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INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR POLICY ANALYSIS

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For two independent principles of intergenerational equity, the implied discount rate equals the growth rate of real per capita income, say, 2%, thus falling right into the range suggested by the U.S. Office of Management and Budget. To prove this, we develop a simple tool to evaluate small policy changes affecting several generations, by reducing the dynamic problem to a static one. A necessary condition is time invariance, which is satisfied by any common solution concept in an overlapping-generations model with exogenous growth. This tool is applied to derive the discount rate for cost–benefit analysis under two different utilitarian welfare functions: classical and relative. It is only with relative utilitarianism, and assuming time-invariance of the set of alternatives (policies), that the discount rate is well defined for a heterogeneous society at a balanced growth equilibrium, is corroborated by an independent principle equating values of human lives, and equals the growth rate of real per-capita income.

Keywords: Overlapping Generations, Policy Reform, Intergenerational Equity, Cost–Benefit Analysis, Discount Rate, Utilitarianism

1. INTRODUCTION

Public decisions often involve trade-offs where economic costs and benefits are spread over time. The choice of discount rate to map policy effects into net present value is then crucial. Arguably, at least for long-term projects, the choice should be governed by principles of intergenerational equity and yet, there is no robust method for deriving the social discount rate from such principles.

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Traditional cost–benefit analysis (CBA) requires using the interest rate for present-value calculations.[1] Although easy to use, this approach is not compatible with a normative one, as required to study intergenerational equity, because typically the only corresponding (utilitarian) social welfare function (SWF) implies that the current equilibrium is optimal. Drèze and Stern (1987) stress the importance of formulating a SWF as a basis for policy analysis, and Mertens and Rubinchik (2010) analyze the implications of consistency between discounting and a SWF—among others, some form of stationarity in the model.

Recent literature [e.g., Stern (2006) and Arrow (2007), following earlier work of Arrow and Kurz (1970)] obtains the discount rate for final consumption from a classical welfarist function. This method requires evaluating the final impact of a policy on individual consumption, and thus a full equilibrium computation. This is why the usual derivations of the discount rate in this framework are for a discount rate on final consumption, and for a single type of agent (or even a single infinitely lived agent). On the other hand, this method does offer a way to relate intergenerational equity requirements (through the SWF) to the discount rate (cf. Section 1.2).

The solution here shares this last advantage with the simplicity of use of the first method; i.e., no equilibrium computation is needed: one can directly discount policy variations, e.g., transfers (endowment perturbations). Further, heterogeneity of agents, even in their attitudes toward risk, is handled correctly.

To keep it simple, this paper presents only the main ideas, relegating all the real difficulties to an assumption of differentiability of the SWF as a function of policy. This differentiability implies in particular both regularity [*a fortiori*, a form of determinacy, contrary to the widespread preconception that overlapping-generations (OG) models are indeterminate] and stability of all balanced growth equilibria. It is shown to hold generically, in the classical particular case of our model (one good, one type of agent, etc.), in Mertens and Rubinchik (2009)—thus providing at least a proof of nonvacuity for the results here. That method also yields much richer and deeper results, such as the full expression of all derivatives—in particular, of all possible impulse responses—and not only the implied discount rate; and this even (but much less explicitly) with nonstationary policies as baseline. But for the moment it is still much harder, by orders of magnitude.

1.1. Intergenerational Equity

Even the U.S. Office of Management and Budget (OMB) (see Note 1) refers explicitly to the requirement of equity vis-à-vis future generations, and acknowledges it by suggesting, for projects with substantial long-term impact, a further analysis at a rate "between 1 and 3%" (p. 36), with no further precision.[2]

The issue of discounting utility and, more broadly, intergenerational justice has been controversial in the literature[3] since, probably, Sidgwick (1874, p. 414).[4] Ramsey (1928) (p. 543) presents discounting future utility ("enjoyments") as a "practice which is ethically indefensible and arises merely from the weakness of the imagination." He suggests a way to overcome technical difficulties of constructing a discount-free utilitarian social welfare criterion using the differences between actual and "bliss" utility levels. Utility discounting is not required per se in our case either, as we evaluate *temporary policy changes*, and thus aggregate utility *differences* from a baseline.[5]

1.2. Welfarist Deduction of the Discount Rate

One could follow Arrow and Kurz (1970), as, e.g., in Stern (2006) and in Dasgupta (2008).

For simplicity, take a discrete-time model where individuals live for just one period, with utility function $U(c) = \frac{c^{1-\rho}}{1-\rho}$, where $\rho > 0$. The economy is on a balanced growth path with per capita consumption growing exponentially at rate $\gamma > 0$, production being black-boxed for now. The baseline per capita consumption at time t is $c_0 e^{\gamma t}$, with $c_0 > 0$. Consider a policy that involves a variation in aggregate consumption δC_t for each generation t. It is to be evaluated at time 0, using the classical criterion, $W = \sum_t \sum_{n \in N_t} e^{-\beta t} U(c_t^n)$, where N_t is the set of individuals at time t. Then the net (social) benefit equals

$$\delta W = \sum_t \sum_{n \in N_t} e^{-\beta t} U'(c_0 e^{\gamma t}) \delta c_t^n = \sum_t c_0^{-\rho} e^{-(\rho\gamma+\beta)t} \delta C_t.$$

In this case, discounting is consistent with a welfare evaluation, and the resulting social discount rate is $\rho\gamma + \beta$, whereas there is no interest rate, because agents live for one period.

There are two major conceptual difficulties with this approach.

First, even if one is to rely on this simplistic model, it provides no guidance for obtaining an intergenerationally fair discount rate.

In a welfarist interpretation, only the indifference map is retained as an individual characteristic, so the choice of utility representation is thought of as a parameter of the welfare function (here, ρ). Hence this theory merely substitutes for a unknown number, the discount rate, an even less known function U and parameter β.

This is illustrated, e.g., by the recent controversies about Stern's report [Arrow (2007); Stern (2007)] and is typically settled by arbitrarily fixing one of those parameters (ρ or β) to get a "reasonable" discount rate.

In contrast, in a utilitarian interpretation, one views ρ as the individual coefficient of relative risk aversion with respect to lifetime income, i.e. (or alternatively), as the income elasticity of the marginal utility of income. Even when interpreting, then, intergenerational equity as meaning $\beta = 0$, reasonable estimates of $\beta + \rho\gamma$ still vary widely (cf. Section 6), and there is evidence of substantial variation of ρ within the population.

We postpone the discussion of the estimates for this social discount rate to Section 6, once it is established.

More importantly, this example is misleadingly simple: in particular, because individuals live only one period, they have no incentive to save, so there can be no capital accumulation and growth. When there is growth and savings, there is also an interest rate, which individuals would use to smooth the shock over their lifetime, each according to their own time preferences, so one might expect the result to be driven back to the interest rate. Thus, to establish such a result, we need at least a growth model (to see the parameter γ coming out), with generations, to be able to talk of intergenerational equity, and those should overlap lest there be no capital accumulation and hence no growth—shortly, an OG model. Finally, nonzero population growth must be used, so γ is unambiguously distinguished from the (e.g., golden rule) interest rate. The reason for using exogenous growth in this model will be explained after Assumption 3 in Section 4.2.

In addition, a major difficulty with the above approach is that for a policy to be evaluated, it has to be translated into changes in personal consumption, which are then discounted; further, just computing the change in aggregate consumption (as in the above example) is not sufficient as soon as individuals differ, whether in their preferences or in their endowment. Hence the method requires a *full equilibrium computation*, taking into account all aftereffects of the policy shock as well as its anticipatory effects. For instance, even for a lump-sum transfer policy, it would be wrong to aggregate changes in individual utility as if individuals consumed their transfer, because the recipient might well experience a welfare loss in competitive equilibrium (transfer paradox).

1.3. The Solution

We start with the simplest general model, which has only two elements: a policy (as a function of time) and an objective function defined over policies. In this model, we ask what property of the objective function ensures that its derivative has a net present value form, i.e., the sum of discounted policy changes at each point in time. The desired property is time invariance (a function over policies is an invariant welfare function (an IWF) when a time shift of policies multiplies welfare by a constant and adds a constant); this fact is established in Theorem 1, which also shows how to calculate the discount rate.

To apply this main result, we consider a growth model (OG), as required (cf. supra), in a general equilibrium fashion, adding the minimal assumptions needed for the existence of balanced growth equilibria, and prove that the composite map—from policies to individual allocations, then to individual utilities, themselves aggregated into welfare—is an IWF. This result stems explicitly from the properties of the individual maps involved in the composition, in particular, that the first map is an *outcome function* (Definition 10), for example, the selection of a locally unique equilibrium in the neighborhood of a balanced growth equilibrium (BGE).[6]

The composite map is an IWF, so, *without any equilibrium computation*, the change in welfare resulting from policy perturbations has a discounted sum form, with an explicit social discount rate.

In sum, our main result rationalizes the commonly practiced net present value calculation for a broad set of economies and policies.

In particular, we allow *heterogeneous preferences* (e.g., ρ), only requiring homogeneity of individual lifetime utility with respect to consumption, which is needed for balanced growth. Time separability is not necessary. *Durable* and *storable* goods are included (Section 3.1.2), so agents can smooth shocks over time, when those are represented as endowment perturbations.

Finally, to describe naturally the anticipation of policy variations, time starts at $-\infty$ rather than 0.[7] Continuous time is at first sight only a matter of convenience, or of more transparency, and a general precaution to avoid pathologies associated with discrete time; however [cf. also Demichelis and Polemarchakis (2007)], it may well prove crucial to avoid indeterminacy,[6] which would have made our result vacuous. The proof of nonvacuity itself is in Mertens and Rubinchik (2009).

1.4. Results for the Social Discount Rate

The two welfare functions we consider yield different social discount rates as applied to monetized policies in the OG model.

Classical utilitarian welfare with homogeneous preferences. Discounting is still valid, and with rate $\rho\gamma + \beta$, exactly as in the example in Section 1.2, though we deal with a very different concept: the endowment equivalent of policies is being discounted, not the final consumption.

Classical utilitarian welfare with heterogeneous preferences. If preferences with respect to lifetime consumption differ across agents, the derivative of the classical utilitarian welfare function with respect to policies does not exhibit the discounted sum form anymore, even when allowing for any form of time-dependent discounting.

Relative utilitarian welfare. This welfare function is the sum of individual von Neumann–Morgenstern (vNM) utilities, 0–1 normalized on the feasible set (which will be assumed time-invariant too).[8] Now the social discount rate is well defined even for heterogeneous preferences, and equals the growth rate of per capita GDP, γ, say, 2% per year.[9]

1.5. Roadmap

Section 2 presents the basic tool for evaluating policy reforms. In Section 2.2, the outcome map (IWF) is fully abstracted, as a map from policies to welfare (as in decision theory); so this would also cover models with a single decision maker, or an infinitely lived agent. In Section 2.3, this is applied to a model with a bit more structure, more appropriate for an economy with finitely lived agents: the map associates with each policy a full profile of individual utilities (as in social choice theory), and the aggregation is done explicitly, enabling use of the previous result. For further usage, the results are particularized to the classical aggregation in Section 2.4 (Theorem 2).

Section 3.1 describes the overlapping-generations economy with exogenous growth, and then Section 3.2 defines outcome maps for this model as having still more structure, being now maps from policies to allocations. The time-invariance requirement on them is carefully justified by exhibiting an automorphism of the economy (uniquely) associated with time shifts. The result of Section 2 is then applied to this economy in Section 4, to derive the discount rates implied by the classical (Section 4.1) and relative utilitarian (Section 4.2) criteria, with quite different implications. In each case, we first compute the derivative of welfare w.r.t. policy variations on an abstract policy space, and then apply this to a specific policy space of lump-sum taxes and subsidies, thought of as representing the monetized value of public projects, to derive the discount rate for cost–benefit analysis.

An alternative derivation of γ as the discount rate, based on the value of a human life, is presented in Section 5. Merits of the two criteria are then discussed in Section 6. Concluding remarks in Section 7 address the issues of evaluating the static component of the derivative of welfare and of nonvacuity of the results.

In the formal treatment below, the proofs are kept to a minimum; longer proofs are deferred to Appendix B.

2. DIFFERENTIATING WELFARE WITH RESPECT TO POLICY VARIATIONS

Here we start with the simplest model that yields a discount rate, including only policies and an objective function defined on them. We formulate a sufficient condition for the derivative of the objective to be of the net-present-value form. Although initially one might find this condition rather abstract, it is satisfied by utilitarian welfare functions in balanced growth equilibria of OG models, as shown in Section 4.

Notation 1. $\overline{\mathbf{R}}$ is the extended real line, $\mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$.

For $f \in E^{\star}$, the dual of a topological vector space E, $\langle f, e\rangle \stackrel{\text{def}}{=} f(e)$.

2.1. The Basic Model

2.1.1. Policies. First we describe a general space, F, of policy changes. An easy example is the space of continuous functions with compact support and the sup norm.

