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The Lane–Emden equation in strips

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We study the Lane–Emden equation in strips.

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1. Introduction

In this work we investigate the celebrated Lane–Emden equation (see (1.1) below). We restrict ourselves to the case in which the domain is a strip: given integers $n \geqslant 2$ and $k \geqslant 1$, we set

$$\Omega = \mathbb{R}^{n-k} \times \omega, \quad \text{where } \omega \subset \mathbb{R}^k \text{ is a smoothly bounded domain.}$$

For $p > 1$, we consider the equation

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

The case of a homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

will also be considered. Strips provide an interesting example of unbounded domains where, as we shall see, rather sharp classification results can be obtained. Let us begin by observing that the symmetries of the domain allow solutions of the form[1]

$$u(x', x'') = u(x'') \quad \text{for } x = (x', x'') \in \Omega \tag{1.3}$$

[1] With a standard abuse of notation, we have used the same letter u to denote distinct functions.

and, more generally,

$$u(x', x'') = u(\rho, x'') \quad \text{for } x = (x', x'') \in \Omega,\ \rho = |x'|. \tag{1.4}$$

Solutions of the form (1.3) are simply solutions in the bounded domain $\omega \subset \mathbb{R}^k$. As can be read in any introductory book on partial differential equations (PDEs), in the simplest situation, for example, when ω is strictly star shaped, such solutions with Dirichlet boundary condition exist if and only if $1 < p < p_S(k)$, where $p_S(k)$ is the classical Sobolev exponent in dimension k. Similarly, if $n - k \geqslant 2$, solutions of the form (1.4) can be constructed when $1 < p < p_S(n)$ by applying the mountain pass lemma in the space H of functions $u \in H_0^1(\Omega)$ having cylindrical symmetry, i.e. such that (1.4) holds almost everywhere (a.e.); see Esteban [4]. The case in which $p \geqslant p_S(n)$ seems rather unexplored for strips. To gain further insight into the problem, we observe that the aforementioned solutions have very distinct stability properties. Recall that a solution of (1.1) is said to be stable in an open subset $\Omega' \subset \Omega$ if

$$\int_{\Omega'} |\nabla\varphi|^2 \,\mathrm{d}x - p\int_{\Omega'} |u|^{p-1}\varphi^2 \,\mathrm{d}x \geqslant 0 \quad \text{for every } \varphi \in C_c^1(\Omega'). \tag{1.5}$$

We prove the following proposition.

Proposition 1.1. *Every solution of the form (1.3) is unstable outside every compact set, while if $n - k \geqslant 2$, the mountain pass solution in the space H of H^1 functions with symmetry (1.4) is stable outside a compact set.*

Remark 1.2. The first part of the proposition is essentially due to Farina (see [5, example 1]).

Remark 1.3. For quite general nonlinearities, in the $n - k \leqslant 2$ case, Dancer [2] proved that every bounded stable solution is, after rotation in the x' variable, a function of x_1, x'' only, is monotone in the x_1 direction, and the limits of $u(x_1, x'')$ as $x_1 \to \pm\infty$ are stable solutions of the problem on ω with the same energy.

The above results suggest that in the supercritical case the class of stable solutions might also be relevant. We obtain the following result, also valid for nodal solutions.

Theorem 1.4. *Assume that $p > 1$. The Dirichlet problem (1.1) has no non-trivial stable solution $u \in C^2(\overline{\Omega})$. More generally, for $p \geqslant p_S(n)$, there is no non-trivial solution that is stable outside a compact set.*

Remark 1.5. Farina [5, theorem 7] proved the above result in the special case in which u is stable and $1 < p < p_c(n - k)$, where $p_c(n - k)$ is the Joseph–Lundgren stability exponent in dimension $n-k$; see, for example, [3] for the definition. Farina also observed that the exponent $p_c(n)$ is sharp when $\Omega = \mathbb{R}^n$. Our result shows that this exponent is irrelevant when Ω is a strip. This also corroborates a recent result of Chen *et al.* [1], who proved, in the case of the half-space $\Omega = \mathbb{R}^n_+$, that given any $p > 1$ there is no positive solution to (1.1). Note that in this case, every positive solution is known to be stable.

Remark 1.6. Recall that for $1 < p < p_S(n)$ there exists a positive solution of the form (1.4) and so, according to proposition 1.1, theorem 1.4 is sharp.

We turn now to the case of homogeneous Neumann boundary conditions. Here, stability in Ω means that (1.5) holds for every $\varphi \in H^1(\Omega)$.

