

LEVEL THEORY, PART 3: A BOOLEAN ALGEBRA OF SETS ARRANGED IN WELL-ORDERED LEVELS

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Abstract. On a very natural conception of sets, every set has an absolute complement. The ordinary cumulative hierarchy dismisses this idea outright. But we can rectify this, whilst retaining classical logic. Indeed, we can develop a boolean algebra of sets arranged in well-ordered levels. I show this by presenting Boolean Level Theory, which fuses ordinary Level Theory (from Part 1) with ideas due to Thomas Forster, Alonzo Church, and Urs Oswald. BLT neatly implement Conway's games and surreal numbers; and a natural extension of BLT is definitionally equivalent with ZF.

Like all walls it was ambiguous,
two-faced. What was inside it
and what was outside it
depended upon which side you
were on.

Le Guin (1974, p. 1)

Building on work by Alonzo Church and Urs Oswald, Thomas Forster has provided a pleasingly different way to think about sets. As in the ordinary cumulative hierarchy, the sets are stratified into well-ordered levels. But, unlike the ordinary cumulative picture, the sets form a boolean algebra. In particular, every set has an absolute complement, in the sense that $\forall a \exists c \forall x (x \in a \leftrightarrow x \notin c)$. In this paper, I develop an axiomatic theory for this conception of set: Boolean Level Theory, or BLT.

I start by outlining the bare-bones idea of a complemented hierarchy of sets, according to which sets are arranged in stages, but where each set is found alongside its complement. I axiomatize this bare-bones story in the most obvious way possible, obtaining Boolean Stage Theory, BST. It is clear that any complemented hierarchy satisfies BST (see Sections 1 and 2). Unfortunately, BST has multiple primitives. To overcome this, I develop Boolean Level Theory, BLT. The only primitive of BLT is \in , but BLT and BST say exactly the same things about sets. As such, any complemented hierarchy satisfies BLT. Moreover, BLT is quasi-categorical (see Sections 3–5). I then provide two interpretations using BLT_{ZF} (an obvious extension of BLT): we can regard ZF as a proper part of BLT_{ZF} ; but ZF is definitionally

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equivalent to BLT_{ZF} (see Sections 6 and 7). I close by explaining how to implement Conway's games and surreal numbers in BLT (see Section 8).

This paper is the third in a triptych. It closely mirrors Part 1, but can be read in isolation. Let me repeat, though, that Part 1 is hugely indebted to the work of Dana Scott, Richard Montague, George Boolos, John Derrick, and Michael Potter; this paper inherits those debts.¹

Some remarks on notation (which is exactly as in Pt.1 Section 0). I use second-order logic throughout. Mostly, though, this is just for convenience. Except when discussing quasi-categoricity (see Section 5), any second-order claim can be replaced with a first-order schema in the obvious way. I use some simple abbreviations (where Ψ can be any predicate whose only free variable is x , and \triangleleft can be any infix predicate):

$$\begin{aligned} (\forall x : \Psi)\phi &:= \forall x(\Psi(x) \rightarrow \phi) & (\forall x \triangleleft y)\phi &:= \forall x(x \triangleleft y \rightarrow \phi) \\ (\exists x : \Psi)\phi &:= \exists x(\Psi(x) \wedge \phi) & (\exists x \triangleleft y)\phi &:= \exists x(x \triangleleft y \wedge \phi). \end{aligned}$$

I also concatenate infix conjunctions, writing things like $a \subseteq r \in s \in t$ for $a \subseteq r \wedge r \in s \wedge s \in t$. And I run these devices together; so $(\forall x \notin x \in a)x \subseteq a$ abbreviates $\forall x((x \notin x \wedge x \in a) \rightarrow x \subseteq a)$. When I announce a result or definition, I list in brackets the axioms I am assuming. For readability, all proofs are relegated to the appendices.

§1. The complemented story. Here is a very natural image of sets: *sets are not just collections of objects; sets partition the universe, and both sides of the partition yield a set.* There is the set of sheep; and there is the set of non-sheep. There is the set of natural numbers; and there is the set of everything else. There is the empty set; and there is the universal set.