DEFINITION 1.

(i) *Let $t_h : t \mapsto t + h$ be the translation by h on $\mathbf{R}$; and let $\mathbf{S}_h : \xi \mapsto \xi \circ t_{-h}$ be the time shift on functions of time.*

(ii) *Fix a Banach space E. K_E is the set of infinitely differentiable functions $\varphi : \mathbf{R} \to E$ with compact support.*[10]

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(iii) *F is a topological vector space of E-valued functions s.t.*
  (1) $\mathbf{S}_h F \subseteq F$.
  (2) *K_E is a dense subset of F.*
  (3) *If $\varphi_n \in K_E$ and its successive derivatives converge uniformly to* 0 *and* $\exists h \in \mathbf{R}\colon |x| \geq h \Rightarrow \varphi_n(x) = 0\ \forall n$, *then* $\varphi_n \to 0$ *in* F.

Remark 1. $K_E = \{\varphi\colon \mathbf{R} \to E \mid \forall f \in E^*,\ f \circ \varphi \in K_{\mathbf{R}}\}$.

Next, we move to the definition of policies. Basic policies are time-independent specifications of government actions. They belong to a Banach space, because income tax schedules, for example, are already in a function space. The baseline is some basic policy kept constant over time. A policy (reform) is a deviation from the baseline.

DEFINITION 2.
(i) $(B, \bar{\pi})$ *is the set B of* basic policies*, open in the Banach space E, together with some point $\bar{\pi} \in B$, called the baseline policy.*
(ii) *P is the set of policies $\pi : t \mapsto \pi(t) \in B$ s.t. $\delta\pi = \pi - \bar{\pi} \in F$.*

The policy space P is shift-invariant, as is F; i.e., policies can be shifted in time. Definition 4 below implies that this shift must be meaningful; so we have to think about a basic policy as expressed in time-invariant terms. This implies, in particular, that a basic policy has to be *unit-free* and *nondiscriminatory*, not prescribing date-specific actions or special treatment of particular individuals or generations, to be applicable at any time. For example, the income-tax part of a policy would satisfy this if brackets of the rate-schedule were indexed to per capita income.

More precisely, in an OG model, a basic policy, if kept constant over time, should lead to balanced growth (Lemma 7).

The policy space is basically unrestricted until now; for instance, with $B = \mathbf{R}$, one can very well have as a space of policies the space of all continuous functions with the topology of uniform convergence on compact sets. Restrictions will come through the following set Z_F, which will be needed throughout the paper, to impose conditions that some parameter belongs to it. Those conditions translate thus as an integrability requirement on policies (think of the case where the parameter is zero), hence excluding permanent deviations, such as a constant policy different from $\bar{\pi}$. If the parameter is non-null, it is roughly the minimal speed of convergence to $\bar{\pi}$ required of policies.

DEFINITION 3. $Z_F = \{\zeta \in \mathbf{R} \mid \forall q \in E^\star,\ f \mapsto \int e^{\zeta t} \langle q, f(t)\rangle dt \in F^\star\}$.[11]

2.1.2. Objective function

DEFINITION 4. $W\colon P \to \overline{\mathbf{R}}$ *is an invariant welfare function (IWF) if* $\exists$ *Lebesgue-measurable* $a_h, b_h > 0\colon \forall h \in \mathbf{R},\ W \circ \mathbf{S}_h = a_h + b_h W$.

Remark 2. That is, VNM preferences on P are shift-invariant if and only if their representation W is an IWF.

Example

Consider again the (nonequilibrium) Arrow–Kurz-like setup described in the Introduction, and, time being discrete, take h integer. Let $E = \mathbf{R}$, with baseline $\bar{\pi} = 0$. There is one individual per period. Policy is a consumption perturbation as a fraction of per capita consumption $\pi(t) = e^{-\gamma t}\delta c_t$ and has finite support:

$$W(\pi) = \sum_{t=-\infty}^{\infty} e^{-\beta t}[U(c_0 e^{\gamma t} + e^{\gamma t}\pi(t)) - U(c_0 e^{\gamma t})],$$

$$W(\mathbf{S}_h\pi) = \sum_{t=-\infty}^{\infty} e^{-\beta t}[U(c_0 e^{\gamma t} + e^{\gamma t}\pi(t-h)) - U(c_0 e^{\gamma t})]$$

$$= e^{(\gamma(1-\rho)-\beta)h} \sum_{-\infty}^{\infty} e^{-\beta t'}[U(c_0 e^{\gamma t'} + e^{\gamma t'}\pi(t')) - U(c_0 e^{\gamma t'})], \quad t' = t - h$$

by homogeneity of U. So $W(\mathbf{S}_h\pi) = e^{(\gamma(1-\rho)-\beta)h} W(\pi)$: W is invariant.

LEMMA 1. *For an IWF W there exist constants ζ and $A \in \mathbf{R}$ s.t. a_h and b_h in Definition 4 can be taken as $a_h = A\frac{e^{\zeta h}-1}{e^{\zeta}-1}$, $b_h = e^{\zeta h}$, the ratio being defined by continuity at $\zeta = 0$. Such a ζ is unique if W takes at least two different real values. ζ is called the parameter of the IWF.*

Proof. Use Lemma 8 in Appendix A, identifying values of W with constant $\overline{\mathbf{R}}$-valued functions of time. ■

2.2. The Main Tool

Recall that a map is Gâteaux-differentiable on F if it has directional derivatives in each direction, which are a continuous linear function of the direction. This is the weakest form of differentiability. We will need an extension of this form of differentiability for correspondences:

DEFINITION 5. *An $\overline{\mathbf{R}}$-valued correspondence Γ with domain in F is G*-differentiable *at x iff every $f\colon F \to \overline{\mathbf{R}}$, s.t. $f(y) \in \Gamma(y)$ when $\Gamma(y)$ is defined and nonempty, is s.t. $f(x) \in \mathbf{R}$, and Gâteaux differentiable at x. Their (common) Gâteaux differential is then the G*-differential *of Γ at x.*

THEOREM 1. *If an IWF W with parameter $\zeta \in Z_F$ is G-differentiable on P at $\bar{\pi}$, then its differential equals $\int e^{\zeta t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^\star$.*

The theorem justifies discounting, i.e., shows that the time component of the derivative of welfare is exponential in time, with a time-independent shadow price q applied to current policy changes $\delta\pi(t)$.

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Mertens and Rubinchik (2009) provide (some) tools to prove the differentiability assumption.

Next we express the discount rate ζ in terms of the parameters of an OG model. The first step is to move from the objective of a single decision maker to a welfare aggregator over individual utilities.

2.3. Constructing an IWF

This section provides sufficient conditions for a welfare function to be an IWF in an OG model. They are twofold. The first relates individual lifetime utilities over policies (Definition 6); the second restricts the aggregator (Definition 7).

2.3.1. Population. Individuals differ by type $\tau \in \Theta$ (Θ finite) and by birth-date, $x \in \mathbf{R}$. They have life length T_τ, and $N_x^\tau dx = N_0^\tau e^{\nu x} dx$ is the number of births in $(x, x+dx)$.

2.3.2. Utilities over policies. Assume now that individual lifetime utility functions are defined over policies. The time-invariance property below requires that whenever any policy π is delayed by h the resulting utility profile $v(\mathbf{S}_h(\pi))$ equals that under π up to an affine transformation *when also* shifting the dates of birth of the agents: shifting policies and agents preserves interpersonal comparisons of utility differences.

DEFINITION 6. *A profile v of $\overline{\mathbf{R}}$-valued functions v_x^τ defined on P is a* valuation *if it is weakly shift-invariant, i.e., $\exists$ Lebesgue-measurable $a_h \in \mathbf{R}^\Theta, b_h > 0$: $\forall h \in \mathbf{R}, v \circ \mathbf{S}_h = a_h + b_h \mathbf{S}_h \circ v$. The profile is a* strict valuation *if it is shift-invariant, i.e., if $a_h = 0, b_h = 1$.*

There is a simple translation of a valuation into a strict one:

LEMMA 2. *For a valuation v there exist constants $A \in \mathbf{R}^\Theta$ and $\varrho \in \mathbf{R}$ s.t. $u_x^\tau = A^\tau \frac{1-e^{-\varrho x}}{e^\varrho - 1} + e^{-\varrho x} v_x^\tau$ is a strict valuation, with x for the ratio at $\varrho = 0$. ϱ is unique except if $\forall \tau$, $v_{x_1}^\tau(\pi_1) = v_{x_2}^\tau(\pi_2)$ whenever $v_{x_i}^\tau(\pi_i) \in \mathbf{R}$. ϱ is called the parameter of the valuation.*

Proof. By Lemma 8, with $n = \#\Theta$, $b_h = e^{\varrho h}$ and $a_h = A \frac{e^{\varrho h}-1}{e^\varrho - 1}$ for some constants A and ϱ, the ratio being h for $\varrho = 0$. The rest is obvious. ■

COROLLARY 1. *For a valuation v and a constant policy π, $v_x^\tau(\pi)$ is of the form $e^{\varrho x} v^\tau(\pi) + C^\tau$.*

Proof. Apply Lemma 2, and for v strict, use the definition. ■

Thus the parameter ϱ is the rate of growth of individual utility scales over policies. It will be further disentangled in Propositions 2 and 3 into growth effects and effects of the utility functions in the OG model.

2.3.3. Aggregation

DEFINITION 7. *An* invariant welfare aggregator *(IWA) is an* $\overline{\mathbf{R}}$*-valued function* V *on* $\overline{\mathbf{R}}^{\Theta\times\mathbf{R}}$ *(utility-profiles), s.t.* $\exists$ *Lebesgue-measurable* $a_h, b_h > 0$: $\forall h \in \mathbf{R}$, $V \circ \mathbf{S}_h = a_h + b_h V$*—i.e.,* V *is weakly shift-invariant.*[12]

A proof similar to that of Lemma 1 yields now

LEMMA 3. a_h *and* b_h *in Definition 7 can be taken as* $a_h = a\frac{e^{ch}-1}{e^c-1}$, $b_h = e^{ch}$, *the ratio being defined by continuity at* $c = 0$. c *is unique if* V *takes at least two different real values.* c *is called the parameter of the IWA.*

Given the goal of evaluating policy changes from the baseline, it is natural to aggregate individual utility *differences* from the baseline. Thus we assume henceforth, for any valuation v, that $v_x^\tau(\bar{\pi}) \in \mathbf{R}\ \forall\tau$. This condition is independent of x by Corollary 1.

LEMMA 4. *Take a valuation* v *with parameter* ϱ, *and a IWA* $V_{c,r}$ *with parameter* c, *positively homogeneous of degree* r.

Then $W(\cdot) \overset{\text{def}}{=} V_{c,r}(v(\cdot) - v(\bar{\pi}))$ *is an IWF with* $\zeta = \varrho r + c$.

If the valuation is strict, homogeneity is not needed, and $\zeta = c$.

We could continue and use general IWAs throughout (homogeneous in Section 4.1); however, for concreteness, and to have an explicit parameter c, we concentrate henceforth on the classical case, and first summarize for future use our results for that case.

2.4. The Utilitarian Aggregator

The two *social welfare functions* (SWF) used in Section 4 are based on the same utilitarian aggregator. It may, however, be just a correspondence, e.g., as the integral in Definition 8 may very well diverge for some policies, so some additional care is required.

DEFINITION 8. *The* utilitarian aggregator S *maps a valuation* v *to* $S(v) = \int_{-\infty}^{\infty} e^{-\beta x} \sum_\tau N_x^\tau (v_x^\tau - v_x^\tau(\bar{\pi}))dx$, *understood as the interval between the lower and upper wide Denjoy integrals [e.g., Gordon (1994)].*[13]

LEMMA 5. *For a valuation* v *with parameter* ϱ, *and* S^* *the upper bound of* S *(the upper integral),* $S^*(v)$ *is an IWF with* $\zeta = \varrho + \nu - \beta$.

Proof. $S^*(v) = W$ of Lemma 4, using the IWA, with degree $r = 1$ and parameter $c = \nu - \beta$, $V_{c,r}: u \mapsto \int^* e^{-\beta x} \sum_\tau N_0^\tau e^{\nu x} u_x^\tau dx$. ■

THEOREM 2. *Let* v *be a valuation with parameter* ϱ. *If* $W = S(v)$ *is G-differentiable on* P *at* $\bar{\pi}$ *and* $\varrho + \nu - \beta \in Z_F$, *then* $\exists q \in E^\star$ *s.t.* W*'s G-differential at* $\bar{\pi}$ *equals* $\int e^{(\varrho+\nu-\beta)t}\langle q, \delta\pi(t)\rangle dt$.

