The incomplete gamma functions

G. J. O. JAMESON

Definitions and elementary properties

Recall the integral definition of the gamma function: $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$ for $a > 0$. By splitting this integral at a point $x \geqslant 0$, we obtain the two *incomplete gamma functions*:

$$\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt, \quad (1)$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}dt, \quad (2)$$

$\Gamma(a, x)$ is sometimes called the *complementary incomplete gamma function*. These functions were first investigated by Prym in 1877, and $\Gamma(a, x)$ has also been called *Prym's function*. Not many books give these functions much space. Massive compilations of results about them can be seen stated without proof in [1, chapter 9] and [2, chapter 8]. Here we offer a small selection of these results, with proofs and some discussion of context. We hope to convince some readers that the functions are interesting enough to merit attention in their own right.

Clearly, $\Gamma(a, 0) = \Gamma(a)$ and

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a) \quad (3)$$

for all $x \geqslant 0$ and $a > 0$. Also, $\Gamma(1, x) = e^{-x}$ and $\gamma(1, x) = 1 - e^{-x}$.

For $x > 0$, the integral (2) converges for all real a, so we regard it as defining $\Gamma(a, x)$ for all such a. In particular, $\Gamma(0, x)$ is the 'exponential integral' $\int_x^\infty t^{-1}e^{-t}dt$. This case has a number of features of its own: some were described in the recent *Gazette* article [3].

The integral (1) only converges for $a > 0$, but in fact the definition of $\gamma(a, x)$ can be extended to negative a, as we see later.

Note on complex a and x: The definition of $\gamma(a, x)$ makes sense for complex a with $\operatorname{Re} a > 0$, and the definition of $\Gamma(a, x)$ for all complex a. Also, with due interpretation of the integrals, one can replace x by a complex variable z. However, in this note we confine ourselves to the case where a and x are real. Some of the results, which interested readers will be able to recognise, apply without change when a is complex.

Of course, $\gamma(a, x)$ and $\Gamma(a, x)$ can be considered both as functions of x (for fixed a) and as functions of a (for fixed x). Our emphasis will be firmly on them as functions of x. First, some simple facts. Since the integrand is non-negative, so are $\gamma(a, x)$ and $\Gamma(a, x)$. For fixed a, $\gamma(a, x)$ is an increasing function of x, with $\lim_{x\to\infty} \gamma(a, x) = \Gamma(a)$, and $\Gamma(a, x)$ is a

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decreasing function of x with $\lim_{x \to \infty} \Gamma(a, x) = 0$ (this applies also for $a \leqslant 0$). By the fundamental theorem of calculus, we have

$$\frac{d}{dx}\gamma(a, x) = -\frac{d}{dx}\Gamma(a, x) = x^{a-1}e^{-x}. \tag{4}$$

If $a > 1$, this is largest when $x = a - 1$.

We record some inequalities that follow very easily from the integrals (1) and (2), and shed some light on their nature for small and large x. Firstly, since $e^{-x} \leqslant e^{-t} \leqslant 1$ for $0 \leqslant t \leqslant x$, we have $t^{a-1}e^{-x} \leqslant t^{a-1}e^{-t} \leqslant t^{a-1}$ for such t. Now $\int_0^x t^{a-1}dt = x^a/a$, so

$$e^{-x}\frac{x^a}{a} \leqslant \gamma(a, x) \leqslant \frac{x^a}{a}. \tag{5}$$

Hence $\gamma(a, x) \sim x^a/a$ as $x \to 0^+$. (Recall that the notation $f(x) \sim g(x)$ as $x \to x_0$ means $f(x)/g(x) \to 1$ as $x \to x_0$.)

Secondly, if $a \geqslant 1$, then $t^{a-1} \geqslant x^{a-1}$ for $t \geqslant x$, hence

$$\Gamma(a, x) \geqslant x^{a-1}\int_x^\infty e^{-t}dt = x^{a-1}e^{-x}, \tag{6}$$

and clearly the opposite holds for $a \leqslant 1$. For $a < 0$, another inequality, comparable to the left-hand side of (5), and stronger than (6) for small x, is

$$\Gamma(a, x) \leqslant e^{-x}\int_x^\infty t^{a-1}dt = e^{-x}\frac{x^a}{-a}. \tag{7}$$

We remark that (5) gives $\gamma(a, a) \geqslant a^{a-1}e^{-a}$, while (6) gives $\Gamma(a, a) \geqslant a^{a-1}e^{-a}$. Hence we have shown, with minimal effort, that $\Gamma(a) \geqslant 2a^{a-1}e^{-a}$ for $a \geqslant 1$, an elementary inequality of the same type as Stirling's formula.