Many will reject this image out of hand. Supposedly, the paradoxes of naïve set theory have taught us that there is no universal set; for if there were a universal set $V = \{x : x = x\}$, then Separation would entail the existence of the Russell set $\{x : x \notin x\}$, which is a contradiction.

That reasoning, though, is too quick. Separation is incompatible with the existence of V .² More generally, Separation is incompatible with the principle of Complementation (i.e., with the principle that every set has an absolute complement). But it does not immediately follow that Complementation is false; only that we must choose between Separation and Complementation.

Both principles are very natural. Separation, however, has the weight of history behind it; and this might not merely be a historical accident. There is a serious argument in favour of Separation and against Complementation, which runs as follows. The paradoxes of naïve set theory forced us to develop a less naïve conception of *set*. The best such conception (according to this

¹See in particular Montague (1965, p. 139), Montague et al. (unpublished, Section 22), Scott (1960, 1974), Boolos (1971, pp. 8–11, 1989), and Potter (1990, pp. 16–22, 2004, Chapter 3).

²NB: I assume classical logic throughout.

argument) is the cumulative iterative conception, as articulated by this bare-bones story (recycled from Pt.1):

The Basic Story. Sets are arranged in stages. Every set is found at some stage. At any stage s : for any sets found before s , we find a set whose members are exactly those sets. We find nothing else at s .

It is easy to see that this conception of set yields Separation rather than Complementation: any subset of a set a occurs at (or before) any stage at which a itself occurs. So (the argument concludes) we should embrace Separation and reject Complementation.

I take this argument very seriously. However, its success hinges on whether the ordinary cumulative iterative conception really is the ‘best’ conception of *set*. Whatever exactly ‘best’ is supposed to mean, the argument lays down a challenge: produce an equally good or better conception of *set*, which accepts Complementation and rejects Separation.

This paper considers a very specific reply to this challenge, due to Forster’s development of work by Church and Oswald.³ Forster’s idea is to make a small tweak to the story of the ordinary hierarchy, so that ‘each time we [find] a new set...we also [find] a companion to it which is to be its complement’.⁴ In slightly more detail, we offer the following bare-bones story:

The Complemented Story. Sets are arranged in stages. Every set is found at some stage. At any stage s : for any sets found before s , we find both

(Lo) a set whose *members* are exactly those sets, and

(Hi) a set whose *non-members* are exactly those sets.

We find nothing else at s .

³Church (1974) and Oswald (1976); see also Mitchell (1976) and Sheridan (2016). Forster (2001) includes a nice summary of the technicalities behind the original Church–Oswald idea.

⁴Forster (2008, p. 100). Note that I speak of ‘finding’ sets, whereas Forster speaks of ‘creating’ them. Talk of ‘creation’ leads Forster to say that the members of V change, stage-by-stage, as more sets are created, so that V is ‘intensional’, in a way that \emptyset is not (2008, p. 100). I think that Forster should regard \emptyset as equally ‘intensional’, since what \emptyset omits changes, stage-by-stage. However, if sets are discovered (rather than created) stage-by-stage, then all issues concerning intensionality can be side-stepped: all that changes, stage-by-stage, is our knowledge about V ’s members and \emptyset ’s non-members.

If we admit contingently-existing urelements, then the discussion of intensionality becomes much more complicated. In the actual world, Boudica $\in \{x : x = x\}$; but in a possible world where she never existed, Boudica $\notin \{x : x = x\}$; by contrast, in all possible worlds, Boudica $\notin \{x : x \neq x\}$. From this, one might infer that V is intensional whereas \emptyset is not. But this inference is not immediate; it requires two substantial, further, assumptions: (1) that the descriptions ‘ $\{x : x \neq x\}$ ’ and ‘ $\{x : x = x\}$ ’ *rigidly* designate \emptyset and V respectively, and (2) that intensionality concerns trans-world variation of *members* rather than trans-world variation of *non-members*. I hope to explore both assumptions elsewhere. (Thanks to James Studd, Timothy Williamson, Stephen Yablo, and an anonymous referee for this journal, for pushing me on this point.)

