{3,u}\|_2 \\
&\leqslant P(2n)\|\phi_1\|_2\|\phi_2\|_2\|\phi_3\|_2.
\end{aligned}$$

□

Proof of Proposition 1.20. Let n be a non-negative integer and let $\phi'_1, \phi'_2, \phi'_3 \in \mathbb{R}_+\mathcal{G}_1 \times \mathcal{G}_2$ such that the support of ϕ'_1 is in $B_L(n)$. Consider the functions $\phi_1, \phi_2, \phi_3 \in \mathbb{R}_+\mathcal{G}_1$ defined by

$$\phi_i(\alpha_1) = \sqrt{\sum_{\alpha_2\in\mathcal{G}_2} \phi_i'^2(\alpha_1, \alpha_2)}, \quad \text{for } i = 1, 2, 3.$$

Notice that the support of ϕ_1 is in $B_{L_1}(n)$ and that $\|\phi_i\|_2 = \|\phi'_i\|_2$ for $i = 1, 2, 3$. Let P_i be the polynomial given by the Haagerup inequality satisfied by $(\mathcal{G}_i, L_i)$, for $i = 1, 2$. We

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then have

$$\sum_{((\alpha_1,\alpha_2);(\beta_1,\beta_2);(\gamma_1,\gamma_2))\in\mathcal{G}_0^{(3)}} \phi_1'(\alpha_1,\alpha_2)\phi_2'(\beta_1,\beta_2)\phi_3'(\gamma_1,\gamma_2)$$

$$\begin{aligned} &= \sum_{(\alpha_1,\beta_1,\gamma_1)\in\mathcal{G}_{1_0}^{(3)}} \sum_{(\alpha_2,\beta_2,\gamma_2)\in\mathcal{G}_{2_0}^{(3)}} \phi_1'(\alpha_1,\alpha_2)\phi_2'(\beta_1,\beta_2)\phi_3'(\gamma_1,\gamma_2) \\ &\leqslant \sum_{(\alpha_1,\beta_1,\gamma_1)\in\mathcal{G}_{1_0}^{(3)}} P_2(n)\phi_1(\alpha_1)\phi_2(\beta_1)\phi_3(\gamma_1) \\ &\leqslant P_1(n)P_2(n)\|\phi_1\|_2\|\phi_2\|_2\|\phi_3\|_2 \\ &= P(n)\|\phi_1'\|_2\|\phi_2'\|_2\|\phi_3'\|_2. \end{aligned}$$

□

1.3. Deformations of triangles

The deformations that are 'compatible' with the Haagerup inequality and that we will consider here are called $\mathcal{B}$-pinching and tetrahedral change of faces. For the following two paragraphs, we fix a length-function L.

1.3.1. *The $\mathcal{B}$-pinching*

To define the $\mathcal{B}$-pinching, we need first to define a family of domains of L-decompositions with respect to which the pinching will be applied.

Definition 1.30.

(i) For any element $\alpha \in \mathcal{G}$, we call a domain of decompositions of α a non-empty set $\mathcal{B}(\alpha)$ of triplets $(u, \tilde{\alpha}, v)$ of elements of $\mathcal{G}$ such that the triplet $(u, \tilde{\alpha}, v^{-1})$ is composable and $\alpha = u\tilde{\alpha}v^{-1}$.

(ii) We call a family of domains of L-decompositions of $\mathcal{G}$, a family $\mathcal{B} = \{\mathcal{B}(\alpha)\}_{\alpha\in\mathcal{G}}$ of domains of decompositions of elements of $\mathcal{G}$ for which there exists a polynomial P_1 such that, for all $\alpha \in \mathcal{G}$ and all $(u, \tilde{\alpha}, v) \in \mathcal{B}(\alpha)$, we have

(1) $L(u) \leqslant P_1(L(\alpha))$,

(2) $(v, \tilde{\alpha}^{-1}, u) \in \mathcal{B}(\alpha^{-1})$.

We say that a triangle $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ is *a $\mathcal{B}$-pinching of a triangle* $(\alpha_1, \alpha_2, \alpha_3)$ if there exist $u_1, u_2, u_3 \in \mathcal{G}$ such that for all $i \in \mathbb{Z}/3\mathbb{Z}$, we have $(u_i, \tilde{\alpha}_i, u_{i+1}) \in \mathcal{B}(\alpha_i)$ (see Figure 1).

Definition 1.31. We say that a set $\mathcal{T}'$ of triangles of $\mathcal{G}$ is a $\mathcal{B}$-pinching of an other set $\mathcal{T}$ of triangles of $\mathcal{G}$ if every triangle of $\mathcal{T}$ has a $\mathcal{B}$-pinching in $\mathcal{T}'$.

Note that Definition 1.31 is a little different from Definition 2.3 given in [**Tal02**]. This is to make results more general.

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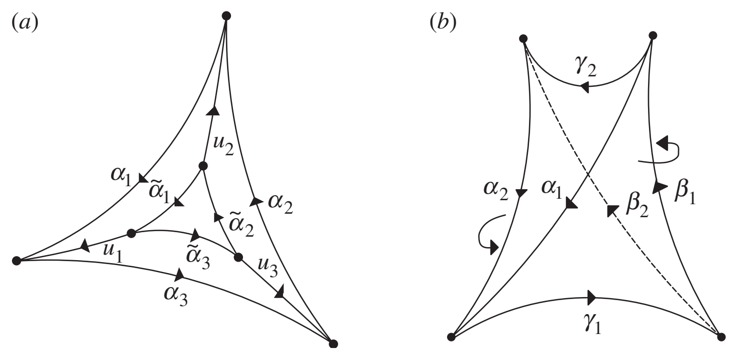

Figure 1. Deformation of triangles: (a) $\mathcal{B}$-pinching; (b) tetrahedral change of faces.

Definition 1.32.

(a) We say that $\mathcal{G}$ is of polynomial $\mathcal{B}$-growth if there exists a polynomial P_2 such that, for all $\alpha \in \mathcal{G}$ and for all $n \in \mathbb{N}$, $\#\{(u,\tilde{\alpha},v) \in \mathcal{B}(\alpha),\ L(u), L(\tilde{\alpha}) \leqslant n\} \leqslant P_2(n)$.

(b) We say that $\mathcal{G}$ is of polynomial rigid $\mathcal{B}$-growth if there exists a polynomial P_2 such that, for all $\alpha \in \mathcal{G}$ and for all $n \in \mathbb{N}$, $\#\{(u,\tilde{\alpha},v) \in \mathcal{B}(\alpha),\ L(u) \leqslant n\} \leqslant P_2(n)$.

Having given these definitions we have the following result.

Proposition 1.33. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two sets of triangles of $\mathcal{G}$ such that $\mathcal{T}'$ is a $\mathcal{B}$-pinching of $\mathcal{T}$ and $\mathcal{T}'$ satisfies the Haagerup inequality. The set $\mathcal{T}$ satisfies the Haagerup inequality if one of the following two additional assumptions is satisfied.*

(1) *$(\mathcal{G}, L)$ is of polynomial $\mathcal{B}$-growth and there exists a polynomial P_3 such that any triangle $(\alpha_1,\alpha_2,\alpha_3)$ in $\mathcal{T}$ has a $\mathcal{B}$-pinching $(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)$ in $\mathcal{T}'$ satisfying*

$$\max_i L(\tilde{\alpha}_i) \leqslant \min_i P_3(L(\alpha_i)), \quad \textit{for } i = 1,2,3.$$

(2) *$(\mathcal{G}, L)$ is of polynomial rigid $\mathcal{B}$-growth.*

Proof. Suppose that one of the two conditions is satisfied. Let P_1, P_2 and P_3 be the three polynomials appearing above and let P_4 be the polynomial which appears in the Haagerup inequality satisfied by $(\mathcal{T}', L)$. Let $n \in \mathbb{N}$ and let ϕ_1, ϕ_2, ϕ_3 be in $\mathbb{R}_+\mathcal{G}$ such that the support of ϕ_1 is in $\mathcal{B}_L(n)$. We have

$$\begin{aligned}
&\sum_{\substack{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{T}\\ L(\alpha_1)\leqslant n}} \phi_1(\alpha_1)\phi_2(\alpha_2)\phi_3(\alpha_3)\\
&\qquad\leqslant \sum_{\substack{(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)\in\mathcal{T}'\\ \text{satisfying } \mathcal{C}_n}} \sum_{\substack{u_1,u_2,u_3\in\mathcal{G}\\ (u_i,\tilde{\alpha}_i,u_{i+1})\in\mathcal{B}(u_i\tilde{\alpha}_i u_{i+1}^{-1})\\ \text{for } i=1,2,3}} \phi_1(u_1\tilde{\alpha}_1 u_2^{-1})\phi_2(u_2\tilde{\alpha}_2 u_3^{-1})\phi_3(u_3\tilde{\alpha}_3 u_1^{-1}),
\end{aligned} \tag{1.1}$$

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where $\mathcal{C}_n$ is either the condition $L(\tilde{\alpha}_1) \leqslant P_3(n)$ in the case of assumption (1) or the condition $\max_i L(\tilde{\alpha}_i) \leqslant P_3(n)$ in the case of assumption (2).

