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τ-TILTING FINITE CLUSTER-TILTED ALGEBRAS

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Abstract We prove if B is a cluster-tilted algebra, then B is τ_B-tilting finite if and only if B is representation-finite.

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1. Introduction

The theory of τ-tilting was introduced by Adachi et al. [**1**] as a far-reaching generalization of classical tilting theory for finite-dimensional associative algebras. One of the main classes of objects in the theory is that of τ-rigid modules: a module M over an algebra A is τ_A-*rigid* if $\mathrm{Hom}_A(M, \tau_A M) = 0$, where $\tau_A M$ denotes the Auslander–Reiten translation of M; such a module M is called τ_A-*tilting* if the number $|M|$ of non-isomorphic indecomposable summands of M equals the number of isomorphism classes of simple A-modules. Recently, a new class of algebras were introduced by Demonet et al. [**10**] called τ_A-*tilting finite* algebras. They are defined as finite-dimensional algebras with only a finite number of isomorphism classes of basic τ_A-tilting modules.

An obvious sufficient condition for an algebra to be τ_A-titling finite is for it to be representation-finite. In general, this condition is not necessary. The aim of this note is to prove for *cluster-tilted algebras*, this condition is in fact necessary.

Tilted algebras are the endomorphism algebras of tilting modules over hereditary algebras, introduced by Happel and Ringel [**11**]. Cluster-tilted algebras are the endomorphism algebras of cluster-tilting objects over cluster categories of hereditary algebras, introduced by Buan et al. [**8**]. The similarity in the two definitions lead to the following precise relation between tilted and cluster-tilted algebras, which was established by Assem et al. [**3**].

There is a surjective map

$$\{\text{tilted algebras}\} \longmapsto \{\text{cluster-tilted algebras}\}$$
$$C \longmapsto B = C \ltimes E$$

where E denotes the C-C-bimodule $E = \mathrm{Ext}^2_C(\mathrm{DC}, C)$ and $C \ltimes E$ is the trivial extension.

This result allows one to define cluster-tilted algebras without using the cluster category. Using this construction, we show the following.

Theorem 1.1. *Let B be a cluster-tilted algebra. Then B is τ_B-tilting finite if and only if B is representation-finite.*

2. Notation and preliminaries

We now set the notation for the remainder of this paper. All algebras are assumed to be finite dimensional over an algebraically closed field k. If A is a k-algebra then denote by $\mathrm{mod}\, A$ the category of finitely generated right A-modules and by $\mathrm{ind}\, A$ a set of representatives of each isomorphism class of indecomposable right A-modules. We denote by $\mathrm{add}\, M$ the smallest additive full subcategory of $\mathrm{mod}\, A$ containing M, that is, the full subcategory of $\mathrm{mod}\, A$ whose objects are the direct sums of direct summands of the module M. Given $M \in \mathrm{mod}\, A$, the projective dimension of M is denoted $\mathrm{pd}_A M$. We let τ_A and τ_A^{-1} be the Auslander–Reiten translations in $\mathrm{mod}\, A$. We let D be the standard duality functor $\mathrm{Hom}_k(-, k)$. Finally, $\Gamma(\mathrm{mod}\, A)$ will denote the Auslander–Reiten quiver of A.

2.1. Tilted algebras

Tilting theory is one of the main themes in the study of the representation theory of algebras. Given a k-algebra A, one can construct a new algebra B in such a way that the corresponding module categories are closely related. The main idea is that of a tilting module.

Definition 2.1. Let A be an algebra. An A-module T is a *partial tilting module* if the following two conditions are satisfied:

(1) $\mathrm{pd}_A T \leq 1$.

(2) $\mathrm{Ext}^1_A(T, T) = 0$.

A partial tilting module T is called a *tilting module* if it also satisfies the following additional condition:

(3) There exists a short exact sequence $0 \to A \to T' \to T'' \to 0$ in $\mathrm{mod}\, A$ with T' and $T'' \in \mathrm{add}\, T$.

We now state the definition of a tilted algebra.

Definition 2.2. Let A be a hereditary algebra with T a tilting A-module. Then the algebra $B = \mathrm{End}_A T$ is called a *tilted algebra*.

2.2. Cluster categories and cluster-tilted algebras

Let $C = kQ$ be the path algebra of the quiver Q and let $\mathcal{D}^b(\mathrm{mod}\, C)$ denote the derived category of bounded complexes of C-modules. The *cluster category* $\mathcal{C}_C$ is defined as the

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orbit category of the derived category with respect to the functor $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}$ is the Auslander–Reiten translation in the derived category and [1] is the shift. Cluster categories were introduced in [7], and in [9] for type $\mathbb{A}$.

