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The Allison–Faulkner construction of E_8

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Abstract. We show that the Tits index $E_{8,1}^{133}$ cannot be obtained by means of the Tits construction over a field with no odd degree extensions. The proof uses a general method coming from the theory of symmetric spaces. We construct two cohomological invariants, in degrees 6 and 8, of the Tits construction and the more symmetric Allison–Faulkner construction of Lie algebras of type E_8 and show that these invariants can be used to detect the isotropy rank.

Tits in [24] proposed a general construction of exceptional Lie algebras over an arbitrary field of characteristic not 2 or 3, now called the Tits construction. The inputs are an alternative algebra and a Jordan algebra, and the result is a simple Lie algebra of type F_4, E_6, E_7, or E_8, depending on the dimensions of the algebras. The construction produces, say, all real forms of the exceptional Lie algebras, and a natural question is if all Tits indices can be obtained this way. Garibaldi and Petersson in [12] showed that it is not the case for type E_6, namely that Lie algebras of Tits index ${}^2E_{6,1}^{35}$ do not appear as a result of the Tits construction. We show a similar result for type E_8, namely that Lie algebras of Tits index $E_{8,1}^{133}$ cannot be obtained by means of the Tits construction, provided that the base field has no odd degree extensions. The proof uses the theory of symmetric spaces and the first author's result with Semenov and Garibaldi about isotropy of groups of type E_7 in terms of the Rost invariant [13].

We prefer to use a more symmetric version of Tits construction due to Allison and Faulkner. Here, the input is a so-called structurable algebra with an involution (say, the tensor product of two octonion algebras) and three constants. The Lie algebra is given by some Chevalley-like relations. The Tits construction and the Allison–Faulkner construction have a large overlap but, strictly speaking, neither one is more general than the other. The Tits construction is capable of producing Lie algebras of type E_8 whose Rost invariant has a nonzero three-torsion part (necessarily using a Jordan division algebra as input), but the Allison–Faulkner construction of E_8 cannot do this—at least when the input is a form of the tensor product of two octonion algebras, because these can always be split by a two-extension of the base field. On the other hand, the Allison–Faulkner construction is capable of producing Lie algebras of type E_8 with the property that the two-torsion part of their Rost invariant has symbol length 3, and this is impossible for the Tits construction (see [8, 11.6]). An E_8 with this

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property would necessarily come from what we call an indecomposable bi-octonion algebra, and these are related to some unusual examples of 14-dimensional quadratic forms discovered by Izhboldin and Karpenko [16].

We produce two new cohomological invariants, one in degree 6 and one in degree 8, and show that these invariants can be used to detect the isotropy rank of either the Tits or the Allison–Faulkner construction (but unlike the results of [12] for groups of type E_6, we give necessary conditions only). The main tool for constructing these invariants is a calculation of the Killing form of an Allison–Faulkner construction which, under a mild condition on the base field, is near to an eight-Pfister form (a so-called Pfister neighbour).

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1 Preliminaries

Let K be a field of characteristic not 2 or 3. If q is a quadratic form, we write $q(x,y) = q(x+y) - q(x) - q(y)$ for the associated symmetric bilinear form. Conversely, if b is a symmetric bilinear form, then $q(x) = \frac{1}{2}b(x,x)$ is the associated quadratic form. This convention agrees with [22] but differs from, say, [17, Section VII.6]. If A is an algebra and $a \in A$, we denote by $L_a, R_a \in \mathrm{End}(A)$ the left- and right-multiplication operators, respectively.

1.1 Bi-octonion algebras

A K-algebra with involution $(A, -)$ is called a *decomposable bi-octonion algebra* if it has two octonion subalgebras C_1 and C_2 that are stabilised by the involution, such that $A = C_1 \otimes_K C_2$. A *bi-octonion algebra* is an algebra with involution $(A, -)$ that becomes isomorphic to a decomposable bi-octonion algebra over some field extension. These are important examples of central simple structurable algebras, as defined by Allison in [1], and they are instrumental in constructing Lie algebras of type E_8 (see Section 1.5).

Any bi-octonion algebra $(A, -)$ is either decomposable or it decomposes over a unique quadratic field extension E/K. In the latter case, there exists an octonion algebra C over E, unique up to K-isomorphism, from which $(A, -)$ can be reconstructed as follows. Let ι be the nonidentity automorphism of E/K, and let ${}^{\iota}C$ be a copy of C as a K-algebra, but with a different E-algebra structure given by $e \cdot z = \iota(e)z$. Then $(A, -)$ is precisely the fixed point set of ${}^{\iota}C \otimes_E C$ under the K-automorphism $x \otimes y \mapsto y \otimes x$, with the involution being the restriction of the tensor product of the canonical involutions on ${}^{\iota}C$ and C [2, Theorem 2.1]. We denote this algebra by $(A, -) = N_{E/K}(C)$.

To unify the description of both decomposable and nondecomposable bi-octonion algebras, if we consider $C = C_1 \times C_2$ as an octonion algebra over the split quadratic étale extension $K \times K$, then $N_{K\times K/K}(C)$ as defined above is just isomorphic to $C_1 \otimes_K C_2$.

1.2 Additive and multiplicative transfer of quadratic forms

Let E/K be a quadratic étale extension and (q, V) an n-dimensional quadratic space over E. The *additive transfer* of (q, V) (also known as the trace or Scharlau transfer) is the $2n$-dimensional K-quadratic space $(\mathrm{tr}_{E/K}(q), V)$ defined by $\mathrm{tr}_{E/K}(q)(v) = \mathrm{tr}_{E/K}(q(v))$ for all $v \in V$.

Rost defined a multiplicative transfer for quadratic forms, and it has been studied by him and his students (e.g., in [18, 26]) and used before to define cohomological invariants. The multiplicative transfer also appeared (independently, it seems) in an old paper of Tignol [23].

If (q, V) is an n-dimensional quadratic space over a quadratic étale extension E/K, one defines the quadratic space $({}^{\iota}q, {}^{\iota}V)$ where ι is the nontrivial automorphism of E/K, ${}^{\iota}V$ is a copy of V as a K-vector space but with the action of E modified by ι, and ${}^{\iota}q(v) = \iota(q(v))$. The *multiplicative transfer* $N_{E/K}(q)$ of q is the n^2-dimensional K-quadratic form obtained by restricting ${}^{\iota}q \otimes_E q$ to the K-subspace of tensors in ${}^{\iota}V \otimes_E V$ fixed by the switch map $x \otimes y \mapsto y \otimes x$.

In the case of a split quadratic étale extension, a quadratic form over $K \times K$ is just a pair (q_1, q_2) where q_1, q_2 are quadratic forms over K of the same dimension, and we have $\mathrm{tr}_{E/K}(q_1, q_2) = q_1 \perp q_2$ and $N_{K\times K/K}(q_1, q_2) = q_1 \otimes q_2$.

