New dualities in convex quadrilaterals

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1. *Introduction*

In [1] de Villiers points out the duality between sides and angles of quadrilaterals. The first objective of this article is make explicit two new dualities in the quadrilaterals. For this we will consider the diagonal segments: if diagonals AC and BD of a quadrilateral $ABCD$ intersect at O, we call the diagonal segments OA, OB, OC, OD. According to [2, pp. 179-188], hierarchical classifications of the convex quadrilaterals are made taking as classification criteria the quantity and position of sides, angles and diagonal segments. In hierarchical classifications the more particular concepts form subsets of the more general concepts. Through classification according to the number and position of equal sides it is possible to define six families: four sides equal, at least three equal, two opposite pairs equal, two consecutive pairs equal, at least one opposite pair equal, at least one consecutive pair equal. In the families two opposite pairs equal and two consecutive pairs equal, the pairs may be equal to each other and then the four sides are equal. So in these two families the possibility $DA = AB$ and $AB = BC$ is excluded. Families analogous to the previous ones can be defined taking as a criterion of hierarchical classification the quantity and position of equal angles or the quantity and position of equal diagonal segments.

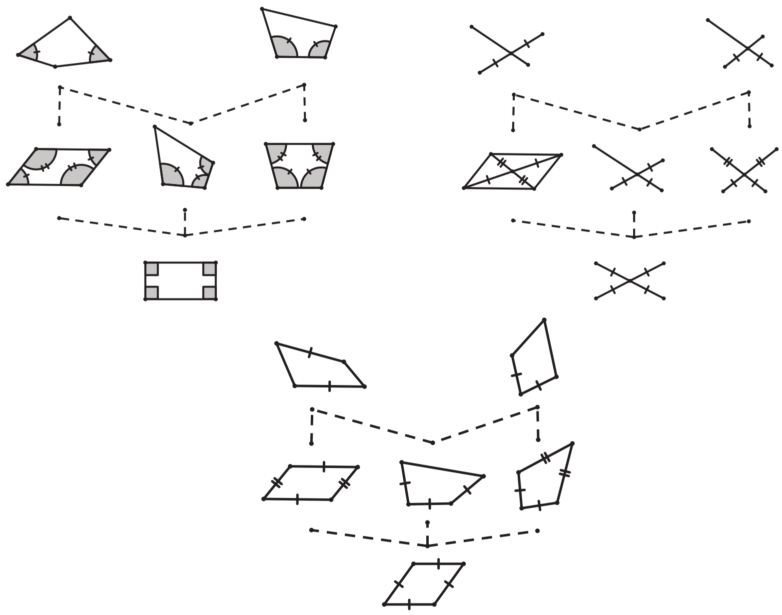

FIGURE 1

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In the following, hierarchical classifications of convex quadrilaterals are made combining the above criteria two by two. Thus three possibilities emerge

- number and position of equal angles;
- number and position of equal diagonal segments;
- number and position of equal sides.

The second objective of this Article is to define new families of quadrilaterals. To do this, we will make explicit the different families of convex quadrilaterals that can be defined by combining at most two of the above conditions. The third objective of the Article is to determine in how many minimum ways each family can be defined by combining at most two of the above conditions.

2. *Quantity and position of equal sides with quantity and position of equal angles*

How does the equality of sides and angles affect each other?

If a quadrilateral has two pairs of equal opposite sides, then it has two pairs of equal opposite angles. (B')

If a quadrilateral has two pairs of equal consecutive sides, then it has at least one pair of equal opposite angles. (C')

If a quadrilateral has two pairs of equal opposite angles, then it has two pairs of equal opposite sides. (E')

If a quadrilateral has two pairs of equal consecutive angles, then it has at least a pair of equal opposite sides. (F')

In Table 1, families 9, 11, 23, 26, 32, 34, 40, 41 support two variants and families 13 and 37 support three variants. With simple proofs it can be verified that the families of quadrilaterals defined by equality of sides and angles are those of Table 1. In Tables 1, 2 and 3 below, the following nine families of quadrilaterals will be distinguished: parallelogram; kite; isosceles trapezium (is tr); kite three equal angles (kite 3∠); kite three equal diagonal segments (kite 3d); isosceles trapezium three equal sides (is tr 3s); rhombus (rhom); rectangle (rect); square.