Proof. $\zeta = \varrho + \nu - \beta \in Z_F$. G-differentiability of W implies that of S^*, whose differential from Theorem 1 is the G-differential of W. ■

We obtain thus $\beta - \nu - \varrho$ as discount rate for policies. Our purpose in the next two sections is to identify the last parameter, ϱ, in terms of economic primitives in an OG model.

3. A GROWING ECONOMY

Valuations with their built-in time invariance might seem confined to stationary economies, but they also arise naturally in models with exogenous growth. We impose only the minimal conditions required for the existence of balanced growth: homogeneity of utility functions with respect to consumption, constant returns to scale in production, absence of land and natural resources, and labor-saving technological growth.

3.1. The Economy

3.1.1. Consumption and labor. Instantaneous consumption is a nonnegative bundle of n consumption goods and h fractions of total time allocated to h different types of labor. Individual preferences over lifetime streams of time allocation and consumption bundles are described by a utility function U^τ, homogeneous of degree $1 - \rho^\tau$ in consumption. There are two interpretations of the parameter ρ^τ: (1) relative risk aversion coefficient and (2) income elasticity of the marginal utility of income. Indeed, by fixing consumption prices and relative wages, labor income varies linearly with the wage level, by homogeneity, so the individual indirect utility function, as a function of labor income, has, by homogeneity, the specified relative risk-aversion coefficient or elasticity.

The fraction of time, $z_i^\tau(s, t)$, devoted at date t to activity i by an agent of type τ and age s is multiplied by a nonnegative and integrable efficiency factor $\varepsilon_i^\tau(s)$ to form effective time. Effective time devoted at date t to any activity is multiplied by $e^{\gamma t}$ to form effective labor input, $e^{\gamma t}\varepsilon_i^\tau(s) z_i^\tau(s, t)$, thus representing labor-saving technological progress.

Example

With $\gamma = 0$ and $\varepsilon(s) = 1$ in the first part of life and zero thereafter, the model is a continuous-time reinterpretation of the standard OG model, as, e.g., in Samuelson (1958) or Gale (1973).

3.1.2. Production. There are m capital goods and a corresponding investment good for each, linked by the usual capital accumulation equation, $K^{i\prime}(t) = I^i(t) - \delta^i K^i(t)$,[14] $K^i \geq 0$ denoting capital and I^i investment of type i, with δ^i as depreciation rate. Consumption and investment goods are manufactured instantaneously by production firms from (the services of) capital and effective labor (and, possibly, from investment), with as instantaneous production set a closed convex cone $Y \subseteq \mathbf{R}_-^h \times \mathbf{R}_-^m \times \mathbf{R}^m \times \mathbf{R}^n$ of production vectors $(-L, -K, I, C)$, satisfying the classical free-disposal and irreversibility (Y contains no straight line) conditions. An investment firm of type i acquires capital $K^i(t_0)$ at time t_0,

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chooses investment flows, rents out accumulated capital to production firms, and sells $K^i(t_1)$ at time $t_1 > t_0$.

Investment goods can be viewed both as outputs and as inputs. E.g., disvestment is crucial to model resource extraction. Or, to model a storable good, introduce a corresponding investment good and capital good ("the good in storage"). A production firm creates the storable investment good, purchased by an intermediary investment ("storage") firm that transforms it into the corresponding capital good, which has no use in production. At the time of consumption, the investment firm disinvests and sells the corresponding investment good to a production ("marketing") firm, which transforms it one to one into the corresponding consumption good. So allow all investment firms to disinvest as well as invest in all goods; restrictions on disinvestment are described by Y.

To include consumer durables, introduce the corresponding investment and capital goods. A production firm creates the durable investment good, purchased by an intermediary investment firm, which rents the capital good out to a leasing production firm, which produces with this capital the consumption good (services), purchased by consumers.

3.1.3. Initial condition. This section ensures that the production set (set of feasible input and output paths) is well defined. Indeed, the formula in Lemma 6 implies that $K^i(\cdot)$ is uniquely determined by $I^i(\cdot)$, but does not suffice for any investment policy (e.g., I is a function of current K instead of time, cf. example infra) to have a well-determined outcome.

To ensure its boundedness, assume capital cannot reproduce itself:

Assumption 1 (No Rabbit Economy). $(0, -K, I, 0) \in Y \Rightarrow I \leq 0$.

Remark 3. Observe that although production of durables, as described before, involves a production of consumption good with only capital and no labor input, it does not violate our assumption on Y that no *investment good* can be produced without some form of labor input. Similarly, production activities (as for storable goods) transforming investment goods one to one into consumption goods, without any capital or labor input, do not violate this assumption.

To see the need for Assumption 1, consider the following "rabbit economy":

Example

Assume a single good, a single type of labor, and a CES production function $(AK^\alpha + BL^\alpha)^{1/\alpha}$, $A^{1/\alpha} \geq R$ with $R = \gamma + \nu + \delta$. In order to get an upper bound on capital and investment, consider a path with all agents working full time and consuming nothing. Note that $L_t = L_0 e^{(\gamma+\nu)t}$, so for $D = BL_0^\alpha$, $K'(t) = [AK^\alpha(t) + De^{\alpha(\gamma+\nu)t}]^{1/\alpha} - \delta K(t)$; or with $x(t) = K(t)e^{-(\gamma+\nu)t}$, $x'(t) = [Ax^\alpha(t) + D]^{1/\alpha} - Rx(t) \geq D^{1/\alpha} > 0$. Because $x(t) \geq 0$, there is no solution; i.e., the upper bound of $K(t)$ is infinity. And even if $B = 0$, the solutions are $x(t) = Ce^{(A^{1/\alpha}-R)t}$, with $C \geq 0$ arbitrarily large, so $K(t)$ is unbounded in this case too.

As for any differential equation, initial conditions are needed. Their natural form is that the capital stock K_t converges at $-\infty$ to given initial values, which become part of the description of the techology. Note that non-null initial values can occur only for capital goods with $\delta^i = 0$, corresponding to land and resources. But, for balanced growth, those initial values must be zero, thus ruling out land and natural resources:

Assumption 2 (Initial Condition). Let $\delta = \min_i \delta^i$. Then $e^{\delta t} K_t$ converges exponentially fast to 0 along some sequence $t \to -\infty$.

Also assume $R \stackrel{\text{def}}{=} \gamma + \nu + \delta > 0$.

LEMMA 6.

(i) $K^i(t) = \int_{-\infty}^{t} e^{\delta^i(s-t)} I^i(s)ds$, *where the* L_1*-norms of all feasible integrands are bounded by a constant times* $e^{(\gamma+\nu)t}$*; in particular, the integral is a Lebesgue integral.*

(ii) *Let* $i_t = e^{-(\gamma+\nu)t} I_t$, $k_t = e^{-(\gamma+\nu)t} K_t$*. There exists* $\bar{K}$ *s.t. along any feasible path,* $\int_a^b \|i_t\| dt \leq \bar{K}(b - a + 1)$ *for any pair* $a \leq b$*, and (hence)* $\|k_t\| \leq \bar{K}$.

Remark 4. As explained and addressed in Appendix C, the initial condition is a bit too stringent conceptually, requiring exponential convergence to 0 instead of just plain convergence. This is not crucial in this paper: land and natural resources being ruled out anyway by the need for balanced growth, it is natural to expect all $\delta^i > 0$, so just K_t bounded at $-\infty$ already ensures exponential convergence to 0.

3.2. Time-Invariant Solution Concepts

To apply the main result, one has to obtain time-invariant profiles of utilities, or a valuation v for this economy.

Consider, to fix ideas, a map from policies to corresponding equilibria. It induces a profile of utility functions over policies. For this to be a valuation, a very natural consistency requirement must hold. When a policy is delayed, the resulting allocation should be the same as under the original policy, only in a transformed economy, with time shifted and quantities scaled up appropriately. Say the policy increases the income tax rate by 1% above a given quantile of the income distribution for 10 years and returns to the baseline thereafter. Delaying it by a year should induce the same response of the economy as applying it today, renaming the affected agents (and dated goods), and rescaling all quantities according to their growth rate. More precisely,

DEFINITION 9. *The transformation* $\mathbf{T}_h$

(i) *applies* $\mathbf{S}_h$ *to allocations and production plans;*
(ii) *multiplies individual consumption bundles by* $e^{\gamma h}$*;*
(iii) *multiplies aggregates—effective labor, capital, investment, and consumption—by* $e^{(\gamma+\nu)h}$.

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Remark 5. Equivalently, $\mathbf{T}_h$ shifts the origin of time back by h, multiplies population measure and thus all aggregates by $e^{\nu h}$, and divides *units* of all nonlabor goods by $e^{\gamma h}$.

DEFINITION 10. *An* outcome function ς *is a map from policies to individual allocations, which is invariant under all* $\mathbf{T}_h$*:* $\mathbf{T}_h \circ \varsigma = \varsigma \circ \mathbf{S}_h$.

Remark 6. We abstract away the exact nature of outcome functions, keeping only the time-invariant structure, to blackbox the policy space.

The rest of this section justifies Definition 10, in Section 3.2.2, gives examples of outcome functions in Section 3.2.3, and further, in Section 3.2.5, defines balanced growth paths and shows that they are the outcomes of constant policies.

3.2.1. Isomorphism between Arrow–Debreu economies. To motivate Definition 9, first define isomorphism between two Arrow–Debreu economies with finitely many goods and individuals. They are isomorphic if there is a linear map ξ from the commodity space of one economy to that of the other and there are one-to-one mappings from the sets of individuals and of firms of one to those of the other such that

(i) the consumption set, preferences, and endowment of any agent in the first economy are mapped by ξ to those of the corresponding agent in the second economy;
(ii) the production set of each firm in one economy is mapped by ξ to that of the corresponding firm in the second;
(iii) shareholdings are preserved.

When consumption sets are the nonnegative orthant, it must be that ξ maps the commodity names in the first economy one to one to those in the second, together with appropriate rescalings (changes of unit).

Another aspect of isomorphism, which is more familiar with a continuum of agents, is to multiply the population measure by a positive constant C. Shareholdings refer then for each firm to a probability distribution over the agents; and the one-to-one mapping of agents has to be understood to be measurable, as well as its inverse, and such that the induced map on measures maps the first population measure to $1/C$ times the second. Further, the firms' production sets, as well as points therein, are multiplied by C (in addition to the above rescalings).

When production has constant returns to scale, as here (capital-accumulation equations are linear, and the instantaneous production sets, cones), shareholdings become irrelevant (profits being zero), and multiplication by C maps the production set onto itself.

The isomorphism is equivalently described by a single linear bijection (with the required structure) between the allocation spaces (product of all consumption sets and production sets) of both economies. For the isomorphism property, it suffices then that it maps allocations *to and onto* allocations, endowments to endowments, and preserves preferences, and that population measures are mapped to each other by the induced map of agents and the multiplication by C, obtaining C from how the map behaves on production sets as compared to consumption sets.

We will use this below with a new twist, in that indeed the mass of each agent is multiplied by C, but with as final effect to *preserve* the population measure, it being σ-finite.

3.2.2. Time-invariance in the OG model. The tranformation $\mathbf{T}_h$ defined above is a particular case of such isomorphisms: it maps an agent of type τ born at time t to an agent of the same type born at $t+h$, multiplying his mass by $e^{\nu h}$, and maps any good dated t to the same good dated $t+h$, multiplying nonlabor quantities by $e^{\gamma h}$, and labor quantities by 1. Individual time is not a good (not marketed), so the linear ξ map of Section 3.2.1 is applied using the equivalent vector of effective time. Thus,

PROPOSITION 1. $\mathbf{T}_h$ *is an automorphism of the economy.*

Now, $\mathbf{T}_h$ mapping the economy to itself h time-units later, and policies being unit-free, the corresponding operation on policies is a pure time-shift, without rescaling. So, natural solution concepts having the invariance properties of the model, $\mathbf{T}_h$ must transform solutions of π to those of $\mathbf{S}_h\pi$. This justifies Definition 10.

3.2.3. Examples of outcome functions. A first example maps a policy π to a locally unique equilibrium close to that of the baseline $\bar{\pi}$, defining the map in an arbitrary invariant way elsewhere.

Indeed, *if* such a selection exists, then it should satisfy invariance: time-shifts map the balanced growth path to itself, so neighboring paths are mapped in its neighborhood; hence, by local uniqueness, the selection is mapped to itself.[15]

The next example is the maximization of a time-invariant *social welfare function*, say a utilitarian one, provided the maximum is unique.

Another example is the "identity map": policies are perturbations of final consumption. This way our results also yield the usual, nonequilibrium approach to discounting.

One way to model *policy surprises* is to assume the contracts signed ("at the beginning of time") in anticipation of the baseline policy cannot be changed, so in the wake of an unexpected policy change, individuals sign additional contracts taking their baseline consumption as new endowment. The initial equilibrium being a balanced growth equilibrium, the economy with that endowment is also time-invariant, and the resulting map from policies is again an outcome function if the final allocation is locally unique. At least when policies are lump sum taxes and benefits (endowment perturbations), this case is particularly simple, as net individual demand under the baseline prices is zero, so income effects disappear and the variation in individual utility depends just on the value of the endowment perturbation.