Lemma 1.1 *Let $(A, -) = N_{E/K}(C)$ for an octonion algebra C over a quadratic étale extension E/K, and let n be the norm of C. Then $N_{E/K}(n)$ equals the normalised trace form $(x, y) \mapsto \frac{1}{64}\, tr(L_{x\bar{y}+y\bar{x}})$.*

Proof Both $N_{E/K}(n)$ and the normalised trace form are invariant symmetric bilinear forms on $(A, -)$ in the sense that Allison defined (see [1, Theorem 17] and [2, Proposition 2.2]). By a theorem of Schafer [19], a central simple structurable algebra has at most one such bilinear form, up to a scalar multiple. (As discussed in [19, pp. 116–117], these facts are valid in characteristic 0 or $p \geq 5$, despite some of the original references being limited to characteristic 0.) ∎

1.3 Lie-related triples

Let $(A, -)$ be a central simple structurable algebra over K. A *Lie related triple* (in the sense of [4, Section 3]) is a triple $T = (T_1, T_2, T_3)$ where $T_i \in \mathrm{End}(A)$ and

$$\overline{T_i(\overline{xy})} = T_j(x)y + xT_k(t)$$

for all $x, y \in A$ and all $(i\ j\ k)$ that are cyclic permutations of $(1\ 2\ 3)$. Define $\mathcal{T}$ to be the Lie subalgebra of $\mathfrak{gl}(A) \times \mathfrak{gl}(A) \times \mathfrak{gl}(A)$ spanned by the set of related triples.

For $a, b \in A$ and $1 \leq i \leq 3$, define

$$T^i_{a,b} = (T_1, T_2, T_3),$$

where (taking indices mod 3):

$$T_i = L_{\bar{b}}L_a - L_{\bar{a}}L_b,$$
$$T_{i+1} = R_{\bar{b}}R_a - R_{\bar{a}}R_b,$$
$$T_{i+2} = R_{\bar{a}b-\bar{b}a} + L_bL_{\bar{a}} - L_aL_{\bar{b}}.$$

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Let $\mathcal{T}_I$ be the subspace of $\operatorname{End}(A)^3$ spanned by $\{T^i_{a,b} \mid a, b \in A, 1 \le i \le 3\}$. Since $(A, -)$ is structurable, $\mathcal{T}_I$ is a Lie subalgebra of $\mathcal{T}$[4, Lemma 5.4]. Finally, denote by $\operatorname{Skew}(A, -) \subset A$ the (-1)-eigenspace of the involution, and let $\mathcal{T}'$ be the subspace of $\operatorname{End}(A)^3$ spanned by triples of the form

$$(D, D, D) + (L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2}), \tag{1.1}$$

where $D \in \operatorname{Der}(A, -)$ and $s_i \in \operatorname{Skew}(A, -)$ with $s_1 + s_2 + s_3 = 0$.

Example 1.2 Let $(C, -)$ be an octonion algebra with norm n. The principle of local triality holds in $\mathcal{T}_I$ in the sense that each of the projections $\mathcal{T}_I \to \mathfrak{gl}(C)$, $(T_1, T_2, T_3) \mapsto T_i$, for $1 \le i \le 3$, is injective [22, Theorem 3.5.5]. The Lie algebra $\mathcal{T}_I$ is isomorphic to $\mathfrak{so}(n)$ [22, Lemma 3.5.2]. The $(i + 2)$th entry of the triple $T^i_{a,b}$ is $R_{\bar{a}b-\bar{b}a} + L_b L_{\bar{a}} - L_a L_{\bar{b}}$, and by [22, pp. 51, 54] this is the map $C \to C$ that sends

$$x \mapsto 2n(x, a)b - 2n(x, b)a. \tag{1.2}$$

Proposition 1.3 *If $(A, -)$ is a bi-octonion algebra of the form $(A, -) = N_{E/K}(C)$ for some quadratic étale extension E/K and some octonion algebra C over E, then $\mathcal{T}_I = \mathcal{T}_0 = \mathcal{T}' \simeq \operatorname{Lie}(R_{E/K}(\mathbf{Spin}(n)))$, where n is the norm of C.*

Proof We have that $\mathcal{T}_I \subset \mathcal{T} \subset \mathcal{T}'$ and $\dim \mathcal{T}' = \dim \operatorname{Der}(A, -) + 2 \dim \operatorname{Skew}(A, -) = 28 + 28 = 56$ by [4, Corollary 3.5]. On the other hand, $\mathcal{T}_I$ (as an E-module) is precisely $\operatorname{Lie}(\mathbf{Spin}(n))$ [22, Theorem 3.5.5] and so $\mathcal{T}_I$ (as a K-vector space) is 56-dimensional and isomorphic to $\operatorname{Lie}(R_{E/K}(\mathbf{Spin}(n)))$. ■

1.4 Local triality

In the context of Proposition 1.3, the Lie algebra $\mathcal{T}_I$ is of type $D_4 + D_4$. Local triality holds here too: the projections $\mathcal{T}_I \to \mathfrak{gl}(A)$, $(T_1, T_2, T_3) \mapsto T_i$ are injective for any $1 \le i \le 3$, and the symmetric group S_3 acts on $\mathcal{T}_I$ by E-automorphisms, where E is the centroid of $\mathcal{T}_I$ (compare with [22, Section 3.5]).

1.5 The Allison–Faulkner construction [4, Section 4]

Let $(A, -)$ be a central simple structurable algebra and let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in K^\times \times K^\times \times K^\times$. For $1 \le i, j \le 3$ and $i \ne j$, define $A[ij] = \{a[ij] \mid a \in A\}$ to be a copy of A, and identify $A[ij]$ with $A[ji]$ by setting $a[ij] = -\gamma_i \gamma_j^{-1} \overline{a}[ji]$. Define the vector space

$$K(A, -, \gamma) = \mathcal{T}_I \oplus A[12] \oplus A[23] \oplus A[31]$$

and equip it with an algebra structure defined by the multiplication:

$$\begin{aligned}
[a[ij], b[jk]] &= -[b[jk], a[ij]] = ab[ik],\\
[T, a[ij]] &= -[a[ij], T] = T_k(a)[ij]\\
[a[ij], b[ij]] &= \gamma_i \gamma_j^{-1} T^i_{a,b}
\end{aligned}$$

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for all $a, b \in A$, $T = (T_1, T_2, T_3) \in \mathcal{T}_I$, and $(i\ j\ k)$ a cyclic permutation of $(1\,2\,3)$. Then $K(A, -, \gamma)$ is clearly a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded algebra, and it is in fact a central simple Lie algebra [4, Theorems 4.1, 4.3, 4.4, and 5.5].

1.6 Relation to the Tits–Kantor–Koecher construction

If the quadratic form $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$ is isotropic then $K(A, -, \gamma) \simeq K(A, -)$ where

$$K(A, -) = \mathrm{Skew}(A) \oplus A \oplus V_{A,A} \oplus A \oplus \mathrm{Skew}(A) \tag{1.3}$$

is the Tits–Kantor–Koecher construction [3, Corollary 4.7]. An isomorphism and its inverse are determined explicitly in [3, Theorem 2.2] in the case where $-\gamma_1\gamma_2^{-1}$ is a square. More generally, if $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$ and $\langle \gamma_1', \gamma_2', \gamma_3' \rangle$ are similar quadratic forms, then $K(A, -, \gamma) \simeq K(A, -, \gamma')$ [3, Proposition 4.1]. In particular, if $(A, -)$ is a bi-octonion algebra, then $K(A, -, \gamma)$ is a central simple Lie algebra of type E_8.