Since $\phi_1(u_1\tilde{\alpha}_1u_2^{-1})$ is zero whenever $L(u_1\tilde{\alpha}_1u_2^{-1}) = L(u_2\tilde{\alpha}_1^{-1}u_1^{-1}) > n$ and since we have $(u_1, \tilde{\alpha}_1, u_2) \in \mathcal{B}(u_1\tilde{\alpha}_1u_2^{-1})$ and $(u_2, \tilde{\alpha}_1^{-1}, u_1) \in \mathcal{B}(u_2\tilde{\alpha}_1^{-1}u_1^{-1})$, we can restrict to the case $L(u_1), L(u_2) \leqslant P_1(n)$. Set

$$\tilde{\phi}_1^2(\tilde{\alpha}_1) = \begin{cases} \displaystyle\sum_{\substack{u_1,u_2\in\mathcal{G},\, L(u_1)\leqslant P_1(n) \\ (u_1,\tilde{\alpha}_1,u_2)\in\mathcal{B}(u_1\tilde{\alpha}_1u_2^{-1})}} \phi_1^2(u_1\tilde{\alpha}_1u_2^{-1}) & \text{if } \tilde{\alpha}_1 \in B_L(P_3(n)), \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\phi_2}^2(\tilde{\alpha}_2) = \begin{cases} \displaystyle\sum_{\substack{u_2,u_3\in\mathcal{G},\, L(u_2)\leqslant P_1(n) \\ (u_2,\tilde{\alpha}_2,u_3)\in\mathcal{B}(u_2\tilde{\alpha}_2u_3^{-1})}} \phi_2^2(u_2\tilde{\alpha}_2u_3^{-1}) & \text{if } \tilde{\alpha}_2 \in B'_L(n), \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\phi_3}^2(\tilde{\alpha}_3) = \begin{cases} \displaystyle\sum_{\substack{u_1,u_3\in\mathcal{G}, L(u_1)\leqslant P_1(n) \\ (u_3,\tilde{\alpha}_3^{-1},u_1)\in\mathcal{B}(u_3\tilde{\alpha}_3u_1^{-1})}} \phi_3^2(u_3\tilde{\alpha}_3u_1^{-1}) & \text{if } \tilde{\alpha}_3 \in B'_L(n), \\ 0 & \text{otherwise,} \end{cases}$$

where $B'_L(n) = B_L(P_3(n))$ in the case of assumption (1) and $B'_L(n) = \mathcal{G}$ in the case of (2). By Lemma 1.29, the expression (1.1) is bounded by

$$\sum_{\substack{(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)\in\mathcal{T}' \\ L(\tilde{\alpha}_1)\leqslant P_3(n)}} \tilde{\phi}_1(\tilde{\alpha}_1)\tilde{\phi}_2(\tilde{\alpha}_2)\tilde{\phi}_3(\tilde{\alpha}_3) \leqslant P_4 \circ P_3(n)\|\tilde{\phi}_1\|_2\|\tilde{\phi}_2\|_2\|\tilde{\phi}_3\|_2.$$

On the other hand, we have, for $i = 1$, 2 or 3,

$$\begin{aligned} \|\tilde{\phi}_i\|_2^2 &\leqslant \sum_{\substack{\tilde{\alpha}_i,u_i,u_{i+1}\in\mathcal{G} \\ L(u_j)\leqslant P_1(n),\, \tilde{\alpha}_i\in B'_L(n) \\ (u_i,\tilde{\alpha}_i,u_{i+1})\in\mathcal{B}(u_i\tilde{\alpha}_iu_{i+1}^{-1})}} \phi_i^2(u_i\tilde{\alpha}_iu_{i+1}^{-1}) \\ &\leqslant \sum_{\alpha\in\mathcal{G}} \#\{(u,\tilde{\alpha},v) \in \mathcal{B}(\alpha),\ L(u) \leqslant P_1(n),\ \tilde{\alpha} \in B'_L(n)\}\phi_i^2(\alpha) \\ &\leqslant P_5(n)\|\phi_i\|_2^2, \end{aligned}$$

where $j = 1$ if $i = 1$ or 3 and $j = 2$ if $i = 2$ and where $P_5(n) = P_2 \circ P_1(n)$ in the case of assumption (1) and $P_5(n) = P_2(\max\{P_1(n), P_3(n)\})$ in the case of (2). This gives the result. □

As a direct application of Proposition 1.33, let us give the following proof.

Proof of Lemma 1.24. We have to show that the set of triangles of the groupoid $\mathcal{G}(X_\Gamma, \Gamma, \alpha)$ satisfies the Haagerup inequality with respect to the length-function L

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defined in Proposition 1.13 whenever the set of triangles of the local groupoid $\mathcal{G}(X,\Gamma,\alpha)$ satisfies it.

Let $C \geqslant 0$ such that for all $x \in X$, $\#\Gamma_x \leqslant C$, where Γ_x is the stabilizer of x. Consider for any $[(x_1,g_1),(x_2,g_2)] \in \mathcal{G}(X_\Gamma,\Gamma,\alpha)$ the following domain of decompositions:

$$\begin{aligned}\mathcal{B}&([(x_1,g_1),(x_2,g_2)])\\ &= \{([(x_1,g_1)h,(x_1h,1)],[(x_1h,1),(x_2h,1)],[(x_2,g_2)h,(x_2h,1)]) \mid h \in \Gamma\}.\end{aligned}$$

Notice that $L([(x_i,g_i)h,(x_ih,1)]) = 0$, $i = 1,2$. It is then easy to see that the family

$$\{\mathcal{B}([(x_1,g_1),(x_2,g_2)])\}_{[(x_1,g_1),(x_2,g_2)]\in\mathcal{G}(X_\Gamma,\Gamma,\alpha)}$$

is a family of domains of L-decompositions.

Moreover, any triangle $([(x_1,g_1),(x_2,g_2)],[(x_2,g_2),(x_3,g_3)],[(x_3,g_3),(x_1,g_1)])$ of the groupoid $\mathcal{G}(X_\Gamma,\Gamma,\alpha)$ has a $\mathcal{B}$-pinching $([(x_1,1),(x_2,1)],[(x_2,1),(x_3,1)],[(x_3,1),(x_1,1)])$ in the set of triangles of the local groupoid $\mathcal{G}(X,\Gamma,\alpha)$.

On the other hand, we have $[(x_i,g_i)h,(x_ih,1)] = [(x_i,g_i),(x_i,\alpha_{x_i,x_ih}h^{-1})]$, $i = 1,2$. Since $\alpha_{x_i,x_ih}h^{-1} \in \Gamma_{x_i}$, $\forall h \in \Gamma$, this implies that $\#\mathcal{B}([(x_1,g_1),(x_2,g_2)]) \leqslant C^2$. This means that the groupoid $\mathcal{G}(X_\Gamma,\Gamma,\alpha)$ is of polynomial rigid $\mathcal{B}$-growth. By using Proposition 1.33, Lemma 1.24 follows. □

1.3.2. *The tetrahedral change of faces*

A quadruplet $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ of elements of $\mathcal{G}$ is *a tetrahedron* if the pairs (α_1,β_1), (α_2,β_2) are composable and have the same product $\alpha_1\beta_1 = \alpha_2\beta_2$. A tetrahedron is obtained from a pair of triangles $\{(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_1)\}$ having a common side γ_1 called *the base*. These two triangles are faces of the tetrahedron. The two other faces are given by the triangles $(\alpha_1^{-1},\alpha_2,\gamma_2)$ and $(\beta_1,\beta_2^{-1},\gamma_2)$. We call this jump from the pair of triangles $\{(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_1)\}$ having a common base γ_1 to the pair $\{(\alpha_1^{-1},\alpha_2,\gamma_2),(\beta_1,\beta_2^{-1},\gamma_2)\}$ with a common base γ_2 *a tetrahedral change of faces* (see Figure 1).

Definition 1.34.

(i) We say that a set of triangles $\mathcal{T}'$ is a tetrahedral change of faces of a set of triangles $\mathcal{T}$ if for any pair of triangles in $\mathcal{T}$ having a common base, the pair of triangles obtained by tetrahedral change of faces is in $\mathcal{T}'$.

(ii) For a given polynomial P, we say that a set of triangles $\mathcal{T}'$ is a tetrahedral P-change of faces of a set of triangles $\mathcal{T}$ if for any $n \in \mathbb{N}$, there exist sets of triangles $\mathcal{T}_n^k$, $1 \leqslant k \leqslant P(n)$, for which $\mathcal{T}'$ is a tetrahedral change of faces and such that $\mathcal{T}_n \subset \bigcup_k \mathcal{T}_n^k$ where $\mathcal{T}_n = \{(\alpha,\beta,\gamma) \in \mathcal{T},\ L(\alpha) \leqslant n\}$.

(iii) We say that a set $\mathcal{T} \subset \mathcal{G}^{(3)}$ is *L-balanced* if there exists a polynomial Q such that for any $(\alpha,\beta,\gamma) \in \mathcal{T}$, $L(\beta) \leqslant Q(L(\alpha))$.

We then have the following proposition.

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Proposition 1.35. *Let P be a given polynomial and let $\mathcal{T}, \mathcal{T}' \subset \mathcal{G}_0^{(3)}$. If $\mathcal{T}'$ is a tetrahedral P-change of faces of $\mathcal{T}$, $(\mathcal{T}', L)$ satisfies the Haagerup inequality and $\mathcal{T}$ is L-balanced, then $(\mathcal{T}, L)$ satisfies the Haagerup inequality.*

Proof. Let $n \in \mathbb{N}$ and let $\phi_1, \phi_2 \in \mathbb{R}_+\mathcal{G}$ with the support of ϕ_1 in $\mathcal{B}_L(n)$. We have

$$\phi_1 *_{\mathcal{T}} \phi_2 = \phi_1 *_{\mathcal{T}_n} \phi_2 \leqslant \sum_{1 \leqslant k \leqslant P(n)} \phi_1 *_{\mathcal{T}_n^k} \phi_2.$$

Now, fix an integer k, $1 \leqslant k \leqslant P(n)$, and denote by $\mathcal{Q}_n^k$ the set of tetrahedrons obtained from the set $\mathcal{T}_n^k$. We have

$$\begin{aligned} \|\phi_1 *_{\mathcal{T}_n^k} \phi_2\|_2^2 &= \sum_{(\alpha_1,\beta_1,\alpha_2,\beta_2) \in \mathcal{Q}_n^k} \phi_1(\alpha_1)\phi_1(\alpha_2)\phi_2(\beta_1)\phi_2(\beta_2) \\ &\leqslant \sum_{\gamma \in \mathcal{G}} \phi_1^* *_{\mathcal{T}'} \phi_1(\gamma) \phi_2 *_{\mathcal{T}'} \phi_2^*(\gamma) \leqslant \|\phi_1^* *_{\mathcal{T}'} \phi_1\|_2 \|\phi_2 *_{\mathcal{T}'} \phi_2^*\|_2. \end{aligned}$$

Since $\mathcal{T}$ is L-balanced, we can assume that the support of ϕ_2 lies in $\mathcal{B}_L(Q(n))$. Thus we have

$$\|\phi_1^* *_{\mathcal{T}'} \phi_1\|_2 \|\phi_2 *_{\mathcal{T}'} \phi_2^*\|_2 \leqslant P_1(n) P_1 \circ Q(n) \|\phi_1\|_2^2 \|\phi_2\|_2^2,$$

where P_1 is the polynomial given by the Haagerup inequality satisfied by $\mathcal{T}'$. We obtain

$$\|\phi_1 *_{\mathcal{T}} \phi_2\|^2 \leqslant P(n)^2 P_1(n) P_1 \circ Q(n) \|\phi_1\|_2^2 \|\phi_2\|_2^2.$$

□

Remark 1.36. Notice that we do not need to assume in this proof that the set $\mathcal{T}$ is L-balanced if it is a $\mathcal{B}$-pinching of another set of triangles and satisfies assumption (1) of Proposition 1.33.

1.4. First applications

The aim of this subsection and the following one is to show how to find, with some improvement, all the known examples of groups satisfying the Haagerup inequality by using these two geometrical tools of manipulating triangles.