3.2.4. The case of indeterminacy. Even if dealing with a situation that does not guarantee local uniqueness, one can choose the prices closest to those in the initial equilibrium in terms of the L_∞-norm $\sum_i \|\ln p_i(t) - \ln \bar{p}_i(t)\|_\infty$, $\bar{p}(t)$ being the baseline price system (or equivalent ones, e.g., the L_∞-norm of the ℓ_2-norm over i of the ln differences). Though the price system does not necessarily specify an equilibrium, it does specify the individual utility levels, which is sufficient for welfare analysis. The logarithms make the distance independent of price normalization, and hence induce a distance between price rays: for any multiple of $\bar{p}_i$, the minimum, over all multiples of p_i, will be achieved at the corresponding multiple, and the value of the minimum is independent of this multiple, and remains the same when the roles of $\bar{p}_i$ and p_i are permuted. Finally, the L_∞-norm being shift-invariant, the selection will be time-invariant. If the set of minimizers is not a singleton, their correspondence can be expected to be sufficiently thin so that hopefully any outcome function obtained as an invariant selection (using the axiom of choice) generically satisfies the differentiability requirement — e.g., as in Mertens and Rubinchik (2009), discussed in Section 7. Finally, because Theorem 2 already allows for a correspondence, one could similarly extend Definition 10, to obviate the need to appeal to the axiom of choice in such cases.

But this is only one example of how to possibly construct outcome functions in the case of indeterminacy (which we do not expect to occur in the model of Section 3.1); there should be a continuum of such outcome functions then. Because our results below hold for any of them, conceivably with a linear functional q depending on the chosen outcome function, the discount rate is established even then.

3.2.5. Balanced growth

DEFINITION 11. *A balanced growth path is a* $\mathbf{T}$*-invariant allocation.*

On a balanced growth path individual labor is independent of the birthdate, individual consumption grows at rate γ, and all aggregate inputs and outputs at rate $\gamma + \nu$, as in the standard (1 type, 1 good) case [e.g., Arrow and Kurz (1970), King et al. (2002)].

The following sharpens the interpretation of basic policies; see Corollary 1:

LEMMA 7. *The outcome of a constant policy is a balanced growth path.*

Proof. By Definition 10, it is mapped to itself by any $\mathbf{T}_h$. ■

4. THE DISCOUNT RATE

The discount rate for cost–benefit analysis depends on the social welfare function. We consider both relative and classical utilitarianism.

Let $v = U \circ \varsigma$ be the profile of utility functions on P induced by the profile of utility functions U and the outcome function ς. Assume $v_x^\tau(\bar{\pi}) \in \mathbf{R}\ \forall \tau$.

4.1. The Classical Utilitarian Approach

In Section 4.1 we assume $(1 - \rho^\tau)\gamma + \nu - \beta \in Z_F\ \forall \tau$.

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4.1.1. Evaluating policies

PROPOSITION 2. *Assume all types have the same parameter ρ. Then v is a valuation with parameter $\varrho = (1-\rho)\gamma$.*

Proof. Let $(c, l) \stackrel{\text{def}}{=} \varsigma(\pi)$. By the time-invariance of ς, $\varsigma(\mathbf{S}_h\pi) = \mathbf{T}_h(c, l) = (e^{\gamma h}\mathbf{S}_h(c), \mathbf{S}_h(l))$. So, by homogeneity of U, $v \circ \mathbf{S}_h = e^{(1-\rho)\gamma h}\mathbf{S}_h(v)$. Thus v is a valuation with $a_h = 0$ and $b_h = e^{(1-\rho)\gamma h}$. ∎

Proposition 2 and Theorem 2 imply now

COROLLARY 2. *Assume all types have the same parameter ρ. If $W = S(v)$ is G-differentiable on P at $\bar{\pi}$, then for some $q \in E^\star$, its differential equals $\int e^{(\nu-\beta+(1-\rho)\gamma)t}\langle q, \delta\pi(t)\rangle dt$.*

In a society with type-dependent ρ, classical utilitarianism leads to questionable implications; besides, it invalidates discounting:

COROLLARY 3. *The welfare differential is*

$$\sum_\tau \int_{-\infty}^{\infty} e^{(\nu-\beta+(1-\rho^\tau)\gamma)t}\langle q^\tau, \delta\pi(t)\rangle dt = \int_{-\infty}^{\infty} e^{(\nu-\beta+\gamma)t}\left\langle \sum_\tau e^{-\rho^\tau\gamma t} q^\tau, \delta\pi(t)\right\rangle dt,$$

and hence the weight in the welfare function of the types with the smallest ρ approaches one as time goes to $+\infty$.

There are other ways to express the same idea; e.g., that along any balanced growth path, in an optimal redistribution of consumption goods (keeping the rest fixed) the fraction allocated to the agents with the smallest ρ converges to 1.

4.1.2. The discount rate for cost–benefit analysis. In cost–benefit analysis, the effects of a variation in public policy are traditionally first “monetized”, i.e., expressed as an equivalent perturbation of individual endowments of consumption goods, here initially 0.

Let thus E be the Banach space M of measures[16] on age-groups and types, i.e., on $\cup_\tau([0, T_\tau] \times \{\tau\})$, with values in $\mathbf{R}^n$ (space of consumption bundles), with $\bar{\pi} = 0$ as baseline, where $b \in B$ determines the endowment perturbation $\omega(t) = e^{(\gamma+\nu)t}b \in M$ at t. Equivalently, express b in a unit-free way, letting, for each set of agents S, $b(S)$ equal the fraction of baseline aggregate consumption S receives, good by good.

Policies are thus endowment perturbations, representing arbitrary flows of lump-sum real taxes and benefits. Then we get $\beta+\rho\gamma$ as the discount rate for “aggregate” resources $\omega(t)$,[17] confirming our simple calculation of Section 1.2:

COROLLARY 4. *Assume all types have the same parameter ρ. If $W = S(v)$ is G-differentiableon P at 0, then its differential equals, for some $q \in M^\star$, $\int e^{-(\beta+\rho\gamma)t}\langle q, \omega(t)\rangle dt$.*

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Proof. By construction, $\bar{\pi} = 0$, so $\delta\pi = \pi$ and $\pi(t) = e^{-(\gamma+\nu)t}\omega(t)$. ■

Remark 7. Clearly, choosing a different growth rate in the definition of ω would lead to the same corollary with a different discount rate. That statement would, however, be empty, because there can be no outcome function: B being a neighborhood of 0, choose a negative measure b s.t. $\forall s \in [0, 1]$, $sb \in B$, and let $\psi = b\phi$ for some $\phi \in K$ with values in [0, 1]. Then ψ is a policy, yet when it is shifted sufficiently to $\pm\infty$, the feasible set under that policy becomes empty, by Lemma 6.

4.2. The Relative Utilitarian Approach

In this section we assume $\nu - \beta \in Z_F$.

As an alternative to classical utilitarianism, we suggest applying relative utilitarianism (RU),[18] the social welfare functional where individual vNM utilities are normalized between zero and one, and then summed. It is stressed in Dhillon and Mertens (1999) that the RU-normalization of individual utilities has to be done on some universal set A of acceptable alternatives, not specific to the problem under consideration, and representing the constraints both of feasibility and of justice.[19]

Assumption 3. The set A of *acceptable* policies is shift-invariant and each individual utility is bounded on A.

The boundedness is a minimal implication of justice; as to the shift-invariance, it is clearly a property of feasibility, but in relation to justice it has a strong meaning, that physical units (such as calories per day) are irrelevant. And without it RU might lead to quite different conclusions. But it is straight in the spirit of exogenous growth models—that (acceptable) policies affect only the height of the growth path, not the growth rate; and it is arguably justified in a world described by such a model: e.g., if the absolute level of calories per day matter, utilities cannot be homogeneous. And the latter is the key assumption ensuring time-invariance. Hence the choice of an exogenous growth model here.

Assume thus vNM utility functions, and that ς is defined on A—and hence v too, by the definitions at the beginning of this section. Let $\mathbf{M}_A$ denote the *RU-normalization*, i.e., the operation on a profile such that each individual utility is normalized to have a range of size 1 on A.

DEFINITION 12. *The relative utilitarian SWF is* $W = S(\mathbf{M}_A(v))$.

RU's anonymity axiom implies $\beta = 0$ in the specification of S, Definition 8. However, to allow for a richer model, incorporating, e.g., a nonzero probability of the world ending tomorrow, β is not restricted here.

4.2.1. Evaluating policies. In a growing economy the RU-normalization yields shift invariance, hence strict valuations:

PROPOSITION 3. $\mathbf{M}_A(v)$ *is a strict valuation.*

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COROLLARY 5. *The RU-normalized utility of an agent of type τ on a balanced growth path is independent of his birthdate.*

Proof. Apply Proposition 3 to the case where the basic policy space is a singleton, which doesn't do anything, and where the outcome function maps to the chosen balanced growth path. ■

COROLLARY 6. *If $W = S(\mathbf{M}_A(v))$ is G-differentiable on P at $\bar{\pi}$, then its differential equals $\int e^{(\nu-\beta)t}\langle q, \delta\pi(t)\rangle dt$ for some $q \in E^\star$.*

Proof. Proposition 3 and Theorem 2. ■

4.2.2. The discount rate for cost–benefit analysis. As in Section 4.1.2, one gets now, using Corollary 6, the discount rate $\beta + \gamma$ for aggregate resources, even for a population with variable ρ^τ:

COROLLARY 7. *If $W = S(\mathbf{M}_A(v))$ is G-differentiable on P at 0, then its differential equals $\int e^{-(\beta+\gamma)t}\langle q, \omega(t)\rangle dt$ for some $q \in M^\star$.*

Restricting basic policies b to have all the same distribution over age-groups and types, and setting $\beta = 0$, yields then the main result in Mertens and Rubinchik (2006).

The derived discount rate, γ, differs generically from the interest rate, even at the golden rule equilibrium if ν is nonzero.

5. A VALUE-OF-LIFE ARGUMENT

One touchstone is the case $\beta = 0$: no discounting of utilities. Do the prescriptions of the theory then indeed correspond to the intuitive meaning of treating individuals of different generations equally?

A compelling implication of equal treatment is to give equal weight to individual lives (cf. note 2), hence, in cost–benefit analysis, to their monetized values, i.e., the change in real consumption equivalent for the individual to an extension of his life.

The monetized value of life, according to any criteria [e.g., each of the four in Mishan's (1971) introduction, or even judicial criteria in assessing damages], is proportional to the individual's lifetime income.[20]

This is also formally true in the above economic model, when allowing for a variable lifespan: individual lifetime utility is homogeneous, so willingness to pay to extend life is proportional to lifetime income.

THEOREM 3. *In the model of Section 3, extended by variable lifetimes, γ is the only discount rate ensuring equal monetary value of human lives.*

Proof. Let the lifetime utility, U^τ, be defined on consumption and labor streams of variable length, $\cup_T(\mathbf{R}^{n+m})^{[0,T]}$. Consider, for an agent of type τ, an optimal lifetime consumption stream c of length T_1 (including the labor coordinates, taken say as negative), and let c' be the restriction of c to $[0, T_0]$, with $T_0 < T_1$. Let $\tilde{c}$ be expenditure-minimizing on $[0, T_1]$ s.t. $U^\tau(\tilde{c}) = U^\tau(c')$. Then $\langle p, c - \tilde{c}\rangle$ is

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the monetary equivalent of the utility loss from (unanticipated) premature death. Further, he leaves a debt (positive or negative) of $\langle p, c' \rangle$; so, because $\langle p, c \rangle = 0$, the net monetary equivalent of the loss equals $\langle p, c' \rangle - \langle p, \tilde{c} \rangle$.

By homogeneity, and because in a BGE relative prices are constant under time shift, when time is shifted by h and c is multiplied by $e^{\gamma h}$, c' and $\tilde{c}$ get multiplied by the same factor. Thus, the willingness to pay to avoid premature death[21] is proportional to $e^{\gamma x}$. The above computation could obviously have been done in several different ways (e.g., for the case of anticipated death, let c' be an optimal plan for a life length of T_0), but all of them would lead to the same conclusion.

So, to treat individuals of all generations equally, future incomes must be discounted *exactly* at rate γ, as implied by RU (Corollary 7). ■

This shows that the conclusions of relative utilitarianism and of Assumption 3 are correct in a world as described by this model.

But maybe the conclusions depend crucially on the special features of the model itself—exogenous growth, homogeneity, balanced growth? Else discounting may no longer be valid as an exact derivative of welfare, but insofar as it is nevertheless used, e.g., "as a first approximation," human lives should still be treated approximately equally. If then "value of life" does not decrease over time[22] exponentially fast to 0 as a proportion of lifetime income, the growth rate of per capita consumption is still the only discount rate treating human lives approximately equally: for any lower (resp. higher) rate, values of future lives would become exponentially higher (resp. lower) than those of present human beings.