The range of Lie algebras of type E_8 that are of the form $K(A, -)$ includes those with Tits index E_8^{91}, E_8^{66}, E_8^{28}, or E_8^{0}, and only those. We formulate a statement to this effect:

Proposition 1.4 *Let L be a Lie algebra of type E_8 corresponding to a class $\varepsilon \in H^1(K, E_8)$. Then $L \simeq K(A, -)$ for some bi-octonion algebra $(A, -)$ if and only if ε is in the image of $H^1(K, \mathbf{Spin}_{14}) \to H^1(K, E_8)$.*

Proof Let $L = \bigoplus_{i=-2}^{2} L_i$ be the split E_8 Lie algebra with the $\mathbb{Z}$-grading indicated by (1.3). The subalgebra L_0 is reductive of type D_7; it is generated by a Cartan subalgebra and the root spaces of roots with α_1-coordinate equal to zero (the dimensions of the components in the $\mathbb{Z}$-grading preclude anything else). If $(A, -)$ is an arbitrary bi-octonion algebra, then $K(A, -)$ has the same $\mathbb{Z}$-grading as L so it must have been twisted by a cocycle coming from $H^1(K, \mathbf{Spin}_{14})$. This proves the "only if" part of the statement.

For the "if" part of the statement, we prefer to make an argument using the Levi subgroup $H \subset E_8$ whose Lie algebra is L_0, rather than its semisimple subgroup $\mathbf{Spin}_{14}$. Nothing is gained or lost this way, because $H^1(K, H)$ and $H^1(K, \mathbf{Spin}_{14})$ have the same image in $H^1(K, E_8)$; see [25, p. 657]. Specifically, H is the group generated by a maximal torus of E_8 and the root groups U_β where β has α_1-coordinate equal to zero. It acts faithfully on L by graded automorphisms, its representation on L_1 has a unique open orbit, and this orbit contains 1 (the identity in the split bi-octonions); see [10, p. 547]. The stabilizer of 1 is the automorphism group G of the split bi-octonion algebra [5, Corollary 8.6]. The map $H^1(K, G) \to H^1(K, H)$ is surjective by the open orbit theorem from [9, pp. 28–29]. Consequently, any cocycle in the image of $H^1(K, H) \to H^1(K, E_8)$ is also in the image of the map $H^1(K, G) \to H^1(K, E_8)$ that sends the class of $(A, -)$ to the class of $K(A, -)$. ■

1.7 Relation to the Tits construction

Tits in [24] defined the following construction of Lie algebras. Let C be an alternative algebra and J be a Jordan algebra. Denote by C° and J° the subspaces of elements of

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generic trace zero and define operations $\circ$ and bilinear forms $(-,-)$ on C° and J° by the formula

$$ab = a \circ b + (a,b)1.$$

Two elements a, b in J and C define an inner derivation $\langle a, b\rangle$ of the respective algebra, namely:

$$\langle a, b\rangle(x) = \tfrac{1}{4}[[a,b],x] - \tfrac{3}{4}[a,b,x].$$

Then there is a Lie algebra structure on the vector space $\mathrm{Der}(J) \oplus J^\circ \otimes C^\circ \oplus \mathrm{Der}(C)$ defined by the formulas

$$\begin{aligned}
&[\mathrm{Der}(J), \mathrm{Der}(C)] = 0;\\
&[B + D, a \otimes c] = B(a) \otimes c + a \otimes D(c);\\
&[a \otimes c, a' \otimes c'] = (c,c')\langle a, a'\rangle + (a \circ a') \otimes (c \circ c') + (a,a')\langle c, c'\rangle
\end{aligned}$$

for all $B \in \mathrm{Der}(J)$, $D \in \mathrm{Der}(C)$, $a, a' \in J^\circ$, and $c, c' \in C^\circ$. If $(A,-) = C_1 \otimes C_2$ is a decomposable bi-octonion algebra, then $K(A,-,\gamma)$ is isomorphic to the Lie algebra obtained via the Tits construction from the composition algebra C_1 and the reduced Albert algebra $\mathcal{H}_3(C_2,\gamma)$ [3, Remark 1.9 (c)].

Proposition 1.5 *Let $(A,-) = C_1 \otimes C_2$ be a decomposable bi-octonion algebra. Then*

$$\mathcal{T}_I \oplus A[ij] \simeq \mathfrak{so}(\langle \gamma_i\rangle n_1 \perp \langle -\gamma_j^{-1}\rangle n_2),$$

where n_ℓ is the norm of C_ℓ.

Proof Consider the quadratic form $Q = \langle \gamma_i\rangle n_1 \perp \langle -\gamma_j^{-1}\rangle n_2$ on the vector space $C_1 \oplus C_2$. The Lie algebra $\mathfrak{so}(Q)$ can be embedded into the Clifford algebra $C(Q)$ as the subspace spanned by elements of the form

$$[u,v]_c, \quad u, v \in C_1 \oplus C_2,$$

where $[-,-]_c$ denotes the commutator in the Clifford algebra (to avoid confusion with the commutators in C_1 and C_2). These generators satisfy the relations [17, p. 232 (30)]:

$$\begin{aligned}
[[u,v]_c,[u',v']_c]_c = &- 2Q(u,u')[v,v']_c + 2Q(u,v')[v,u']_c\\
&+ 2Q(v,u')[u,v']_c - 2Q(v,v')[u,u']_c.
\end{aligned}$$

If $z, z' \in C_1$ and $w, w' \in C_2$, this becomes

$$[[z,w]_c,[z',w']_c]_c = -2\gamma_i n_1(z,z')[w,w']_c + 2\gamma_j^{-1} n_2(w,w')[z,z']_c. \tag{1.4}$$

This implies that the 64-dimensional subspace spanned by

$$[z,w]_c, \quad z \in C_1, w \in C_2$$

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generates the Lie algebra $\mathfrak{so}(Q)$. Now define a linear bijection $\theta : \mathfrak{so}(Q) \to \mathcal{T}_I \oplus A[ij]$ by

$$[z, z']_c \mapsto \gamma_i T^i_{z,z'},$$
$$[w, w']_c \mapsto -\gamma_j^{-1} T^i_{w,w'},$$
$$[z, w]_c \mapsto zw[ij]$$

for all $z, z' \in C_1$ and $w, w' \in C_2$. By [17, p. 232 (31)] and (1.2), the restriction of θ to the subalgebra $[C_1, C_1]_c \oplus [C_2, C_2]_c \simeq \mathfrak{so}(\langle\gamma_i\rangle n_1) \times \mathfrak{so}(\langle-\gamma_j^{-1}\rangle n_2)$ is a homomorphism.

