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Large deviations in Axiom A endomorphisms

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By applying a general large-deviation theorem of Kifer and Ruelle's Smale space technique, some large-deviation estimates are proved for Axiom A endomorphisms.

1. Introduction and statement of main results

Consider a discrete time dynamical system generated by a measurable self-map $f : X \hookleftarrow$ of some measurable space $(X, \mathcal{B})$. Let P be a reference probability measure on $(X, \mathcal{B})$ and let $\psi : X \to \boldsymbol{R}$ be an observable. If $(1/n)\sum_{k=0}^{n-1} \psi \circ f^k$ converges to some constant $\psi^* P$-a.e. as $n \to \infty$, then, for given $\varepsilon > 0$,

$$Q_n(\varepsilon) := \left\{ x \in X : \left| \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) - \psi^* \right| > \varepsilon \right\}$$

satisfies $P(Q_n(\varepsilon)) \to 0$ as $n \to +\infty$. Large-deviation theory in this set-up deals with estimates of the exponential speed of this last convergence to zero. More precisely and more generally, large-deviation questions concern estimates of the following form,

$$\limsup_{n \to +\infty} \frac{1}{n} \log P\left\{ x \in X : \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) \in K \right\} \leqslant -\inf_{z \in K} I(z) \tag{1.1}$$

for any closed set $K \subset \boldsymbol{R}$ and

$$\liminf_{n \to +\infty} \frac{1}{n} \log P\left\{ x \in X : \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) \in G \right\} \geqslant -\inf_{z \in G} I(z) \tag{1.2}$$

for any open set $G \subset \boldsymbol{R}$, where $I : \boldsymbol{R} \to [0, +\infty)$ is a lower semi-continuous function and is called *a rate function*. Such questions have been well studied by Orey and Pelikan [9] for Anosov diffeomorphisms and by Young [13], among other things, for Axiom A attractors. Developing the ideas of [2, 4, 5, 12], Kifer [7] presents a unified approach to large deviations of dynamical systems and stochastic processes based on the existence of a pressure function and on the uniqueness of equilibrium states

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for certain potentials, and this approach enables one to generalize results from [9] and [13] and to recover the large-deviation estimates in Donsker and Varadhan [3]. In this paper, we apply Kifer's results in [7], together with Ruelle's Smale space technique in [11], to give some large-deviation estimates for Axiom A endomorphisms.

Our set-up and main results are as follows. Let M be a Riemannian manifold without boundary, O an open subset of M with compact closure and $f : O \to M$ a C^r $(r \geqslant 1)$ map. Let $\Lambda = f(\Lambda) \subset O$ be a compact invariant set of f and let

$$\Lambda^f := \{\bar{x} = (x_i)_{-\infty}^{+\infty} : x_i \in \Lambda,\ f(x_i) = x_{i+1},\ i \in \boldsymbol{Z}\}$$

be the orbit space of (Λ, f) with $\sigma_f : \Lambda^f \to \Lambda^f$ denoting the left-shift operator on Λ^f. Write $E = p^* T_\Lambda M$ for the pull-back bundle of $T_\Lambda M$ via the natural projection $p : \Lambda^f \to \Lambda$, $\bar{x} \mapsto x_0$, and write

$$E_{\bar{x}} = p_{\bar{x}}^* T_{x_0} M \rightleftharpoons_{p_{\bar{x}}^*}^{p_*} T_{x_0} M$$

for the natural isomorphisms between the fibres $E_{\bar{x}}$ and $T_{x_0} M$. A fibre-preserving map on E that covers σ_f can be defined by

$$p_{\sigma_f(\bar{x})}^* \circ Tf \circ p_* : E_{\bar{x}} \to E_{\sigma_f(\bar{x})}$$

for all $\bar{x} \in \Lambda^f$, and, for simplicity of notation, we will denote it still by Tf.

Definition 1.1. Λ is called a hyperbolic set of f if there is a continuous splitting $E = E^{\mathrm{s}} \oplus E^{\mathrm{u}}$, together with constants $C > 0$ and $0 < \lambda < 1$, such that

$$Tf E^{\mathrm{s}} \subset E^{\mathrm{s}}, \qquad Tf E^{\mathrm{u}} = E^{\mathrm{u}}$$

and, for all $n \geqslant 0$, $|Tf^n \xi| \leqslant C\lambda^n |\xi|$ for $\xi \in E^{\mathrm{s}}$, $|Tf^n \eta| \geqslant C^{-1}\lambda^{-n}|\eta|$ for $\eta \in E^{\mathrm{u}}$.