We mention some equivalent forms given by simple substitutions. For $c > 0$, the substitution $ct = u$ gives

$$\int_0^x t^{a-1}e^{-ct}dt = \int_0^{cx}\left(\frac{u}{c}\right)^{a-1}e^{-u}\frac{1}{c}\,du = \frac{1}{c^a}\gamma(a, cx), \tag{8}$$

and similarly $\int_x^\infty t^{a-1}e^{-ct}dt = \frac{1}{c^a}\Gamma(a, cx)$. Note the case $x = 1$:

$$\gamma(a, c) = c^a\int_a^1 t^{a-1}e^{-ct}dt.$$

The substitution $t = x + u$ gives

$$\Gamma(a, x) = e^{-x}\int_0^\infty (x + u)^{a-1}e^{-u}du. \tag{9}$$

The substitution $u = t^n$ gives

$$\int_0^x e^{-t^n}dt = \frac{1}{n}\int_0^{x^n} e^{-u}u^{\frac{1}{n}-1}du = \tfrac{1}{n}\gamma\left(\tfrac{1}{n}, x^n\right).$$

Integration by parts; two basic identities; evaluation for positive integer a

The most basic property of the gamma function is the identity $\Gamma(a + 1) = a\Gamma(a)$. We now show how this identity decomposes into two companion ones for the incomplete gamma functions. This is achieved by a very simple integration by parts. Clarity and simplicity are gained by stating the basic result for general integrals of the same type. Given a function $f(t)$ on $(0, \infty)$, write

$$I_f(x) = \int_0^x f(t)e^{-t}dt, \qquad J_f(x) = \int_x^\infty f(t)e^{-t}dt$$

if these integrals exist.

If $I_f(x)$ and $I_{f'}(x)$ exist and $f(0) = 0$, then for $x > 0$,

$$I_f(x) = \left[-f(t)e^{-t}\right]_0^x + \int_0^x f'(t)e^{-t}dt = -f(x)e^{-x} + I_{f'}(x). \tag{10}$$

Similarly, if $J_f(x)$ and $J_{f'}(x)$ exist and $f(x)e^{-x} \to 0$ as $x \to \infty$, then

$$J_f(x) = f(x)e^{-x} + J_{f'}(x). \tag{11}$$

If $f(t) = t^a$, then $I_f(x) = \gamma(a + 1, x)$ and $I_{f'}(x) = a\gamma(a, x)$, and similarly for J_f and $J_{f'}$. Also, $f(0) = 0$ if $a > 0$, and $\lim_{x \to \infty} f(x)e^{-x} = 0$ for any a, so we conclude:

Theorem 1: For $x > 0$ and $a > 0$,

$$\gamma(a + 1, x) = a\gamma(a, x) - x^a e^{-x}. \tag{12}$$

For $x > 0$ and all a,

$$\Gamma(a + 1, x) = a\Gamma(a, x) + x^a e^{-x}. \tag{13}$$

Added together, (12) and (13), with (3), reproduce the identity $\Gamma(a + 1) = a\Gamma(a)$.

The process in (10) and (11) can be repeated by application to f' and higher derivatives. In the case of (11), the statement is:

Theorem 2: Suppose that $J_{f^r}(x)$ exists for $0 \leqslant r \leqslant k$ and that $f^{(r)}(x)e^{-x} \to 0$ as $x \to \infty$ for $0 \leqslant r \leqslant k - 1$. Then

$$J_f(x) = e^{-x}\left[f(x) + f'(x) + \ldots + f^{(k-1)}(x)\right] + J_{f^{(k)}}(x). \tag{14}$$

With $f(x) = x^a$, the condition $\lim_{x \to \infty} f^{(r)}(x)e^{-x} = 0$ is, of course, satisfied for all r. The corresponding development of (10) is trickier to apply, because it requires the condition $f^{(r)}(0) = 0$ for successive r, which will fail for some r.

One can write out explicitly what (14) says for $\Gamma(a, x)$ in general: the outcome is a fairly complicated asymptotic expression effective for large x. However, for positive integers a, it simplifies and delivers at once a closed expression for $\Gamma(a, x)$. For more pleasant notation, we will state this with $a = n + 1$: note that $\Gamma(n + 1, x) = \int_x^\infty t^n e^{-t} dt$. Of course, once we have evaluated $\Gamma(n + 1, x)$, the value of $\gamma(n + 1, x)$ is given by $\gamma(n + 1, x) = n! - \Gamma(n + 1, x)$.