Now we calculate using (1.4) that

$$\begin{aligned} \theta([[z,w]_c,[z',w']_c]_c) &= \theta(-2\gamma_i n_1(z,z')[w,w']_c + 2\gamma_j^{-1} n_2(w,w')[z,z']_c) \\ &= -2\gamma_i n_1(z,z')\theta([w,w']_c) + 2\gamma_j^{-1} n_2(w,w')\theta([z,z']_c) \\ &= 2\gamma_i\gamma_j^{-1}\big(n_1(z,z')T^i_{w,w'} + n_2(w,w')T^i_{z,z'}\big). \end{aligned} \tag{1.5}$$

Meanwhile, we have

$$[\theta([z,w]_c), \theta([z',w']_c)] = [zw[ij], z'w'[ij]] = \gamma_i\gamma_j^{-1} T^i_{zw,z'w'}. \tag{1.6}$$

To complete the proof that θ is an isomorphism, we show that the triples (1.5) and (1.6) are equal. It suffices to compare the ith entries of each triple (by §1.4). After recalling that

$$L_x L_{\overline{x'}} + L_{x'} L_{\overline{x}} = L_{\overline{x}} L_{x'} + L_{\overline{x'}} L_x = n_i(x, x')\,\mathrm{id}$$

for all $x \in C_\ell$ [22, Lemma 1.3.3 (iii)], the ith entry of (1.5) is

$$\begin{aligned} &2\gamma_i\gamma_j^{-1}\big(n_1(z,z')(L_{\overline{w'}}L_w - L_{\overline{w}}L_{w'}) + n_2(w,w')(L_{\overline{z'}}L_z - L_{\overline{z}}L_{z'})\big) \\ &= 2\gamma_i\gamma_j^{-1}\big((L_{\overline{z}}L_{z'} + L_{\overline{z'}}L_z)(L_{\overline{w'}}L_w - L_{\overline{w}}L_{w'}) + (L_{\overline{w}}L_{w'} + L_{\overline{w'}}L_w)(L_{\overline{z'}}L_z - L_{\overline{z}}L_{z'})\big) \\ &= 2\gamma_i\gamma_j^{-1}(2L_{\overline{z'}}L_zL_{\overline{w'}}L_w - 2L_{\overline{z}}L_{z'}L_{\overline{w}}L_{w'}) = 4\gamma_i\gamma_j^{-1}(L_{\overline{z'}}L_zL_{\overline{w'}}L_w - L_{\overline{z}}L_{z'}L_{\overline{w}}L_{w'}). \end{aligned}$$

In the last line, we have used (multiple times) the fact that C_1 and C_2 commute and associate with each other in A. Using this fact a few more times, the ith entry of (1.6) is just

$$4\gamma_i\gamma_j^{-1}(L_{\overline{z'w'}}L_{zw} - L_{\overline{zw}}L_{z'w'}) = 4\gamma_i\gamma_j^{-1}(L_{\overline{z'}}L_zL_{\overline{w'}}L_w - L_{\overline{z}}L_{z'}L_{\overline{w}}L_{w'}). \qquad \blacksquare$$

2 The Killing form of $K(A, -, \gamma)$

By our convention, the Killing form (as a quadratic form) on a Lie algebra L is the form $x \mapsto \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_x{}^2)$. For any quadratic form $q = \langle x_1, \dots, x_n\rangle$, the Killing form of $\mathfrak{so}(q)$ is

$$\langle 2 - n\rangle\lambda^2(q), \tag{2.1}$$

where $\lambda^2(q) = \perp_{i<j}\langle x_i x_j\rangle$ [9, Exercise 19.2].

Lemma 2.1 *Let $A = N_{E/K}(C)$ as before, and let $\rho_{ij} : R_{E/K}(\mathbf{Spin}(n)) \to \mathrm{GL}(A[ij])$ be the representation lifted from the representation of $\mathcal{T}_I$ in $A[ij]$. Every quadratic form q on A invariant under this action of $R_{E/K}(\mathbf{Spin}(n))$ is a scalar multiple of*

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the multiplicative transfer $N_{E/K}(n)$ (equivalently, a scalar multiple of the trace form $(x,y) \mapsto tr(L_{x\bar{y}+y\bar{x}})$).

Proof We can extend scalars from K to E, and then q_E is a quadratic form on $A_E = C \otimes_E {}'C$ which is invariant under the action of $R_{E/K}(\mathbf{Spin}(n)) \times_K E = \mathbf{Spin}(n) \times \mathbf{Spin}('n)$. Then clearly q_E decomposes as $q_1 \otimes q_2$ for some $\mathbf{Spin}(n)$-invariant form q_1 on C and some $\mathbf{Spin}('n)$-invariant form q_2 on ${}'C$. This implies $q_1 \simeq \langle \lambda_1 \rangle n$ and $q_2 \simeq \langle \lambda_2 \rangle 'n$ for certain scalars $\lambda_i \in E^\times$, and therefore, $q_E = \langle \lambda_1 \lambda_2 \rangle n \otimes 'n$. However, since (q_E, A_E) is extended from (q, A) and $n \otimes 'n(1 \otimes 1) = 1$, we have $\lambda_1 \lambda_2 \in K^\times$. Therefore, $q = q_E|_A = \langle \lambda_1 \lambda_2 \rangle N_{E/K}(n)$. ∎

We can now calculate the Killing form of $K(A,-,\gamma)$ in the case where $(A,-)$ is a bi-octonion algebra.

Proposition 2.2 *If $(A,-) = N_{E/K}(C)$, then the Killing form on $K(A,-,\gamma)$ is*

$$\langle -30 \rangle \big(tr_{E/K}(\lambda^2(n)) \perp \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1} \rangle N_{E/K}(n) \big). \tag{2.2}$$

Proof Let κ be the Killing form of $K(A,-,\gamma)$. If $x, y \in K(A,-,\gamma)$ are from different homogeneous components in the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading, then $\mathrm{ad}_x \, \mathrm{ad}_y$ shifts the grading and consequently $\kappa(x,y) = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y) = 0$.

Let τ be the Killing form of $\mathcal{T}_I$. The Killing form of $\mathrm{Lie}(\mathbf{Spin}(n))$ is $\langle -6 \rangle \lambda^2(n)$; see (2.1). Since $\mathcal{T}_I \simeq \mathrm{Lie}(R_{E/K}(\mathbf{Spin}(n))$ by Proposition 1.3, we have

$$\tau = \mathrm{tr}_{E/K}(\langle -6 \rangle \lambda^2(n)) = \langle -6 \rangle \, \mathrm{tr}_{E/K}(\lambda^2(n)).$$

There is an automorphism of $K(A,-,\gamma) \otimes_K K^{\mathrm{alg}}$ that swaps the two simple subalgebras of $\mathcal{T}_I \otimes_K K^{\mathrm{alg}}$, and this implies $\kappa|_{\mathcal{T}_I}$ is a scalar multiple of τ; say $\kappa|_{\mathcal{T}_I} = \langle \phi_0 \rangle \langle -6 \rangle \, \mathrm{tr}_{E/K}(\lambda^2(n))$ for some $\phi_0 \in K^\times$.