According to our new story, we find each set using either clause (*Lo*) or clause (*Hi*). Moreover, if we find a set using clause (*Lo*), then we find its absolute complement using clause (*Hi*), and vice versa. This is the absolute complement since, in clause (*Hi*), we quantify over all sets that will ever be discovered, not just those discovered before stage *s*. This story therefore secures Complementation; it describes the bare idea of a *complemented* hierarchy of sets. But it only describes the *bare* idea, since, for example, it says nothing about the height of the hierarchy.

In what follows, I will develop an axiomatic theory of this story, and explore that theory's behaviour. To be clear: I am not claiming that we should reject the ordinary hierarchy in favour of the complemented. My aim is only to provide a coherent (and surprisingly elegant) conception of *set* which allows for Complementation rather than Separation.

In what follows, I will speak of *low* sets and *high* sets.⁵ A set is low iff we find it using clause (*Lo*); we characterize low sets by saying 'exactly these things, which we found earlier, are this set's members'. The limiting case of a low set is the empty set, \emptyset . A set is high iff we find it using clause (*Hi*); we characterize high sets by saying 'exactly these things, which we found earlier, are omitted from this set'. The limiting case of a high set is the universe, V . (Note that low sets can have high sets as members, e.g., $\{V\}$ would be a low set with a high member.)

§2. Boolean Stage Theory. Given a model of ZF, there are simple methods for constructing models of the complemented hierarchy.⁶ However, if the idea of a complemented hierarchy is genuinely to rival that of the ordinary hierarchy, it cannot remain parasitic upon ZF; it needs a fully autonomous theory. I will provide such a theory over the next two sections.⁷

The Complemented Story, which introduces the bare-bones idea of a complemented hierarchy, speaks of both stages and sets. To begin, then, I will present a theory which quantifies distinctly over both sorts of entities. Boolean Stage Theory, or BST, has two distinct sorts of first-order variable, for *sets* (lower-case italic) and for **stages** (lower-case bold). It has five primitive predicates:

- \in : a relation between sets; read ' $a \in b$ ' as ' a is in b '
- $<$: a relation between stages; read ' $\mathbf{r} < \mathbf{s}$ ' as ' \mathbf{r} is before \mathbf{s} '
- \preceq : a relation between a set and a stage; read ' $a \preceq \mathbf{s}$ ' as ' a is found at \mathbf{s} '

⁵Note that every set will be low or high. This terminology departs somewhat from Church's. Church (1974, p. 298) defined 'a *low* set as a set which has a one-to-one relation with a well-founded set' and 'a *high* set as a set which is the complement of a low set'. This leaves logical space for sets which are neither low nor high (in Church's terms), and Church (1974, p. 305) used such sets to provide a Frege–Russell definition of cardinal numbers.

⁶See Forster (2001, Section 1–2, 2008, pp. 106–8); and my interpretation *I* in Appendix D.1.

⁷The approach in this section follows Scott and Boolos, but in the setting of complemented hierarchies rather than the ordinary hierarchies; see Pt.1 Sections 1 and 8.

Lo: a property of sets; read ‘*Lo*(*a*)’ as ‘*a* is low’, i.e., we find *a* using clause (*Lo*)

Hi: a property of sets; read ‘*Hi*(*a*)’ as ‘*a* is high’, i.e., we find *a* using clause (*Hi*)

For brevity, I write $a \prec \mathbf{s}$ for $\exists \mathbf{r}(a \preceq \mathbf{r} < \mathbf{s})$, i.e., *a* is found before *s*. Then BST has eight axioms:⁸

- Extensionality** $\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b)$
- Order** $\forall \mathbf{r} \forall \mathbf{s} \forall \mathbf{t} (\mathbf{r} < \mathbf{s} < \mathbf{t} \rightarrow \mathbf{r} < \mathbf{t})$
- Staging** $\forall a \exists \mathbf{s} a \preceq \mathbf{s}$
- Cases** $\forall a (Lo(a) \vee Hi(a))$
- Priority_{Lo}** $\forall \mathbf{s} (\forall a : Lo)(a \preceq \mathbf{s} \rightarrow (\forall x \in a) x \prec \mathbf{s})$
- Priority_{Hi}** $\forall \mathbf{s} (\forall a : Hi)(a \preceq \mathbf{s} \rightarrow (\forall x \notin a) x \prec \mathbf{s})$
- Specification_{Lo}** $\forall F \forall \mathbf{s} ((\forall x : F) x \prec \mathbf{s} \rightarrow (\exists a : Lo)(a \preceq \mathbf{s} \wedge \forall x (F(x) \leftrightarrow x \in a)))$
- Specification_{Hi}** $\forall F \forall \mathbf{s} ((\forall x : F) x \prec \mathbf{s} \rightarrow (\exists a : Hi)(a \preceq \mathbf{s} \wedge \forall x (F(x) \leftrightarrow x \notin a)))$