THEOREM 1.7. *Assume that $1 < p < p_c(n-k)$. Then the Neumann problem has no non-trivial stable solution $u \in C^2(\overline{\Omega})$. More generally, if $p_S(n-k) < p < p_c(n-k)$, there is no non-trivial solution that is stable outside a compact set. Finally, if $p = p_S(n-k)$, any solution that is stable outside a compact set is a function of x' only and it has finite energy*

$$\int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1} < \infty.$$

REMARK 1.8. We do not know whether the second statement of theorem 1.7 remains true in the case in which $1 < p < p_S(n-k)$. But the theorem is sharp in the $p \geqslant p_S(n-k)$ case. Indeed, there exists a (radial) stable solution of the Lane–Emden equation in $\mathbb{R}^{n-k}$ whenever $p \geqslant p_c(n-k)$, which is also clearly[2] a stable solution of the Neumann problem of the form (1.4). Similarly, when $p = p_S(n-k)$, the standard bubble in $\mathbb{R}^{n-k}$ is a solution of Morse index 1 in $\mathbb{R}^{n-k}$. As such, it is stable outside a compact set of $\mathbb{R}^{n-k}$ and so it is also stable outside a compact set of Ω.

NOTATION. Without further notice, we shall use the following notation. A point $x \in \mathbb{R}^n$ is written $x = (x', x'') \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. The same applies to the operators $\nabla = (\nabla', \nabla'')$ and $\Delta = \Delta' + \Delta''$. Polar coordinates in the x' variable are written $x' = \rho\theta$, where $\rho = |x'|$ and $\theta = x'/|x'|$ whenever $x' \neq 0$. In particular, $\partial_\rho = \nabla \cdot x'/\rho$ differentiates functions in the ρ-variable, and $\Delta' = \rho^{1+k-n}\partial_\rho(\rho^{n-k-1}\partial_\rho) + \rho^{-2}\Delta_\theta$, where Δ_θ is the Laplace–Beltrami operator on the unit sphere S^{n-k-1} of $\mathbb{R}^{n-k}$.

2. Proof of proposition 1.1

As mentioned in the introduction, the fact that solutions of the form (1.3) are unstable outside any compact set is an obvious generalization of [5, example 1], so we skip it.

Turning to mountain pass solutions in the space H of functions in $H^1_0(\Omega)$ having the cylindrical symmetry (1.4), we recall that, thanks to Solimini's work [8], they have Morse index 1 in H. In particular, there exists a function $\phi \in H$ such that $Q_u(\phi) < 0$, where the second variation of the energy is defined as usual by

$$Q_u(\phi) = \int_\Omega |\nabla \phi|^2 \, \mathrm{d}x - p \int_\Omega |u|^{p-1}\phi^2 \, \mathrm{d}x.$$

Without loss of generality, we may assume in addition that $\phi \in C^\infty_c(\Omega)$. Fix $R > 0$ large enough so that ϕ is supported in $B'(0,R) \times \omega$. We claim that u is stable outside $\overline{B'(0,R) \times \omega}$. To see this, assume by contradiction that a function $\varphi \in C^\infty_c(\Omega \setminus \overline{B'(0,R) \times \omega})$ satisfies $Q_u(\varphi) < 0$. Let

$$\chi(\rho, x'') = \left[\frac{1}{|S^{n-k-1}|}\int_{S^{n-k-1}} \varphi^2(\rho\theta, x'') \, \mathrm{d}\sigma(\theta)\right]^{1/2}$$

[2] The stability in Ω is a direct consequence of the stability in $\mathbb{R}^{n-k}$ and Fubini's theorem.

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denote the quadratic average of φ on the unit sphere of $\mathbb{R}^{n-k}$, and let $\phi_0(x', x'') = \chi(|x'|, x'')$. Clearly, $\phi_0 \in H$ has disjoint support from ϕ. We are going to prove that $Q_u(\phi_0) < 0$, so that u would have Morse index in H at least equal to 2, which is a contradiction. Note that

$$\int_\Omega |u|^{p-1}\varphi^2 = \int_\Omega |u|^{p-1}\phi_0^2. \tag{2.1}$$

For $\epsilon > 0$, let

$$\chi_\epsilon(\rho, x'') = [\chi^2(\rho, x'') + \epsilon]^{1/2} \quad \text{and} \quad \phi_\epsilon = \chi_\epsilon(|x'|, x'').$$