Let us determine ϕ_0. The grading on $K(A,-,\gamma)$ makes it a sum of four $\mathcal{T}_I$-modules. For $T, S \in \mathcal{T}_I$ and $a \in A$,

$$[T,[S,a[ij]]] = T_k(S_k(a))[ij].$$

Therefore,

$$\kappa(T,S) = \mathrm{tr}(\mathrm{ad}_T \, \mathrm{ad}_S) = \tau(T,S) + \mathrm{tr}(T_1S_1) + \mathrm{tr}(T_2S_2) + \mathrm{tr}(T_3S_3).$$

The trace forms of the irreducible representations $\mathcal{T}_I \to \mathfrak{gl}(A)$, $T \mapsto T_\ell$ for $1 \le \ell \le 3$ are all equal (despite them being inequivalent representations) and so $\mathrm{tr}(T_1S_1) = \mathrm{tr}(T_2S_2) = \mathrm{tr}(T_3S_3)$ for all $T, S \in \mathcal{T}_I$. Moreover, $\mathrm{tr}(T_1S_1)$ is a scalar multiple of $\tau(T,S)$.

To determine the ratio between $\mathrm{tr}(T_1S_1)$ and $\tau(T,S)$, we can assume $A = C_1 \otimes C_2$ is decomposable, and consider the subalgebra $\mathfrak{so}(n_1) \subset \mathfrak{so}(n_1) \times \mathfrak{so}(n_2) \simeq \mathrm{Lie}(R_{E/K}(\mathbf{Spin}(n))$, where n_ℓ is the norm on C_ℓ. It is well-known that the Killing form κ_1 on $\mathfrak{so}(n_1)$ is 6 $(= 8 - 2)$ times the trace form of its vector representation $\mathfrak{so}(n_1) \to \mathfrak{gl}(C_1)$, while the trace form of the representation $\mathfrak{so}(n_1) \to \mathfrak{gl}(C_1 \otimes C_2)$ is clearly eight times the trace form of the vector representation. But κ_1 is equal to the restriction of the Killing form τ on $\mathfrak{so}(n_1) \times \mathfrak{so}(n_2)$, so this means that (if $T \in \mathcal{T}_I$

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belongs to the $\mathfrak{so}(n_1)$ subalgebra) we have $\operatorname{tr}(T_1^2) = 8\operatorname{tr}(T_1|_{C_1}{}^2) = \frac{8}{6}\kappa_1(T) = \frac{8}{6}\tau(T)$. In conclusion, $\phi_0 = 5$, so $\kappa|_{\mathcal{T}_I} = \langle -30\rangle \operatorname{tr}_{E/K}(\lambda^2(n))$.

The restriction $\kappa|_{A[ij]}$ is an invariant form under the action of $R_{E/K}(\mathbf{Spin}(n))$, which means it is proportional to $N_{E/K}(n)$, by Lemma 2.1. Say $\kappa|_{A[ij]} = \langle\phi_{ij}\rangle N_{E/K}(n)$. To determine the ϕ_{ij}, it suffices to calculate $\kappa(1[ij])$, since $\kappa(1[ij]) = \phi_{ij}N_{E/K}(n)(1) = \phi_{ij}$. By definition $\kappa(1[ij])$ is half the trace of $\operatorname{ad}_{1[ij]}{}^2$. The graded components of $K(A,-,\gamma)$ are invariant under $\operatorname{ad}_{1[ij]}{}^2$, so we work out the trace separately for each of these components.

For all $b \in A$, we have

$$[1[ij],[1[ij],b[jk]]] = [1[ij],b[ik]] = -\gamma_i\gamma_j^{-1}[1[ji],b[ik]] = -\gamma_i\gamma_j^{-1}b[jk],$$

so $\operatorname{ad}_{1[ij]}{}^2|_{A[jk]} = -\gamma_i\gamma_j^{-1}\operatorname{id}$, and $\operatorname{tr}(\operatorname{ad}_{1[ij]}{}^2|_{A[jk]}) = -64\gamma_i\gamma_j^{-1}$. Similarly, for all $b \in A$,

$$\begin{aligned}[1[ij],[1[ij],b[ki]]] &= (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})[1[ij],[1[ji],\bar{b}[ik]]]\\ &= (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})[1[ij],\bar{b}[jk]]\\ &= (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})[\bar{b}[ik]] = (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})(-\gamma_i\gamma_k^{-1})b[ki]\\ &= -\gamma_i\gamma_j^{-1}b[ki],\end{aligned}$$

so $\operatorname{ad}_{1[ij]}{}^2|_{A[ki]} = -\gamma_i\gamma_j^{-1}\operatorname{id}$, and $\operatorname{tr}(\operatorname{ad}_{1[ij]}{}^2|_{A[ki]}) = -64\gamma_i\gamma_j^{-1}$. In contrast, for all $b \in A$,

$$\begin{aligned}[1[ij],[1[ij],b[ij]]] &= [1[ij],\gamma_i\gamma_j^{-1}T^i_{1,b}] = -\gamma_i\gamma_j^{-1}(T^i_{1,b})_k(1)\\ &= -\gamma_i\gamma_j^{-1}(R_{b-\bar{b}} + L_b - L_{\bar{b}})(1) = -2\gamma_i\gamma_j^{-1}(b-\bar{b}).\end{aligned}$$

Therefore, $\operatorname{ad}_{1[ij]}{}^2|_{A[ij]}$ has a 50-dimensional kernel $\{a[ij] \mid \bar{a} = a\}$ and a 14-dimensional eigenspace $\{a[ij] \mid \bar{a} = -a\}$ with eigenvalue $-4\gamma_i\gamma_j^{-1}$. This proves that $\operatorname{tr}(\operatorname{ad}_{1[ij]}{}^2|_{A[ij]}) = -56\gamma_i\gamma_j^{-1}$.