Theorem 1.37. *In all of the following cases, the group Γ satisfies the Haagerup inequality:*

(i) *Γ is a finitely generated free Abelian group;*

(ii) *Γ is a free non-Abelian group, not necessarily finitely generated, with respect to its canonical word length;*

(iii) *Γ is a word hyperbolic group;*

(iv) *Γ is a group of polynomial growth.*

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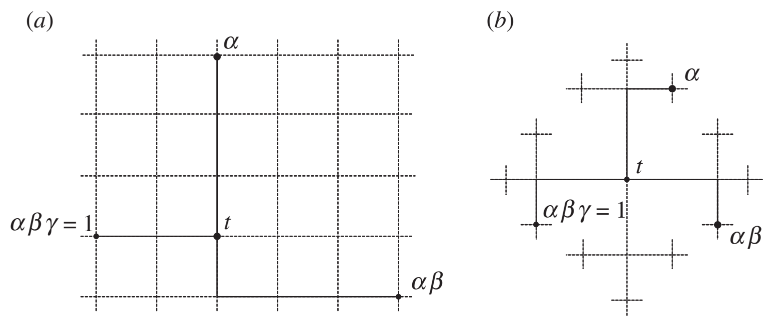

Figure 2. Free cases: (*a*) $\mathbb{Z}^2$; (*b*) $\mathbb{F}_2$.

Proof. In any of these cases, it is possible to choose a suitable domain $\mathcal{B}$ for which the group Γ is of polynomial $\mathcal{B}$-growth and any triangle of Γ has a degenerate $\mathcal{B}$-pinching. By using Propositions 1.28 and 1.33, where $\mathcal{T}$ is taken to be the set of all triangles of Γ and $\mathcal{T}'$ is taken to be the set of degenerate triangles, the theorem follows.

In the first two cases, choose for the domain $\mathcal{B}$ the following:

$$\forall g \in \Gamma, \quad \mathcal{B}(g) = \{(g_1, 1, g_2) \mid g = g_1 g_2^{-1} \text{ and } |g| = |g_1| + |g_2|\}.$$

In both these cases, the group Γ is of polynomial $\mathcal{B}$-growth. Indeed, for the free Abelian group $\mathbb{Z}^d$, take for P_2 a suitable polynomial of degree d, while for any free non-Abelian group, take for P_2 the polynomial $P_2 = X + 1$.

On the other hand, any triangle in Γ in both these two cases admits a degenerated $\mathcal{B}$-pinching. Indeed, consider for a triangle (α, β, γ) of Γ the element t, as shown in Figure 2, defined by

(1) in the Abelian case $\mathbb{Z}^d$, the jth component t_j of t is the medium integer by ordering the jth components α_j, β_j, γ_j for $j = 1, \dots, n$;

(2) in the non-Abelian case, t the common prefix of the minimal decompositions of α and γ^{-1}.

In both these cases, one has

$$\alpha = t1t'^{-1}, \qquad \beta = t'1t''^{-1}, \qquad \gamma = t''1t^{-1}.$$

For (iii) and (iv), we give the proof in more general cases. To do this, let us recall some definitions.

Definition 1.38. Let (X, d) be a pseudo-metric space.

(1) The space (X, d) is said to be weakly geodesic if there exists a constant $\delta \geqslant 0$ such that for every pair (x, y) of points of X and every $t \in [0, d(x, y)]$ there exists a point $w \in X$ such that $d(x, w) \leqslant t + \delta$ and $d(y, w) \leqslant d(x, y) - t + \delta$.

(2) The space (X, d) is said to be hyperbolic if there exists a constant $\delta \geqslant 0$ such that for every quadruplet (x_1, x_2, x_3, x_4) of points of X we have

$$d(x_1, x_2) + d(x_3, x_4) \leqslant \max\{d(x_1, x_4) + d(x_2, x_3), d(x_1, x_3) + d(x_2, x_4)\} + \delta.$$

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(3) The space (X, d) is said to be uniformly locally finite if, for any $r \geqslant 0$, there exist a constant $K(r)$ such that $\#B(x, r) \leqslant K(r)$ for all $x \in X$.

Recall that a word hyperbolic group is a finitely generated group which is hyperbolic in the sense of (2) of Definition 1.38 when equipped with its word distance. Any such group is weakly geodesic, take for example $\delta = \frac{1}{2}$, and uniformly locally finite since it is finitely generated. The following proposition, extracted from already known results, gives two essential geometric properties inducing the Haagerup inequality.

Proposition 1.39. *Let (X, d) be a pseudo-metric space. If (X, d) is weakly geodesic, hyperbolic and uniformly locally finite, then there exist two constants $C, \delta \geqslant 0$ such that*

(1) *for all $x_1, x_2, x_3 \in X$, there exists $w \in X$ such that*

$$d(x_i, w) + d(w, x_{i+1}) \leqslant d(x_i, x_{i+1}) + \delta, \quad \text{for } i = 1, 2, 3;$$

(2) *for all $x, y \in X$ and for all $r \in \mathbb{R}_+$,*

$$\#\{w \in X \mid d(x, w) + d(w, y) \leqslant d(x, y) + \delta,\ d(x, w) \leqslant r\} \leqslant C(r + 1).$$

Proof of Proposition 1.39. Let $\delta_1, \delta_2 \geqslant 0$ be constants in (1) and (2) of Definition 1.38, respectively, and set $\delta = 2\delta_1 + 2\delta_2$ and $C = K(1+2\delta)$, given in (3) of the same definition.

(1) Let $x_1, x_2, x_3 \in X$ and consider a point $w \in X$ such that

$$d(x_1, w) \leqslant (x_2 \mid x_3)_{x_1} + \delta_1 \quad \text{and} \quad d(x_2, w) \leqslant d(x_1, x_2) - (x_2 \mid x_3)_{x_1} + \delta_1,$$

where $(x_2 \mid x_3)_{x_1} = \frac{1}{2}(d(x_1, x_2) + d(x_1, x_3) - d(x_2, x_3))$. We have

$$\begin{aligned} d(x_3, w) + d(x_1, x_2) &\leqslant \max\{d(x_2, x_3) + d(x_1, w), d(x_1, x_3) + d(x_2, w)\} + \delta_2 \\ &\leqslant \tfrac{1}{2}(d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1)) + \delta_1 + \delta_2. \end{aligned}$$

By simplification and easy computations we can prove (1).

(2) Let $x, y \in X$ and fix for any integer $n \in [0, d(x, y)]$ a point $w_n \in X$ such that $d(x, w_n) \leqslant n + 1 + \delta_1$ and $d(y, w_n) \leqslant d(x, y) - n + \delta_1$. For any point $w \in X$ such that $d(x, w) \leqslant n + 1 + \delta$ and $d(y, w) \leqslant d(x, y) - n + \delta$, we have

$$\begin{aligned} d(x, y) + d(w, w_n) &\leqslant \max\{d(x, w_n) + d(y, w), d(x, w) + d(y, w_n)\} + \delta_2 \\ &\leqslant d(x, y) + 1 + 2\delta. \end{aligned}$$

This proves that

$$\{w \in X \mid d(x, w) + d(w, y) \leqslant d(x, y) + \delta,\ d(x, w) \leqslant r\} \subset \bigcup_{0 \leqslant n < r} B(w_n, 1 + 2\delta)$$

and then we have (2).

□

Theorem 1.40. *Any discrete groupoid $\mathcal{G}$ acting freely and by isometries on a weakly geodesic, hyperbolic and uniformly locally finite pseudo-metric space (X, d) satisfies the Haagerup inequality.*

This theorem is a generalization of (iii) of Theorem 1.37 since the action by left translation of any word hyperbolic group on its self is freely and by isometries.

Proof of Theorem 1.40. By Proposition 1.21, it suffices to prove it for the local groupoid $\mathcal{G}(X,\mathcal{G})$ together with the length-function given by d. Let δ be the constant given by Proposition 1.39. For any $x,y \in X$, let $\mathcal{B}([x,y]) = \{([x,w],[w,w],[y,w]) \mid d(x,w)+d(w,y) \leqslant d(x,y)+\delta\}$. By (ii) of Proposition 1.39, the local groupoid $\mathcal{G}(X,\mathcal{G})$ is of polynomial $\mathcal{B}$-growth, and by (i), any triangle in $\mathcal{G}(X,\mathcal{G})$ admits a degenerate $\mathcal{B}$-pinching. □

For a generalization of (iv) of Theorem 1.37, let us give the following definition.

Definition 1.41. We say that a groupoid $\mathcal{G}$ is of polynomial growth with respect to a length-function L if there exists $a,d>0$ such that for all $u \in \mathcal{G}^{(0)}$ and for all $r \in \mathbb{R}_+$,

$$\#(\Gamma_u \cap \mathcal{B}_L(r)) \leqslant a(r+1)^d,$$

where $\Gamma_u = \mathcal{G}(u)$ is the isotropy group at u.

This definition coincides with the notion of polynomial growth in the case of a finitely generated group. It is a refinement of Definition 4.2.9 given in [**Tal01**].

When the groupoid $\mathcal{G}$ is a disjoint union of groups, Definition 1.41 coincides with the one given by Black [**Bla98**] for an infinite family of finite groups endowed with a uniformly bounded generator system family. Black proved in that paper (Theorem 2.1) that the notion of polynomial growth of such a family does not depend on the choice of the generator system family, although the associated algebraic lengths are not necessarily equivalent. These are, in fact, the families of groups $\mathcal{S} = \{\Gamma_i\}_{i\in I}$ for which there exist two constants $c,l>0$ such that any group $\Gamma_i \in \mathcal{S}$ admits a normal subgroup of a nilpotence class at most c and of index at most l in Γ_i.

When the groupoid is $\mathcal{G}(X,\Gamma)$, where the group Γ acts freely and by isometries on the discrete metric space (X,d) as in Definition 1.2, this definition is equivalent to saying that there exists a polynomial P such that

$$\#B_{L_x}(r) \leqslant P(r), \quad \forall x \in X,\ r \in \mathbb{R}_+,$$

where $B_{L_x}(r)$ designs the ball in Γ of radius r with respect to the length L_x defined in Proposition 1.13.

Proposition 1.42. *If $\mathcal{G}$ is a groupoid of polynomial growth with respect to a length-function L, then it satisfies the Haagerup inequality for this length.*

Proof. By Proposition 1.17, it is enough to show that there exist $C,r>0$ such that, for all $u \in \mathcal{G}^{(0)}$, the isotropy groups Γ_u satisfy the Haagerup inequality with respect to the restriction of L for the same constants C, r. But constants C, r depend only on the constants a, d given in Definition 1.41.