Via a change of Riemannian metric we may and will assume that $C = 1$. Note that there may be points in Λ at which Tf is degenerate, and that the splitting $E_{\bar{x}} = E_{\bar{x}}^{\mathrm{s}} \oplus E_{\bar{x}}^{\mathrm{u}}$ may depend on the past of $\bar{x}$, i.e. it may happen that $p_* E_{\bar{x}}^{\mathrm{u}} \neq p_* E_{\bar{y}}^{\mathrm{u}}$ while $p(\bar{x}) = p(\bar{y})$. In what follows we denote by S_f the set of points in O at which Tf is degenerate, and by m the Lebesgue measure on M.

A hyperbolic set Λ is said to be an *Axiom A basic set* if Λ is locally maximal (i.e. there exists a neighbourhood U of Λ such that $\bigcap_{n=-\infty}^{+\infty} f^n U = \Lambda$) and f is positively topologically transitive on it (i.e. $(f^n x_0)_{n \geqslant 0}$ is dense in Λ for some $x_0 \in \Lambda$). (It can be shown that periodic points are dense in an Axiom A basic set.) If an Axiom A basic set Λ has arbitrarily small open neighbourhood U such that $f\bar{U} \subset U$ and $\bigcap_{n=0}^{+\infty} f^n U = \Lambda$, it is then called an *Axiom A attractor*, and U is called a *basin of attraction of* Λ. Applying Ruelle's Smale space technique, Qian and Zhang [10] presents an ergodic theory of such an Axiom A basic set Λ. In particular, they proved that (Λ, f) admits a unique equilibrium state μ_ϕ for each Hölder continuous $\phi : \Lambda \to \boldsymbol{R}$ and, in case of Λ being an attractor of $f \in C^2(O, M)$ with basin of attraction U and $m(S_f) = 0$, Λ supports a unique f-invariant measure ρ, called the *SRB measure*, which is generic with respect to Lebesgue measure in the following sense: for m-a.e. $x \in \bar{U}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k x) = \int_\Lambda \psi \,\mathrm{d}\rho \quad \text{for all } \psi \in C(\bar{U}).$$

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Our main results of this paper are as follows, where $\mathcal{P}(X)$ denotes the space of Borel probability measures on a compact metric space X endowed with the topology of weak convergence.

THEOREM 1.2.

(1) *Let Λ be an Axiom A basic set of $f \in C^1(O, M)$, let $\phi : \Lambda \to \boldsymbol{R}$ be Hölder continuous and let μ_ϕ be the unique equilibrium state. Then there hold*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu_\phi \left\{ x \in \Lambda : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \in K \right\} \leqslant -\inf\{J(\nu) : \nu \in K\} \tag{1.3}$$

for any closed $K \subset \mathcal{P}(\Lambda)$ and

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mu_\phi \left\{ x \in \Lambda : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \in G \right\} \geqslant -\inf\{J(\nu) : \nu \in G\} \tag{1.4}$$

for any open $G \subset \mathcal{P}(\Lambda)$, where

$$J(\nu) = \begin{cases} P_f(\phi) - \int \phi \, \mathrm{d}\nu - h_\nu(f) & \text{if } \nu \in \mathcal{P}_f(\Lambda), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.5}$$

$\mathcal{P}_f(\Lambda)$ is the set of f-invariant measures on Λ, $P_f(\phi)$ is the pressure of f for ϕ and $h_\nu(f)$ is the entropy of (f, ν).

(2) *Let Λ be an Axiom A attractor of $f \in C^2(O, M)$ and let ρ be the SRB measure on Λ. Then (1.3) and (1.4) hold true with μ_ϕ being replaced by ρ and with $J(\cdot)$ being defined by*

$$J(\nu) = \begin{cases} \int \sum_i \lambda^{(i)}(x)^+ m^{(i)}(x) \, \mathrm{d}\nu - h_\nu(f) & \text{if } \nu \in \mathcal{P}_f(\Lambda), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.6}$$

where $\lambda^{(i)}(x)$, $1 \leqslant i \leqslant r(x)$, are the Lyapunov exponents of f at x, $m^{(i)}(x)$ is the multiplicity of $\lambda^{(i)}(x)$ and $a^+ := \max\{a, 0\}$.