Clearly, χ_ϵ is differentiable in $[0; +\infty[\times\omega$,

$$\frac{\partial \chi_\epsilon}{\partial \rho}(\rho; x'') = \frac{1}{\chi_\epsilon}\left(\frac{1}{|S^{n-k-1}|}\int_{S^{n-k-1}} \varphi \frac{\partial \varphi}{\partial \rho}(\rho\theta, x'')\,\mathrm{d}\sigma(\theta)\right),$$

and

$$\frac{\partial \chi_\epsilon}{\partial x_j}(\rho, x'') = \frac{1}{\chi_\epsilon}\left(\frac{1}{|S^{n-k-1}|}\int_{S^{n-k-1}} \varphi \frac{\partial \varphi}{\partial x_j}(\rho\theta, x'')\,\mathrm{d}\sigma(\theta)\right), \quad n-k+1 \leqslant j \leqslant n.$$

Therefore, $\phi_\epsilon \in C^1(\overline{\Omega})$ and, by Hölder's inequality, we have

$$|\nabla \phi_\epsilon(x', x'')|^2 \leqslant \frac{1}{|S^{n-k-1}|}\int_{S^{n-k-1}} |\nabla \varphi(\rho\theta, x'')|^2\,\mathrm{d}\sigma(\theta),$$

which also implies that

$$\int_\Omega |\nabla \phi_\epsilon|^2 \leqslant \int_\Omega |\nabla \varphi|^2.$$

So, (ϕ_ϵ) is bounded in H^1 and converges weakly in H^1 and a.e. to ϕ_0 as ϵ converges to 0. In particular, $\phi_0 \in H^1(\Omega)$ and

$$\int_\Omega |\nabla \phi_0|^2 \leqslant \int_\Omega |\nabla \varphi|^2.$$

So, $Q_u(\phi_0) < 0$, and we obtain the desired contradiction. □

3. Proof of theorem 1.4

Step 1 (capacitary estimate). The following estimate is due to Farina (see [5, proposition 6]). Let u denote a solution that is stable outside a compact set. For every $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$, there exist constants $C_1, C_2 > 0$ such that, for every $R > 0$,

$$\int_{B'_R\times\omega} (|\nabla |u|^{(\gamma-1)/2} u|^2 + |u|^{p+\gamma})\,\mathrm{d}x \leqslant C_1 + C_2 R^{(n-k)-2(p+\gamma)/(p-1)}. \tag{3.1}$$

Step 2. In fact, if u is stable, one can choose $C_1 = 0$. As in Farina's work, we readily deduce, by letting $R \to +\infty$, that there is no non-trivial stable solution of (1.1) in the special case in which $1 < p < p_c(n-k)$.

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STEP 3. If $1 < p \leqslant p_s(n-k)$ and u is a solution of (1.1) that is stable outside a compact set, then in view of (3.1), we obtain

$$\int_\Omega |\nabla u|^2 + |u|^{p+1} < \infty. \tag{3.2}$$

By dominated convergence, it follows from (3.2) that if $A_R = \{R < |x'| < 2R\}$, then

$$\int_{A_R\times\omega} |\nabla u|^2 + |u|^{p+1} = o(1) \quad \text{as } R \to \infty. \tag{3.3}$$

Consider a test function $\phi_R(x) = \phi(|x'|/R)$, where $\phi \in C_c^2([0;\infty[)$ is a standard cut-off function satisfying $\phi(t) = 1$ if $0 \leqslant t \leqslant 1$, and $\phi(t) = 0$ if $t \geqslant 2$. Multiplying equation (1.1) by $u\phi_R$ and integrating by parts over $B'_{2R} \times \omega$, we get

$$\int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R - \int_{B'_{2R}\times\omega} |u|^{p+1}\phi_R = \frac{1}{2}\int_{B'_{2R}\times\omega} u^2\Delta\phi_R.$$

Using Hölder's inequality and (3.3), we conclude that

$$\int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R - \int_{B'_{2R}\times\omega} |u|^{p+1}\phi_R = o(1) \quad \text{as } R \to \infty.$$

Hence, letting $R \to \infty$, we get

$$\int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1}. \tag{3.4}$$

To complete the proof of theorem 1.4 for $p_s(n) \leqslant p \leqslant p_s(n-k)$, we need the following Pohozaev identity.

PROPOSITION 3.1. *Let u be a solution of (1.1) that is stable outside a compact set and let $z = (0, z) \in \mathbb{R}^{n-k} \times \omega$. Then, if $1 < p \leqslant p_s(n-k)$, we have*

$$\left(1 - \frac{2n}{(n-2)(p+1)}\right)\int_\Omega |u|^{p+1} = -\frac{1}{n-2}\int_{\mathbb{R}^{n-k}\times\partial\omega} |\nabla'' u|^2 (x-z)'' \cdot \nu'', \tag{3.5}$$

where ν denotes the outward unit vector normal on $B'_{2R} \times \partial\omega$.