An object T in $\mathcal{C}_C$ is called *cluster-tilting* if $\mathrm{Ext}^1_{\mathcal{C}_C}(T,T)=0$ and T has $|Q_0|$ non-isomorphic indecomposable direct summands where $|Q_0|$ is the number of vertices of Q. The endomorphism algebra $\mathrm{End}_{\mathcal{C}_C} T$ of a cluster-tilting object is called a *cluster-tilted algebra* [8].

2.3. Relation extensions

Let C be an algebra of global dimension at most 2 and let E be the C-C-bimodule $E=\mathrm{Ext}^2_C(\mathrm{D}C,C)$.

Definition 2.3. The *relation extension* of C is the trivial extension $B=C\ltimes E$, whose underlying C-module structure is $C\oplus E$, and multiplication is given by $(c,e)(c',e')=(cc',ce'+ec')$.

Relation extensions were introduced in [3]. In the special case where C is a tilted algebra, we have the following result.

Theorem 2.4 (Cf. [3, Theorem 3.4]). *Let C be a tilted algebra. Then $B=C\ltimes \mathrm{Ext}^2_C(\mathrm{D}C,C)$ is a cluster-tilted algebra. Moreover, all cluster-tilted algebras are of this form.*

2.4. Slices and local slices

Definition 2.5. Let B be an algebra. A *slice* Σ in $\Gamma(\mathrm{mod}\, B)$ is a set of indecomposable B-modules such that

(1) Σ is sincere.

(2) Any path in $\mathrm{mod}\, B$ with source and target in Σ consists entirely of modules in Σ.

(3) If M is an indecomposable non-projective B-module then at most one of M , $\tau_B M$ belongs to B.

(4) If $M\to S$ is an irreducible morphism with $M,S\in \mathrm{ind}\, B$ and $S\in\Sigma$, then either M belongs to Σ or M is non-injective and $\tau_B^{-1}M$ belongs to Σ.

The existence of slices is used to characterize tilted algebras in the following way.

Theorem 2.6 (see [12]). *Let A be a hereditary algebra, T a tilting A-module, and $B=\mathrm{End}_A T$ a tilted algebra. Then the class of B-modules $\mathrm{Hom}_A(T,DA)$ forms a slice in $\mathrm{mod}\, B$. Conversely, any slice in any module category is obtained in this way.*

The following notion of local slices was introduced in [2] in the context of cluster-tilted algebras. Let A be an algebra. We say a path $X=X_0\to X_1\to\cdots\to X_s=Y$ in $\Gamma(\mathrm{mod}\, A)$ is *sectional* if, for each i with $0<i<s$, we have $\tau_A X_{i+1}\neq X_{i-1}$.

Definition 2.7. Let A be an algebra. A *local slice* Σ in $\Gamma(\operatorname{mod} A)$ is a set of indecomposable A-modules inducing a connected full subquiver of $\Gamma(\operatorname{mod} A)$ such that

(1) If $X \in \Sigma$ and $X \to Y$ is an arrow in $\Gamma(\operatorname{mod} A)$, then either Y or $\tau_A Y \in \Sigma$.

(2) If $Y \in \Sigma$ and $X \to Y$ is an arrow in $\Gamma(\operatorname{mod} A)$, then either X or $\tau_A^{-1} X \in \Sigma$.

(3) For every sectional path $X = X_0 \to X_1 \to X_2 \to \cdots \to X_s = Y$ in $\Gamma(\operatorname{mod} A)$ with $X, Y \in \Sigma$, we have $X_i \in \Sigma$, for $i = 0, 1, \ldots, s$.

(4) The number of indecomposable A-modules in Σ equals the number of non-isomorphic summands of T, where T is a tilting A-module.

There is a relationship between tilted and cluster-tilted algebras given in terms of slices and local slices.

Theorem 2.8 (Cf. [2, Corollary 20]). *Let C be a tilted algebra and $B = C \ltimes \operatorname{Ext}^2_C(\mathrm{D}C, C)$ its relation extension. Then any slice in $\operatorname{mod} C$ embeds as a local slice in $\operatorname{mod} B$ and any local slice Σ in $\operatorname{mod} B$ arises in this way.*

The existence of local slices in a cluster-tilted algebra gives rise to the following definition. The unique connected component of $\Gamma(\operatorname{mod} B)$ that contains local slices is called the *transjective component*.

The next result says a slice in a tilted algebra together with its τ and τ^{-1} translates full embeds in the cluster-tilted algebra.