I will now explain how to justify each axiom.

The first two axioms make implicit assumptions explicit. Whilst I did not mention Extensionality when I told the story of the complemented hierarchy, I take it as analytic that sets are extensional.⁹ Similarly, Order records the analytic fact that ‘before’ is transitive. Note, though, that I do not explicitly assume that the stages are well-ordered,¹⁰ as it is unclear at this point what would justify that assumption. (After all, if we are willing to countenance entities as ill-founded as *V*, then it is not immediately obvious that we should refuse to countenance a hierarchy with infinite descending chains of stages. And the Complemented Story does not explicitly require that the stages be well-ordered.)

Informally, Staging says that every set is discovered at some stage; this claim appears verbatim in the Complemented Story. Likewise, Cases says that every set is either low or high, and this is immediate from the fact that every set is discovered using either clause (*Lo*) or clause (*Hi*). (Note, though, that I do not assume at the outset that this is an exclusive disjunction; initially, we should be open to the thought that one set could be discovered using both clauses.)¹¹

Next, Priority_{Lo} and Priority_{Hi} say that if we find a low set at a stage, then we find all its members earlier, and if we find a high set at a stage, then we find all its non-members earlier; both claims follow from clauses (*Lo*) and (*Hi*). Finally, Specification_{Lo} and Specification_{Hi} say that if every *F* was

⁸Using classical logic yields ‘cheap’ proofs of the existence of a stage, an empty set, and a universal set, via Staging, Specification_{Lo} and Specification_{Hi}. Those who find such proofs *too* cheap might wish to add some explicit existence axioms. (Cf. Pt.1 footnote 2.)

⁹For brevity of exposition, I am considering hierarchies of pure sets.

¹⁰Here I part company with Forster (2008, p. 100), who explicitly stipulates that the stages are well-ordered. Ultimately, BST proves a well-ordering result (Theorem 4.1).

¹¹Ultimately, BST proves that no set is discovered using both clauses (Lemma B.7).

found before a certain stage, then at that stage we find both the low set of all *F*s, and the high set of all non-*F*s; again, both claims follow from (*Lo*) and (*Hi*).

Since all eight axioms hold of the Complemented Story, any complemented hierarchy satisfies BST.

§3. Boolean Level Theory. Unfortunately, BST contains rather a lot of primitives. Fortunately, most of them can be eliminated. In this section, I present Boolean Level Theory, or BLT. This theory’s only primitive is \in , but it makes exactly the same claims about sets as BST does.¹² I start with a key definition:¹³

DEFINITION 3.1. For any *a*, let *a*’s absolute complement be $\bar{a} = \{x : x \notin a\}$, if it exists. Let $\mathbf{P}a = \{x : (\exists c \notin c \in a)(x \subseteq c \vee \bar{x} \subseteq c)\}$, if it exists.¹⁴

The definition of \bar{a} needs no comment, but the definition of $\mathbf{P}a$ merits explanation. It turns out that BST proves that *a* is low iff $a \notin a$, and *a* is high iff $a \in a$ (see Lemma B.7). Seen in this light, $\mathbf{P}a$ collects together all the subsets of low members of *a*, and all the complements of such subsets. As a specific example, if *b* is low, then $\mathbf{P}\{b\} = \{x : x \subseteq b \vee \bar{x} \subseteq b\}$, i.e., it is the result of closing *b*’s powerset under complements. We use this operation in this next definition (where ‘bistory’ is short for ‘boolean-history’, and ‘bevel’ is short for ‘boolean-level’):¹⁵

DEFINITION 3.2. Say that *h* is a *bistory*, written $Bist(h)$, iff $h \notin h \wedge (\forall x \in h)x = \mathbf{P}(x \cap h)$. Say that *s* is a *bevel*, written $Bev(s)$, iff $(\exists h : Bist) s = \mathbf{P}h$.