The duality between sides and angles of quadrilaterals pointed out in [1] by de Villiers can be seen between the properties B' and E', C' and F'. In their statements, if we change sides by angles and angles by sides, the properties are still valid. Also in the symmetry with respect to a diagonal of Table 1, in the families that it is possible to construct: square (self-dual), rectangle and rhombus, parallelogram (self-dual), isosceles trapezium and kite, isosceles trapezium three equal sides and kite three equal angles, 13 and 37, 14 and 44, 33 (self-dual), 34 and 40, 35 and 47, 41 (self-dual), 42 and 48. This duality can also be seen in families that admit more than one possibility, such as 9, 11 and 23, 13 and 37, 26 and 32, 34 and 40, 41. The

Number and position of equal ANGLES								
	Convex quad	43 rhom	44 3 s	45 parallel	46 kite	47 1 ops	48 1 cps	49 convex quad
	At least 1 consecutive pair	36 square	37 is tr 3s is tr 3s 3s-1cp∠	38 rect	39 kite 3∠	40 is tr 1ops-1cp∠	41 1 cps-1 cp∠ 2 cases	42 1 cp∠
	At least 1 opposite pair	29 rhomb	30 rhomb	31 parallel	32 rhomb kite	33 1 ops-1 op∠	34 kite 1 cps-1 op∠	35 1 op∠
	2 consecutive pairs	22 square	23 square is tr 3 s	24 rect	25 square	26 rect is tr	27 is tr 3 s	28 is tr
	2 opposite pairs	15 rhomb	16 rhomb	17 parallel	18 rhomb	19 parallel	20 rhomb	21 parallel
	At least 3	8 square	9 square	10 rect	11 square kite 3∠	12 rect	13 kite 3∠ kite 3∠ 1cps-3∠	14 3∠s
	4	1 square	2 square	3 rect	4 square	5 rect	6 square	7 rect
		4	at least 3	2 opposite pairs	2 consec pairs	at least 1 opposite pair	at least 1 consec pair	convex quad
		Number and position of equal SIDES						

TABLE 1: Sides-angles hierarchical classification

proofs that family 12 is rectangle and family 30 is rhombus are dual too. This derives from the duality between triangles with at least two equal sides and triangles with at least two equal angles.

Family 12: at least $DA = BC$ and at least $A = B = C$.

In Figure 2(a), if $A < 90°$ or $A > 90°$ and DA and BC intersect at E, as $A = B$, then $\angle BAE = \angle ABE$, from where $AE = BE$ and as $DA = CB$, it is true that $DE = CE$, so $D = C$. Therefore $ABCD$ has four equal angles. Then $ABCD$ is rectangle. If $A = 90°$, then $A = B = C = D = 90°$.

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Family 30: at least $DA = AB = BC$ and at least $A = C$.

In Figure 2(b), $AB = BC$, then $\angle CAB = \angle ACB$ and as $A = C$, so $\angle DAC = \angle DCA$, and then $DA = DC$. Therefore $ABCD$ has four equal sides. Then $ABCD$ is rhombus.

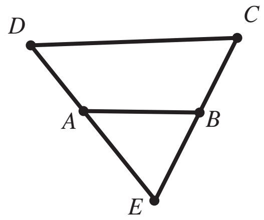

FIGURE 2(a)

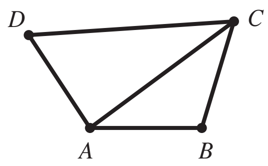

FIGURE 2(b)

Dual side-angle proofs of existence can also be done for families 9 (self-dual), two of the three possibilities of 13 and 37, one of the two possibilities of 11 and 23, 26 and 32, 34 and 40.

3. *Number and position of equal sides with number and position of equal diagonal segments*

How the equality of sides and diagonal segments affect each other?

If a quadrilateral has two pairs of equal opposite sides, then it has two pairs of equal opposite diagonal segments. (H')

If a quadrilateral has two pairs of equal consecutive sides, then it has at least a pair of equal opposite diagonal segments. (I')

If a quadrilateral has two pairs of equal opposite diagonal segments, then it has two pairs of opposite equal sides. (K')

If a quadrilateral has two pairs of equal consecutive diagonal segments, then it has at least a pair of equal opposite sides. (L')

In Table 2, families 59, 61, 73, 76, 82, 84, 90, 91 support two variants and families 63 and 87 support three variants. With simple proofs it can be verified that the families of quadrilaterals defined by equality of sides and diagonal segments are those of Table 2.