If $f(x) = x^n$, then $f^{(n+1)}(x) = 0$, so the expression in (14) terminates, and we have

$$\Gamma(n+1, x) = J_f(x) = e^{-x}\left[f(x) + f'(x) + \ldots + f^{(n)}(x)\right]. \tag{15}$$

The bracketed sum simply comprises successive derivatives until they become zero. With no further effort, we can write down the first few cases:

$$\Gamma(2, x) = e^{-x}(x + 1), \qquad \Gamma(3, x) = e^{-x}(x^2 + 2x + 2),$$
$$\Gamma(4, x) = e^{-x}(x^3 + 3x^2 + 6x + 6).$$

We now give an expression for the general case. For this purpose, write

$$e_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!},$$

the exponential series truncated after $n + 1$ terms.

Theorem 3: For integers $n \geqslant 1$ and $x \geqslant 0$,

$$\Gamma(n + 1, x) = n!\, e_n(x) e^{-x}. \tag{16}$$

Proof: For $f(x) = x^n$, we have

$$f^{(k)}(x) = n(n - 1)\ldots(n - k + 1)x^{n-k} = \frac{n!}{(n - k)!} x^{n-k}.$$

Now applying (15) and substituting r for $n - k$, we obtain

$$\Gamma(n + 1, x) = n!\, e^{-x} \sum_{k=0}^{n} \frac{x^{n-k}}{(n - k)!} = n!\, e^{-x} \sum_{r=0}^{n} \frac{x^r}{r!}.$$

This displays $\Gamma(n + 1, x)$ in a rather pleasing way as a fraction of $\Gamma(n + 1) = n!$, and shows that $\Gamma(n + 1, x)/n! \to 1$ as $n \to \infty$ for any fixed x. It also shows that $\Gamma(n + 1, x) \sim x^n e^{-x}$ as $x \to \infty$, supplementing (6).

While this derivation is, surely, attractive enough, it is instructive to see a second, equally efficient proof.

Alternative proof of (16): By (9), the binomial expansion and the fact that $\Gamma(n - r + 1) = (n - r)!$, we have

$$\Gamma(n + 1, x) = e^{-x} \int_0^\infty (x + u)^n e^{-u} du$$

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$$= e^{-x}\sum_{r=0}^{n}\binom{n}{r}x^r\int_0^\infty u^{n-r}e^{-u}du$$

$$= e^{-x}\sum_{r=0}^{n}\binom{n}{r}x^r\Gamma(n-r+1)$$

$$= e^{-x}\sum_{r=0}^{n}\frac{n!}{r!}x^r.$$

Note: Write I_n for $\gamma(n+1, 1) = \int_0^1 t^n e^{-t}dt$. It has been known for students (including my students) to be set the exercise of evaluating, say, I_3 by repeated application of the recurrence relation implied by (12), starting with I_0. After at least as much work as either of the methods just described, the student arrives (barring accidents) at the answer $I_3 = 6 - 16e^{-1}$, but this approach does little to reveal the pattern and the formula for general n.

I am indebted to the referee for pointing out that Theorem 3 leads to the following neat proof of the irrationality of e. By (16), $\Gamma(n+1, 1) = B_n e^{-1}$, where $B_n = \sum_{r=0}^{n}\frac{n!}{r!}$, which is an integer. So $\gamma(n+1, 1) = A_n - B_n e^{-1}$, where $A_n = n!$. Note that $\gamma(n+1, 1) > 0$. Suppose now that $e = p/q$, where p and q are integers. Then $p\gamma(n+1, 1) = A_n p - B_n q$: for each n, this is positive and an integer, hence at least 1. But by (5), $\gamma(n+1, 1) \leqslant \frac{1}{n+1}$, so $p\gamma(n+1, 1) \to 0$ as $n \to \infty$, a contradiction.

It is clear from (5) and (6) that functions like $x^{-a}\gamma(a, x)$ and $e^x\Gamma(a, x)$ are particularly relevant. Using (12) and (13), we can derive very satisfactory expressions for the derivatives of these functions. By (4) and (12), we have

$$\frac{d}{dx}\frac{\gamma(a, x)}{x^a} = -\frac{a\gamma(a, x)}{x^{a+1}} + \frac{x^{a-1}e^{-x}}{x^a}$$

$$= \frac{1}{x^{a+1}}\left[-a\gamma(a, x) + x^a e^{-x}\right]$$

$$= -\frac{\gamma(a+1, x)}{x^{a+1}},$$

and similarly for $\Gamma(a, x)$. So if $f(a, x) = x^{-a}\gamma(a, x)$, then $\frac{d}{dx}f(a, x) = -f(a+1, x)$. We can deduce at once that the nth derivative is $(-1)^n f(a+n, x)$.