Indeed, let us consider, for any $u \in \mathcal{G}^{(0)}$, the set $\mathcal{B}_u(g)$, for $g \in \Gamma_u$, given by

$$\mathcal{B}_u(g) = \{(h,1_u,g^{-1}h) \mid h \in \Gamma_u,\ L(h) \leqslant L(g)\}.$$

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It is clear that (Γ_u, L) is of polynomial $\mathcal{B}_u$-growth where all polynomials do not depend on u but only on a, d. On the other hand, any triangle (g_1, g_2, g_3) in Γ_u admits a degenerate $\mathcal{B}_u$-pinching. Indeed, let us fix $k \in \mathbb{Z}/3\mathbb{Z}$ such that $L(g_k) \geqslant L(g_{k\pm1})$. We have the following decompositions:

$$g_{k+1} = g_{k+1}1_u1_u, \qquad g_{k-1} = 1_u1_ug_{k-1}, \qquad g_k = g_{k-1}^{-1}1_ug_{k+1}^{-1}.$$

This means that the degenerate triangle $(1_u, 1_u, 1_u)$ is a $\mathcal{B}_u$-pinching of (g_1, g_2, g_3). And by Propositions 1.28 and 1.33, we obtain the desired result. □

□

1.5. Connection with Lafforgue's properties H_δ and K_δ

In this paragraph, we give definitions of Lafforgue's properties H_δ and K_δ in terms of pinching and tetrahedral change of faces to show how to deduce the analytical part of his results.

Definition 1.43. Let $\delta \geqslant 0$, let (X, d) be a metric space, let Γ be a group acting freely and by isometries on (X, d), and consider for any $\alpha \in \mathcal{G}(X, \Gamma)$ the following domain of L-decompositions, where L is the length given in Proposition 1.13:

$$\mathcal{B}(\alpha) = \{(u, \tilde{\alpha}, v),\ \alpha = u\tilde{\alpha}v^{-1},\ L(u) + L(\tilde{\alpha}) + L(v) \leqslant L(\alpha) + \delta\}.$$

(1) We say that X satisfies (H_δ) if there exists a polynomial P such that, for any $r \in \mathbb{R}_+$, $x, y \in X$, one has $\#\{t \in X,\ d(x,t) + d(t,y) \leqslant d(x,y) + \delta,\ d(x,t) \leqslant r\} \leqslant P(r)$.

(2) We say that X and Γ satisfy (K_δ) if there exist two sets $\mathcal{T}$, $\mathcal{T}'$ of triangles of $\mathcal{G}(X, \Gamma)$ and $k \in \mathbb{N}$ such that

$(K_\delta(\mathrm{a}))$ there exists $C \in \mathbb{R}_+$ such that any triangle $(\alpha_1, \alpha_2, \alpha_3)$ of $\mathcal{G}(X, \Gamma)$ has a $\mathcal{B}$-pinching $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ in $\mathcal{T}$ such that

$$\max_i L(\tilde{\alpha}_i) \leqslant C \min_i L(\alpha_i), \quad \text{for } i = 1, 2, 3;$$

$(K_\delta(\mathrm{b}))$ the set $\mathcal{T}'$ is a tetrahedral k-change of faces of the set $\mathcal{T}$ and has a degenerate $\mathcal{B}$-pinching.

Notice that property (H_δ) implies that the groupoid $\mathcal{G}(X, \Gamma)$ is of polynomial $\mathcal{B}$-growth. Thus, if, for some $\delta \geqslant 0$, X and Γ satisfy (H_δ) and (K_δ) and $\mathcal{T}$ and $\mathcal{T}'$ are such as above, then $(\mathcal{T}', L)$ satisfies the Haagerup inequality by Propositions 1.33 and 1.28. But the set $\mathcal{T}'$ is a tetrahedral k-change of faces of $\mathcal{T}$ and $\mathcal{T}$ is a $\mathcal{B}$-pinching of the set of triangles of $\mathcal{G}(X, \Gamma)$. By Proposition 1.35 and Remark 1.36, $(\mathcal{T}, L)$ satisfies the Haagerup inequality. By applying Proposition 1.33 once more we deduce Theorem 2.5 in [**Laf00**].

Theorem 1.44. *If for some $\delta \geqslant 0$, X and Γ satisfy (H_δ) and (K_δ), then the groupoid $\mathcal{G}(X, \Gamma)$ satisfies the Haagerup inequality with respect to L and therefore Γ satisfies the Haagerup inequality.*

Using Theorem 1.44, Lafforgue proved in [**Laf00**] that cocompact lattices in $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{C})$ satisfy the Haagerup inequality. In [**Cha01**], Chatterji generalized his result by using the same theorem to cocompact lattices of $\mathrm{SL}_3(\mathcal{H})$ and $\mathrm{SL}_3(\mathcal{O})$ and their products.

2. The Haagerup inequality and affine buildings

Let Δ be an affine combinatorial building and let $\mathcal{V}_\Delta$ be the set of its vertices. We provide $\mathcal{V}_\Delta$ with the theoretic metric d given by the 1-skeleton of the building Δ and let Γ be a group acting freely and by isometries on $(\mathcal{V}_\Delta, d)$. In what follows, we will apply the geometrical deformations tools developed before to sets of triangles of the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ defined in Definition 1.2 with respect to the length-function given by d to show the Haagerup inequality for $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$. By using Proposition 1.21 we deduce this inequality for the group Γ.

To prove some technical lemmas, we need to consider the geometric realization $|\Delta|$ of the building Δ. This is the affine realization of the building endowed with a metric ∂ which, restricted to the affine realization of any apartment, is Euclidean [**Bro89**, § VI.3]. The set $\mathcal{V}_\Delta$ can be seen as a subset of $|\Delta|$ and is therefore provided with another metric ∂. For any two vertices $x, y \in \mathcal{V}_\Delta$ we will denote by $[x, y]_\partial$ the unique geodesic line in $|\Delta|$ between x and y, to distinguish it from the element $[x, y]$ of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$.

2.1. For a suitable domain $\mathcal{B}$

We have to consider for any element $[x, y]$ of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ a suitable domain $\mathcal{B}([x, y])$ such that the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ is of polynomial $\mathcal{B}$-growth. Until now, all the domains $\mathcal{B}$ produced in the previous sections were given by the length. If we do the same in this case and consider for example the set $\mathcal{C}([x, y]) = \{([x, z], [z, t], [y, t]) \in \mathcal{G}(\mathcal{V}_\Delta, \Gamma)^3,\ d(x, y) = d(x, z) + d(z, t) + d(t, y)\}$, the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ will not be of polynomial $\mathcal{C}$-growth with respect to the metric d when the building Δ is thick and will be of type other than $\tilde{A}_1$, $\tilde{A}_2$ or their product. This is because the set $C(x, y) = \{z \in \mathcal{V}_\Delta,\ d(x, y) = d(x, z) + d(z, y)\}$ is not, in general, contained in an apartment.

Indeed, let us consider a thick affine building Δ of irreducible type other than $\tilde{A}_1$ or $\tilde{A}_2$. A classification of affine buildings can be found in [**Bou68**, Chapter VI, Theorem 4]. Let C be a chamber of Δ contained in an apartment Σ. Let H and H' be two orthogonal walls of C in Σ. These walls exist because we can find two vertices in the Coxeter graph of all affine buildings of irreducible type, excepting those of type $\tilde{A}_1$ or $\tilde{A}_2$, which are not connected by an edge. Let S be the maximal face of C contained in $H \cap H'$. The simplex S is of codimension 2. Now consider the two vertices x, y contained in the wall H such that $S \cup \{x\}$ and $S \cup \{y\}$ are simplices, as shown in Figure 3.

It is clear that the set $C(x, y)$ contains, in addition to the two vertices in H' that are not in H, other vertices that are not in the apartment Σ. On the other hand, if we take two apartments Σ and Σ' that contain x and y, any isomorphism between Σ and Σ' that fixes x and y transforms $C(x, y) \cap \Sigma$ to $C(x, y) \cap \Sigma'$.

This phenomenon makes the map $n \mapsto \max_{x,y \in \mathcal{V}_\Delta} \operatorname{card}\{z \in C(x, y),\ d(x, z) \leqslant n\}$ grow exponentially, since the building is thick. However, if the building Δ is of type

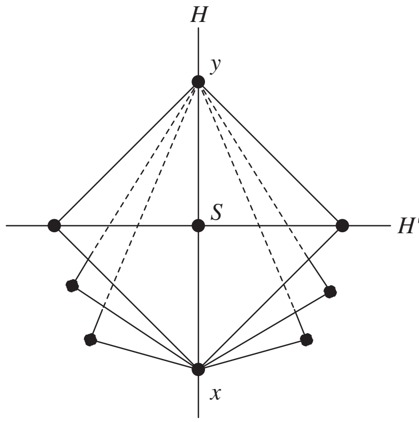

Figure 3. Exponential $\mathcal{C}$-growth.

Figure 4. Case $\tilde{A}_1$.

$\tilde{A}_{k_1} \times \cdots \times \tilde{A}_{k_n}$, where $k_i \in \{1, 2\}$, this problem does not appear and the domain $\mathcal{C}([x, y])$ is suitable.

In order to avoid any restriction to some specific affine building, we need to consider for $\mathcal{B}$ another domain which will be defined by using the combinatorial structure of the building rather than its metric one. To do this, we need the following domain.

Definition 2.1. Fix for any two vertices $x, y \in \mathcal{V}_\Delta$ an apartment Σ containing both of them. Denote by $B(x, y)$ their combinatorial convex closure in Σ. That is, let $B(x, y)$ be the set of vertices contained in all the roots (half-spaces) of Σ containing both x and y.

When the building Δ is of type $\tilde{A}_1$, i.e. a tree without ends, $B(x, y)$ is nothing but the set of vertices between x and y, as shown in Figure 4.

When the building Δ is of dimension 2 (see [**Car97**] for the form of the apartments), Figure 5 illustrates some examples of $B(x, y)$.

Remark 2.2. If the building Δ is of type $W_1 \times W_2$, the set $B(x, y)$ is identified with the set $B(x_1, y_1) \times B(x_2, y_2)$, where the $B(x_i, y_i)$ is the combinatorial convex closure of the vertices x_i, y_i in a building of type W_i, $i = 1, 2$.