(3) *Assume the circumstances of (2). Let U be a sufficiently small basin of attraction of Λ and let $\bar{m}$ be the normalized Lebesgue measure on $\bar{U}$. Then (1.3) and (1.4) hold true with μ_ϕ and Λ being replaced by $\bar{m}$ and $\bar{U}$, respectively, and with $J(\cdot)$ being given by (1.6).*

The proof of this theorem will be given in § 2. From theorem 1.2 and the contraction principle, we have the following result.

COROLLARY 1.3. *Let $\psi : O \to \boldsymbol{R}$ be a continuous function and let us be in the circumstances of theorem 1.2 (3). For $J(\cdot)$ given by (1.6), put*

$$I(z) = \inf\left\{ J(\nu) : \int \psi \, \mathrm{d}\nu = z \right\}.$$

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Then (1.1) and (1.2) hold true with P and X being taken, respectively, as $\bar{m}$ and $\bar{U}$. In particular, for $\varepsilon > 0$, there exists $h > 0$ such that

$$\bar{m}\biggl\{x \in \bar{U} : \biggl|\frac{1}{n}\sum_{k=0}^{n-1}\psi(f^k x) - \int \psi\,\mathrm{d}\rho\biggr| \geqslant \varepsilon\biggr\} \leqslant \mathrm{e}^{-hn}$$

when n is sufficiently large.

Similar results hold true in the circumstances of (1) or (2) of theorem 1.2.

2. Proof of theorem 1.2

2.1. A large-deviation theorem from Kifer [7]

Let (X, d) be a compact metric space, $f : (X, d) \hookleftarrow$ a continuous map, and, as before, $\mathcal{P}(X)$ the space of Borel probabilities on X endowed with the weak convergence topology and $\mathcal{P}_f(X)$ the set of those elements in $\mathcal{P}(X)$ that are f-invariant. For $x \in X$, $\varepsilon > 0$ and $n \in \boldsymbol{N}$, put

$$B_f(x, \varepsilon, n) = \{y \in X : d(f^k x, f^k y) \leqslant \varepsilon,\ 0 \leqslant k \leqslant n-1\}.$$

The following theorem is a special case of the general large-deviation results of [7], and we will apply it to Axiom A endomorphisms in this paper.

THEOREM 2.1. *Suppose that $\mu \in \mathcal{P}(X)$, the support of μ is the whole X and there is $\phi \in C(X)$ such that, for any given small $\varepsilon > 0$ and for all $n \geqslant 1$, $x \in X$,*

$$A_\varepsilon(n)^{-1} \leqslant \mu(B_f(x, \varepsilon, n)) \exp\biggl(-\sum_{k=0}^{n-1}\phi(f^k x)\biggr) \leqslant A_\varepsilon(n), \tag{2.1}$$

where $A_\varepsilon(n) > 0$ is a constant satisfying $(1/n)\log A_\varepsilon(n) \to 0$ as $n \to +\infty$. Then, for any $\psi \in C(X)$, there holds

$$\lim_{n\to\infty}\frac{1}{n}\log\int \exp\biggl(\sum_{k=0}^{n-1}\psi(f^k x)\biggr)\mathrm{d}\mu(x) = P_f(\phi + \psi) = P_{f|_Y}(\phi + \psi), \tag{2.2}$$

where $P_f(\cdot)$ denotes the pressure of f and Y is the closure of $\bigcup_{\nu\in\mathcal{P}_f(X)}\operatorname{supp}\nu$. Suppose further that the entropy $h_\nu(f)$ is upper semicontinuous at all $\nu \in \mathcal{P}_f(X)$ and define

$$J(\nu) = \begin{cases} -\displaystyle\int \phi\,\mathrm{d}\nu - h_\nu(f) & \text{if } \nu \in \mathcal{P}_f(X), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

Then the above conclusion implies

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mu\biggl\{x : \sum_{k=0}^{n-1}\delta_{f^k x} \in K\biggr\} \leqslant -\inf\{J(\nu) : \nu \in K\} \tag{2.4}$$

for any closed set $K \subset \mathcal{P}(X)$. Moreover, if there exist a countable number of functions $\psi_1, \psi_2, \ldots \in C(X)$ such that their span

$$\varGamma = \biggl\{\sum_{i=1}^{n}\beta_i\psi_i : \beta_i \in \boldsymbol{R},\ n \in \boldsymbol{N}\biggr\}$$

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is dense in $C(X)$ with respect to the supremum norm, and that, for each $\psi \in \Gamma$, there is a unique $\nu_\psi \in \mathcal{P}(X)$ satisfying

$$P_f(\phi + \psi) = \int \psi \, \mathrm{d}\nu_\psi - J(\nu_\psi), \tag{2.5}$$

then one has, for any open $G \subset \mathcal{P}(X)$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mu\biggl\{x : \sum_{k=0}^{n-1} \delta_{f^k x} \in G\biggr\} \geqslant -\inf\{J(\nu) : \nu \in G\}. \tag{2.6}$$

2.2. Smale spaces

Here we recall the notion and some properties of Smale spaces from Ruelle [11].