Proof of proposition 3.1. Multiplying equation (1.1) by $\nabla u\cdot(x-z)\phi_R(x')$ and integrating over $B'_{2R} \times \omega$, we obtain

$$\int_{B'_{2R}\times\omega} -\Delta u \nabla u \cdot (x-z)\phi_R = \int_{B'_{2R}\times\omega} |u|^{p-1}u\nabla u\cdot(x-z)\phi_R. \tag{3.6}$$

For the left-hand side of (3.6), integrating by parts yields

$$\begin{aligned} I_1(R) &:= \int_{B'_{2R}\times\omega} -\Delta u\nabla u\cdot(x-z)\phi_R \\ &= \frac{1}{2}\int_{B'_{2R}\times\omega} \nabla(|\nabla u|^2)\cdot(x-z)\phi_R + \int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R \\ &\quad + \int_{B'_{2R}\times\omega} (\nabla u\cdot\nabla\phi_R)(\nabla u\cdot(x-z)) - \int_{B'_{2R}\times\partial\omega} (\nabla u\cdot\nu)(\nabla u\cdot(x-z))\phi_R. \end{aligned}$$

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Integrating again by parts the first term of the last equality, we get

$$I_1(R) = \frac{2-n}{2}\int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R + \frac{1}{2}\int_{B'_{2R}\times\partial\omega} |\nabla u|^2 (x-z)\cdot\nu\phi_R \\ - \int_{B'_{2R}\times\partial\omega} (\nabla u\cdot\nu)(\nabla u\cdot(x-z))\phi_R + o(1), \quad (3.7)$$

where, in view of (3.3) and the definition of ϕ_R,

$$o(1) = \int_{B'_{2R}\times\omega} (\nabla u\cdot\nabla\phi_R)(\nabla u\cdot(x-z)) + \frac{1}{2}\int_{B'_{2R}\times\omega} |\nabla u|^2\nabla\phi_R\cdot(x-z) \quad \text{as } R\to\infty.$$

Taking into account that $\nu = (0,\nu'')$, and $u = 0$ on $\mathbb{R}^{n-k}\times\partial\omega$, at any point $x\in\mathbb{R}^{n-k}\times\partial\omega$ where $\nabla'' u \neq 0$, we have $\nu'' = \epsilon(\nabla'' u/|\nabla'' u|)$, where $\epsilon\in\{-1,1\}$. Therefore, (3.7) becomes

$$I_1(R) = \frac{2-n}{2}\int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R - \frac{1}{2}\int_{B'_{2R}\times\partial\omega} |\nabla u|^2 (x-z)\cdot\nu\phi_R + o(1).$$

Now, integrate by parts the right-hand side of (3.6) to obtain

$$\begin{aligned} I_2(R) &:= \int_{B'_{2R}\times\omega} |u|^{p-1}u\nabla u.(x-z)\phi_R \\ &= -\frac{n}{p+1}\int_{B'_{2R}\times\omega} |u|^{p+1}\phi_R - \frac{1}{p+1}\int_{B'_{2R}\times\omega} |u|^{p+1}\nabla\phi_R\cdot(x-z) \\ &= -\frac{n}{p+1}\int_{B'_{2R}\times\omega} |u|^{p+1}\phi_R + o(1). \end{aligned} \quad (3.8)$$

Since $I_1(R) = I_2(R)$, combining (3.7), (3.8), and letting $R\to\infty$, from (3.6) we get

$$\frac{n-2}{2}\int_{\Omega} |\nabla u|^2 - \frac{n}{p+1}\int_{\Omega} |u|^{p+1} = -\frac{1}{2}\int_{\mathbb{R}^{n-k}\times\partial\omega} |\nabla'' u|^2 (x-z)''\cdot\nu''.$$

From (3.4), we derive the desired result. □

As a consequence of proposition 3.1, if ω is strictly star shaped with respect to z, it holds that $(x-z)''\cdot\nu'' > 0$ on $\mathbb{R}^{n-k}\times\partial\omega$, and so $u\equiv 0$ when $p_s(n) < p \leqslant p_s(n-k)$.