Now if $T = (T_1,T_2,T_3) \in \mathcal{T}_I$, then

$$[1[ij],[1[ij],T]] = [1[ij],-T_k(1)[ij]] = -\gamma_i\gamma_j^{-1}T^i_{1,T_k(1)}.$$

We can use (1.1) to write $T = (D,D,D) + (L_{s_2}-R_{s_3}, L_{s_3}-R_{s_1}, L_{s_1}-R_{s_2})$ for some unique $D \in \operatorname{Der}(A,-)$ and $s_i \in \operatorname{Skew}(A,-)$ such that $s_1+s_2+s_3 = 0$. Note that the kth entry of T is $L_{s_i} - R_{s_j}$. Then $T_k(1) = D(1) + L_{s_i}(1) - R_{s_j}(1) = s_i - s_j$, so $\operatorname{ad}_{1[ij]}{}^2(T) = -\gamma_i\gamma_j^{-1}T^i_{1,T_k(1)}$ is the triple whose kth entry is

$$\begin{aligned}-\gamma_i\gamma_j^{-1}(R_{1T_k(1)-\overline{T_k(1)}1} + L_{T_k(1)}L_1 - L_1L_{\overline{T_k(1)}}) &= -\gamma_i\gamma_j^{-1}(R_{2(s_i-s_j)} + L_{2(s_i-s_j)})\\ &= -2\gamma_i\gamma_j^{-1}\big((L_{s_i}-R_{s_j}) - (L_{s_j}-R_{s_i})\big).\end{aligned}$$

This shows $\ker(\operatorname{ad}_{1[ij]}{}^2|_{\mathcal{T}_I})$ is the 42-dimensional subspace of $\mathcal{T}_I$ whose kth projection is

$$\{D + L_s - R_s \mid D \in \operatorname{Der}(A,-), s \in \operatorname{Skew}(A,-)\}.$$

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And the subspace of $\mathcal{T}_I$ whose kth projection is

$$\{L_s + R_s \mid s \in \mathrm{Skew}(A,-)\}$$

is a 14-dimensional eigenspace of $\mathrm{ad}_{1[ij]}{}^2|_{\mathcal{T}_I}$ with eigenvalue $-4\gamma_i\gamma_j^{-1}$. This proves that $\mathrm{tr}(\mathrm{ad}_{1[ij]}{}^2|_{\mathcal{T}_I}) = -56\gamma_i\gamma_j^{-1}$. Therefore,

$$\phi_{ij} = \kappa(1[ij]) = \tfrac{1}{2}\,\mathrm{tr}(\mathrm{ad}_{1[ij]}{}^2) = -32\gamma_i\gamma_j^{-1} - 32\gamma_i\gamma_j^{-1} - 28\gamma_i\gamma_j^{-1} - 28\gamma_i\gamma_j^{-1} = -120\gamma_i\gamma_j^{-1},$$

and we can simplify to get (2.2) because 30 is in the same square class as 120. ∎

If $\mathrm{char}(K) = 5$, then the Killing form on E_8 is zero. However, if $(A,-) = N_{E/K}(C)$ then the symmetric bilinear form on $K(A,-,\gamma)$ associated to

$$\mathrm{tr}_{E/K}(\lambda^2(n)) \perp \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/K}(n) \tag{2.3}$$

is nondegenerate and Lie invariant. This can be proved in at least two ways: one can factor out $\langle -30\rangle$ in the Killing form of the Chevalley Lie algebra of type E_8 defined over $\mathbb{Z}$, extend the new bilinear form to the split E_8 over K, and then twist it to get the form (2.3) on $K(A,-,\gamma)$. This form is clearly invariant and nondegenerate (its radical is a nonzero ideal and E_8 is a simple Lie algebra in all characteristics). Alternatively, one use the hint from [11, Exercise 27.21 (2)]: lift the Killing form of $K(A,-,\gamma)$ to the ring of Witt vectors, divide by -30 up there, and reduce modulo 5 to get (2.3).

Lemma 2.3 *Let $(A,-) = N_{E/K}(C)$, and let κ' be a nondegenerate Lie invariant bilinear form on $K(A,-,\gamma)$. If -1 is a sum of two squares in K, then $\kappa' \in I^6(K)$ and there is a unique 64-dimensional form $q \in I^6(K)$ such that $q + \kappa' \in I^8(K)$.*

Proof Since κ' is unique up to a scalar multiple, we can assume without loss of generality that

$$\kappa' = \mathrm{tr}_{E/K}(\lambda^2(n)) \perp \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/K}(n).$$

The assumption that -1 is a sum of two squares is equivalent to the identity $4 = 0$ in the Witt ring $W(K)$. This assumption implies that $\mathrm{tr}_{E/K}(\lambda^2(n)) = 0$ [9, Lemma 19.8] and also that $N_{E/K}(n) \in I^6(K)$ [18, 26, Satz 2.16 (ii)], hence $\kappa' \in I^6(K)$. Setting $q = N_{E/K}(n)$ yields

$$q + \kappa' = \langle 1, \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/K}(n) = \langle\!\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}\rangle\!\rangle N_{E/K}(n) \in I^8(K).$$

The uniqueness of q follows from the Arason–Pfister Hauptsatz. ∎

Let $Q(*) \subset R(*) \subset H^1(*, E_8)$ be the functors $\mathrm{Fields}_{/K} \to \mathrm{Sets}$ such that for all fields F/K:

(1) $Q(F)$ is the set of isomorphism classes of Lie algebras of type E_8 that are isomorphic to $K(A,-,\gamma)$ for some bi-octonion algebra $(A,-)$ over F and some $\gamma = (\gamma_1,\gamma_2,\gamma_3) \in (K^\times)^3$; i.e. $Q(F)$ is the image of the Allison–Faulkner construction

$$H^1(F, (G_2 \times G_2 \rtimes \mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^3) \to H^1(F, E_8).$$

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(2) $R(F)$ is the set of isomorphism classes of Lie algebras L of type E_8 such that the class of L is contained in $Q(F')$ for some odd-degree extension F'/F.

Recall from 1.7 that $Q(*)$ contains all Lie algebras of type E_8 that are obtainable using the Tits construction from a reduced Albert algebra and an octonion algebra. Whereas, $R(*)$ strictly contains all Lie algebras of type E_8 that are obtainable using the Tits construction from an Albert algebra (even a division algebra) and an octonion algebra. Any cohomological invariant $Q(*) \to \bigoplus_{i\geq 0} H^i(*, \mathbb{Z}/2\mathbb{Z})$ can be extended uniquely to a cohomological invariant $R(*) \to \bigoplus_{i\geq 0} H^i(*, \mathbb{Z}/2\mathbb{Z})$ [9, Section 7].

By applying the quadratic form invariants $e_n : I^n(*) \to H^n(*, \mathbb{Z}/2\mathbb{Z})$ for $n = 6$ and 8, we obtain cohomological invariants of the Tits construction and the Allison–Faulkner construction.

Corollary 2.4 *If -1 is a sum of two squares in K, then there exist nontrivial cohomological invariants*

$$h_6 : R(*) \to H^6(*, \mathbb{Z}/2\mathbb{Z}),$$
$$h_8 : R(*) \to H^8(*, \mathbb{Z}/2\mathbb{Z}),$$

such that if $(A, -) = N_{E/F}(C)$, then

$$h_6(K(A, -, \gamma)) = e_6(N_{E/F}(n)),$$
$$h_8(K(A, -, \gamma)) = (-\gamma_1\gamma_2^{-1}) \cup (-\gamma_2\gamma_3^{-1}) \cup e_6(N_{E/F}(n)).$$