Definition 2.2. Suppose that (X, d) is a compact metric space and $f : X \to X$ a homeomorphism. (X, d, f) is said to be a Smale space if, for suitable $\varepsilon > 0$, $0 < \delta < \varepsilon$, $0 < \lambda < 1$, there exists a continuous map

$$[\cdot,\cdot] : \{(x, y) \in X \times X : d(x, y) < \varepsilon\} \to X$$

with the following properties.

(1) $[x, x] = x$ and $[[x, y], z] = [x, z], [x, [y, z]] = [x, z]$ when the two sides of these relations are well defined.

(2) $f[x, y] = [fx, fy]$ when both sides are well defined and

$$d(f^n y, f^n z) \leqslant \lambda^n d(y, z) \quad \text{for } y, z \in V_x^+(\delta), \quad n > 0,$$
$$d(f^{-n} y, f^{-n} z) \leqslant \lambda^n d(y, z) \quad \text{for } y, z \in V_x^-(\delta), \quad n > 0,$$

where

$$V_x^+(\delta) = \{u : u = [u, x],\ d(x, u) < \delta\}$$

and

$$V_x^-(\delta) = \{v : v = [x, v],\ d(x, v) < \delta\}.$$

Here are some properties of a Smale space (X, d, f). Define $C^f(X)$ to be the space of functions $\phi \in C(X)$ that satisfy the following conditions: there exist $\delta > 0$ and $K \geqslant 0$ such that, if $d(f^k x, f^k y) < \delta$ for $k = 0, 1, \ldots, n$, then

$$\biggl|\sum_{k=0}^{n} \phi(f^k x) - \sum_{k=0}^{n} \phi(f^k y)\biggr| \leqslant K \tag{2.7}$$

($\phi \in C^f(X)$ if it is Hölder continuous (see [11, p. 136])). If (X, f) is positively topologically transitive, then it has a unique equilibrium state μ_ϕ for each $\phi \in C^f(X)$. (X, d, f) is expansive. In the case of (X, d, f) being topologically mixing, it has the specification property, and hence there is the following proposition, which follows from Katok and Hasselblatt [6, lemma 20.3.4 and theorem 20.3.7]. For the transitive case, the proposition can be reduced to the mixing case by the spectral

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decomposition of Smale spaces (see also [11]) and by considering $(f^l, \sum_{i=0}^{l-1} \phi \circ f^i)$ restricted to one of the basic sets, say X_0, in the spectral decomposition, where l is the number of the basic sets (note that $\sum_{i=0}^{l-1} \phi \circ f^i \in C^{f^l}(X_0)$ if $\phi \in C^f(X)$; when f is not Hölderian, nor is $\sum_{i=0}^{l-1} \phi \circ f^i$, even if ϕ is).

Proposition 2.3. *Assume that (X, d, f) is a positively topologically transitive Smale space and $\phi \in C^f(X)$. Let μ_ϕ be the unique equilibrium state of (X, d, f) for ϕ. Then, for small $\varepsilon > 0$, there exist $A_\varepsilon, B_\varepsilon > 0$ such that, for $x \in X$ and $n \in \boldsymbol{N}$, one has*

$$A_\varepsilon \leqslant \mu_\phi(B_f(x, \varepsilon, n)) \exp\bigg\{ -\sum_{k=0}^{n-1} \phi(f^k x) + nP_f(\phi) \bigg\} \leqslant B_\varepsilon.$$

2.3. Smale space property of locally maximal hyperbolic sets

We first collect in the following proposition some properties of local stable and unstable manifolds of a hyperbolic set, for details see Qian and Zhang [10] which is the first paper to apply the Smale space technique in the study of Axiom A endomorphisms.