The above argument is valid even with variable or stochastic growth; it does, however, refer to "average human life" at any given time.[23] Else further qualifications would be needed in case income distribution became more and more disperse. Thus we have the following theorem:

THEOREM 4. *In a world where the ratio of average value of life to per capita income is bounded away from* 0 *and* ∞*, discounting at the growth rate of per capita income ensures that the present values of average human lives in different periods are of the same order of magnitude.*

6. THE CHOICE OF SOCIAL WELFARE FUNCTION

Given that classical and relative utilitarianism—together with Assumption 3—have such different implications for discounting, we discuss some of the underlying principles of equitable treatment of different generations that each incorporates.

Interestingly, in relative utilitarianism, the implication of a time-invariant set of alternatives consistent with accepted public policy: the rate based on the relative utilitarian criterion, $\gamma \approx 2\%$, falls exactly in the range, "between 1 and 3%" (cf. Section 1.1), mandated by the U.S. OMB.

Remarkably, relative utilitarianism is also consistent with the "balanced generational policy" as presented in Kotlikoff (2002, p. 1905), requiring ". . . that the generational accounts [lifetime net tax burdens] of all future generations are equal, except for a growth adjustment."

The discount rate $\beta + \rho\gamma$ based on classical utilitarianism (Corollary 4) is well known in the applied literature. Even if individual risk attitudes were identical [far from the empirical findings; see Einav and Cohen (2007)], based on reported estimates [e.g., Drèze (1981)], implied discount rates would be far above the range suggested by the OMB. To get acceptable conclusions one has to set ρ close to unity [e.g., Stern (2007, pp. 6–11)]. However, imposing individual elasticity or risk preferences (ρ) contradicts any utilitarian foundation, because only the indifference map is retained as an individual characteristic. But (neglecting society's rationality over risky prospects), it is consistent with a welfarist approach, ρ being then viewed as a parameter of aggregation rather than an individual characteristic. Note that imposing $\rho = 1$ means forcing the discount rate implied by Assumption 3 under RU.

The discount rate under classical utilitarianism depends on ρ because of the presumption that marginal utility of income is independent of the environment surrounding the individual. In particular, a 1% increase in real income of any of our contemporaries has the same effect as it would 100 years ago for the same individual *with the same real income*.

In contrast, RU, in the context of a growing economy, implies that to compare individual utility differences, utilities must first be normalized on the feasible policies (consumption paths), which is time-invariant by Assumption 3. So the social value of a 1% increase in real income of an individual *at a given quantile* of the income distribution is independent of the date. Forcing logarithmic utilities, as in Stern (2006), amounts to choosing the best possible approximation to this under traditional welfarism, given his restriction to $\beta \sim 0$.

However, relaxing this restriction, and viewing β as just an arbitrary parameter of the welfare aggregator, one could obtain exactly the RU welfare function, without distorting individual utility functions (ρ), by using a type-dependent (even if probably negative!) discount factor on utilities, $\beta^\tau = \delta + (1 - \rho^\tau)\gamma$, with δ the "probability that the world ends tomorrow" (compare Corollaries 3 and 6). Adjusting β rather than ρ was advocated by Arrow[24] (in the single type context of the Stern report).

So RU provides a consistent methodology to aggregate correctly arbitrary (and heterogeneous) individual lifetime preferences over lotteries, while keeping the "ethical judgment" input completely independent of those, in the set of "feasible and just alternatives"—i.e., in the realm of ethics and political philosophy, where it belongs.

And it is easier and more objective to consider what are "just" laws rather than to assign millions of individual utility weights. For example, for transfers, in the form of income taxes, let y denote individual income as a percentage of per capita income, and let t_y denote the net tax (positive or negative), in the same units. (So t_y integrates to 0 under the income distribution μ.) Let the set of alternative tax rates t_y consist of $\{t \mid -t_0 \geq m,\ t' \leq M,\ t \text{ convex and monotone}\}$,[25] with the

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minimal income $m > 0$ and the maximal tax rate $M < 1$,[26,27] and conduct a sensitivity analysis in terms of the parameters m and M—it is typically for those that different interpretations of justice may give different bounds. This is easier and clearer than in terms of the millions of welfare weights: the advantage of the anonymity requirement on laws.

7. CONCLUDING REMARKS

In general, welfare evaluations of policy changes in a growing OG economy entail further challenges.

7.1. The Static Component of the Derivative

The problem of evaluating small policy changes, i.e., finding the derivative of social welfare with respect to policy variations, is now reduced to the static one of computing the linear functional q on the space of basic policies B.[28]

Observe that the result can hold only along a balanced growth path: else the direction of q would be time-dependent, contrary to even a very broad definition of "discounting" as in Mertens and Rubinchik (2010).

7.2. The Differentiability Assumption

Applicability of the results hinges on the existence of differentiable outcome functions, as illustrated by Remark 7. For the case of endowment perturbations (Corollaries 4 and 7), this would be a straight extension of Debreu's classical generic regularity theorem (1976). There are, however, several aspects that make such an extension highly nontrivial. First, it is well known that OG models can give rise to indeterminacy; see, e.g., Kehoe and Levine (1984) and Geanakoplos and Brown (1985). Next, even if regularity is ensured, already for the welfare function to be well-defined, the equilibrium has to be stable: the perturbed equilibrium has to converge sufficiently fast back to the unperturbed solution, both at $+\infty$ and at $-\infty$.

This program was successfully completed in Mertens and Rubinchik (2009) for the most classical case (1 good, 1 type, etc.) of our model, ensuring thus at least nonvacuity of our results. (Time starting at $-\infty$ seems crucial there too.) We think this should be extendable to the full model of Section 3.1, with policies as in Section 4.1.2.

7.3. Permanent Changes

Our exclusion of permanent policy deviations, e.g., to a different constant policy from the beginning of time (Section 2.1.1), does not necesarily exclude permanent policy changes, e.g., switching at some time forever from the baseline policy to

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some other constant policy; however, to satisfy the assumption $\nu - \beta \in Z_F$ (in Section 4.2), one then needs a natural restriction $\beta > \nu$.

For evaluating permanent policy deviations, or the above when $\beta \leq \nu$, one should reinterpret welfare functions in this paper as normalized, e.g., in per capita terms, such as $\lim_{T\to\infty} (1/N_T)\int_{-T}^{T} e^{\nu x} \sum_\tau N_0^\tau U_x^\tau dx$ [where $N_T = \int_{-T}^{T} e^{\nu x} dx = 2\sinh(\nu T)/\nu$]. Welfare per capita is our preferred interpretation of a social welfare function, as would have been Harsanyi's, if we are reading him correctly, e.g., when thinking of it as the expected utility of an unidentified individual. Welfare as a sum, as in this paper, is then only a higher-order term (of order $1/N_T$) in the expansion of the above (w.r.t. N_T), to fine tune transitions. In fact, our restriction $\nu - \beta \in Z_F$ just ensures that the other terms vanish. This whole area remains to be explored; we have no idea about the form of an asymptotic expansion, nor *a fortiori* about the right way to differentiate it.

Because under RU, for constant policies π, $U_x^\tau(\pi)$ is independent of x (Corollary 1 and Proposition 3), the above average immediately gives the SWF on constant policies. But a conjecture that this U^τ might yield q, say as a derivative, cannot work: it is independent of β, where as q should in general depend on β: e.g., policies favoring the old will come out better with high discounting, because the old were born earlier. This is why the heavier approach in Mertens and Rubinchik (2009) was needed to get a handle on q analytically.

NOTES

1. Circular A-4 of the OMB [U.S. Office of Management and Budget (2003)] mandates that all executive agencies and establishments conduct a "regulatory analysis" for any new proposal, and more specifically (pp. 33–36), a cost–benefit analysis, at rates of both 3% and 7%. Both rates are rationalized there as "the" interest rate: the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

2. Other practitioners share this view; e.g., "Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable" [Wildi et al. (2000)].

3. And it is not our purpose here to argue in favor or against. There may very well be good arguments, e.g., for rather using the population growth rate to discount.

4. "How far we are to consider the interests of posterity when they seem to conflict with those of existing human beings? Perhaps, however, it is clear that the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries, except in so far as the effect of his actions on posterity—and even the existence of human beings to be affected—must necessarily be more uncertain."

5. Aggregation of utility differences is also why strong Pareto and Ramsey's anonymity can be combined here, avoiding the impossibility results of, e.g., Basu and Mitra (2003) and Crespo et al. (2009). The literature in welfare economics and social choice offers diverse ways to construct welfare criteria by weakening one of the two desiderata. Koopmans (1960) axiomatizes discounting utilities, or "social impatience." Several authors are concerned with incorporating intergenerational justice principles into a social welfare criterion. Chichilnisky (1996) offers the "no dictatorship of the past" and "no dictatorship of the future" axioms (describing "sustainable preferences") and shows that the resulting welfare criterion is inconsistent with a sum of discounted utilities. Asheim et al. (2006) and

d'Aspremont (2007) show the existence of welfare functions satisfying some of Koopmans's postulates of intergenerational equity, but still, in particular, Chichilnisky's axioms. For alternative formulations of ethically acceptable allocations see, e.g., Asheim (1991) and Fleurbaey and Michel (2003).

6. Determinacy (cf. Section. 7) is a must for any form of comparative statics.

7. This poses novel questions concerning the above model, especially how to specify correctly initial conditions at $-\infty$. This is addressed in Section. 3.1, because those initial conditions are crucial to our argument (in Proposition. 1); informally, there can be no balanced growth in the presence of natural resources.

8. See Dhillon and Mertens (1999) for axiomatization.

9. For the United States, e.g., according to Johnston and Williamson (2007), the average until 2006 is 2.1% since 1950, 1.9% since 1900 or 1850, 2% since 1869, the first year when data become reliable (loc. cit.), especially for growth computations, because by then both colonial expansion and the immediate aftermath of the Civil War were over.

10. $K[= K_{\mathbf{R}}]$ is defined in Schwartz (1957–1959) or Gel'fand and Shilov (1959).

11. Implying that the integrals exist, as improper wide Denjoy integrals (cf. note. 13).

12. One can define in the same way IWAs on any shift-invariant subspace of $\overline{\mathbf{R}}^{\Theta \times \mathbf{R}}$ (e.g., integrable functions). This may sometimes be more convenient, but we will not need this generalization here.

13. The reader is welcome to think of any other type of integral (say, Lesbegue), as no properties possessed solely by the wide Denjoy integral are used in the paper. The basic reason for using Denjoy integration is the capital-accumulation equation in Section. 3.1 below, to be sure any solution of its differential equation form is also one of the integral form, and then to systematically use always the same integration theory on $\mathbf{R}$. No harm is done by sticking with the most encompassing one, in particular, in this case, where a requirement of absolute summability would have no economic meaning whatsoever.

14. Assumed to hold a.e., and implying the conditions for it to be meaningful: K_t^i is assumed locally a Denjoy primitive and I_t^i locally Denjoy-integrable.

15. Clearly this also needs some form of stability, else as the amount of shifting grows, the corresponding equilibria might slowly get out of the specified neighborhood. However, for the welfare differential to exist in this model, much more stringent stability properties are needed anyway.

16. Or the absolutely continuous measures (L_1), or those with continuous densities.

17. More precisely, the "welfare value" of the aggregate endowment perturbation at time t, $\sum_\tau \langle q^\tau, \omega_t^\tau \rangle$ [heuristically, $\sum_\tau \int q^\tau(s)\omega_t^\tau(ds)$], is discounted at this rate.

18. The axiomatization [Dhillon and Mertens (1999)] is for a finite set of agents.

19. Justice including in particular the anonymity requirement on laws. Diamond's critique was probably treated too lightly loc. cit., by essentially siding with Harsanyi, and is a source of recurrent questions and objections. It rests upon a confusion between on one side intuitive concepts of justice—e.g., in this case, anonymity of laws—, which are described by the "set of feasible and just allocations," and on the other requirements for correct aggregation, which are described by the RU axioms. The "wrong" alternative in his example is dismissed essentially because it involves unjust (non anonymous) laws, not because it involves an incorrect utility calculus.

20. Even a claim that from the point of view of society, it would be proportional to average lifetime income at his time would leave our argument below intact.

21. Oher ways to compute a "value of life," such as fully anticipated lifespans, or adding the welfare effect on the rest of society, lead to the same theorem.

22. Spending for life extension cannot be invoked as a measure of its value, because it might very well increase with the probability of success of treatments. But because this probability is bounded above, it can be invoked for the asymptotic behavior.

23. As opposed to the "social value of a specific individual's life" at that time, which presumably depends also on his contribution to society in the rest of his life.