In a similar way, we can define the combinatorial convex closure of any two simplices of the same dimension, or polysimplices if the building is not of irreducible type, by fixing an apartment Σ containing both of the two simplices and considering the set of all simplices of the same dimension contained in the roots (half-spaces) of Σ containing both the two initial simplices. In particular, for two chambers C_1, C_2, it is well known that the convex combinatorial closure $B(C_1, C_2)$ coincides with the set of chambers of Σ lying in minimal galleries between C_1 and C_2 in Σ. But minimal galleries between C_1 and C_2 in Σ are exactly minimal galleries in the building Δ, which means that $B(C_1, C_2)$

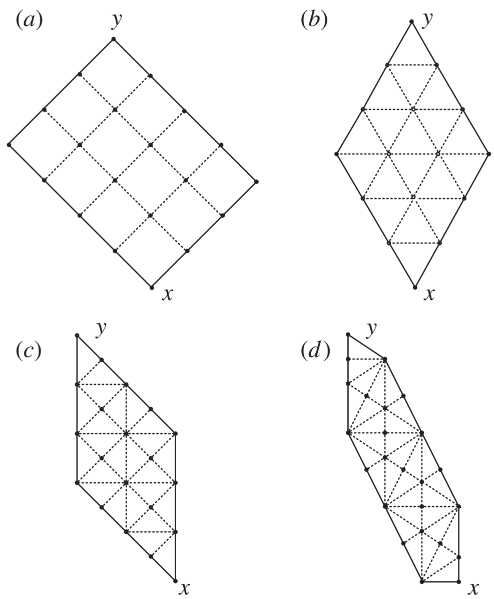

Figure 5. Cases of dimension 2: (*a*) $\tilde{A}_1 \times \tilde{A}_1$; (*b*) $\tilde{A}_2$; (*c*) $\tilde{B}_2 = \tilde{C}_2$; (*d*) $\tilde{G}_2$.

does not depend on the choice of the apartment Σ. The same thing goes for the vertices x, y since we have the following lemma.

Lemma 2.3. *The set $B(x,y)$ does not depend on the choice of the apartment Σ.*

Notice that if two vertices u, v are in $B(x,y)$, then $B(u,v) \subset B(x,y)$. This justifies using the notion of convexity. In a general way, a subset X of $\mathcal{V}_\Delta$ will be called convex in the sense of buildings if for any $x, y \in X$, $B(x,y) \subset X$. The convex closure of a subset X of $\mathcal{V}_\Delta$ in the sense of buildings, shall mean the smallest convex set containing X.

Proof of Lemma 2.3. By Remark 2.2 it suffices to prove this lemma in the irreducible case. First, consider the set $B'(x,y)$ to be the smallest subset of $\mathcal{V}_\Delta$ containing $\{x,y\}$ and satisfying the property that, if $x', y' \in B'(x,y)$ and if z is a vertex of a simplex S such that the geodesic line $[x',y']_\partial$ crosses the affine realization $|S|$ (i.e. $[x',y']_\partial \cap |S| \neq \emptyset$), then $z \in B'(x,y)$. The set $B'(x,y)$ is contained in all apartments Σ containing $\{x,y\}$. This is because if an apartment Σ contains $\{x',y'\}$, then its affine realization $|\Sigma|$ contains $[x',y']_\partial$ (see [**Bro89**, p. 152]) and therefore contains all the vertices of simplices crossed by $[x',y']_\partial$.

On the other hand, we have $B'(x,y) = B(x,y)$. Indeed, if $x', y' \in B(x,y) \subset \Sigma$, any half-space of Σ containing x', y' must contain any simplex S crossed by the line $[x',y']_\partial$. This proves that $B'(x,y) = B(x,y)$. Conversely, consider a vertex $z \in \mathcal{V}_\Delta$ satisfying $d(z, B'(x,y)) = 1$ and consider the set V of vertices $z' \in B'(x,y)$ such that $d(z,z') = 1$. The set V is contained in a wall which does not contain z. If this is not true, there exist two vertices z_1, z_2 in V such that the geodesic line $[z_1,z_2]_\partial$ crosses a simplex containing

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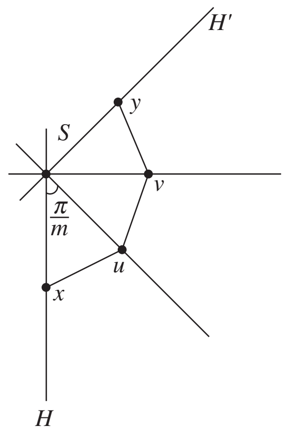

Figure 6. Example of $B(x, y)$ not contained in $C(x, y)$.

z which contradicts the fact that $z \notin B'(x, y)$. This wall separates the vertex z from x, y and then $z \notin B(x, y)$. □

As seen above, the set $C(x, y)$ is in general not contained in $B(x, y)$ since the latter is contained in an apartment. Conversely, the set $B(x, y)$ is in general not contained in $C(x, y)$. This is not true for example whenever an integer $m \geqslant 4$ appears in the Coxeter matrix of the building Δ.

Indeed, let us consider in an apartment of such a building a simplex S contained in the intersection of two hyperplanes H and H' of angle $3\pi/m$. Take two vertices x in $H \setminus H'$ and y in $H' \setminus H$ such that $S \cup \{x\}$ and $S \cup \{y\}$ are simplices and such that $S \cup \{x, y\}$ is not a simplex.

Let u and v be vertices not contained in $H \cup H'$ such that $S \cup \{x, u\}$, $S \cup \{u, v\}$ and $S \cup \{v, y\}$ are simplices as shown in Figure 6. The set $B(x, y)$ contains u and v while the set $C(x, y)$ does not contain them. This problem appears in buildings of type $\tilde{B}_n$, $\tilde{C}_n$ for $n \geqslant 2$, $\tilde{G}_2$ and $\tilde{F}_4$ (see [**Bou68**, Chapter VI, Theorem 4]).

When the building is of type $\tilde{A}_{n_1} \times \cdots \times \tilde{A}_{n_k}$ we have $B(x, y) \subset C(x, y)$ and the equality holds when $n_i \leqslant 2$, $i = 1, \ldots, k$. However, in the general case we still have the following remark.

Remark 2.4. If x, y are vertices of an affine building and if $u \in B(x, y)$, then $d(x, u) \leqslant d(x, y)$.

With the aim of defining a suitable domain $\mathcal{B}$, we need to consider the following lemma.

Lemma 2.5. *If Γ is a group acting by isometries on $(\mathcal{V}_\Delta, d)$, for any pair of vertices x, y and any element $\alpha \in \Gamma$, we have $B(x, y)\alpha = B(x\alpha, y\alpha)$.*

Proof. If the building Δ is of type $\tilde{A}_{n_1} \times \cdots \times \tilde{A}_{n_k}$, $n_i \in \{1, 2\}$, this lemma is obvious since $B(x, y) = C(x, y)$, $\forall x, y \in \mathcal{V}_\Delta$. In the general case, notice that a chamber of Δ is well defined by the metric d. This means that the action of Γ on $(\mathcal{V}_\Delta, d)$ induces an action on the building Δ. If Δ is provided with the complete system of apartments [**Bro89**, § IV.4],

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any element of Γ transforms an apartment to an apartment and a root to a root, which proves the lemma. $\square$

In view of all this, we can now define our domain $\mathcal{B}$ by considering, for any $[x,y] \in \mathcal{G}(\mathcal{V}_\Delta,\Gamma)$, the set $\mathcal{B}([x,y]) = \{([x,x'],[x',y'],[y,y']) \mid x',y' \in B(x,y)\}$. By Lemma 2.5, the set $\mathcal{B}([x,y])$ is well defined since it does not depend on the choice of (x,y). By Remark 2.4, $\mathcal{B}([x,y])$ is a domain, since it satisfies the axioms of Definition 1.30. And we have the following proposition.

Proposition 2.6. *The groupoid $\mathcal{G}(\mathcal{V}_\Delta,\Gamma)$ is of polynomial $\mathcal{B}$-growth.*

To prove Proposition 2.6, we first need to make the following remark.

Remark 2.7. There exist two constants $a,b>0$ such that $a\partial \leqslant d \leqslant b\partial$ in $\mathcal{V}_\Delta$. For example,

$$a = \frac{1}{\min_{d(x',y')=1} \partial(x',y')}, \qquad b = k \max_{d(x',y')=1} \partial(x',y'),$$

where k is the dimension of the building Δ.

By this remark, if we fix an apartment Σ containing x, y, the set

$$\{z \in \Sigma,\ d(x,z) \leqslant n\} \subset \left\{z \in \Sigma,\ \partial(x,z) \leqslant \frac{1}{a} n\right\}$$

has a volume less than or equal to a polynomial in n since (Σ,∂) has an isometric embedding into the Euclidean space $\mathbb{R}^k$. But $B(x,y)$ is contained in Σ. Then the volume of $\{z \in B(x,y),\ d(x,z) \leqslant n\}$ is less than or equal to a polynomial in n and the proposition follows.

2.2. Reduced triangles

To state one of the two important results of this section, we first need to define reduced triangles.

Definition 2.8.

(1) A triplet (x_1,x_2,x_3) in $\mathcal{V}_\Delta$ is said to be reduced if, for all $i \in \mathbb{Z}/3\mathbb{Z}$, we have $B(x_i,x_{i+1}) \cap B(x_i,x_{i-1}) = \{x_i\}$.

(2) A triangle $([x_1,x_2],[x_2,x_3],[x_3,x_1])$ in $\mathcal{G}(\mathcal{V}_\Delta,\Gamma)$ is said to be reduced if the triplet (x_1,x_2,x_3) in $\mathcal{V}_\Delta$ is reduced.

(3) Denote by $\mathcal{R}$ the set of reduced triangles of $\mathcal{G}(\mathcal{V}_\Delta,\Gamma)$.

Notice that, because of Lemma 2.5, part (2) of this definition does not depend on the choice of the triplet (x_1,x_2,x_3). To make us more familiar with this definition, let us illustrate it with examples in some special cases.

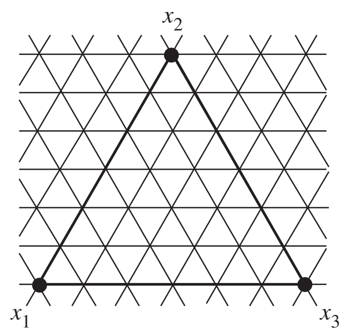

Figure 7. Case $\tilde{A}_2$.

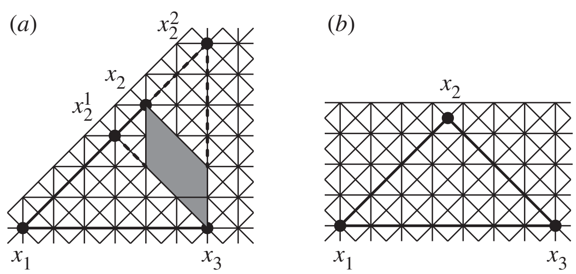

Figure 8. Case $\tilde{B}_2 = \tilde{C}_2$: ($a$) reduced triplets with three special vertices; (b) reduced triplets with only two special vertices.

When the building Δ is of type $\tilde{A}_1$, reduced triplets are precisely degenerate ones. This is because triangles in a tree are tripods.

When the building Δ is of type $\tilde{A}_2$, the non-degenerate reduced triplets contained in an apartment are exactly equilateral triangles with the form given by Figure 7.