By (4) and (13), we have

$$\frac{d}{dx}\left[e^x\Gamma(a, x)\right] = e^x\left[\Gamma(a, x) - x^{a-1}e^{-x}\right] = e^x(a-1)\Gamma(a-1, x),$$

and similarly for $\gamma(a, x)$. In particular, $e^x\Gamma(a, x)$ increases with x when $a \geqslant 1$, and decreases when $a < 1$.

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Series expression for $\gamma(a, x)$ and extension to $a < 0$

We now give an explicit power-type series expression for $\gamma(a, x)$.

Theorem 4: For $a > 0$ and $x > 0$,

$$\gamma(a,x) = x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\,(a+n)} = x^a \left(\frac{1}{a} - \frac{x}{a+1} + \frac{x^2}{2!\,(a+2)} - \dots \right). \tag{17}$$

Proof: This expression is obtained at once by termwise integration on $[0, x]$ of the series

$$t^{a-1}e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{a+n-1}.$$

Termwise integration, for any readers who care, is justified by uniform convergence of the power series for e^{-t} on bounded intervals (after separating out the first term if $a < 1$).

In principle, (17) enables us to calculate $\gamma(a,x)$, although in practice the calculation is only pleasant for fairly small x.

For a fixed $x > 0$, the series (17) converges for all a except 0 and negative integers, so we take it as the definition of $\gamma(a,x)$ for such a. We will show that the extended function still satisfies the basic identity (12).

For the gamma function itself, the usual procedure is to extend the definition by the identity $\Gamma(a+1) = a\Gamma(a)$: given $\Gamma(a+1)$, this defines $\Gamma(a)$. Meanwhile, $\Gamma(a,x)$ is already defined for all a. We show that the two extensions are compatible, in the sense that the identity (3) still holds.

Theorem 5: For all a except 0 and negative integers, and all $x > 0$,

$$\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x}. \tag{18}$$

If the definition of $\Gamma(a)$ is extended in the way just stated, then $\gamma(a, x) + \Gamma(a, x) = \Gamma(a)$ for all such a.

Proof: We have

$$\begin{aligned} a\gamma(a, x) &= x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \frac{a}{a+n} \\ &= x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \left(1 - \frac{n}{a+n}\right) \\ &= x^a e^{-x} - x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n-1)!\,(a+n)} \\ &= x^a e^{-x} + x^a \sum_{m=0}^{\infty} \frac{(-1)^m x^{m+1}}{m!\,(a+m+1)} \\ &= x^a e^{-x} + a\gamma(a+1, x). \end{aligned}$$

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Now write $\gamma(a, x) + \Gamma(a, x) = F(a, x)$. We know that $F(a, x) = \Gamma(a)$ for $a > 0$. By (13) and (18), $F(a + 1, x) = aF(a, x)$. Hence if we know that $F(a+1, x) = \Gamma(a+1)$, it follows that $F(a, x) = \Gamma(a)$. By repeated backwards steps of length 1, it now follows that $F(a, x) = \Gamma(a)$ for all a except 0 and negative integers.

A satisfying application of Theorem 5 is that it gives an explicit formula for the extended function $\Gamma(a)$. For this purpose, we only need to take $x = 1$, obtaining

$$\Gamma(a) = \gamma(a, 1) + \Gamma(a, 1) = \int_1^\infty t^{n-1}e^{-t}dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(a + n)}.$$

In fact, as an alternative to the procedure described above, one can adopt this as the definition of $\Gamma(a)$ for such a. If this approach is chosen, then the conclusion from (18) is that the gamma function (extended in this way) still satisfies $\Gamma(a + 1) = a\Gamma(a)$.

Since the identity $\gamma(a, x) + \Gamma(a, x) = \Gamma(a)$ still applies, so does the identity $\frac{d}{dx}\gamma(a, x) = -\frac{d}{dx}\Gamma(a, x) = x^{a-1}e^{-x}$. In particular, $\gamma(a, x)$ is an increasing function of x. However, it may well be negative: indeed (18) shows that it is certainly negative when $-1 < a < 0$.

Some integrals

We now evaluate some integrals of expressions involving the incomplete gamma functions. Since these are already defined as integrals, the integrals considered will appear as double integrals, and evaluation will be achieved by reversing them (which is valid, because the integrands are positive). The answers will be in terms of the gamma function itself.