24. Personal recollection from private conversation.

25. Thus $t_y \leq 0$ for $y \leq 1$ (we assume feasibility, i.e., $m \leq M$).

26. To illustrate the positive aspect of this approach, consider the change of attitudes toward justice reflected by the passage of the sixteenth amendment in the United States.

27. So, to normalize utilities, the minimal and maximal after-tax incomes at y are $\underline{w}_y = y + X(1-y)$ with $X = M$ for $y \geq 1$ and $X = m$ else, and $\bar{w}_y = y + M\int(x-y)^+\mu(dx) - [m - M\int(x-y)^+\mu(dx)]^+\frac{\int(y-x)^+\mu(dx)}{\int \min(x,y)\mu(dx)}$, where $z^+ \stackrel{\text{def}}{=} \max(z, 0)$.

28. Mertens and Rubinchik (2009) obtain q in the classical case of our model, with policies as in Section 4.1.2.

29. Conceptually our "initial condition" is best thought of as a pair: on the one hand, a general form, say something like K_t bounded at $-\infty$, provided one can prove from this convergence at $-\infty$, and on the other hand, a specific assumption to ensure balanced growth, i.e., that the limit is 0.

30. Independently of the natural requirement that for natural resources (e.g., mining), Y should force $I^i \leq 0$, and for land (raw acreage), $I^i = 0$.

31. Note that for $m = 1$ this bound is attained, so the strong no-rabbit condition is best possible: else, under the "weak" initial condition, there exist feasible paths with $\|k_t\|$ unbounded at $-\infty$, and for any fixed t the set of feasible K_t is unbounded.

REFERENCES

Arrow, K.J. (2007) Global climate change: A challenge to policy. *The Economists' Voice 4*(3), article 2. Available at http://www.bepress.com/ev/vol4/iss3/art2.

Arrow, K.J. and M. Kurz (1970) *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. Baltimore and London: Johns Hopkins Press.

Asheim, G.B. (1991) Unjust intergenerational allocations. *Journal of Economic Theory* 54, 350–371.

Asheim, G.B., T. Mitra, and B. Tungodden (2006) Sustainable Recursive Social Welfare Functions. Mimeo, Oslo University, Department of Economics.

Basu, K. and T. Mitra (2003) Aggregating infinite utility streams with intergenerational equity: The impossibility of being Paretian. *Econometrica* 71(5), 1557–1563.

Chichilnisky, G. (1996) An axiomatic approach to sustainable development. *Social Choice and Welfare* 13, 231–257.

Crespo, Juan, Carmelo Núñez, and Juan Rincón-Zapatero (2009). On the impossibility of representing infinite utility streams. *Economic Theory* 40(1), 47–56.

Dasgupta, P. (2008) Discounting climate change. *Journal of Risk and Uncertainty 37*(2–3), Special Issue on Discounting Dilemmas, 141–169.

d'Aspremont, C. (2007) Formal welfarism and intergenerational equity. In J. Roemer and K. Suzumura (eds.), *Intergenerational Equity and Sustainability, Conference Proceedings of the IEA Roundtable Meeting on Intergenerational Equity*, pp. 113–130. Houndsmills, Basingstoke, UK: Palgrave Macmillan.

Debreu, G. (1976, May) Regular differentiable economies. *American Economic Review 66*(2), Papers and Proceedings of the Eighty-eighth Annual Meeting of the American Economic Association, 280–287.

Demichelis, S. and H. M. Polemarchakis (2007) The determinacy of equilibrium in economies of overlapping generations. *Economic Theory* 32(3), 3461–3475.

Dhillon, A. and J.-F. Mertens (1999) Relative utilitarianism. *Econometrica* 67(3), 471–498.

Drèze, J. and N. Stern (1987) The theory of cost–benefit analysis. In A. J. Auerbach and M. Feldstein (eds.), *Handbook of Public Economics*, Volume II, Chapter 14. Amsterdam: Elsevier Science Publishers.

Drèze, J.H. (1981) Inferring risk tolerance from deductibles in insurance contracts. *Geneva Papers on Risk and Insurance*, 48–52.

Edwards, R.E. (1965) *Functional Analysis*. Chicago: Holt, Rinehart and Winston.

https://doi.org/10.1017/S1365100510000386 Published online by Cambridge University Press

Einav, L. and A. Cohen (2007) Estimating risk preferences from deductible choice. *American Economic Review* 97(3), 745–788.

Fleurbaey, M. and P. Michel (2003) Intertemporal equity and the extension of the Ramsey criterion. *Journal of Mathematical Economics* 39, 777–802.

Fréchet, M. (1913) Pri la funkcia ekvacio $f(x+y) = f(x) + f(y)$. *L'Enseignement Mathématique* 15, 390–393.

Gale, D. (1973) Pure exchange equilibrium of dynamic economic models. *Journal of Economic Theory* 6, 12–36.

Geanakoplos, J.D. and D.J. Brown (1985) Comparative Statics and Local Indeterminacy in OLG Economies: An Application of the Multiplicative Ergodic Theorem. Cowles Discussion Paper 773.

Gel'fand, I.M. and G.E. Shilov (1959) *Obobshennuye funkzii i deistviya nad nimi*, 2nd ed. Moscow: Fizmatgiz.

Gordon, R. A. (1994) *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*. Providence, RI: American Mathematical Society.

Johnston, L.D. and S.H. Williamson (2007) Source Note for US GDP, 1789–Present. Economic History Services. Available at http://www.eh.net/hmit/gdp/.

Kehoe, T.J. and D.K. Levine (1984) Regularity in overlapping generations exchange economies. *Journal of Mathematical Economics* 13(1), 69–93.

Kelley, J. and I. Namioka (1963) *Linear Topological Spaces*. The University Series in Higher Mathematics. Princeton, NJ: Van Nostrand.

King, R.G., C.I. Plosser, and S.T. Rebelo (2002) Production, growth and business cycles: Technical appendix. *Computational Economics* 20(1–2), 87–116.

Koopmans, T.C. (1960) Stationary ordinal utility and impatience. *Econometrica* 28, 287–309.

Kotlikoff, L.J. (2002) Generational policy. In A. J. Auerbach and M. Feldstein (eds.), *Handbook of Public Economics*, Vol. 4, Chap. 27, pp. 1873–1932. Amsterdam: Elsevier.

Mertens, J.-F. and A. Rubinchik (2006) Intergenerational Equity and the Discount Rate for Cost–Benefit Analysis. Discussion Paper 2006/91, CORE, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Mertens, J.-F. and A. Rubinchik (2009) Regularity and Stability of Equilibria in an Overlapping Generations Model with Exogenous Growth. Discussion Paper 2009/5, CORE, Université Catholique de Louvain, Louvain-la-Neuve, Belgium. Prelimary Version 0. Most recent version available at http://econ.haifa.ac.il/~arubinchik/papers/Reg_Stab_OLG.pdf.

Mertens, J.-F. and A. Rubinchik (2010) Discounting, Stationarity, and Welfare Evaluation of Policies. Mimeo, University of Haifa.

Mishan, E. J. (1971) Evaluation of life and limb: A theoretical approach. *Journal of Political Economy* 79(4), 687–705.

Ramsey, F.P. (1928) A mathematical theory of saving. *Economic Journal* 38(152), 543–559.

Samuelson, P.A. (1958) An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy* 66(6), 467–482.

Schwartz, L. (1957-1959) *Théorie des distributions*, Vols. I and II. Paris: Hermann.

Sidgwick, H. (1874) *The Methods of Ethics*. London: Macmillan.

Stern, N. (2006) Stern Review on the Economics of Climate Change. Available at http://www.hm-treasury.gov.uk/independent_reviews/stern_review_economics_climate_change/stern_review_report.cfm.

Stern, N. (2007) Value Judgements, Welfare Weights and Discounting: Issues and Evidence. Available at http://www.hm-treasury.gov.uk/media/C/8/Paper_B.pdf.

U.S. Office of Management and Budget (2003) OMB Circular A-4: Regulatory Analysis. Available at http://www.whitehouse.gov/OMB/circulars/a004/a-4.pdf.

Wildi, W., D. Appel, M. Buser, F. Dermange, A. Eckhardt, P. Hufschmied, and H.-R. Keusen (2000) Disposal Concepts for Radioactive Waste. Written on Behalf of the Federal Department for the Environment, Transport, Energy and Communication of Switzerland.

APPENDIX A: AN EQUATION WITH SHIFT OPERATORS

LEMMA 8. *Let E be a set, with maps $\mathbf{S}_h : E \to E$ s.t. $\mathbf{S}_{h_1} \circ \mathbf{S}_{h_2} = \mathbf{S}_{h_1+h_2}$. Let V be the space of functions of time with values in $\overline{\mathbf{R}}^n$; note that $\mathbf{S}_h$ operates on V by Definition 2.1. For $\varphi : E \to V$, assume that $\forall h\ \exists a_h \in \mathbf{R}^n$, $b_h \in \mathbf{R}_{++}$, both Lebesgue measurable in h, s.t. $\varphi \circ \mathbf{S}_h = a_h + b_h \mathbf{S}_h \circ \varphi$.*

Then $\exists \zeta \in \mathbf{R}$ and $A \in \mathbf{R}^n$ such that $\forall h$, one can take $b_h = e^{\zeta h}$, $a_h = A\frac{e^{\zeta h}-1}{e^{\zeta}-1}$, the fraction being defined by continuity if $\zeta = 0$.

ζ is not unique iff $\exists \alpha \in \mathbf{R}^n : \varphi_e^i(t) \in \mathbf{R} \Rightarrow \varphi_e^i(t) = \alpha^i$, the superscript denoting the coordinate. When ζ is unique, A^i is unique iff $\exists e, t : \varphi_e^i(t) \in \mathbf{R}$.

Proof. Let $h = h_1 + h_2$. Then $\varphi \circ \mathbf{S}_h = a_{h_2} + b_{h_2}[\mathbf{S}_{h_2} \circ (a_{h_1} + b_{h_1}\mathbf{S}_{h_1} \circ \varphi)]$. So $a_h + b_h \mathbf{S}_h \circ \varphi = a_{h_2} + b_{h_2} a_{h_1} + b_{h_2} b_{h_1} \mathbf{S}_h \circ \varphi$.

If, for some pair h_1, h_2, $b_h \neq b_{h_1} b_{h_2}$, then whenever $(\mathbf{S}_h \circ \varphi)_e^i(t) \in \mathbf{R}$, the above equation determines its value, say α^i. The same obviously holds then for φ itself. For such a φ, one can set $a_h = 0$, $b_h = 1\ \forall h$; thus we can always assume $b_{h_1+h_2} = b_{h_1} b_{h_2}$. Because $b_h > 0$, taking logarithms reduces it to $f(x+y) = f(x) + f(y)$, of which it is well known that any Lebesgue-measurable solution is linear [Fréchet (1913)]. Thus $b_h = e^{\zeta h}$.

As to a_h, for each i, if $\exists e, t : (\mathbf{S}_h \circ \varphi)_e^i(t) \in \mathbf{R}$, then our above equation simplifies, after substituting the b's, to $a_{h_1+h_2}^i = a_{h_2}^i + a_{h_1}^i e^{\zeta h_2}$, and otherwise one can set $a_h^i = 0\ \forall h$, so again we can assume that the above equation holds always. The same argument as above implies then the result in the case $\zeta = 0$. And for $\zeta \neq 0$, we get $a_{h_2} + a_{h_1} e^{\zeta h_2} = a_{h_1} + a_{h_2} e^{\zeta h_1}$, i.e., $a_{h_1}(e^{\zeta h_2} - 1) = a_{h_2}(e^{\zeta h_1} - 1)$. This implies first $a_0 = 0$, hence the result for $h = 0$, and next that, for all h_1, h_2 different from 0, $a_{h_i}/(e^{\zeta h_i} - 1)$ is independent of i, so $\frac{a_h}{e^{\zeta h}-1}$ is constant over all $h \neq 0$. Because $\zeta \neq 0$, we can write this constant as $\frac{A}{e^{\zeta}-1}$, thus finishing the proof, the uniqueness part being elementary. ■

APPENDIX B: PROOFS

B.1. THE SPACE K_E

This section is to enable a more convenient laguage (in Lemma 9) for the statement of the main result: else all topological concepts there (like continuity, Gâteaux-differentiability, density) would have to be given a more complex "sequential" reinterpretation, entailing in addition a slight loss of generality (any case covered by that theorem would also be by the present one, endowing F with the weak topology generated by the sequentially continuous linear functionals).

Proof of Remark 1. Obviously, for $\varphi \in K_E$, $f \circ \varphi \in K_{\mathbf{R}}$. Conversely, because $f \circ \varphi$ is C^∞ for all $f \in E^*$, φ is C^∞ with values in E [e.g., Edwards (1965, Ex. 8.14, p. 609)]. Because further each $f \circ \varphi$ has compact support, it is elementary that φ has compact support. ■

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DEFINITION 13. *The topology on K_E is the strongest locally convex topology s.t. φ_n in K_E converges to 0 iff φ_n and its successive derivatives converge uniformly to* 0 *and* $\exists h \in \mathbf{R}$: $\varphi_n(x) = 0$ *for* $|x| \geq h$.