Proposition 2.4. *Let Λ be a hyperbolic set of $f \in C^r(O, M)$, $r \geqslant 1$. Then there exist in M a continuous family of C^r-embedded $\dim E^{\mathrm{s}}$-dimensional discs $\{W^{\mathrm{s}}_{\mathrm{loc}}(x_0)\}_{x_0 \in \Lambda}$ and a continuous family of C^r-embedded $\dim E^{\mathrm{u}}$-dimensional discs $\{W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})\}_{\bar{x} \in \Lambda^f}$ with the following properties.*

(1) *For each $\bar{x} = (x_i)_{i \in \boldsymbol{Z}} \in \Lambda^f$, both $W^{\mathrm{s}}_{\mathrm{loc}}(x_0)$ and $W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})$ contain x_0, and*

$$fW^{\mathrm{s}}_{\mathrm{loc}}(x_0) \subset W^{\mathrm{s}}_{\mathrm{loc}}(fx_0), \qquad fW^{\mathrm{u}}_{\mathrm{loc}}(\bar{x}) \supset W^{\mathrm{u}}_{\mathrm{loc}}(\sigma_f \bar{x}).$$

(2) *There is a $0 < \bar{\lambda} < 1$ such that, for any $\bar{x} \in \Lambda^f$,*

$$d(fy, fz) \leqslant \bar{\lambda} d(y, z) \quad \text{if } y, z \in W^{\mathrm{s}}_{\mathrm{loc}}(x_0)$$

and, for each $y_0 \in W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})$, there exists a unique $y_{-1} \in W^{\mathrm{u}}_{\mathrm{loc}}(\sigma_f^{-1}\bar{x})$ with $fy_{-1} = y_0$. Furthermore, this y_{-1} and the similar z_{-1} for $z_0 \in W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})$ satisfy $d(y_{-1}, z_{-1}) \leqslant \bar{\lambda} d(y_0, z_0)$.

(3) *There is $\delta > 0$ such that, for any $x_0 \in \Lambda$, $\bar{y} \in \Lambda^f$ with $d(x_0, y_0) < \delta$, $W^{\mathrm{s}}_{\mathrm{loc}}(x_0)$ intersects transversely with $W^{\mathrm{u}}_{\mathrm{loc}}(\bar{y})$ at a unique point $[x_0, \bar{y}]$, which depends continuously on $(x_0, \bar{y}) \in \{(u_0, \bar{v}) \in \Lambda \times \Lambda^f : d(u_0, v_0) < \delta\}$, and, furthermore, if Λ is locally maximal, then there is a unique $\bar{z} \in \Lambda^f$ satisfying $z_0 = [x_0, \bar{y}]$ and $z_i \in W^{\mathrm{u}}_{\mathrm{loc}}(\sigma_f^{-i}\bar{y})$.*

We remark that the local stable manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(x_0)$, $x_0 \in \Lambda$, can be constructed by the usual standard argument, but, since Λ may contain degenerate points, the local unstable manifolds $W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})$, $\bar{x} \in \Lambda^f$, cannot be constructed similarly. However, one may construct $W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})$ in the following (standard as well) way. Let $\bar{x} = (x_i)_{i \in \boldsymbol{Z}} \in \Lambda^f$. For small $r > 0$, let $h_{\bar{x}} : E^{\mathrm{u}}_{\bar{x}}(r) \to E^{\mathrm{s}}_{\bar{x}}(r)$ be a Lipschitz map with $h_{\bar{x}}(0) = 0$ and $\mathrm{Lip}(h_{\bar{x}}) \leqslant 1$, where

$$E^a_{\bar{x}}(r) = \{\xi \in E^a_{\bar{x}} : |\xi| < r\}, \quad a = \mathrm{u,s}.$$

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Then one can show that there is a similar map $h_{\sigma_f\bar{x}} : E^{\mathrm{u}}_{\sigma_f\bar{x}}(r) \to E^{\mathrm{s}}_{\sigma_f\bar{x}}(r)$ such that

$$(\exp_{x_1}^{-1} \circ f \circ \exp_{x_0}) \operatorname{Graph}(h_{\bar{x}}) \supset \operatorname{Graph}(h_{\sigma_f\bar{x}}) \tag{2.8}$$

(see [8, proposition 2.6] for details). Starting from

$$h_{\sigma_f^{-n}\bar{x}} : E^{\mathrm{u}}_{\sigma_f^{-n}\bar{x}}(r) \to E^{\mathrm{s}}_{\sigma_f^{-n}\bar{x}}(r), \qquad \xi \mapsto 0,$$

via the relation (2.8) one ends by succession with a C^r function

$$h^{(n)}_{\bar{x}} : E^{\mathrm{u}}_{\bar{x}}(r) \to E^{\mathrm{s}}_{\bar{x}}(r)$$

with $h^{(n)}_{\bar{x}}(0) = 0$ and $\operatorname{Lip}(h^{(n)}_{\bar{x}}) \leqslant 1$. It is easy to show that $h^{(n)}_{\bar{x}}$ converges as $n \to +\infty$ uniformly to a similar function

$$h^{(\infty)}_{\bar{x}} : E^{\mathrm{u}}_{\bar{x}}(r) \to E^{\mathrm{s}}_{\bar{x}}(r)$$

whose graph gives $W^{\mathrm{u}}_{\mathrm{loc}}(\bar{x})$ under the exponential map.