2.1 Comparison with invariants of $G_2 \times F_4$

Since $R(K)$ contains the image of the Tits construction $H^1(K, G_2 \times F_4) \to H^1(K, E_8)$, there are unique cohomological invariants

$$h_i^* : H^1(*, G_2 \times F_4) \to H^i(*, \mathbb{Z}/2\mathbb{Z}) \quad i = 6, 8,$$

such that $h_i^*(C, J) = h_i(L)$ where L is the Lie algebra of type E_8 constructed from the octonion algebra C and Albert algebra J. The cohomological invariants of G_2 and F_4 are classified [11]. The unique nontrivial invariant e_3 of G_2 assigns an octonion algebra C to the class $e_3(C) = (\alpha_1) \cup (\alpha_2) \cup (\alpha_3)$, where $\langle\langle \alpha_1, \alpha_2, \alpha_3 \rangle\rangle$ is the norm of C. The unique nontrivial mod 2 invariants f_3, f_5 of F_4 assign a reduced Jordan algebra $\mathcal{H}_3(C, \gamma)$ to the classes $f_3(\mathcal{H}_3(C, \gamma)) = e_3(C)$ and $f_5(\mathcal{H}_3(C, \gamma)) = (-\gamma_1\gamma_2^{-1}) \cup (-\gamma_2\gamma_3^{-1}) \cup e_3(C)$, respectively (see [11, Section 22] and [22, p. 118]). Comparing with Corollary 2.4 and using the fact that $e_6(N_{K\times K/K}(n_1, n_2)) = e_6(n_1 \otimes n_2) = e_3(n_1) \cup e_3(n_2)$ yields

$$h_i^*(C_1, \mathcal{H}_3(C_2, \gamma)) = h_i(K(C_1 \otimes C_2, -, \gamma)) = e_3(C_1) \cup f_{i-3}(\mathcal{H}_3(C_2, \gamma))$$

for all pairs of octonion algebras C_1, C_2 and scalars $\gamma_1, \gamma_2, \gamma_3$. If two invariants with values in $H^i(*, \mathbb{Z}/2\mathbb{Z})$ agree up to odd-degree extensions, then they are equal, so it follows that

$$h_6^*(C, J) = e_3(C) \cup f_3(J),$$
$$h_8^*(C, J) = e_3(C) \cup f_5(J)$$

for all octonion algebras C and Albert algebras J.

3 Isotropy of Tits construction

In this section, we continue to assume that the base field K is of characteristic not 2 or 3.

3.1 Generalities on symmetric spaces

We use the basics of the theory of symmetric spaces over arbitrary fields; we refer to [15] for the generalities.

Let G be a (connected) split reductive algebraic group over a field K and σ be an involution on G (that is an automorphism of order 2). Then the fixed point subgroup $H = G^\sigma$ has a reductive connected component H°; in the case when σ is from $G(K)$ and the commutator subgroup of G is simply connected, H is connected and has the same rank as G (see [14, Théorème 3.1.5]). We state some facts about its normalizer in the lemma below.

Lemma 3.1 (1) *$N_G(H) = N_G(H^\circ)$;*

(2) *$g \in G$ belongs to $N_G(H)$ if and only if $\sigma(g)g^{-1}$ belongs to the center of G;*

(3) *If σ is from $G(K)$ and T is a σ-stable split maximal torus in G, then the map*

$$N_H(T)/T \to H/H^\circ$$

is surjective.

Proof The first two items are from [15, Corollary 1.3], and the third is [14, Lemme 3.1.4]. Note that T as above always exists by [15, Proposition 2.3]. ∎

A torus S in G (not necessary maximal) is called *σ-split* if $\sigma(t) = t^{-1}$ for all $t \in S$. In the particular case $S = \mathbb{G}_m$, S defines two opposite parabolic subgroups in G; they are also called σ-split and are characterized by the fact that σ sends a σ-split parabolic subgroup to an opposite parabolic subgroup. Possible types of σ-split maximal parabolic subgroups correspond to the white vertices on the Satake diagram of (G, σ), see [21, Lemma 2.9 and 2.11].

The quotient variety G/H is called a *symmetric space*. It is known to be spherical, that is for any parabolic subgroup P in G, H acts on G/P with a finite number of orbits. In particular, there is an open orbit; it consists of all σ-split parabolic subgroups of the same type as P (provided they exist).

Let us state a general lemma that will be applied to the case of E_8 below.

Lemma 3.2 *Let G be a split adjoint semisimple group over an infinite field K of characteristic not 2, $H = G^\sigma$ be the fixed point subgroup of an involution σ on G, P be a parabolic subgroup of G, C be the stabilizer of a point from the open orbit of the action of H on G/P, $[\xi]$ be an element from $H^1(K, G)$. Assume that the twisted form ${}_\xi G$ contains (over the base field K) a parabolic subgroup P' of the same type as P and a subgroup H' that is conjugate to H over a separable closure of K. Then $[\xi]$ comes from some $[\zeta] \in H^1(K, C)$ such that ${}_\zeta H$ is isomorphic to H'.*

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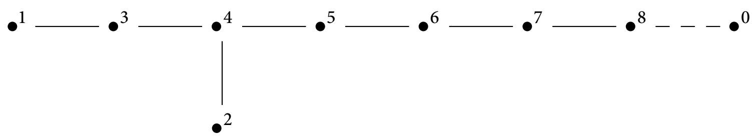
Proof By Lemma 3.1 (ii) $N_G(H) = H$, since the center of G is trivial. Note that H' corresponds to a rational point on ${}_\xi(G/H)$, and by [20, Proposition 37] $[\xi]$ comes from some $[\xi'] \in H^1(K, N_G(H))$ with ${}_{\xi'}H = H'$.

Now ${}_\xi(G/P)$ is a smooth compactification of its open subvariety $U = {}_{\xi'}(H/C)$ and by the assumption has a rational point. The unipotent radical of a parabolic subgroup opposite to P' defines an open subvariety in ${}_\xi(G/P)$ isomorphic to the affine space $\mathbb{A}^N$ for some N. Since the base field is infinite, there is a rational point in $\mathbb{A}^N \cap U$. Applying [20, Proposition 37] to ${}_{\xi'}(H/C)$ we obtain the claim. ∎

3.2 E_8/P_8 as a compactification of $D_8/N(A_7)$

Let G be the split group of type E_8 over K and σ be the involution whose fixed point subgroup is D_8 obtained by erasing vertex 1 from the extended Dynkin diagram:



More precisely, σ is the inner automorphism defined by $\omega_1^\vee(-1)$, where α_i are the fundamental roots and $\omega_j^\vee$ are coweights defined by $\omega_j^\vee(\alpha_i) = \delta_{ij}$.

All vertices of the Satake diagram for the symmetric space E_8/D_8 are white; in particular, there is a parabolic subgroup P of type P_8 such that $\sigma(P)$ is opposite to P. It is not difficult to construct such a parabolic subgroup directly: it is defined by $S = \mathbb{G}_m$ which is the image of $\alpha_1^\vee$ in the maximal torus T (note that $\alpha_1^\vee$ is Weyl-conjugate to $\omega_8^\vee$ and so it has type P_8 indeed).