The intuitive idea behind Definition 3.2 is that the bevels go proxy for the stages of the Complemented Story, and each bistory is an initial sequence of bevels. (It is far from obvious that these definitions work as described, but we will soon see that they do.) Using these definitions, BLT has just four axioms:¹⁶

- Extensionality** $\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b)$
- Complements** $\forall a (\exists c = \bar{a})(a \notin a \leftrightarrow c \in c)$
- Separation** \notin $\forall F (\forall a \notin a) (\exists b \notin b) \forall x (x \in b \leftrightarrow (F(x) \wedge x \in a))$
- Stratification** \notin $(\forall a \notin a) (\exists s : Bev) a \subseteq s$

¹²The approach in this section mirrors Pt.1 Sections 2–4, which builds on work by Montague, Scott, Derrick and Potter; see also Pt.1 Section 8.

¹³Compare Montague’s and Scott’s \mathbb{P} -operation, presented in Pt.1 Definition 2.1.

¹⁴By the notational conventions, $\mathbf{P}a = \{x : \exists c (c \in a \wedge c \notin c \wedge (x \subseteq c \vee \bar{x} \subseteq c))\}$. BLT’s axiom Complements guarantees that \bar{a} exists for every *a*. However, we do not initially assume that $\mathbf{P}a$ exists for every *a*; instead, we initially treat every expression of the form ‘ $b = \mathbf{P}a$ ’ as shorthand for ‘ $\forall x (x \in b \leftrightarrow (\exists c \notin c \in a)(x \subseteq c \vee (\exists z \subseteq c) \forall y (y \in z \leftrightarrow y \notin x)))$ ’, and must double-check whether $\mathbf{P}a$ exists. Ultimately, though, BLT proves that $\mathbf{P}a$ exists for every *a*: if $a \notin a$ then $\mathbf{P}a \subseteq \mathbf{B}a$ (see Definition 4.3); if $a \in a$ then $\mathbf{P}a = V$.

¹⁵Compare Pt. 1 Definition 2.2, which simplifies the Derrick–Potter definition of ‘level’.

¹⁶As in footnote 8, classical logic yields a ‘cheap’ proof of the existence of \emptyset and *V*.

Intuitively, Complements tells us that every set has a complement, and a set is low iff its complement is high; Separation_≠ tells us that arbitrary subsets of low sets exist (and are low); and Stratification_≠ tells us that every low set is a subset of some bevel (which corresponds to the thought that it is found at some stage). These axioms and definitions are vindicated by this next result, which shows that BLT has exactly the same set-theoretic content as BST (see Appendix B for the proof):

THEOREM 3.3. $BST \vdash \phi$ iff $BLT \vdash \phi$, for any BLT-sentence ϕ .

Otherwise put: no information about sets is gained or lost by moving between BST and BLT. Moreover, since every complemented hierarchy satisfies BST, every complemented hierarchy satisfies BLT. In what follows, then, I will treat BLT as the canonical theory of complemented hierarchies.

§4. Characteristics and extensions of BLT. To give a sense of how BLT behaves, I will state some of its ‘characteristic’ results (the proofs are in Appendix A). The first two results allow us to characterize BLT with a simple slogan: *a boolean algebra of sets arranged in well-ordered levels*.

THEOREM 4.1 (BLT). *The bevels are well-ordered by \in .*

THEOREM 4.2 (BLT). *The sets form a boolean algebra under complementation, \cap and \cup .*

This first result is quite surprising:¹⁷ the Complemented Story does not *explicitly* specify that the stages must be well-ordered (see Section 2); but, since every complemented hierarchy satisfies BLT (see Section 3), every complemented hierarchy has well-ordered levels.