Proposition 2.9 (Cf. [4, Proposition 3]). *Let C be a tilted algebra, Σ a slice, $M \in \Sigma$, and $B = C \ltimes \operatorname{Ext}^2_C(\mathrm{D}C, C)$ its relation extension.*

(1) $\tau_C M \cong \tau_B M$.

(2) $\tau_C^{-1} M \cong \tau_B^{-1} M$.

In [2], the authors gave an example of an indecomposable transjective module over a cluster-tilted algebra that does not lie on a local slice. It was proved in [5] the number of such modules is finite.

Proposition 2.10 (Cf. [5, Corollary 3.8]). *Let B be a cluster-tilted algebra. Then the number of isomorphism classes of indecomposable transjective B-modules that do not lie on a local slice is finite.*

2.5. τ-tilting finite algebras

Following [1] we state the following definition. Let A be an algebra.

Definition 2.11. An A-module M is τ_A-*rigid* if $\operatorname{Hom}_A(M, \tau_C M) = 0$. A τ_A-rigid module M is τ_A-*tilting* if the number of pairwise, non-isomorphic, indecomposable summands of M equals the number of isomorphism classes of simple A-modules.

It follows from the Auslander–Reiten formulas that any τ_A-rigid module M is rigid, that is, $\operatorname{Ext}^1_A(M, M) = 0$ and the converse holds if the projective dimension is at most 1. In particular, any partial tilting module is a τ_A-rigid module, and any tilting module is a τ_A-tilting module. Thus, we can regard τ_A-tilting theory as a generalization of classic tilting theory. Following [10], we have the following definition.

Definition 2.12. Let A be an algebra. We say that A is *τ_A-tilting finite* if there are only finitely many isomorphism classes of basic τ_A-tilting A-modules.

The authors of [10] provide several equivalent conditions for an algebra A to be τ_A-tilting finite. In particular, we need the following.

Lemma 2.13 (Cf. [10, Corollary 2.9]). *An algebra A is τ_A-tilting finite if and only if there are only finitely many isomorphism classes of indecomposable τ_A-rigid A-modules.*

2.6. A criterion for representation-finiteness

We will need the following criterion for-an algebra to be representation-finite.

Theorem 2.14 (Cf. [6, IV Theorem 5.4]). *Assume A is a basic and connected finite dimensional algebra. If $\Gamma(\operatorname{mod} A)$ admits a finite connected component $\mathcal{C}$, then $\mathcal{C} = \Gamma(\operatorname{mod} A)$. In particular, A is representation-finite.*

3. Main result

We are now ready to prove our main theorem.

Theorem 3.1. *Let B be a cluster-tilted algebra. Then B is τ_B-tilting finite if and only if B is representation-finite.*

Proof. The sufficiency is obvious so we prove the necessity. Assume B is τ_B-tilting finite but representation-infinite. By Theorems 2.6 and 2.8, we know the transjective component of $\Gamma(\operatorname{mod} B)$ exists. Since B is representation-infinite, Theorem 2.14 guarantees the transjective component must be infinite. By Proposition 2.10 and the fact that the transjective component is infinite, we must have an infinite number of indecomposable transjective B-modules which lie on a local slice. Let M be such a B-module. Theorem 2.8 guarantees there exists a tilted algebra C and a slice Σ such that M is a C-module and $M \in \Sigma$. It follows from parts (2) and (3) of the definition of a slice that M is τ_C-rigid. By Proposition 2.9, we know $\tau_C M \cong \tau_B M$. This implies M is τ_B-rigid. Since M was arbitrary, we have shown there exists an infinite number of indecomposable transjective B-modules which are τ_B-rigid. This is a contradiction to our assumption that B was τ_B-tilting finite and Lemma 2.13. We conclude B must be representation-finite. □

Remark 3.2. By Theorem 2.4, every cluster-tilted algebra B is the relation extension of some tilted algebra C. Thus, it is natural to ask whether a τ_C-tilting finite tilted algebra C is also representation-finite. We recall a connected component $\mathcal{P}$ of $\Gamma(\operatorname{mod} C)$ is called a *preprojective component* if $\mathcal{P}$ does not contain an oriented cycle and each indecomposable module $X \in \mathcal{P}$ is of the form $\tau_C^{-r} P$ for some $r \in \mathbb{N}$ and an indecomposable projective C-module P. By [13], tilted algebras have a preprojective component $\mathcal{P}$. Since $\mathcal{P}$ is acyclic, we have $\operatorname{Hom}_C(X, \tau_C X) = 0$ for every indecomposable $X \in \mathcal{P}$. Thus, if C is τ_C-tilting finite, it must be representation-finite.

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