When the building Δ is of type $\tilde{B}_2 = \tilde{C}_2$, we have two possible cases for the non-degenerate reduced triplets contained in an apartment. The first case is when the three vertices of the triangle are specials (i.e. any one of them is contained in four walls of the apartment) and the second one is when only two of them are specials. In both these two cases, the form of the reduced triangle is given in Figure 8 up to permutations.

Notice that in the first case of Figure 8, the vertex x_2 can be any of the special vertices between x_1^1 and x_2^2 on the same wall.

When the building Δ is of type $\tilde{G}_2$, Figure 9 illustrates some examples of non-degenerate reduced triplets contained in an apartment. In contrast to the previous case, these are more complicated since special vertices are contained in six walls of a fixed apartment.

When the building Δ is of type $\tilde{A}_3$, the non-degenerate reduced triplets contained in an apartment are given by the following.

If the triplet is contained in a wall, it has the form of an isosceles triangle in the geometric realization of the apartment with the cosine of the principal angle α equal to

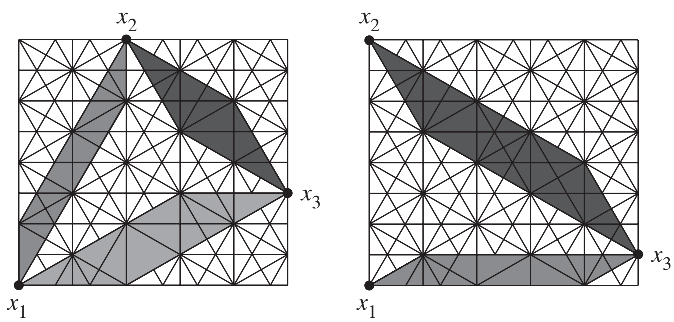

Figure 9. Case $\tilde{G}_2$.

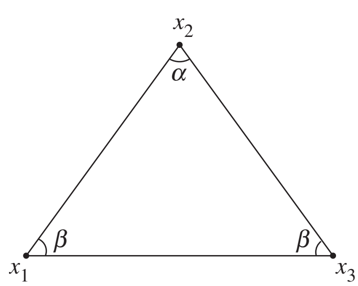

Figure 10. Case $\tilde{A}_3$.

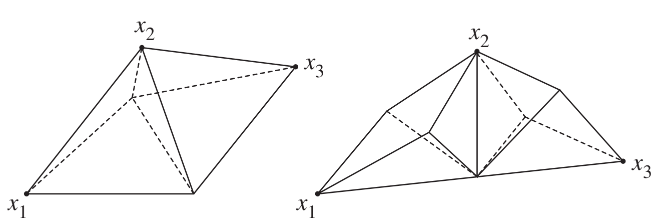

Figure 11. Case $\tilde{A}_3$.

$\frac{1}{3}$ and the cosine of the two other angles β equal to $1/\sqrt{3}$, as shown in Figure 10. Notice that we have $\alpha > \frac{1}{3}\pi > \beta > \frac{1}{4}\pi$.

When the support of the convex combinatorial closure of only a pair of vertices of the triplet is of dimension 2, and the dimension of the other pairs is equal to 1, the triplets have the form given in the first case of Figure 11 up to permutation of vertices. But when the support of the combinatorial closure of two pairs of vertices of the triplet is of dimension 2, the triplets have the form given in the second case of Figure 11 up to permutation of vertices.

In the first case of Figure 11, the triplet has the form of a pentahedron obtained by gluing along a same face two tetrahedra with two orthogonal dihedral opposite angles. In the second case, it has the form of two pentahedra glued along the same edge.

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Lemma 2.9. *In an affine building of type $W_1 \times W_2$, any reduced triangle $T = (x, y, z)$ is identified with a triangle of the form $((x_1, x_2), (y_1, y_2), (z_1, z_2))$ where the $T_i = (x_i, y_i, z_i)$ are reduced triangles of a building of type W_i, for $i = 1, 2$. We will denote this by $T = T_1 * T_2$.*

This means that to know the form of reduced triplets of any affine building it is sufficient to know their form in the irreducible case. In particular, we have the following remark.

Remark 2.10. For any building Δ of type $\tilde{A}_1 \times \cdots \times \tilde{A}_1$, reduced triangles of the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ are exactly degenerate triangles.

Proof of Lemma 2.9. Indeed, let (x, y, z) be a reduced triplet and let (x_1, x_2), (y_1, y_2), (z_1, z_2) be the coordinates of vertices x, y, z in the decomposition $\Delta = \Delta_1 \times \Delta_2$ associated to $W = W_1 \times W_2$. Since $\mathcal{B}(x, y) = \mathcal{B}(x_1, y_1) \times \mathcal{B}(x_2, y_2)$ and the same thing occurs for $\mathcal{B}(x, z)$, we have $\mathcal{B}(x_1, y_1) \times \{x_2\} \cap \mathcal{B}(x_1, z_1) \times \{x_2\} = \{(x_1, x_2)\}$ due to the fact that the triplet (x, y, z) is reduced. This gives us $\mathcal{B}(x_1, y_1) \cap \mathcal{B}(x_1, z_1) = \{x_1\}$. We do the same for the other vertices and we obtain the result. □

The following lemma gives a useful estimate of the size of the sides of a reduced triangle.

Lemma 2.11. *For any affine building Δ, there exists a constant C such that for any reduced triangle $([x_1, x_2], [x_2, x_3], [x_3, x_1])$ we have*

$$\max_i L([x_i, x_{i+1}]) \leqslant C \min_i L([x_i, x_{i+1}]).$$

Proof of Lemma 2.11. By Remark 2.7 it is equivalent to prove that there exists a constant C' such that for any reduced triplet (x_1, x_2, x_3) of $\mathcal{V}_\Delta$ we have

$$\max_i \partial(x_i, x_{i+1}) \leqslant C' \min_i \partial(x_i, x_{i+1}).$$

To do this, let us first define a notion of angles of any non-trivial triplet (x_1, x_2, x_3) of $\mathcal{V}_\Delta$. These are the angles between the geodesic lines $[x_1, x_2]_\partial$, $[x_2, x_3]_\partial$ and $[x_3, x_1]_\partial$ in the geometric realization $|\Delta|$. They are defined for any vertex x_i of such a triplet as follow: fix an apartment Σ_i containing both the two simplices S_i^{i+1}, S_i^{i-1} containing $\{x_i\}$ and crossed, respectively, by $]x_i, x_{i+1}[_\partial$ and by $]x_i, x_{i-1}[_\partial$. Define the angle $(\widehat{x_{i-1}, x_i, x_{i+1}})$ to be the angle between these two geodesic lines in the geometric realization $|\Sigma_i|$ of this apartment. If Σ_i' is another apartment containing S_i^{i+1}, S_i^{i-1}, its geometric realization is isometric to $|\Sigma_i|$ and therefore gives the same angle.

Remark 2.12. For any affine building Δ of irreducible type, there exists $\alpha > 0$ such that the angles of non-degenerate reduced triplets of $\mathcal{V}_\Delta$ are greater than or equal to α. Denote by α_Δ the maximal such α.

This comes from the fact that for any vertex x_i of such a triplet (x_1, x_2, x_3), the intersection of the two simplices S_i^{i+1}, S_i^{i-1} defined in the proof of Lemma 2.11 is reduced to $\{x_i\}$. If the building Δ is of type $\tilde{A}_2$, $\alpha_\Delta = \frac{1}{3}\pi$. If it is of type $\tilde{B}_2$ or $\tilde{A}_3$, $\alpha_\Delta = \frac{1}{4}\pi$. If it is of type $\tilde{G}_2$, $\alpha_\Delta = \frac{1}{6}\pi$, etc.

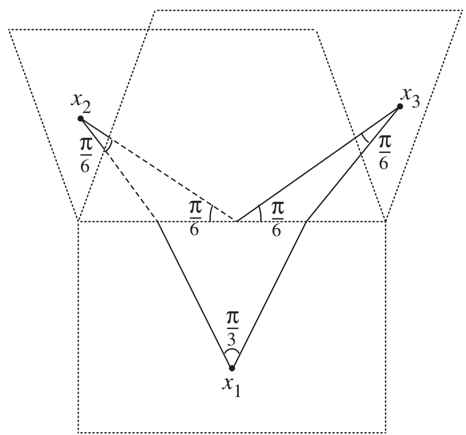

Figure 12. Case $\tilde{G}_2$.

Now, consider a non-degenerate reduced triplet (x_1, x_2, x_3) and fix $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\partial(x_{i-1}, x_{i+1}) = \min_j \partial(x_j, x_{j+1})$. Fix an apartment Σ_i containing the two simplices S_i^{i+1}, S_i^{i-1} defined above and fix a chamber C_i in Σ_i containing x_i. Consider the retraction $\rho_i = \rho_{\Sigma_i, C_i}$ of the building Δ onto Σ_i centred at C_i [**Bro89**, § IV.3]. Set $x'_j = \rho_i(x_j)$ for $j = 1, 2, 3$. Since $x_i \in C_i$, we have $x'_i = x_i$ and $\partial(x_i, x'_{i\pm 1}) = \partial(x_i, x_{i\pm 1})$ and we have $\partial(x'_{i-1}, x'_{i+1}) \leqslant \partial(x_{i-1}, x_{i+1})$ [**Bro89**, § VI.3]. On the other hand, we have

$$(\widehat{x'_{i-1}, x_i, x'_{i+1}}) = (\widehat{x_{i-1}, x_i, x_{i+1}}) \geqslant \alpha_\Delta,$$

and since $(|\Sigma|, \partial)$ is isometric to the Euclidean space $\mathbb{R}^k$, this give us the following inequality: $\max\{\partial(x_i, x'_{i-1}), \partial(x_i, x'_{i+1})\} \leqslant C'\partial(x'_{i-1}, x'_{i+1})$, where $C' = 1/(1 - \cos\alpha_\Delta)$. We deduce from this that $\max_i \partial(x_i, x_{i+1}) \leqslant C' \min_i \partial(x_i, x_{i+1})$, which proves the lemma. □

The following proposition gives a perfect characterization of reduced triplets in an affine building of some special type.

Proposition 2.13. *In an affine building of type $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_2$ or their product with the complete system of apartments, any reduced triplet is contained in an apartment.*

This means that reduced triplets in cases $\tilde{A}_2$ and $\tilde{B}_2$ are precisely those given in Figures 7 and 8.

Proposition 2.13 is in general not true for an affine building of type $\tilde{G}_2$. This is because the angle $\alpha_\Delta = \frac{1}{6}\pi$ is very small. Figure 12 provides us with an example of a reduced triplet that cannot be contained in an apartment.