The $p = p_s(n)$ case requires more analysis. In fact, from (3.5) one has $\partial u/\partial\nu = 0$ on $\partial\Omega$. In addition, applying (3.1) with $\gamma = p$ and recalling that

$$n-k-2\frac{2p}{p-1} < n-k-2\frac{p+1}{p-1} < 0$$

for $p = p_s(n)$, we find that

$$\int_{\Omega} |u|^{2p} < \infty, \quad \text{i.e. } |u|^{p-1}u \in L^2(\Omega).$$

Recall that Poincaré's inequality holds in $H_0^1(\Omega)$. By L^p-elliptic theory, we deduce that $u\in H^2\subset L^{2N/(N-4)}$, and by a standard boot-strap argument, one has $u\in$

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$W^{2,r}$ with $r > n/2$; then $u \in L^\infty(\Omega)$. Therefore, (1.1) becomes $\Delta u + qu = 0$, where $q = |u|^{p-1} \in L^\infty(\Omega)$ with $u \in H^2(\Omega)$ satisfying $u = \partial u/\partial\nu = 0$ on $\partial\Omega$. By the unique continuation principle for Cauchy data (see [6]), it follows that $u \equiv 0$.

Step 4. We derive a variant of a monotonicity formula due to Pacard [7]; see also Wang [9]. For $\lambda > 0$, define u^λ by

$$u^\lambda(x) = \lambda^{2/(p-1)} u(\lambda x', x'') \quad \text{for all } x = (x', x'') \in \mathbb{R}^{n-k} \times \omega$$

and

$$E(u;\lambda) = \int_{B_1' \times \omega} \left(\tfrac{1}{2}[|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2] - \frac{1}{p+1}|u^\lambda|^{p+1} \right) \mathrm{d}x + \frac{1}{p-1} \int_{\partial B_1' \times \omega} |u^\lambda|^2 \,\mathrm{d}\sigma.$$

We claim that E is a non-decreasing function of λ. Furthermore, E is differentiable and

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} = \lambda \left[\int_{\partial B_1' \times \omega} |\partial_\lambda u^\lambda|^2 \,\mathrm{d}\sigma + \int_{B_1' \times \omega} |\nabla'' u^\lambda|^2 \,\mathrm{d}x \right]$$

To prove our claim, we note that u^λ solves

$$-\Delta' u^\lambda - \lambda^2 \Delta'' u^\lambda = |u^\lambda|^{p-1} u^\lambda \quad \text{in } \mathbb{R}^{n-k} \times \omega, \tag{3.9}$$

that

$$E(u;\lambda) = E(u^\lambda; 1), \tag{3.10}$$

and that

$$\lambda \partial_\lambda u^\lambda = \frac{2}{p-1} u^\lambda + \rho \partial_\rho u^\lambda \quad \text{for } x \in \Omega,\ \lambda > 0, \tag{3.11}$$

where we recall that $\rho = |x'|$ and $\partial_\rho = \nabla \cdot (x'/\rho)$. So, if

$$E_1 = \int_{B_1' \times \omega} \left(\tfrac{1}{2} |\nabla' u^\lambda|^2 - \frac{1}{p+1} |u^\lambda|^{p+1} \right) \mathrm{d}x, \tag{3.12}$$

then

$$\frac{\mathrm{d}E_1}{\mathrm{d}\lambda} = \int_{B_1' \times \omega} (\nabla' u^\lambda \cdot \nabla' \partial_\lambda u^\lambda - |u^\lambda|^{p-1} u^\lambda \partial_\lambda u^\lambda) \,\mathrm{d}x.$$

Integrating by parts and using (3.9), we have

$$\frac{\mathrm{d}E_1}{\mathrm{d}\lambda} = \lambda^2 \int_{B_1' \times \omega} \Delta'' u^\lambda \partial_\lambda u^\lambda + \int_{\partial B_1' \times \omega} \partial_\rho u^\lambda \partial_\lambda u^\lambda.$$

Integrating by parts again and using the boundary condition, the first addend is equal to

$$\begin{aligned}\lambda^2\int_{B_1'\times\omega}\Delta'' u^\lambda\partial_\lambda u^\lambda &= -\lambda^2\int_{B_1'\times\omega}\nabla'' u^\lambda\cdot\nabla''\partial_\lambda u^\lambda\\ &= -\frac{\lambda^2}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{B_1'\times\omega}|\nabla'' u^\lambda|^2\\ &= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\lambda^2}{2}\int_{B_1'\times\omega}|\nabla'' u^\lambda|^2\right]+\lambda\int_{B_1'\times\omega}|\nabla'' u^\lambda|^2.\end{aligned}$$

Thanks to (3.11), the second addend is equal to

$$\begin{aligned}\int_{\partial B_1'\times\omega}\partial_\rho u^\lambda\partial_\lambda u^\lambda &= \int_{\partial B_1'\times\omega}\left(\lambda\partial_\lambda u^\lambda-\frac{2}{p-1}u^\lambda\right)\partial_\lambda u^\lambda\\ &= \lambda\int_{\partial B_1'\times\omega}|\partial_\lambda u^\lambda|^2-\frac{1}{p-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\partial B_1'\times\omega}|u^\lambda|^2\end{aligned}$$

and the result follows.