Consider $\int_0^\infty x^{p-1}\Gamma(a, x)\,dx$, possibly with $a < 0$. If $a > 0$, then convergence at 0 requires $p > 0$. For $a < 0$, one can show, building on (7), that $\Gamma(a, x) \sim -x^a/a$ as $x \to 0^+$, so convergence at 0 requires $a + p > 0$. The need for these conditions also shows up clearly in the following proof.

Theorem 6: Suppose that either (i) $a > 0$ and $p > 0$ or (ii) $a \leqslant 0$ and $p > -a$. Then

$$\int_0^\infty x^{p-1}\Gamma(a, x)\,dx = \frac{1}{p}\Gamma(a + p). \tag{19}$$

In particular, for $a > -1$,

$$\int_0^\infty \Gamma(a, x)\,dx = \Gamma(a + 1). \tag{20}$$

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Proof: Under either condition, $p > 0$. Reversing the double integral, we have

$$\begin{aligned}\int_0^\infty x^{p-1}\Gamma(a, x)\,dx &= \int_0^\infty x^{p-1}\int_x^\infty t^{a-1}e^{-t}dt\,dx \\ &= \int_0^\infty t^{a-1}e^{-t}\left(\int_0^t x^{p-1}dx\right)dt \\ &= \frac{1}{p}\int_0^\infty t^{a+p-1}e^{-t}\,dt \\ &= \frac{1}{p}\Gamma(a+p).\end{aligned}$$

This is (19). The case $p = 1$ is (20).

We leave it to the reader to prove in similar style the following companion result for $\gamma(a, x)$: for $0 < p < a$,

$$\int_0^\infty \frac{1}{x^{p+1}}\gamma(a, x)\,dx = \frac{1}{p}\Gamma(a-p). \tag{21}$$

Note that the case $p = a - 1$ simplifies to $\int_0^\infty \frac{1}{x^a}\gamma(a, x)\,dx = \frac{1}{a-1}$ for $a > 1$.

An alternative method for both (19) and (21) is integration by parts, using (4) for $\gamma'(a, x)$ and $\Gamma'(a, x)$. One needs (5) and (7) to identify the limits at 0.

Theorem 7: For $a > 0$ and $c > 0$,

$$\int_0^\infty e^{-cx}\gamma(a, x)\,dx = \frac{\Gamma(a)}{c(c+1)^a}, \tag{22}$$

$$\int_0^\infty e^{-cx}\Gamma(a, x)\,dx = \frac{\Gamma(a)}{c}\left(1 - \frac{1}{(c+1)^a}\right). \tag{23}$$

Proof: Note that the substitution $bt = u$ gives $\int_0^\infty t^{a-1}e^{-bt}dt = \frac{\Gamma(a)}{b^a}$. Using this, we have

$$\begin{aligned}\int_0^\infty e^{-cx}\gamma(a, x)\,dx &= \int_0^\infty e^{-cx}\int_0^x t^{a-1}e^{-t}\,dt\,dx \\ &= \int_0^\infty t^{a-1}e^{-t}\left(\int_t^\infty e^{-cx}dx\right)dt \\ &= \int_0^\infty t^{a-1}e^{-t}\,\frac{e^{-ct}}{c}\,dt\end{aligned}$$

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$$= \frac{1}{c}\int_0^\infty t^{a-1}e^{-(c+1)t}dt$$

$$= \frac{\Gamma(a)}{c(c+1)^a},$$

which is (22). We deduce (23) using (3) and $\int_0^\infty e^{-cx}dx = \frac{1}{c}$.

In particular, $\int_0^\infty e^{-x}\gamma(a, x)\,dx = \frac{1}{2^a}\Gamma(a)$.

The case $a = 0$ in (23) is special. To solve it, we need the following integral: for positive a, b,

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}dx = \log b - \log a, \tag{24}$$

which can be proved by the method of [3, Proposition 2], or, very neatly, by expressing the integrand as $\int_a^b e^{-xy}dy$ and reversing the double integral obtained.

Theorem 8: For $c > -1$,

$$\int_0^\infty e^{-cx}\Gamma(0, x)\,dx = \frac{1}{c}\log(1 + c).$$

Proof: The integral is

$$\int_0^\infty e^{-cx}\int_x^\infty \frac{e^{-t}}{t}\,dt\,dx = \int_0^\infty \frac{e^{-t}}{t}\left(\int_0^t e^{-cx}dx\right)dt$$

$$= \int_0^\infty \frac{e^{-t}\left(1 - e^{-ct}\right)}{ct}\,dt.$$

By (24), this equals $\frac{1}{c}\log(1 + c)$.

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G. J. O. JAMESON

Dept. of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF

e-mail: *g.jameson@lancaster.ac.uk*