We think, however, that, with the exception of the case of type $\tilde{G}_2$ above, this proposition is still true for all other affine buildings of irreducible type and their products.

Proof of Proposition 2.13. The case $\tilde{A}_1$ is obvious. For the other cases, it is enough to prove the proposition in the cases of buildings of type $\tilde{A}_2$ and $\tilde{B}_2$. The general case is deduced from these cases by Lemma 2.9.

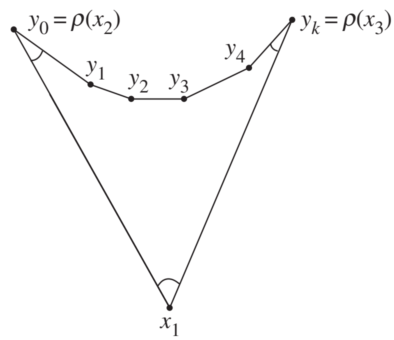

Figure 13. The broken line $\rho([x_2, x_3]_\partial)$.

Let us consider for a reduced triplet (x_1, x_2, x_3) the simplices S_1^2, S_1^3 defined in the proof of Lemma 2.11. Fix an apartment Σ containing S_1^2, S_1^3 and a chamber C in Σ containing the vertex x_1. Denote by $\rho = \rho_{\Sigma,C}$ the retraction of the building Δ onto Σ centred at C.

It is enough to prove that $\partial(\rho(x_2), \rho(x_3)) = \partial(x_2, x_3)$. This implies that the triplet (x_1, x_2, x_3) is contained in an apartment. Indeed, let us consider the set $Y = \overline{|C|} \cup \{x_2, x_3\}$, where $\overline{|C|}$ is the closure of the affine realization of the chamber C in $|\Delta|$. The restriction of ρ to Y is an isometry to a subset of $|\Sigma|$ which is isometric to $\mathbb{R}^k$, since, $\forall u \neq v \in Y$, if u or v are in $\bar{C}$, then $\partial(\rho(u), \rho(v)) = \partial(u, v)$ [**Bro89**, § VI.3]. But the interior of Y is equal to $|C|$, which is non-empty. By [**Bro89**, § VI.7, Theorem 2], Y is contained in an apartment and therefore (x_1, x_2, x_3).

Let us consider $t \in [0, 1] \mapsto p_t \in [x_2, x_3]_\partial$, the geodesic line between x_2 and x_3. Since $x_1 \in C$, we have for all $t \in [0, 1]$, $\partial(x_1, p_t) = \partial(x_1, \rho(p_t))$. There exists a finite subdivision $(t_0, \dots, t_k)$ of $[0, 1]$ such that the portion of the segment $p_{[t_i, t_{i+1}]}$ is contained in the closure of the affine realization of a chamber C_i. Since the restriction of the retraction ρ to any chamber is an isometry, $\rho(p_{[t_i, t_{i+1}]})$ is a geodesic line in $|\Sigma|$ (i.e. a segment). This implies that there exists a net of vertices $y_0 = \rho(x_2), \dots, y_k = \rho(x_3)$ such that the image of $[x_2, x_3]_\partial$ by ρ is a union of lines $[y_0, y_1]_\partial \cup \cdots \cup [y_{k-1}, y_k]_\partial$. Take k to be the smallest such integer. To prove that $\partial(\rho(x_2), \rho(x_3)) = \partial(x_2, x_3)$, it is equivalent to prove that $k = 1$.

If $k > 1$, this means that $\rho([x_2, x_3]_\partial)$ is a broken line. When the dimension of the building Δ is 2, by the negative curvature inequality [**Bro89**, § VI.3] the broken line $\rho([x_2, x_3]_\partial)$, which is planar, does not change the concavity, as shown in Figure 13.

Now consider two walls H_1, H_2 in Σ separating $[y_0, x_1]$ and $[y_0, y_1]$ such that H_1 is closer to $[y_0, x_1]$ than H_2. Both H_1 and H_2 exist since the triplet (x_1, x_2, x_3) is reduced and they are secant in y_0. Consider, on the other hand, two walls H_3, H_4 separating $[y_k, x_1]$ and $[y_k, y_{k-1}]$ such that H_3 is closer to $[x_3, x_1]$ than H_4. They are secant in y_k. It is clear that H_1, H_2, H_3 are pairwise secant and that H_4 is either equal to H_2 or secant to all the three walls.

When the building Δ is of type $\tilde{A}_2$, it is not possible to have more than three pairwise secant walls. This means that the walls H_2 and H_4 are equal and therefore $\rho([x_2, x_3]_\partial)$ is a segment and $\partial(\rho(x_2), \rho(x_3)) = \partial(x_2, x_3)$.

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When the building Δ is of type $\tilde{B}_2$, it is not possible to have more than four pairwise secant walls. If $k > 1$, let H_5 be the wall of Σ such that $y_1 \in |H_5|$, $[y_1, y_2]_\partial$ and $[y_0, y_1]_\partial$ lie on the same side of $|H_5|$ and the reflection of $[y_1, y_2]_\partial$ through $|H_5|$ is aligned with $[y_0, y_1]_\partial$. The wall H_5 exists since it is the support of a panel containing y_1 and it cannot be parallel to any of the four walls H_1, H_2, H_3, H_4. But this is not possible and this proves that $k = 1$ and therefore $\partial(\rho(x_2), \rho(x_3)) = \partial(x_2, x_3)$. □

2.3. The Haagerup inequality and the set of reduced triangles

In this subsection we will make the connection between the set of all triangles of the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ and the one of the reduced triangles and simplify the verification of the Haagerup inequality for any affine building to the set of reduced triangles. To do this, let us first give the following two lemmas.

Lemma 2.14. *Let x, y be two vertices of the affine building Δ. For any vertex $u \in B(x, y)$ such that $u \neq y$ we have $y \notin B(x, u)$.*

Proof. Fix an apartment Σ containing x, y and consider a wall H containing u but not y. This wall divides Σ in to two half-spaces (roots) Σ^+ containing y and Σ^-. The vertex x is not contained in Σ^+, if not, this implies that $B(x, y)$ is contained in Σ^+, whereas the vertex u is not contained in it. Since both the vertices x and u are in the closed half-space $H \cup \Sigma^+$, this closed half-space consequently contains $B(x, u)$ but not $B(x, y)$. □

Lemma 2.15. *Let (x_1, x_2, x_3) be a triplet in $\mathcal{V}_\Delta$. There exists a reduced triplet (x'_1, x'_2, x'_3) such that $x'_i \in B(x_i, x_{i+1}) \cap B(x_i, x_{i-1})$, for all $i \in \mathbb{Z}/3\mathbb{Z}$.*

We will say in this case that (x'_1, x'_2, x'_3) is an associated reduced triplet to the triplet (x_1, x_2, x_3).

Proof. To determine the triplet (x'_1, x'_2, x'_3) let us first apply the following process to the vertex x_1 of the triplet (x_1, x_2, x_3).

Set $x_1^1 = x_1$ and suppose that the vertex x_1^k, $k \geqslant 1$, has already been defined such that the intersection $B(x_1^k, x_2) \cap B(x_1^k, x_3)$ is contained in $B(x_1, x_2) \cap B(x_1, x_3)$. If this intersection is reduced to the vertex x_1^k, we can set $x'_1 = x_1^k$, and the process stops here. If not, set x_1^{k+1} to be any vertex of this intersection different from the vertex x_1^k. By Lemma 2.14, the intersection $B(x_1^{k+1}, x_2) \cap B(x_1^{k+1}, x_3)$ is strictly contained in $B(x_1^k, x_2) \cap B(x_1^k, x_3)$. Since $B(x_1, x_2) \cap B(x_1, x_3)$ is a finite set, this process may stop at the end and we obtain our vertex $x'_1 \in B(x_1, x_2) \cap B(x_1, x_3)$ such that $B(x'_1, x_2) \cap B(x'_1, x_3) = \{x'_1\}$.

We apply the same process to the vertex x_2 of the new triplet (x'_1, x_2, x_3) to obtain x'_2 and then to the vertex x_3 of the triplet (x'_1, x'_2, x_3) to obtain our reduced triplet (x'_1, x'_2, x'_3). □

Notice that the vertex x'_1 in the above proof is in general not unique, as shown in the example given by Figure 14.

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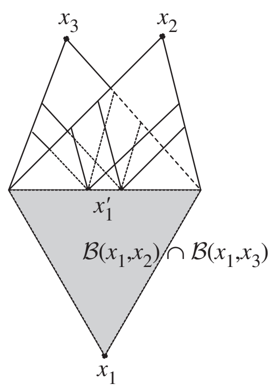

Figure 14. Associated reduced triplets in type $\tilde{A}_2$.

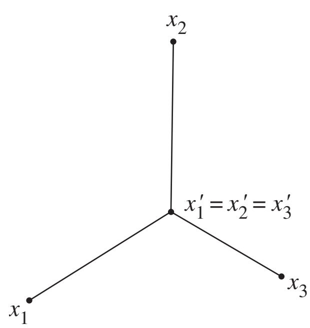

Figure 15. Case $\tilde{A}_1$.

When the building is of type $\tilde{A}_2$, if the vertex x'_1 is not unique, the associated reduced triplets to the triplet (x_1, x_2, x_3) are degenerates. This corresponds to the diagram $\mathcal{D}'(m, n)$ of Paragraph 3 of [**RRS98**].

When the building is of type $\tilde{A}_1$, the reduced triplet associated to any triplet (x_1, x_2, x_3) is given by the common vertex to the geodesics between the three vertices x_1, x_2, x_3, as shown in Figure 15.

Figure 16 gives a construction of reduced triplets (x'_1, x'_2, x'_3) associated to triplets (x_1, x_2, x_3) contained in the same apartment of an affine building of dimension 2.

Now we are ready to give the first important result of this section.

Theorem 2.16. *Let Δ be any affine building. The groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ satisfies the Haagerup inequality with respect to $L : [x, y] \mapsto d(x, y)$ if and only if the set $\mathcal{R}$ of reduced triangles of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ satisfies this inequality with respect to L.*

Proof. This theorem is a direct consequence of what has been established before. Indeed, in § 2.1 we have constructed a suitable domain $\mathcal{B}$ with respect to which, by Proposition 2.6, the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ is of polynomial $\mathcal{B}$-growth. By Lemma 2.15, the set $\mathcal{R}$ of reduced triangles of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ is a $\mathcal{B}$-pinching of the set of triangles of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$. By

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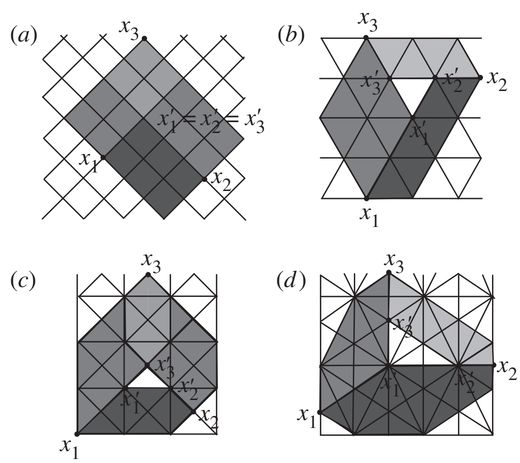

Figure 16. (*a*) $\tilde{A}_1 \times \tilde{A}_1$. (*b*) $\tilde{A}_2$. (*c*) $\tilde{B}_2$. (*d*) $\tilde{G}_2$.