Step 5 (blow-down analysis for stable solutions). From step 1 (applied to u on a ball of radius λR), we know that, given $R>0$,

$$\int_{B_R'\times\omega}(|\nabla' u^\lambda|^2+\lambda^2|\nabla'' u^\lambda|^2+|u^\lambda|^{p+1})\,\mathrm{d}x\leqslant CR^{(n-k)-2(p+1)/(p-1)}. \tag{3.13}$$

So, $(u^\lambda)_{\lambda\geqslant 1}$ is uniformly bounded in $H^1\cap L^{p+1}(B_R'\times\omega)$ for any $R>0$. In particular, a sequence (u^{λ_n}) converges weakly to some function u^∞ in $H^1\cap L^{p+1}(B_R'\times\omega)$ for every $R>0$ as $\lambda_n\to+\infty$. Note also that u^λ satisfies the following PDE:

$$-\Delta'' u^\lambda=\lambda^{-2}(\Delta' u^\lambda+|u^\lambda|^{p-1}u^\lambda). \tag{3.14}$$

Taking limits in the sense of distributions, it follows that

$$-\Delta'' u^\infty=0\quad\text{in }\mathcal{D}'(\mathbb{R}^{n-k}\times\omega).$$

The maximum principle applied for almost every $x'\in\mathbb{R}^{n-k}$ to the function $u^\infty(x',\cdot)$ implies that $u^\infty\equiv 0$.

Actually, the full family (u^λ) converges *strongly* to $u^\infty=0$ in $L^{p+1}(B_R'\times\omega)$. Indeed, by Rellich's theorem, (u^λ) is compact in $L^2(B_R'\times\omega)$, while it remains bounded in $L^{p+\gamma}(B_R'\times\omega)$ for some $\gamma>1$, thanks to step 1. By Hölder's inequality, (u^λ) is compact in $L^{p+1}(B_R'\times\omega)$; $u^\infty=0$ being its only cluster point, the claim follows.

Now, multiply (1.1) by $pu\zeta^2$, where $\zeta\in C_c^1(\overline{\Omega})$ is a cut-off function to be specified soon.[3] We find that

$$p\int_\Omega\nabla u\cdot\nabla(u\zeta^2)=p\int_\Omega|u|^{p+1}\zeta^2.$$

[3] Such test functions can indeed be used in the stability inequality, thanks to the Dirichlet boundary condition; see [5, remark 5].

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The left-hand side is equal to

$$p\int_{\Omega}|\nabla(u\zeta)|^2-u^2|\nabla\zeta|^2,$$

while the right-hand side is bounded above by $\int_{\Omega}|\nabla(u\zeta)|^2$, since u is stable. It follows that

$$(p-1)\int_{\Omega}|\nabla(u\zeta)|^2\leqslant p\int u^2|\nabla\zeta|^2. \tag{3.15}$$

Now choose $\zeta(x)=\zeta_0(x'/\lambda)$, where $\zeta_0\equiv 1$ in B_1' and $\zeta_0\equiv 0$ outside B_2'. Then,

$$\int_{B_\lambda'\times\omega}|\nabla u|^2\leqslant C\lambda^{-2}\int_{B_{2\lambda}'\times\omega}u^2.$$

Going back to u^λ, we arrive at

$$\int_{B_1'\times\omega}|\nabla' u^\lambda|^2+\lambda^2|\nabla'' u^\lambda|^2\leqslant C\int_{B_2'\times\omega}|u^\lambda|^2.$$

Recalling that (u^λ) converges to zero in $L^{p+1}(B_R'\times\omega)$, and thus also in $L^2(B_2'\times\omega)$, we conclude that

$$\lim_{\lambda\to+\infty}E_2(u;\lambda)=\lim_{\lambda\to+\infty}E_2(u^\lambda;1)=0,$$

where E_2 is given by

$$E_2(u^\lambda;1)=\int_{B_1\times\omega}\left[\tfrac{1}{2}(|\nabla' u^\lambda|^2+\lambda^2|\nabla'' u^\lambda|^2)-\frac{1}{p+1}|u^\lambda|^{p+1}\right]\mathrm{d}x.$$