LEMMA 9. *The topology on K_E is uniquely defined, and K_E is an LF space [e.g., Kelley and Namioka (1963; 22C p. 218)]. A sequence in K_E converges to* 0 *iff it is as in Definition 13. A linear map to a locally convex space is continuous iff it is sequentially so.*

Proof. The $K_n = \{\varphi \in K_E \mid |x| \geq n \Rightarrow \varphi(x) = 0\}$, endowed with the topology of uniform convergence of all derivatives, are an increasing sequence of Fréchet spaces (using Remark 1, or directly), with K_n topologically a closed subspace of K_{n+1}. Thus the specified topology is the strongest locally convex topology s.t. the inclusions $K_n \subseteq K_E$ are continuous: this is the inductive limit topology $\mathscr{T}$ [Kelley and Namioka (1963; 16Cb p. 149)], and also [Kelley and Namioka (1963; 22C p. 218)] the inductive limit $(K_E, \mathscr{T})$ is an *LF* space, and is a *strict inductive limit* [Kelley and Namioka (1963; 17Gb p. 164)].

For the second sentence: a convergent sequence is bounded, hence contained in some K_n [Kelley and Namioka (1963; 17Gb.iii p. 164)]. Hence it is as specified, using [Kelley and Namioka (1963; 17Gb.i p. 164)]. The last sentence follows now straight from Definition 13. ∎

B.2. THE MAIN THEOREM

Proof of Theorem 1. If ζ is not uniquely determined, Lemma 1 implies that W is, on every straight line through $\bar{\pi}$, constant in a neighborhood of $\bar{\pi}$. Letting $q = 0$ thus makes the result true for any ζ.

Else there exists, by definition of a Gâteaux differential, $\mu \in F^*$ s.t.

$$DW_{\bar{\pi}}(\delta\pi) = \lim_{\varepsilon \to 0} \frac{W(\bar{\pi} + \varepsilon\delta\pi) - W(\bar{\pi})}{\varepsilon} = \langle \mu, \delta\pi \rangle. \qquad \textbf{(B.1)}$$

Start with the particular case $E = \mathbf{R}$ and $F = K$, using K for $K_{\mathbf{R}}$.

By Lemma 1, $W \circ \mathbf{S}_h = e^{\zeta h} W + A \frac{e^{\zeta h} - 1}{e^{\zeta} - 1}$; hence by constancy of $\bar{\pi}$ ($\mathbf{S}\bar{\pi} = \bar{\pi}$), and (B.1),

$$\langle \mu, \mathbf{S}_h(\delta\pi) \rangle = e^{\zeta h} \langle \mu, \delta\pi \rangle.$$

Because B is a neighborhood of $\bar{\pi}$, every $\varphi \in K$ is a multiple of some $\delta\pi$; hence the following holds for all $h \in \mathbf{R}$ and all $\varphi \in K$:

$$\langle \mu, \varphi - e^{-\zeta h} \mathbf{S}_h \varphi \rangle = 0.$$

Dividing by h and taking the limit (in K !) as $h \to 0$ yields

$$\langle \mu, \varphi' + \zeta\varphi \rangle = 0.$$

The definition of the derivative of a generalized function, $\mu \in K^*$,

$$\langle \mu', \varphi \rangle = -\langle \mu, \varphi' \rangle, \forall \varphi \in K,$$

yields then

$$\langle \zeta\mu - \mu', \varphi \rangle = 0, \forall \varphi \in K.$$

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By Gel'fand and Shilov (1959; p. 53), the equation $\zeta\mu - \mu' = 0$ has $\mu = qe^{\zeta t}$ for some $q \in \mathbf{R}$ as only solutions in K^*, so

$$DW_{\bar{\pi}}(\delta\pi) = \langle qe^{\zeta t}, \delta\pi\rangle = q\int e^{\zeta t}\delta\pi_t dt, \forall\delta\pi \in K. \qquad \textbf{(B.2)}$$

The next step is to extend the result to any Banach space E and $F = K_E$.

LEMMA 10. *Any function $\varphi \in K_E$ can be approximated in K_E by functions with finite-dimensional range.*

Proof. Let $D_n = \{\varphi^{(i)}\left(\frac{j}{n!}\right) \mid 0 \leq i \leq n,\ j \in \mathbf{Z}\}$. D_n is an increasing sequence of finite subsets of E. Let F_n be the subspace spanned by D_n and p_n a projector from E to F_n (i.e., $p_n\colon E \to F_n$ is the identity on F_n). Its existence follows from Hahn–Banach, F_n being finite-dimensional. Then $\varphi_n = p_n \circ \varphi \in K_{F_n}$ and converges in K_E to φ. ■

Consider policy variations $\delta\pi \in K_E$ of the form $b\phi\colon t \mapsto b\phi(t)$ with $b \in E$ and $\phi \in K$. By (B.1) and (B.2), $\forall b \in E\ \exists q_b \in \mathbf{R}$ s.t. $\langle\mu, b\phi\rangle = q_b\int\phi(t)e^{\zeta t}dt\ \forall\phi \in K$. So, for $I_\phi = \int\phi(t)e^{\zeta t}dt \neq 0$, the map $b \mapsto q_b = \frac{\langle\mu, b\phi\rangle}{I_\phi}$ is in E^*, i.e., $q_b = \langle q, b\rangle$ with $q \in E^*$.

So, for any φ of the form $b\phi$,

$$\langle\mu, \varphi\rangle = \int\langle q, \varphi(t)\rangle e^{\zeta t}dt. \qquad \textbf{(B.3)}$$

Because any $\varphi \in K_E$ with finite-dimensional range is a sum of policy variations of the form $b\phi$, (B.3) remains true for them by linearity. They being dense in K_E by Lemma 10, (B.3) extends by continuity to K_E.

Finally, we extend the result to arbitrary F.

Because Gâteaux differentials are continuous linear functionals, all assumptions remain true and Z_F is unchanged with the weak topology on F, which is locally convex. The assumption on F (Definition 1.1) and Lemma 9 imply then that the inclusion map $K_E \subseteq F$ is continuous. Hence, Gâteaux differentiability on F implies that on K_E. Thus the assumptions of the theorem hold on K_E too. So the differential is a continuous linear functional on F, given on K_E by the formula $\int\langle q, \varphi(t)\rangle e^{\zeta t}dt$. This being by assumption continuous on F, the differential on F must coincide with it, K_E being dense in F. ■

B.3. OTHER PROOFS

Proof of Lemma 4. Because $\mathbf{S}_h\bar{\pi} = \bar{\pi}$, $W(\mathbf{S}_h\pi) = V_{c,r}(v(\mathbf{S}_h\pi) - v(\mathbf{S}_h\bar{\pi}))$; by Definition 6, Lemma 2, homogeneity of $V_{c,r}$, Definition 7, and Lemma 3,

$$\begin{aligned} W(\mathbf{S}_h(\pi)) &= V_{c,r}(e^{\varrho h}\mathbf{S}_h(v(\pi) - v(\bar{\pi}))) = e^{\varrho hr}V_{c,r}(\mathbf{S}_h(v(\pi) - v(\bar{\pi}))) \\ &= a_h e^{\varrho hr} + e^{(\varrho r + c)h}V_{c,r}(v(\pi) - v(\bar{\pi})) = a'_h + e^{(\varrho r + c)h}W(\pi). \end{aligned} \qquad ■$$

Proof of Lemma 6. Because the closed convex cone Y is pointed (irreversibility), there exists a linear functional α whose unique maximizer on Y is 0. Then $\langle\alpha, y\rangle \leq -\varepsilon\|y\|$ on Y; i.e., by rescaling α, $\langle\alpha, y\rangle \leq -\|y\|$. Observe too that free disposal implies that $\alpha \gg 0$. Write α as $(\alpha^L, \alpha^K, \alpha^I, \alpha^C)$.

The first step is to establish the bound on K sub (ii).

Fix a vector $\bar{L} \in \mathbf{R}^h$ s.t. any feasible vector of labor inputs $L_t \leq \bar{L}e^{(\gamma+\nu)t}$ (i.e., to compute a given coordinate of $\bar{L}$, assume that all agents spend 100% of their time on that activity).

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Allow perfect substitution at rates α^I between all investment goods and between all capital goods: let $F\colon \mathbf{R}^2_+ \to \mathbf{R}_+\colon (\bar\kappa, \lambda) \mapsto \sup\{\langle \alpha^I, I\rangle \mid \exists K \geq 0, \langle \alpha^I, K\rangle \leq \bar\kappa, (-\lambda \bar L, -K, I, 0) \in Y\}$.

The sup is finite, because $\langle \alpha^I, I\rangle \leq \langle \alpha^K, K\rangle + \lambda\langle \alpha^L, \bar L\rangle$ and $\langle \alpha^K, K\rangle$ is bounded on the compact set $K \geq 0, \langle \alpha^I, K\rangle \leq \bar\kappa$ (recall $\alpha \gg 0$). Further the sup is achieved, the sets $\{y \in Y \mid \langle \alpha, y\rangle \geq -M\}$ being compact (because then $\|y\| \leq M$), so that the sup is effectively over a compact set. Clearly F is positively homogeneous of degree 1 and concave, and is continuous again because locally everything happens within a compact subset of Y.

Thus by homogeneity $F(\bar\kappa, \lambda) = \lambda\varphi(\frac{\bar\kappa}{\lambda})$, where $\varphi(x) = F(x, 1)$ is concave, ≥ 0, and continuous. Further, by the "no-rabbit" assumption, $F(\bar\kappa, 0) = 0$, so, by continuity of F, $\frac{\varphi(x)}{x} \to 0$ at ∞.

For any feasible path (L_t, K_t, I_t, C_t), let $\tilde\imath_t = \langle \alpha^I, I_t\rangle$ and $\tilde\kappa_t = \langle \alpha^I, K_t\rangle$. Then, because $\delta \leq \delta^i$ and $K \geq 0$, the capital-accumulation equation implies that $\tilde\kappa_t' \leq \tilde\imath_t - \delta\tilde\kappa_t$; further, $L_t \leq \bar L e^{(\gamma+\nu)t}$ implies (free-disposal) $\tilde\imath_t \leq e^{(\gamma+\nu)t}\varphi(e^{-(\gamma+\nu)t}\tilde\kappa_t)$. So with $(l_t, \kappa_t, \iota_t, c_t) = e^{-(\gamma+\nu)t}(L_t, \tilde\kappa_t, \tilde\imath_t, C_t)$:

$$\kappa_t' \leq \varphi(\kappa_t) - R\kappa_t.$$

To bound $\|k_t\|$, it suffices to prove from this that κ_t is bounded by some constant independent of the feasible path, because $\alpha^I \gg 0$.

Also, the initial condition yields that $e^{Rt}\kappa_t$ converges exponentially fast to 0 at $-\infty$; i.e., because $R > 0$, there exists $\varepsilon\colon 0 < \varepsilon < R$ such that, with $r = R - \varepsilon > 0$, $e^{rt}\kappa_t \to 0$ at $-\infty$ along a subsequence. Because $\frac{\varphi(x)}{x} \to 0$ at ∞, there exists A s.t., $\forall x, \varphi(x) \leq A + \varepsilon x$; so $\kappa_t' \leq A - r\kappa_t$.

The next step is to prove from this that $\kappa_t \leq \bar K$, with $\bar K = A/r$.

Otherwise $\kappa_{t_0} > \bar K$ for some t_0; because $\kappa_t' < 0$ for $\kappa_t > \bar K$, this implies that $\kappa_t > \bar K$ and is decreasing for $t \leq t_0$. Define y by $y_t' = A - ry_t$ with the prescribed terminal value κ_{t_0} at t_0. The relations for κ and y are equivalent to $\frac{r\,d}{de^{rt}}(e^{rt}\kappa_t) \leq A$ and $\frac{r\,d}{de^{rt}}(e^{rt}y_t) = A$, so, because $\frac{de^{rt}}{rdt} > 0$, $\frac{d}{dt}(e^{rt}\kappa_t) \leq \frac{d}{dt}(e^{rt}y_t)$: for $t \leq t_0, \kappa_t \geq y_t = A/r + (\kappa_{t_0} - A/re^{r(t_0-t)}$, contradicting that $e^{rt}\kappa_t \to 0$ at $-\infty$ along a subsequence.

Hence the uniform bound on $\|k_t\|$.