There is a duality between sides and diagonal segments of quadrilaterals. This duality can be seen between the properties H' and K', I' and L'. If we change sides by diagonal segments and diagonal segments by sides, the properties are still valid. Also in the symmetry with respect to a diagonal of Table 2, it is possible to recognise: square (self-dual), rectangle and rhombus, parallelogram (self-dual), isosceles trapezium and kite, isosceles trapezium 3 sides and kite 3 angles, 63 and 87, 64 and 94, 83 (self-dual), 84 and 90, 85 and 97, 91 (self-dual), 92 and 98. This duality can also be seen in families that admit more than one possibility, such as 59, 61 and 73, 63 and 87, 76 and 82, 84 and 90, 91. We have found no references for the duality between sides and diagonal segments in convex quadrilaterals.

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Number and position of equal DIAGONAL SEGMENTS	*Convex quad*	93 rhomb	94 3 s	95 parallel	96 kite	97 1 ops	98 1cps	99 convex quad
	At least 1 consecutive pair	86 square	87 is tr 3 s is tr 3 s 3 s-1 cpd	88 rect	89 kite 3d	90 is tr 1 ops- 1 cpd	91 1 cps- 1 cpd 2 cases	92 1 cpd
	At least 1 opposite pair	79 rhom	80 rhom	81 parallel	82 rhomb kite	83 1 ops- 1 opd	84 kite 1 cps- 1 opd	85 1 opd
	2 consecutive pairs	72 square	73 square is tr 3 s	74 rect	75 square	76 rectan is tr	77 is tr 3 s	78 is tr
	2 opposite pairs	65 rhomb	66 rhomb	67 parallel	68 rhomb	69 parallel	70 rhomb	71 parallel
	At least 3	58 square	59 square square	60 rect	61 square kite 3 d	62 rect	63 kite 3 d kite 3 d 1 cps-3 d	64 3 d
	4	51 square	52 square	53 rect	54 square	55 rect	56 square	57 rect
		4	At least 3	2 opposite pairs	2 consecutive pairs	at least 1 opposite pair	at least 1 consecutive pair	convex quad
		Number and position of equal SIDES						

TABLE 2: Sides-diagonal segments hierarchical classification

Referring to the analogy in a general sense, Polya says ‘Similar objects agree with each other in some respect, analogous objects *agree in certain relations* of their respective parts’ (see [3, p.37]). One of the three most important cases presented by Polya in which analogy attains the precision of mathematical ideas is the next: ‘There is a one-one correspondence between the objects of the two systems S and S', preserving certain relations. That is, if such a relation holds between the objects of one system, the same relation holds between the corresponding objects of the other system. Such a connection between two systems is a very precise sort of analogy; it is called

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isomorphism' (see [3, p. 48]). The side-angle duality and the side-diagonal segment duality are analogous in this sense. This analogy can be seen between the properties B' and H', C' and I', E' and K', F' and L'. If we change angles by diagonal segments and diagonal segments by angles, the properties are still valid. Families that exist and families with more than one possibility have similar locations in Table 1 and Table 2 (adding 50 to each family in Table 1 is obtain the analogous family in Table 2). We have found no references for this analogy.

4. *Number and position of equal diagonal segments with number and position of equal angles*

How do the equality of diagonal segments and angles affect each other?

If a quadrilateral has two pairs of equal opposite diagonal segments, then it has two pairs of equal opposite angles. (N')

If a quadrilateral has two pairs of equal consecutive diagonal segments, then it has two pairs of equal consecutive angles. (O')

If a quadrilateral has two pairs of equal opposite angles, then it has two pairs of equal opposite diagonal segments. (Q')

If a quadrilateral has two pairs of equal consecutive angles, then it has two pairs of equal consecutive diagonal segments. (R')

In Table 3, families 109, 112, 113, 125, 130, 133, 137, 141 support two variants. With simple proofs it can be verified that the families of quadrilaterals defined by equality of diagonal segments and angles are those of Table 3.

There is a duality between diagonal segments and angles. This duality can be seen between the properties N' and Q', O' and R'. If in statements are changed diagonal segments for angles and angles for diagonal segments, the properties are still valid. Also in the symmetry with respect to a diagonal of Table 3, it is possible to recognise: kite three equal angles and kite three equal diagonal segments, rectangle (self-dual), parallelogram (self-dual), 114 and 144, 134 and 140, 135 and 147, 144 and 148, 141 (self-dual). This duality can also be seen in families that admit more than one possibility, such as 112 and 130, 113 and 137. Families 109, 133, 141, which also admit two possibilities each one, are self-dual. We have found no references for the diagonal segments-angles duality.