We claim that the same holds true for E. To see this, simply observe that since E is non-decreasing,

$$\begin{aligned}E(u^\lambda,1)&=E(u,\lambda)\\&\leqslant\frac{1}{\lambda}\int_\lambda^{2\lambda}E(u,t)\,\mathrm{d}t\\&=\frac{1}{\lambda}\int_\lambda^{2\lambda}E_2(u,t)\,\mathrm{d}t+\frac{1}{p-1}\lambda^{-1}\int_\lambda^{2\lambda}t^{n-k-1-4/(p-1)}\int_{\partial B_t'\times\omega}|u|^2\\&\leqslant\sup_{t\geqslant\lambda}E_2(u,t)+C\int_{B_2'\times\omega}|u^\lambda|^2.\end{aligned} \tag{3.16}$$

Thanks to this, we deduce that

$$\lim_{\lambda\to+\infty}E(u,\lambda)=\lim_{\lambda\to+\infty}E(u^\lambda,1)=0.$$

In addition, since u is C^2, one easily verifies that

$$E(u,0)=0.$$

Then $E(u,\lambda)\equiv 0$, since E is non-decreasing, and so $\mathrm{d}E/\mathrm{d}\lambda=0$, which means that u is homogeneous and independent of x''. Thanks to the boundary condition, we readily deduce that $u\equiv 0$.

STEP 6 (blow-down analysis for solutions that are stable outside a compact set). We assume that $p > p_S(n-k)$. As before, by step 1, (u^λ) is uniformly bounded in $L^{p+\gamma}(B'_R \times \omega)$ for some $\gamma > 1$. In addition, $|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2$ is bounded in $L^1(B'_R \times \omega)$, for any $R > 0$. As in step 5, this is enough to conclude that (u^λ) converges strongly to $u^\infty \equiv 0$ in $L^{p+1}(B'_R \times \omega)$. In addition, $E_2(u^\lambda; 1)$ remains bounded. This time, however, (3.15) remains valid only for cut-off functions $\zeta \in C^1_c(\overline{\Omega} \setminus B_{R_0} \times \omega)$, for R_0 sufficiently large. So, choose $\zeta(x) = \zeta_0(x'/\lambda)$, where $\zeta_0 \equiv 0$ in $B'_{\varepsilon/2}$, $\zeta_0 \equiv 1$ in $B'_1 \setminus B'_\varepsilon$ and $\zeta_0 \equiv 0$ outside B'_2. Then, for $\lambda > R_0/\varepsilon$,

$$\int_{B'_\lambda \setminus B'_{\varepsilon\lambda} \times \omega} |\nabla u|^2 \leqslant C\lambda^{-2} \int_{B'_{2\lambda} \times \omega} u^2.$$

Going back to u^λ yields

$$\int_{B'_1 \setminus B'_\varepsilon \times \omega} |\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2 \leqslant C \int_{B'_2 \times \omega} |u^\lambda|^2,$$

and so

$$\begin{aligned} E_2(u^\lambda; 1) &= \int_{B'_1 \times \omega} \left(\tfrac{1}{2}[|\nabla' u^\lambda|^2 + \lambda^2 |\nabla'' u^\lambda|^2] - \frac{1}{p+1} |u^\lambda|^{p+1} \right) \mathrm{d}x \\ &= \int_{B'_\varepsilon \times \omega} + \int_{B'_1 \setminus B'_\varepsilon \times \omega} \\ &= \varepsilon^{n-k-2(p+1)/(p-1)} E_2(u; \lambda\varepsilon) + \int_{B'_1 \setminus B'_\varepsilon \times \omega} \\ &\leqslant C\left(\varepsilon^{n-k-2(p+1)/(p-1)} + \int_{B'_2 \times \omega} |u^\lambda|^2 \right). \end{aligned}$$

Letting $\lambda \to +\infty$ and then $\varepsilon \to 0$, we deduce that $\lim_{\lambda \to +\infty} E_2(u; \lambda) = 0$. The remaining part of the proof of step 5 can be used unchanged. □

4. Proof of theorem 1.7

We indicate here how to adapt the proof of theorem 1.4 to this case.