Next, $\langle \alpha, y\rangle \leq -\|y\|$ yields $\int_a^b \|i_t\|dt \leq \int_a^b(\langle \alpha^L, l_t\rangle + \langle \alpha^K, k_t\rangle - \langle \alpha^I, i_t\rangle - \langle \alpha^C, c_t\rangle)dt$. The last inner product is nonnegative, and the capital-accumulation equation yields $k_t'^j = i_t^j - R^j k_t^j$ with $R^j = \gamma + \nu + \delta^j$ as before, so that $\int_a^b \langle \alpha^I, i_t\rangle dt = \int_a^b \langle \alpha^I, k_t'\rangle dt + \sum_j \alpha_j^I R^j \int_a^b k_t^j dt = \langle \alpha^I, k_b - k_a\rangle + \sum_j \alpha_j^I R^j \int_a^b k_t^j dt \geq -\langle \alpha^I, k_a\rangle$, because $R^j > 0$. Thus our bounds on k_t and l_t imply that $\int_a^b \|i_t\|dt \leq \bar K(b - a + 1)$ for some constant $\bar K$.

Thus point (ii). For (i), $e^{-(\gamma+\nu)t}\int_{-\infty}^t e^{\delta^j(s-t)}|I_s^j|ds = \int_0^\infty e^{-R^j x}|i_{t-x}^j|dx \leq \sum_{n=0}^\infty e^{-R^j n}\int_n^{n+1}|i_{t-x}^j|dx$ is uniformly bounded by (ii). For $M_t^i = e^{\delta^i t}K_t^i$, the differential equations become $M_t^{i\,\prime} = h_t^i$, with $h_t^i \overset{\text{def}}{=} e^{\delta^i t}I_t^i$; hence, by the integrability, $M_t^i = M_{-\infty}^i + \int_{-\infty}^t h_s^i ds$. And the initial condition yields $\lim_{t\to-\infty} M_t^i = 0$, so $M_{-\infty}^i = 0$; hence (i). ■

Proof of Proposition 1. Comparing the rescaling of consumption goods in the consumption sets (ii) and in the production set (iii) shows that the mass of any agent is to be multiplied by $C = e^{\nu h}$. For labor goods, this ratio is correct too, given the labor-saving technological growth included in aggregate effective labor. By (i), the "induced map of agents" maps an individual of type τ born at time t to an individual of the same type born at $t + h$, so

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the population of the new economy at time t is that of the old at time $t-h$ multiplied by $e^{\nu h}$, and hence equals that of the original economy at t: the population measure is preserved. It thus remains only to prove that preferences are preserved and that allocations are mapped to allocations: the one-to-one and onto aspect will then follow from the same property for the inverse $\mathbf{T}_{-h}$. Consumption sets are unchanged: at any t nonnegativity constraints are preserved by the rescalings (ii), also, time fractions are not rescaled, so the constraint that their sum be ≤ 1 is preserved too. Preferences are homogeneous in the consumption goods, so they are preserved by rescaling (ii). As for production, capital accumulation equations are linear in capital and investment, so they are preserved given (iii), as well as the initial condition (also in its weak form of Proposition. 4): both convergence to 0 and exponential convergence to 0 are preserved under shifting and multiplication by a constant. And Y is unchanged under scaling by $e^{(\gamma+\nu)h}$ (iii). ■

Proof of Corollary 3. Let $V_c = \sum_\tau V_c^\tau$, where $V_c^{\tau_0}$ with $c = \nu - \beta$ is the utility aggregator defined as $\int_{-\infty}^{\infty} e^{-\beta t} \sum_\tau \mathbf{1}_{\tau_0} N_t^\tau \left[v_t^\tau(\pi) - \bar{u}_t^\tau\right] dt$. Applying Corollary 2 to each V_c^τ, i.e., to the economy in which utilities of all types but τ are identically zero, one obtains the differential of W:

$$\sum_\tau \int_{-\infty}^{\infty} e^{(\nu-\beta+(1-\rho^\tau)\gamma)t} \langle q^\tau, \delta\pi(t)\rangle dt = \int_{-\infty}^{\infty} e^{(\nu-\beta+\gamma)t} \left\langle \sum_\tau e^{-\rho^\tau \gamma t} q^\tau, \delta\pi(t)\right\rangle dt.$$

Visibly the criterion q^τ of the types with the smallest ρ asymptotically gets all the weight. ■

Proof of Proposition 3. Shift invariance of A implies $\mathbf{T}$-invariance of the set of the induced (acceptable) allocations under an outcome function ς; thus if $\varsigma(\pi) = (c, l)$ for some $\pi \in A$, then $\forall h\ \varsigma(\mathbf{S}_h\pi) = (e^{\gamma h}\mathbf{S}_h c, \mathbf{S}_h l)$, and $\mathbf{S}_h\pi \in A$. So the utility difference between the worst and best acceptable allocations for an agent of type τ is, by homogeneity of utilities, proportional to $e^{(1-\rho^\tau)\gamma x}$: this is the normalization factor. Thus, again by homogeneity, $\mathbf{M}_A[v_x^\tau(\pi)] = w^\tau U^\tau(e^{-\gamma x} c_x^\tau, l_x^\tau)$. Hence, because $\varsigma \circ \mathbf{S}_h = \mathbf{T}_h \circ \varsigma$ and $e^{-\gamma x}\mathbf{T}_h c_x^\tau = \mathbf{S}_h(e^{-\gamma x} c_x^\tau)$, one gets $\mathbf{M}_A[v_x^\tau(\mathbf{S}_h\pi)] = w^\tau U^\tau(\mathbf{S}_h(e^{-\gamma x} c_x^\tau, l_x^\tau)) = \mathbf{S}_h \mathbf{M}_A[v_x^\tau(\pi)]$. ■

APPENDIX C: INITIAL CONDITIONS

Even with the weakest initial condition, say K_t bounded at $-\infty$, one should expect K_t^i to converge to 0 at $-\infty$ if $\delta^i > 0$. But land and natural resources are the typical examples of goods with $\delta^i = 0$, so the natural value for δ in a general form of the initial condition and Lemma 6 is 0.[29] The initial condition is thus a bit too stringent conceptually, requiring exponential convergence to the initial value 0 instead of just plain convergence. An additional reason to want just plain convergence there is that then the "initial condition" becomes equivalent to the initial condition for the integral formula of K_t in terms of I_t: even with $\delta^i = 0$, $K_t^i = K_{-\infty}^i + \int_{-\infty}^t e^{\delta^i(s-t)} I^i(s) ds$ implies $K_t^i \to K_{-\infty}^i$ at $-\infty$.[30] We make a first attempt here to address this issue.

LEMMA 11.

(i) *Every $\beta \gg 0$ in $\mathbf{R}^m$ is the α^I of some linear functional α having a unique maximizer on Y.*

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(ii) *For $\alpha \gg 0$, let $\psi_\alpha(x) = \sup\{\langle \alpha, I\rangle \mid \exists K \geq 0, \|K\| \leq x,$ $(-(1,1,1,\dots), -K, I, 0) \in Y\}$. Then $\int_1^\infty \frac{\psi_\alpha(x)}{x^2} dx$ is finite iff the same integral is finite replacing $\psi_\alpha(x)$ by $\varphi_\alpha(x) \stackrel{\text{def}}{=} \sup\{\langle \alpha, I\rangle \mid \exists K \geq 0, \langle \alpha, K\rangle \leq x, (-\bar{L}, -K, I, 0) \in Y\}$.*

Proof. For point (i), let E be the commodity space $\mathbf{R}^h \times \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^n$, containing Y, with vectors typically denoted $(-L, -K, I, C)$. Let F be the subspace where $L = C = 0$, and let β' extend β with arbitrary positive K-coordinates. Let $G = \{x \in F \mid \langle \beta', x\rangle = 0\}$. By the no-rabbit assumption, $G \cap Y = \{0\}$. By irreversibility, there exists a linear functional γ on E having a unique maximizer on Y; so $Y' = \{y \in Y \mid \langle \gamma, y\rangle \leq -1\}$ and G are disjoint closed convex sets, with disjoint asymptotic cones: their difference is a closed convex set disjoint from 0; hence they can be strictly separated. So there is a linear functional α with $\langle \alpha, G\rangle > \langle \alpha, Y'\rangle$. G being a subspace, this implies that α vanishes on G, and has 0 as unique maximizer on Y. Thus some positive multiple of α coincides with β' on F; in particular, $\alpha^I = \beta$.

For point (ii), observe first that the integrability condition on $\varphi_\alpha(x)$ is equivalent to that on $\varphi_\alpha(cx)$, for any $c > 0$. Now, $K \geq 0$ and $\alpha \gg 0$ imply that $\langle \alpha, K\rangle$ is a norm, so for any norm there exist $\underline{c} > 0$ and $\bar{c}$ such that $\underline{c}\|K\| \leq \langle \alpha, K\rangle \leq \bar{c}\|K\|$. The independence from c of the condition on φ_α allows then to replace that inner product by $\|K\|$. Similarly, to replace $\bar{L}$ by a vector of 1's, first majorize and minorize it by a multiple of this vector. ■

As the proof of Lemma 6 shows, the "no-rabbit" condition is equivalent to $\varphi(x)/x \to 0$, so the condition $\int_1^\infty [\varphi(x)/x^2] dx < \infty$ appears as a very slight strengthening. This justifies the following:

DEFINITION 14. *The "strong no-rabbit" condition on Y is that $\int_1^\infty \frac{\psi_\alpha(x)}{x^2} dx < \infty$ for some $\alpha \gg 0$.*

PROPOSITION 4. *Provided that the strong no-rabbit condition holds, the conclusions of Lemma 6 remain true when weakening the exponentially fast convergence to plain convergence in the initial condition.*

Proof. Fix α according to the strong no-rabbit condition, and, using Lemma 11.i to find a corresponding α in the proof of Lemma 6, follow that proof until where φ $(= \varphi_\alpha)$ is majorized by $A + \varepsilon x$, and now let $f(x) = Rx - \varphi(x)$, $\bar{K} = \inf\{k \mid f(k) > 0\}$, and fix a corresponding κ_{t_0}.

Then, prove first that, for $t \leq t_0$, $\kappa_t \geq y_t$, with y_t the solution of $y_t' = \varphi(y_t) - Ry_t$ with prescribed value at t_0: reversing time, and translating t_0 to 0, we have, using x_t for κ_t, $x_t' \geq f(x_t)$ a.e., $y_t' = f(y_t)$ a.e., $x_0 = y_0$, $f(x_0) > 0$, f is increasing for $x \geq x_0$, and need to show $x_t \geq y_t$ for $t > 0$. Translating f and the functions x, y, we can even assume that $x_0 = y_0 = 0$, $f(0) > 0$, so f is positive and increasing on $\mathbf{R}_+$. So $H(x) = \int_0^x [1/f(y)] dy$ is well-defined, positive, C^1, concave, and increasing on $\mathbf{R}_+$. Assuming the chain rule for differentiation established for the composition $H \circ x$ of such an H with a Denjoy primitive like x_t, we obtain $(H \circ x)_t' = H'(x_t)x_t' = \frac{x_t'}{F(x_t)} \geq 1$, and similarly $(H \circ y)_t' = 1$; hence, for $t \geq 0$, $H(x_t) \geq H(y_t)$ and so $x_t \geq y_t$ by strict monotonicity of H.

Thus indeed $\kappa_t \geq y_t$ for $t \leq t_0$. Because further κ_t and y_t are decreasing and continuous on that range, they have continuous and decreasing inverse functions t^κ and t^y defined on $[\kappa_{t_0}, \infty[$ and values in $]-\infty, t_0]$, and there $t^\kappa \geq t^y$. Now $y_t' = -f(y_t)$ means that $\frac{dy}{f(y)} = -dt$; hence, because $y_{t_0} = \kappa_{t_0}$, $t^y(x) = t_0 - \int_{\kappa_{t_0}}^x \frac{dz}{f(z)}$. So $t^\kappa(x) \geq t_0 - \int_{\kappa_{t_0}}^x \frac{dz}{f(z)}$.

But the "weak" initial condition is that $e^{Rt}\kappa(t) \xrightarrow[t\to-\infty]{} 0$; so $e^{Rt^{\kappa}(x)}x \xrightarrow[x\to\infty]{} 0$, i.e., $\ln(x) + Rt^{\kappa}(x) \xrightarrow[x\to\infty]{} -\infty$, and thus, by our bound on t^{κ},[31] and replacing $\ln$ by its integral definition, neglecting constants, $\int_a^x \frac{dz}{z} - R\int_a^x \frac{dz}{f(z)} \xrightarrow[x\to\infty]{} -\infty$, with $a = \max\{1, \kappa_{t_0}\}$. Given the formula for f, this means $\int_a^x \frac{\varphi(z)}{z(Rz-\varphi(z))} \to \infty$, and hence, $\varphi(z)$ being negligible compared to z, and $R > 0$, that $\int_1^x \frac{\varphi(z)}{z^2} \to \infty$, contradicting the strong no-rabbit condition by Lemma 11.ii. Thus indeed $\kappa_t \leq \bar{K}\ \forall t$. The rest of the proof of Lemma 6 remains as is. ■

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