Assume in what follows that Λ is locally maximal. Let $0 < \bar{\lambda} < 1$ be as given in proposition 2.4 and define a metric on Λ^f by

$$d_f(\bar{x}, \bar{y}) = \left(\sum_{i=-\infty}^{+\infty} 2^{-|i|} d(x_i, y_i)^N \right)^{1/N}, \quad \bar{x}, \bar{y} \in \Lambda^f,$$

where $N > 0$ is an integer such that $\bar{\lambda}^N < \frac{1}{2}$.

For sufficiently small $\varepsilon > 0$, define

$$[\cdot, \cdot] : \{(\bar{x}, \bar{y}) \in \Lambda^f \times \Lambda^f : d_f(\bar{x}, \bar{y}) < \varepsilon\} \to \Lambda^f, (\bar{x}, \bar{y}) \mapsto \bar{z},$$

where $\bar{z}$ is the unique point in Λ^f given in proposition 2.4 (3) corresponding to x_0 and $\bar{y}$. It is then easy to derive the following result.

PROPOSITION 2.5. *$(\Lambda^f, d_f, \sigma_f)$ is a Smale space.*

Clearly, when (Λ, f) is positively topologically transitive, so is (Λ^f, σ_f).

2.4. Proof of theorem 1.2

In what follows, we will always endow Λ^f with the metric $d_f(\cdot, \cdot)$.

Proof of theorem 1.2 (1). Each Hölder continuous $\phi : \Lambda \to \boldsymbol{R}$ gives a Hölder continuous $\bar{\phi} = \phi \circ p : \Lambda^f \to \boldsymbol{R}$, which belongs to $C^{\sigma_f}(\Lambda^f)$. By results in the last two subsections, there is a unique equilibrium state $\bar{\mu}_{\bar{\phi}}$ of σ_f for $\bar{\phi}$ and $\mu_\phi = p\bar{\mu}_{\bar{\phi}}$ gives the unique equilibrium state of f for ϕ. Noting that there is a countable set of Hölder continuous functions which is dense in $C(\Lambda^f)$ and the entropy map of (Λ^f, σ_f) is upper semicontinuous, by theorem 2.1 and proposition 2.3, one has (1.3) and (1.4) for $(\Lambda^f, \sigma_f, \bar{\mu}_{\bar{\phi}})$, with rate function

$$\bar{J}(\bar{\nu}) = \begin{cases} P_{\sigma_f}(\bar{\phi}) - \displaystyle\int \bar{\phi} \, \mathrm{d}\bar{\nu} - h_{\bar{\nu}}(\sigma_f) & \text{if } \bar{\nu} \in \mathcal{P}_{\sigma_f}(\Lambda^f), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.9}$$

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Since $P_{\sigma_f}(\bar{\phi}) = P_f(\phi)$ and, for any $\nu \in \mathcal{P}_f(\Lambda)$, there is a unique $\bar{\nu} \in \mathcal{P}_{\sigma_f}(\Lambda^f)$ such that $p\bar{\nu} = \nu$, and this $\bar{\nu}$ satisfies $h_{\bar{\nu}}(\sigma_f) = h_\nu(f)$, one has, for any $\nu \in \mathcal{P}(\Lambda)$,

$$\inf_{p\bar{\nu}=\nu} \bar{J}(\bar{\nu}) = J(\nu),$$

where $J(\nu)$ is given by (1.5). One then obtains theorem 1.2 (1) by the contraction principle. □

Proof of theorem 1.2 (2). Define

$$\bar{\phi}^{\mathrm{u}}(\bar{x}) = -\log|\det(T_{x_0} f|_{E^{\mathrm{u}}_{\bar{x}}})|, \quad \bar{x} \in \Lambda^f. \tag{2.10}$$