CASE 1 ($p > p_S(n-k)$). Here the only difference comes from the classification of the blow-down limit u^∞. In fact, multiplying (3.9) by $u_\infty \phi_R(x')$, we see easily that

$$\int_{B'_{2R} \times \omega} (\nabla'' u_{\lambda_j} \cdot \nabla'' u_\infty) \phi_R$$

converges to 0 as $j \to \infty$. Since (u_{λ_j}) converges weakly to u_∞ in $H^1(B'_{2R} \times \omega)$, we deduce that

$$\int_{B'_{2R} \times \omega} |\nabla'' u_\infty|^2 \phi_R = 0 \quad \forall R > 0.$$

In other words, u^∞ is a function of x' only. But integrating (3.9) in the x'' variable and passing again to the weak limit then implies that $u^\infty = (1/|\omega|) \int_\omega u^\infty \,\mathrm{d}x''$ is an energy solution of

$$-\Delta' u^\infty = |u^\infty|^{p-1}u^\infty \quad \text{in } \mathbb{R}^{n-k}.$$

In addition, since u is stable outside a compact set, u^∞ is stable outside the origin. If $n-k \geqslant 2$, points have zero Newtonian capacity and so u^∞ is stable in all of $\mathbb{R}^{n-k}$. By Farina's theorem 1 of [5], which still holds for energy solutions, $u^\infty \equiv 0$. If $n-k = 1$, then u^∞ is stable only outside the compact set $\{0\}$. But $p_S(n-k) = p_c(n-k) = +\infty$, so we can apply, for example, [5, theorem 2] to arrive at the same conclusion.

Case 2 ($p = p_S(n-k)$). First we need the following version of Pohozaev's identity.

Proposition 4.1. *Let u be a solution of (1.1) that is stable outside a compact set. Then we have*

$$\left(1 - \frac{2(n-k)}{(p+1)(n-k-2)}\right)\int_\Omega |u|^{p+1} = -\frac{2}{n-k-2}\int_\Omega |\nabla'' u|^2.$$

Proof of proposition 4.1. Note that (3.2)–(3.4) hold for (1.2). As in the proof of proposition 3.1, multiplying (1.2) by $\nabla' u \cdot (x-z)'\phi_R(|x'|)$ and integrating over $B'_{2R}\times\omega$, we get

$$\int_{B'_{2R}\times\omega} -\Delta u \nabla' u \cdot (x-z)\phi_R = \int_{B'_{2R}\times\omega} |u|^{p-1}u\nabla' u \cdot (x-z)\phi_R. \tag{4.1}$$

For the left-hand side of (4.1), integrating by parts, we obtain

$$\begin{aligned} J_1(R) &:= \int_{B'_{2R}\times\omega} -\Delta u \nabla' u \cdot (x-z)'\phi_R \\ &= \frac{1}{2}\int_{B'_{2R}\times\omega} \nabla'(|\nabla u|^2)\cdot (x-z)'\phi_R + \int_{B'_{2R}\times\omega} |\nabla' u|^2\phi_R + o(1). \end{aligned}$$

Integrating again by parts the first term of the last equality, we get

$$\begin{aligned} J_1(R) &= \frac{k-n}{2}\int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R + \int_{B'_{2R}\times\omega} |\nabla' u|^2\phi_R + o(1) \\ &= -\frac{n-k-2}{2}\int_{B'_{2R}\times\omega} |\nabla u|^2\phi_R - \int_{B'_{2R}\times\omega} |\nabla'' u|^2\phi_R + o(1). \end{aligned} \tag{4.2}$$

Now, integrating by parts the right-hand side of (4.1) to obtain

$$J_2(R) := \int_{B'_{2R}\times\omega} |u|^{p-1}u\nabla' u \cdot (x-z)'\phi_R = -\frac{n-k}{p+1}\int_{B'_{2R}\times\omega} |u|^{p+1}\phi_R + o(1). \tag{4.3}$$

Since $J_1(R) = J_2(R)$, combining (4.2) and (4.3), and letting $R \to \infty$, from (4.1) we obtain

$$\frac{n-k-2}{2}\int_\Omega |\nabla u|^2 - \frac{n-k}{p+1}\int_\Omega |u|^{p+1} = -\int_\Omega |\nabla'' u|^2. \tag{4.4}$$

Therefore, (4.4) becomes

$$\int_\Omega |\nabla u|^2 - \frac{2(n-k)}{(p+1)(n-k-2)}\int_\Omega |u|^{p+1} = -\frac{2}{n-k-2}\int_\Omega |\nabla'' u|^2.$$

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From (3.4), we derive that

$$\left(1-\frac{2(n-k)}{(p+1)(n-k-2)}\right)\int_{\Omega}|u|^{p+1}=-\frac{2}{n-k-2}\int_{\Omega}|\nabla''u|^{2}. \tag{4.5}$$

If $p=p_s(n-k)$, from (4.5) one has

$$\int_{\Omega}|\nabla''u|^{2}=0.$$

This gives the following classification: $u(x',x'')=u(x')$ in Ω and u satisfies

$$-\Delta' u=|u|^{p-1}u \quad \text{in } \mathbb{R}^{n-k}, \text{ with } p=p_s(n-k). \qquad \square$$

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