6. *First conclusion*

The hierarchical classification of the convex quadrilaterals according to the criteria explained in Tables 1 to 3 allowed us to observe the sides-diagonal segments duality and the angles-diagonal segments duality. It also made it possible to establish the analogy between the sides-angles duality and the sides-diagonal segments duality, as can be seen in Figure 3.

Number and position of equal ANGLES	*Convex quad*	143 rect	144 3d	145 parallel	146 is tr	147 1 opd	148 1 cpd	149 convex quad
	At least 1 consecutive pair	136 rect	137 rect rect	138 rect	139 is tr	140 1opd-1cp∠	141 tr is 1cpd-1cp∠	142 1cp∠
	At least 1 opposite pair	129 rect	130 rect kite 3d	131 parallel	132 rect	133 parallel kite	134 1cpd-1op∠	135 1op∠
	2 consecutive pairs	122 rect	123 rect	124 rect	125 rect tr is	126 rect	127 is tr	128 is tr
	2 opposite pairs	115 rect	116 rect	117 parallel	118 rect	119 parallel	120 rect	121 parallel
	At least 3	108 rect	109 rect rect	110 rect	111 rect	112 rect kite 3∠	113 rect rect	114 3∠
	4	101 rect	102 rect	103 rect	104 rect	105 rect	106 rect	107 rect
		4	at least 3	2 opposite pairs	2 consecutive pairs	at least 1 opposite pair	at least 1 consecutive pair	convex quad
		Number and position of equal DIAGONAL SEGMENTS						

TABLE 3: Diagonal segments-angles hierarchical classification

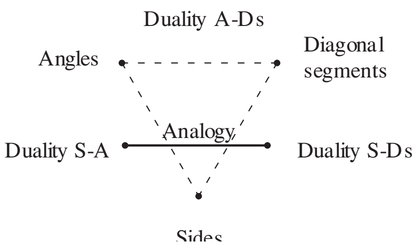

FIGURE 3

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7. *Second conclusion*

In Table 4 we distinguish the different families that are defined in a hierarchical way. The following nine families of quadrilaterals were designated by their names: parallelogram, kite, isosceles trapezium (is tr), kite three equal angles (kite 3∠), kite three equal diagonal segments (kite 3d), isosceles trapezium three equal sides (is tr 3s), rhombus, rectangle, square. In the right column we indicate the number of minimum definitions for each family. The nine named families are those that admit more than one minimal hierarchical definition.

Combining at most two of the three conditions considered can be defined in a hierarchical minimal way thirty-five families of convex quadrilaterals. In [4] they are defined with the names side quad, angle quad, bisecting quad, skew kite, skew isosceles quad, bi-diagonal quad (see respectively families 22, 2, 4, 27, 26, 3 of [4]) the same first six families from Table 4 and in the same way as here. In [5] the family at least one pair of opposite equal angles (1op∠) is named tilted kite and five characterizations of the family are given. In [6] the family at least one pair of opposite equal diagonal segments (1opd) is named bisect-diagonal quadrilateral and four characterizations of the family are given. In [7] the same family is named scalene kite. In [4], [8] and [9] the families kite three equal angles (kite 3∠) and isosceles trapezium three equal sides (is tr 3s) are named triangular kite and trilateral trapezium respectively. In this work, as in [2], we did not start from any set of quadrilaterals but from a classification criterion and the classification was made with a central intention to create families. And new families were obtained: the three families at least three equal (3s, 3∠, 3d), the thirteen families at least one pair and at least one pair, the four families at least one pair and at least three, kite with at least three equal diagonal segments (kite 3d). Analogous from this last family are triangular kite and trilateral trapezium that appear in [4].

8. *Third conclusion*

This work allows us to know sixty-five minimum hierarchical definitions for the nine quadrilaterals previously distinguished by their names. Most of these minimal definitions are new or infrequent. The large number of minimal definitions for each of these families is surprising: 4 for kite, 4 for isosceles trapezium, 4 for kite three equal angles, 4 for kite three equal diagonal segment, 6 for isosceles trapezium three equal sides, 7 for rhombus, 18 for rectangle, 14 for square. The only 4 definitions for the parallelogram is surprising too and one of them is peculiar: at least one pair of equal opposite angles and at least one pair of equal opposite diagonal segments (133 (1)). The only reference found for this last definition was [10, p. 36, Figure 5].