It is Hölder continuous (see [10]) and the unique equilibrium state $\bar{\mu}_{\bar{\phi}^{\mathrm{u}}}$ of σ_f for $\bar{\phi}^{\mathrm{u}}$ projects under p to the SRB measure ρ, and, in this case,

$$\inf_{p\bar{\nu}=\nu} \bar{J}^{\mathrm{u}}(\bar{\nu}) = \begin{cases} \displaystyle\int \sum_i \lambda^i(x)^+ m^{(i)}(x)\,\mathrm{d}\nu - h_\nu(f) & \text{if } \nu \in \mathcal{P}_f(\Lambda), \\ +\infty & \text{otherwise,} \end{cases}$$

since $P_{\sigma_f}(\bar{\phi}^{\mathrm{u}}) = 0$, where $\bar{J}^{\mathrm{u}}(\bar{\nu})$ is given by (2.9) corresponding to $\bar{\phi}^{\mathrm{u}}$. This proves theorem 1.2 (2). □

Proof of theorem 1.2 (3). Let $\bar{\phi}^{\mathrm{u}}$ be given by equation (2.10). The following result is from [10].

LEMMA 2.6 (volume lemma). *Let Λ be a hyperbolic set of $f \in C^2(O, M)$. Then, for small $\varepsilon > 0$, $\delta > 0$, there is a constant $A_{\varepsilon,\delta} > 0$ such that, for $\bar{x} \in \Lambda^f$, $n \geqslant 0$, and $y_0 \in B_f(x_0, \varepsilon, n)$, one has*

$$A_{\varepsilon,\delta}^{-1} \leqslant m(B_f(y_0, \delta, n)) \exp\biggl[-\sum_{k=0}^{n-1} \bar{\phi}^{\mathrm{u}}(\sigma_f^k(\bar{x}))\biggr] \leqslant A_{\varepsilon,\delta}. \tag{2.11}$$

LEMMA 2.7. *Let Λ be as given in the last lemma. Then each Hölder continuous $\bar{\phi} : \Lambda^f \to \boldsymbol{R}$ is homologous to some $\hat{\phi} \in C(\Lambda^f)$ that satisfies $\hat{\phi}(\bar{x}) = \hat{\phi}(\bar{y})$ whenever $x_i = y_i$ for $i \leqslant 0$, i.e. there is $\bar{u} \in C(\Lambda^f)$ such that*

$$\bar{\phi} = \hat{\phi} + \bar{u} - \bar{u} \circ \sigma_f.$$

Proof of lemma 2.7. For each $x_0 \in \Lambda$, pick $(z_{i,x_0})_{i\in\boldsymbol{Z}} \in \Lambda^f$ with $z_{0,x_0} = x_0$. Define $r : \Lambda^f \to \Lambda^f$ by $r(\bar{x}) = \bar{x}^* = (x_i^*)_{i\in\boldsymbol{Z}}$, where

$$x_i^* = \begin{cases} x_i & \text{for } i \geqslant 0, \\ z_{i,x_0} & \text{for } i < 0. \end{cases}$$

Let $\bar{u} : \Lambda^f \to \boldsymbol{R}$ be defined by

$$\bar{u}(\bar{x}) = \sum_{j=0}^{+\infty} [\bar{\phi}(\sigma_f^j \bar{x}) - \bar{\phi}(\sigma_f^j r(\bar{x}))].$$

Then $\hat{\phi} = \bar{\phi} + \bar{u} \circ \sigma_f - \bar{u}$ satisfies the requirements (see [1, lemma 1.6] for a similar argument). □

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Now let Λ be an Axiom A attractor of $f \in C^2(O, M)$. Lemma 2.7 tells us that $\bar{\phi}^{\mathrm{u}}$ is homologous to $\phi^{\mathrm{u}} \circ p$ for some $\phi^{\mathrm{u}} \in C(\Lambda)$. Extend ϕ^{u} to a continuous function $\Phi^{\mathrm{u}} : V \to \boldsymbol{R}$, where V is a neighbourhood of Λ. Take $\varepsilon_0 > 0$ such that lemma 2.6 holds for $\varepsilon = \varepsilon_0$ and for all small $\delta > 0$. Now let U be a basin of attraction of Λ such that

$$\bar{U} \subset \bigcup_{x_0 \in \Lambda} [W^{\mathrm{s}}_{\mathrm{loc}}(x_0) \cap B(x_0, \varepsilon_0)]$$

(this union contains an open neighbourhood of Λ (see [10])), $\bar{U} \subset V$ and, moreover, $U_1 \supset \bar{U} \supset U \supset \bar{U}_2$ for two other basins of attraction U_1, U_2 of Λ. Put

$$a_n = \sup\{|\Phi^{\mathrm{u}}(x) - \Phi^{\mathrm{u}}(y)| : x, y \in \bar{U},\ d(x, y) \leqslant \varepsilon_0 \bar{\lambda}^n\}.$$