Lemma 2.11, we deduce from part (2) of Proposition 1.33 that if the set $\mathcal{R}$ satisfies the Haagerup inequality with respect to L, the set of triangles of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ satisfies it. □

As a direct corollary of this theorem we have the following result, which generalizes the case of affine buildings of type $\tilde{A}_1 \times \tilde{A}_1$ obtained in [**RRS98**].

Theorem 2.17. *Let Δ be a building of type $\tilde{A}_1 \times \cdots \times \tilde{A}_1$ and let Γ be a discrete group acting freely on the* 1*-skeleton of Δ. The groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ satisfies the Haagerup inequality with respect to $L : [x, y] \mapsto d(x, y)$ and, therefore, the group Γ satisfies the Haagerup inequality.*

Proof. This is a direct result of Theorem 2.16, because the set of reduced triangles of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ for such a building coincides with the set of its degenerate triangles, by Remark 2.10. This set of degenerate triangles satisfies the Haagerup inequality by Proposition 1.28. By using Proposition 1.21, we deduce that the group Γ satisfies the Haagerup inequality. □

2.4. The Haagerup inequality and buildings of type $\tilde{A}_{k_1} \times \cdots \times \tilde{A}_{k_n}$ with $k_i \in \{1, 2\}$

Before giving the second important result of this section, let us first denote by $\mathcal{D}$, for an affine building of any type, the set of triangles of $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ admitting a degenerate $\mathcal{B}$-pinching. These are triangles $([x_1, x_2], [x_2, x_3], [x_3, x_1])$ such that $\mathcal{B}(x_1, x_2) \cap \mathcal{B}(x_2, x_3) \cap \mathcal{B}(x_3, x_1) \neq \emptyset$. We have the following result.

Lemma 2.18. *The set of triangles $\mathcal{D}$ satisfies the Haagerup inequality with respect to the length-function $L : [x, y] \mapsto d(x, y)$.*

Proof. Since the set of degenerate triangles satisfies the Haagerup inequality with respect to L by Proposition 1.28 and obviously satisfies condition (2) of Proposition 1.33,

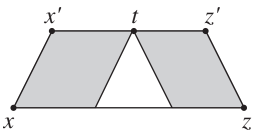

Figure 17. The trapezium $xx'z'z$.

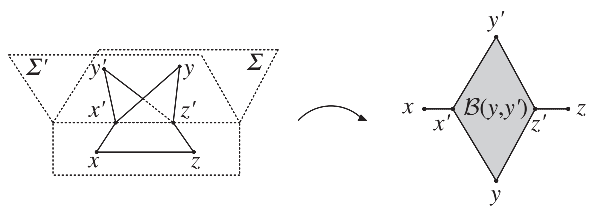

Figure 18. Case $\tilde{A}_2$.

and since it is a $\mathcal{B}$-pinching of the set $\mathcal{D}$ and the groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ is of polynomial $\mathcal{B}$-growth by Proposition 2.6, this lemma follows directly from Proposition 1.33. □

Lemma 2.19. *In an affine building Δ of type $\tilde{A}_{k_1} \times \cdots \times \tilde{A}_{k_n}$, with $k_i \in \{1, 2\}$, the set of triangles $\mathcal{D}$ is a tetrahedral change of faces of the set of triangles $\mathcal{R}$.*

Proof. Notice first that if two pairs of triangles $T_1'^i$, $T_2'^i$, for $i = 1, 2$, are, respectively, tetrahedral change of faces of two pairs of triangles T_1^i, T_2^i, for $i = 1, 2$, in buildings Δ_1, Δ_2, the pair of triangles $T_1'^1 * T_1'^2$, $T_2'^1 * T_2'^2$ is a tetrahedral change of faces of the pair $T_1^1 * T_1^2$, $T_2^1 * T_2^2$ in the building $\Delta_1 \times \Delta_2$. Notice, on the other hand, that if a triangle T_1 of an affine building Δ_1 belongs to the corresponding set $\mathcal{D}_1$ and if T_2 of an affine building Δ_2 belongs to the corresponding set $\mathcal{D}_2$, then the triangle $T_1 * T_2$ of the building $\Delta = \Delta_1 \times \Delta_2$ belongs to the corresponding set $\mathcal{D}$. It is then enough to prove this lemma in the case of an affine building of the irreducible type $\tilde{A}_2$.

Let $([x, y], [y, z], [z, x])$, $([x, y'], [y', z], [z, x])$ be two reduced triangles having a common base $[z, x]$. The corresponding triplets (x, y, z), (x, y', z) are, by Proposition 2.13, contained, respectively, in apartments Σ, Σ' and have the form of the equilateral triangles given in Figure 7. Let t be a vertex contained in the intersection $\Sigma \cap \Sigma'$. The intersection $\Sigma \cap \Sigma'$ must then contain the convex combinatorial closures $\mathcal{B}(x, t)$ and $\mathcal{B}(y, t)$. Thus, the intersection of the convex combinatorial closure of two sets $\{x, y, z\}$, $\{x, y', z\}$ which is contained in $\Sigma \cap \Sigma'$ has the form of a trapezium $xx'z'z$ given by Figure 17, where $\mathcal{B}(x, x') = \mathcal{B}(x, y) \cap \mathcal{B}(x, y')$ and $\mathcal{B}(z, z') = \mathcal{B}(z, y) \cap \mathcal{B}(z, y')$.

By applying the tetrahedral change of faces to the pair of triangles

$$([x, y], [y, z], [z, x]), \quad ([x, y'], [y', z], [z, x]),$$

we obtain the pair of triangles

$$([y, x], [x, y'], [y', y]), \quad ([y, z], [z, y'], [y', y]),$$

as shown in Figure 18.

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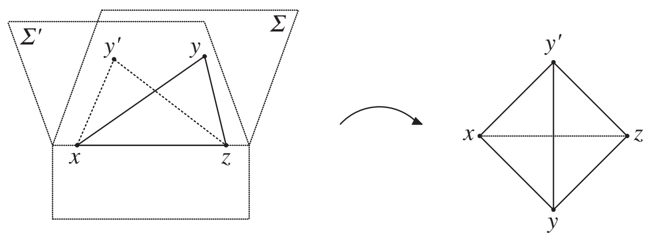

Figure 19. Case $\tilde{B}_2$.

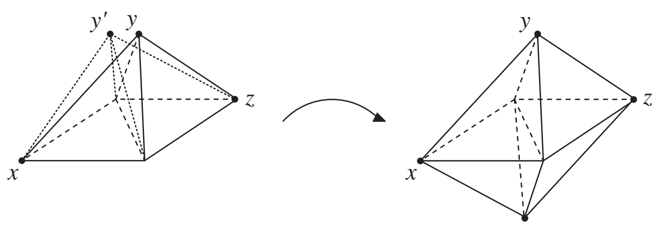

Figure 20. Case $\tilde{A}_3$.

But it is clear from Figure 18 that $\mathcal{B}(y,x)\cap\mathcal{B}(x,y')\cap\mathcal{B}(y',y)=\{x'\}\neq\emptyset$ and $\mathcal{B}(y,z)\cap\mathcal{B}(z,y')\cap\mathcal{B}(y',y)=\{z'\}\neq\emptyset$. This means that both triangles $([y,x],[x,y'],[y',y])$ and $([y,z],[z,y'],[y',y])$ belong to $\mathcal{D}$. This proves the lemma. □

If the building Δ is neither of type $\tilde{A}_1$ nor of type $\tilde{A}_2$, there exists no polynomial P for which the set of triangles $\mathcal{D}$ is a tetrahedral P-change of faces of the set of reduced triangles $\mathcal{R}$. Figures 19 and 20 explain this fact when the building is of type $\tilde{B}_2$ or $\tilde{A}_3$.

In Figure 19 the triplets (y,x,y'), (y,z,y') obtained by tetrahedral change of faces of two reduced triplets having a common base (x,y,z), $(x,y'z)$ are themselves reduced and not degenerate. This is because the walls containing $\{x,z\}$ are orthogonal to the walls containing, respectively, $\{y,y'\}$. This phenomenon appears whenever an even integer appears in the Coxeter matrix associated to the affine building and this is the case for all affine buildings excepting the two cases $\tilde{A}_1$ and $\tilde{A}_2$. Figure 20 shows the same problem when the building is of type $\tilde{A}_3$.

Now we are ready to give the second important result of this section, which generalizes the case $\tilde{A}_2$ in [**RRS98**].

Theorem 2.20. *Let Δ be an affine building of type $\tilde{A}_{k_1}\times\cdots\times\tilde{A}_{k_n}$, where $k_i\in\{1,2\}$, and let Γ be a discrete group acting freely on the 1-skeleton of Δ. The groupoid $\mathcal{G}(\mathcal{V}_\Delta,\Gamma)$ satisfies the Haagerup inequality with respect to $L:[x,y]\mapsto d(x,y)$ and, therefore, the group Γ satisfies the Haagerup inequality.*

Proof. By Lemma 2.18, the set $\mathcal{D}$ satisfies the Haagerup inequality with respect to L and by Lemma 2.19 it is a tetrahedral change of faces of the set $\mathcal{R}$. By Lemma 2.11 the set of triangles $\mathcal{R}$ is L-balanced. By using Proposition 1.35 we deduce that the set $\mathcal{R}$ satisfies the Haagerup inequality with respect to L. Now, by using Theorem 2.16, the

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groupoid $\mathcal{G}(\mathcal{V}_\Delta, \Gamma)$ satisfies the Haagerup inequality with respect to L and therefore, by Proposition 1.21, the group Γ satisfies the Haagerup inequality. $\square$

Acknowledgements. This paper was written while the author was a postdoc funded by the 'European Union Network in Quantum Spaces–Noncommutative Geometry' at the University of Münster, contract no. HPRN-CT-2002-00280. The author thanks Thierry Fack for proposing this subject to me and for his guidance and support during my PhD thesis at the University of Lyon 1. The author also thanks Siegfried Echterhoff for his friendly hospitality and Heath Emerson for reading this paper.

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