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	Table 1	Table 2	Table 3	
	Sides-angles	Sides-diag. segs	Diag. segs-angles	
1 ops	47	97		1
1 op∠	35		135	1
1 opd		85	147	1
1 cps	48	98		1
1 cp∠	42		142	1
1 cpd		92	148	1
3 s	44	94		1
3∠	14		114	1
3d		64	144	1
1 ops-1 cp∠	40(2)			1
1 ops-1 op∠	33			1
1 cps-1 op∠	34(2)			1
1 cps-1 cp∠(1)	41(1)			1
1 cps-1 cp∠(2)	41(2)			1
1 ops-1 cpd		90		1
1 ops-1 opd		83		1
1 cps-1 opd		84		1
1 cps-1 cpd(1)		91(1)		1
1 cps-1 cpd(2)		91(2)		1
1 opd-1 cp∠	140			1
1 cpd-1 op∠	134			1
1 cpd-1c p∠	141(2)			1
3 s-1 cp∠	37(3)			1
1 cps-3∠	13(3)			1
3 s-1 cpd		87		1
1 cps-3 d		63		1
parallelogram	21, 45	71, 95	121, 133(1), 145	4
kite	34(1), 46	84(1), 96	133(2)	4
is tr	28, 40(1)	78, 90(1)	128, 141(1), 146	4
kite 3∠	13(1), 13(2), 39		112(2)	4
kite 3 d		63(1), 63(2), 89	130(2)	4
is tr 3 s	27, 37(1), 37(2)	77, 87(1), 87(2)		6
rhombus	20, 30, 32(1), 43	70, 80, 82(1), 93		7
rectangle	7, 12, 26(1), 38	57, 62, 76(1), 88	107, 109(1), 109(2), 112(1), 113(1), 113(2), 120, 125(1), 126, 130(1), 132, 137(1), 137(2), 138, 143	8
square	6, 9(1), 9(2), 11(1), 23(1), 25, 36	56, 59(1), 59(2), 61(1), 73(1), 75, 86		14

TABLE 4: The families with their respective minimal hierarchical definitions

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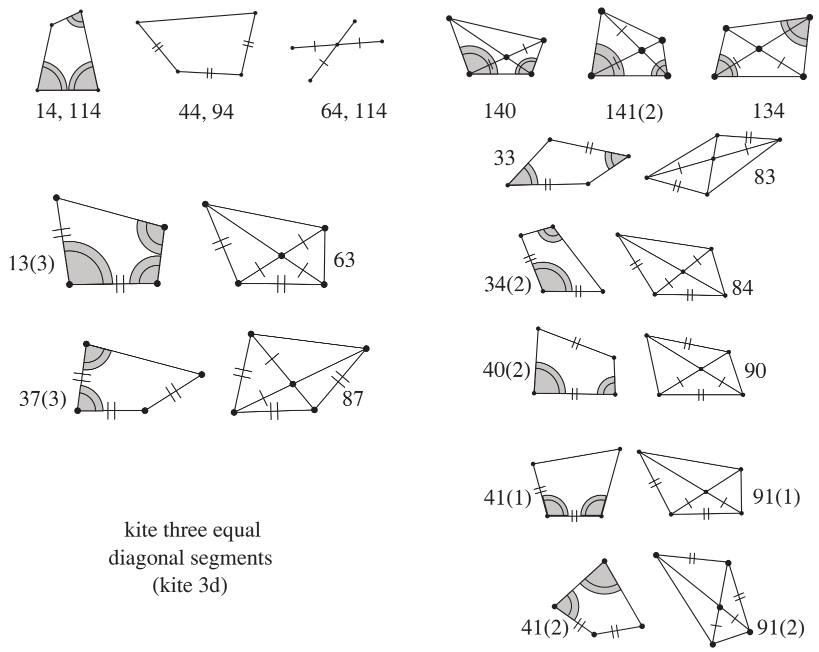