From lemma 2.6, it follows that, for any $y_0 \in \bar{U}$, small $\delta > 0$ and all $n \geqslant 0$, one has

$$A_\delta(n)^{-1} \leqslant m(B_f(y_0, \delta, n)) \exp\biggl(-\sum_{k=0}^{n-1} \Phi^{\mathrm{u}}(f^k y_0)\biggr) \leqslant A_\delta(n),$$

where

$$A_\delta(n) = A_{\varepsilon_0, \delta} \exp\biggl(\sum_{k=0}^{n-1} a_k\biggr),$$

which clearly satisfies $(1/n) \log A_\delta(n) \to 0$ as $n \to \infty$. Then, by a minor modification of the proof of [7, proposition 3.2], one can prove that, for any $\psi \in C(\bar{U})$,

$$P_{f|_{U_2}}(\Phi^{\mathrm{u}} + \psi) \leqslant \lim_{n\to\infty} \frac{1}{n} \log \int \exp\biggl(\sum_{k=0}^{n-1} \psi(f^k x)\biggr) \mathrm{d}\bar{m}(x) \leqslant P_{f|_{U_1}}(\Phi^{\mathrm{u}} + \psi),$$

which implies

$$\lim_{n\to+\infty} \frac{1}{n} \int \exp\biggl(\sum_{k=0}^{n-1} \psi(f^k x)\biggr) \mathrm{d}\bar{m}(x) = P_{f|_\Lambda}(\phi^{\mathrm{u}} + \psi),$$

where $\bar{m}$ is the normalized Lebesgue measure on $\bar{U}$ and, when working on U_1, one may take continuous extensions of Φ^{u} and ψ. Noting that, for each Hölder continuous $\psi : \bar{U} \to \boldsymbol{R}$, there is a unique equilibrium state of $f|_{\bar{U}}$ for $\Phi^{\mathrm{u}} + \psi$ and the entropy map of $f|_U$ is upper semicontinuous, one obtains theorem 1.2 (3) by theorem 2.1 for $X = \bar{U}$, $\mu = \bar{m}$ and by

$$\int \Phi^{\mathrm{u}}\, \mathrm{d}\nu = \int \sum_i \lambda^{(i)}(x)^+ m^{(i)}(x)\, \mathrm{d}\nu(x) \quad \text{for all } \nu \in \mathcal{P}_f(\bar{U}).$$

□

References

1 R. Bowen. *Equilibrium states and ergodic theory of Anosov diffeomorphisms*. Lecture Notes in Mathematics, vol. 470 (Springer, 1975).

2 A. de Acosta. Upper bounds for large deviations of dependent random vectors. *Z. Wahrsch. Verw. Gebiete* **69** (1985), 551–565.

https://doi.org/10.1017/S0308210500002997 Published online by Cambridge University Press

3 M. D. Donsker and S. R. S. Varadhan. Asymptotic evaluation of certain Markov processes expectations for large time. I. *Commun. Pure Appl. Math.* **28** (1975), 1–47.

4 R. S. Ellis. Large deviation for a general class of random vectors. *Ann. Prob.* **12** (1984), 1–12.

5 J. Gärtner. On large deviations from the invariant measure. *Theory Prob. Applic.* **22** (1977), 24–39.

6 A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems* (Cambridge University Press, 1995).

7 Y. Kifer. Large deviations in dynamical systems and stochastic processes. *Trans. Am. Math. Soc.* **321** (1990), 505–524.

8 P.-D. Liu. Stability of orbit spaces of endomorphisms. *Manuscr. Math.* **93** (1997), 109–128.

9 S. Orey and S. Pelikan. Deviations of trajectory averages and the defect in Pesin's formula for Anosov diffeomorphisms. *Trans. Am. Math. Soc.* **315** (1989), 741–753.

10 M. Qian and Z.-S. Zhang. Ergodic theory for Axiom A endomorphisms. *Ergod. Theory Dynam. Syst.* **15** (1995), 161–174.

11 D. Ruelle. *Thermodynamic formalism* (Addison-Wesley, 1978).

12 Y. Takahashi. Entropy functional (free energy) for dynamical systems and their random perturbations. In *Taniguchi Symp., Katata, Japan, 1982*, pp. 437–467.

13 L.-S. Young. Some large deviation results for dynamical systems. *Trans. Am. Math. Soc.* **318** (1990), 525–543.

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