Lemma 3.3 *The stabilizer of a point from the open orbit of the action of D_8 on E_8/P_8 is $N(A_7)$, the normalizer of the maximal subgroup of type A_7 in the simply connected group of type E_7.*

Proof To check the claim we may pass to the algebraic closure of K. The stabilizer of the point corresponding to P is L^σ, where $L = P \cap \sigma(P)$. It contains the A_7 subgroup generated by root subgroups corresponding to $\pm\alpha_2, \pm\alpha_4, \dots, \pm\alpha_8$ and $\pm\alpha_0$ (where α_0 stands for the negative maximal root). Since A_7 is maximal in the commutator subgroup E_7 of L and L is an almost direct product of E_7 and the σ-split torus S, we see that the connected component of L^σ is A_7.

It is known (and can be deduced from Lemma 3.1) that A_7 has index 2 in $N(A_7)$, so it remains to present an element from L^σ not lying in A_7. Consider any lifting $\tilde{w}_0$ of the longest element in the Weyl group of E_7. Note that $\tilde{w}_0$ normalizes A_7 but cannot belong to L^σ, otherwise the fixed point subgroup E_7^σ would be not connected. Lemma 3.1 implies that $\sigma(\tilde{w}_0) = \tilde{w}_0\alpha_1^\vee(-1)$, for the second factor is the only nontrivial element in the center of E_7. Now $\tilde{w}_0\alpha_1^\vee(i)$, where i is a square root of -1, is an element from $L^\sigma \cap N(A_7)$ not belonging to A_7. ∎

One can show that $N(A_7)$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by SL_8/μ_2, which is split if and only if -1 is a square in K.

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Lemma 3.4 *Let $[\xi] \in H^1(K, PGO_{2n})$ be in the image of $H^1(K, GL_n/\mu_2 \rtimes \mathbb{Z}/2\mathbb{Z})$. Then there exists a quadratic field extension E/K such that the orthogonal involution corresponding to ξ_E is hyperbolic.*

Proof Consider the following short exact sequence:

$$H^1(K, \mathrm{GL}_n/\mu_2) \to H^1(K, \mathrm{GL}_n/\mu_2 \rtimes \mathbb{Z}/2\mathbb{Z}) \to H^1(K, \mathbb{Z}/2\mathbb{Z}),$$

and take E/K corresponding to the image in $H^1(K, \mathbb{Z}/2\mathbb{Z})$ of $[\zeta]$ in $H^1(K, \mathrm{GL}_n/\mu_2 \rtimes \mathbb{Z}/2\mathbb{Z})$ whose image in $H^1(K, \mathrm{PGO}_{2n})$ is $[\xi]$. Passing to E we see that $[\xi_E]$ comes from $H^1(E, \mathrm{GL}_n/\mu_2)$ and so produces a hyperbolic involution. ■

Theorem 3.5 *Let K be a two-special (that is with no odd degree extensions) field of characteristic not 2 and 3, L be a Lie algebra of type E_8 over K obtained via the Tits construction. Then the group corresponding to L is not of Tits index $E_{8,1}^{133}$.*

Proof Assume the contrary. Obviously the base field is infinite, for there are only split groups of type E_8 over finite fields. Let L be obtained via the Tits construction from C_1 and $\mathcal{H}_3(C_2, \gamma)$ for some octonion algebras C_1 and C_2, i.e., is $K(A, -, \gamma)$ for $(A, -) = C_1 \otimes C_2$. Denote by $[\xi]$ the class corresponding to L in $H^1(K, E_8)$. By Proposition 1.5 L contains a Lie subalgebra of type D_8, namely $\mathfrak{so}(\langle\gamma_i\rangle n_1 \perp \langle -\gamma_j^{-1}\rangle n_2)$, and so the corresponding group contains a subgroup H' of type D_8 with the same Lie algebra (see [7, Exposé XXII, Corollaire 5.3.4]), that is corresponding to the quadratic form $\langle\gamma_i\rangle n_1 \perp \langle -\gamma_j^{-1}\rangle n_2$.

Applying Lemma 3.2 to the case $G = E_8$, $H = D_8$ and H' as above, we see that $[\xi]$ comes from some $[\zeta] \in H^1(K, N(A_7))$ such that H' is isomorphic to ${}_\zeta D_8$. Now the image of $N(A_7)$ in PGO_{16}^+ normalizes SL_8/μ_2 and so is contained in $\mathrm{GL}_8/\mu_2 \rtimes \mathbb{Z}/2\mathbb{Z}$. Applying Lemma 3.4, we see that the quadratic form $\langle\gamma_i\rangle n_1 \perp \langle -\gamma_j^{-1}\rangle n_2$ becomes hyperbolic over a quadratic field extension E/K. It follows that $e_3(n_1) + e_3(n_2)$ is trivial over E, hence $n_1 - n_2$ belongs to I^4 and so is hyperbolic over E. Now $n_1 - n_2$ is divisible by the discriminant of E and so $e_3(n_1) + e_3(n_2)$ is a sum of two symbols with a common slot. But the Rost invariant of the anisotropic kernel of type E_7 is $e_3(n_1) + e_3(n_2)$, and applying [13, Theorem 10.18] we see that this group must be isotropic, a contradiction. ■

Note that [9, Appendix A] provides an example of a strongly inner group of type E_7 over a two-special field, hence an example of a group of Tits index $E_{8,1}^{133}$ over such a field, which is not obtained via the Tits construction.

Corollary 3.6 *Suppose K is a field such that -1 is a sum of two squares, and let L be a Lie algebra over K of type E_8 obtained via the Tits construction.*

(1) *If $h_8(L) \neq 0$ then L is anisotropic.*
(2) *If -1 is a square in K and $h_6(L) \neq 0$ then L has K-rank ≤ 1.*

Proof It suffices to prove both items in case K is two-special.

(i) Suppose L is isotropic. We can assume that L does not have Tits index $E_{8,1}^{133}$ by Theorem 3.5. Using [6, Table 10] we see that L corresponds to a class in the image of

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$H^1(K, \mathbf{Spin}_{14}) \to H^1(K, E_8)$, which implies by Proposition 1.4 that it is isomorphic to $K(A, -) \simeq K(A, -, (1, -1, 1))$ for some bi-octonion algebra A. Then clearly $h_8(L) = 0$.

(ii) Suppose L has K-rank ≥ 2. Then L corresponds to a class in the image of $H^1(K, \mathbf{Spin}_{12}) \to H^1(K, E_8)$. Its anisotropic kernel is a subgroup of $\mathbf{Spin}(q)$ for some 12-dimensional form q belonging to $I^3(K)$, and by a well-known theorem of Pfister (see [9, Theorem 17.13]) q is similar to $n_1 - n_2$ for a pair of three-Pfister forms n_i with a common slot, say $n_i = \langle\langle x, y_i, z_i \rangle\rangle$. If C_i is the octonion algebra corresponding to n_i then we have $L \simeq K(C_1 \otimes C_2, -)$, and since -1 is a square,

$$h_6(L) = e_6(\langle\langle x, y_1, z_1, x, y_2, z_2 \rangle\rangle) = (-1) \cup (x) \cup (y_1) \cup (y_2) \cup (z_1) \cup (z_2) = 0. \quad \blacksquare$$

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