FIGURE 4: Twenty-one new families of convex quadrilaterals

9. *Final comments*

As in other classifications, well-known quadrilateral families remained outside of it. Of the seventeen convex quadrilaterals that appear in de Villiers' classification made in [9], only nine appear in this one. Of the eighteen most important quadrilaterals considered by M. Josefsson in the excellent [11], only six appear in this. Just to mention one, the trapezium defined as quadrilateral with at least one pair of parallel sides not appear here sides, since parallelism was not a criterion used. But one that does appear is the isosceles trapezium defined hierarchically as a quadrilateral with two pairs of equal consecutive angles or as a quadrilateral with two pairs of equal consecutive diagonal segments, which include the rectangle as a particular case when the four angles are equal or the four segments of diagonal are equal. Both definitions make clear the angles-diagonal segments duality and the duality of both with the definition of kite as a quadrilateral with two pairs of equal consecutive sides, which includes the rhombus as a particular case when all four sides are equal. In a future article we will make a diagram that includes the thirty-five families of convex quadrilaterals that were defined in this one. The diagram will also include the ninety-one minimum definitions generated for these quadrilateral families. It will make it easier to appreciate the dualities and links between these quadrilateral families.

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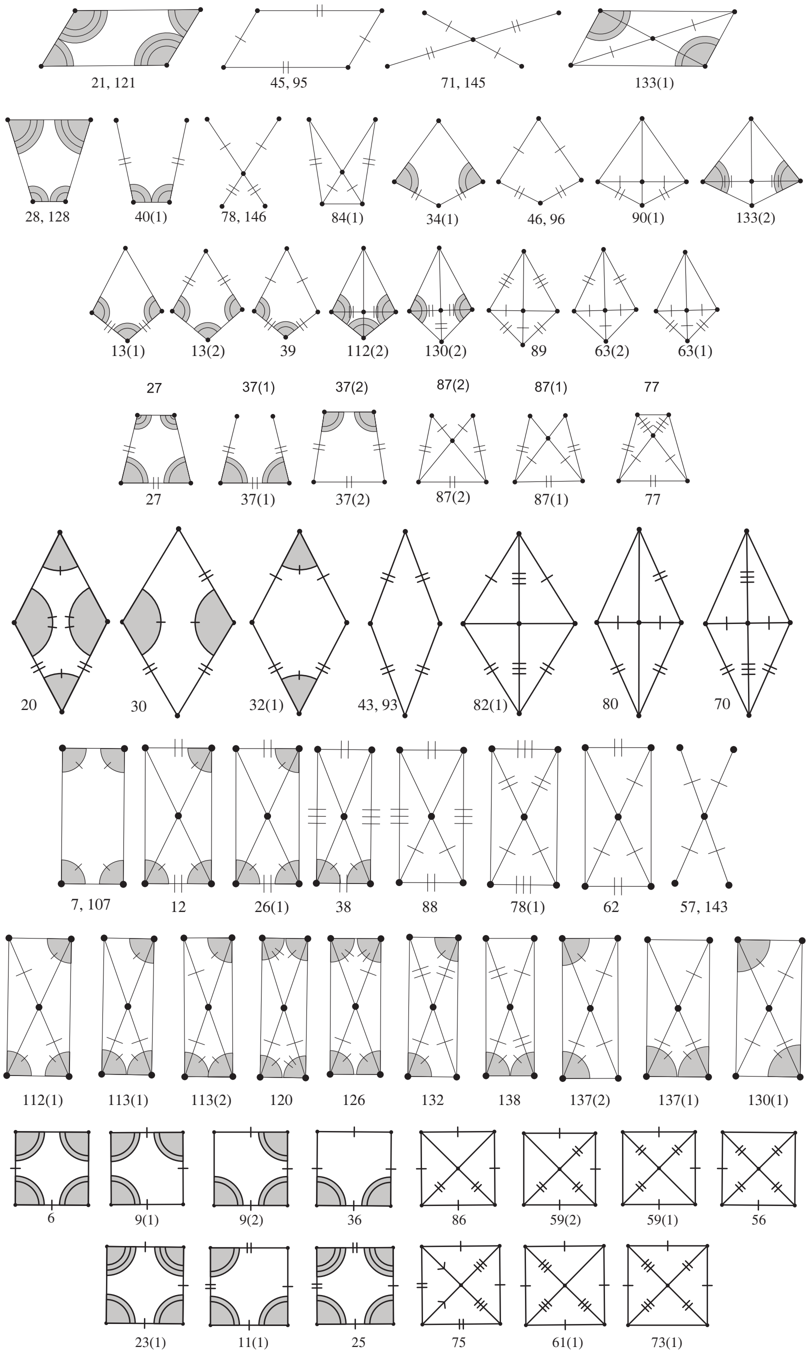

FIGURE 5: Sixty-five minimal hierarchical definitions of nine families

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