The well-ordering of the bevels yields a powerful tool, which intuitively allows us to consider the bevel at which a set is first found:

DEFINITION 4.3 (BLT). If $a \notin a$, let $\mathbf{B}a$ be the \in -least bevel with a as a subset; i.e., $a \subseteq \mathbf{B}a$ and $\neg(\exists s : Bev)a \subseteq s \in \mathbf{B}a$. If $a \in a$, let $\mathbf{B}a = \mathbf{B}\bar{a}$.

Note that $\mathbf{B}a$ exists for any a , by Stratification_≠, Complements and Theorem 4.1.

A third characteristic result is that there is a *contra-automorphism* on the universe.¹⁸ Roughly put: replacing membership with non-membership (and vice versa) yields an isomorphic universe. Formally:

DEFINITION 4.4. We recursively define a 's *negative*, written $-a$, as follows:

$$-a := \overline{\{-x : x \in a\}}, \text{ if } a \notin a \quad -a := \{-x : x \notin a\}, \text{ if } a \in a$$

THEOREM 4.5 (BLT). $\forall a \forall b (a \in b \leftrightarrow -a \notin -b)$

This immediately yields a nice duality:

¹⁷It will be much less surprising for those who have read Pt.1 Section 5.

¹⁸See Forster (2001, Definition 16 and subsequent comments). This result inspires my epigraph, from Le Guin. I owe the point to Brian King: in 2006, he arrived at an idea like the Complemented Story (independently of Forster) and explained it using Le Guin's image.

COROLLARY 4.6 (BLT). $\phi \leftrightarrow \phi^\circ$, for any BLT-sentence ϕ , where ϕ° is the sentence which results from ϕ by replacing every ‘ \in ’ with ‘ \notin ’ and vice versa.

These results highlight some of BLT’s deductive strengths. Now let me comment on its (deliberate) weakness. By design, BLT axiomatizes only the *bare* idea of a complemented hierarchy, and so makes no comment on the hierarchy’s height.¹⁹ If we want to ensure that our hierarchy is reasonably tall, three axioms suggest themselves (where ‘ P ’ is a second-order function-variable in the statement of Unbounded_\notin):

- Endless $_\notin$** $(\forall s : \text{Bev})(\exists t : \text{Bev})s \in t$
- Infinity $_\notin$** $(\exists s : \text{Bev})((\exists q : \text{Bev})q \in s \wedge (\forall q : \text{Bev})(q \in s \rightarrow (\exists r : \text{Bev})q \in r \in s))$
- Unbounded $_\notin$** $\forall P(\forall a \notin a)(\exists s : \text{Bev})(\forall x \in a)P(x) \in s$

Endless_\notin says there is no last bevel. Infinity_\notin says that there is an infinite bevel, i.e., a bevel with no immediate predecessor. Unbounded_\notin states that the hierarchy of bevels is so tall that no low set can be mapped unboundedly into it (recall that the low sets are precisely the non-self-membered sets).

To make all of this more familiar, here are some simple facts relating BLT to ZF. Let BLT_+ stand for $\text{BLT} + \text{Endless}_\notin$, and BLT_{ZF} stand for $\text{BLT} + \text{Infinity}_\notin + \text{Unbounded}_\notin$; then:²⁰

PROPOSITION 4.7.

- (1) BLT proves the Axiom of Empty Set, i.e., $\exists a \forall x \ x \notin a$.
- (2) BLT proves Union, i.e. $\forall a (\bigcup a \text{ exists})$.
- (3) BLT_+ proves Pairing, i.e., $\forall a \forall b (\{a, b\} \text{ exists})$, but BLT does not.
- (4) BLT_+ proves Powersets-restricted-to-low-sets, i.e., $(\forall a \notin a)(\wp a \text{ exists})$, but BLT does not.
- (5) BLT contradicts Powersets, i.e., it proves $\exists a \neg \exists b \forall x (x \in b \leftrightarrow x \subseteq a)$.
- (6) BLT proves Foundation-restricted-to-high-sets, i.e., $(\forall a \in a)(\exists x \in a) a \cap x = \emptyset$.
- (7) BLT_+ contradicts Foundation, i.e., it proves $(\exists a \neq \emptyset)(\forall x \in a) a \cap x \neq \emptyset$.
- (8) BLT_{ZF} proves Endless_\notin .

If we want to state this result with maximum shock value: of the standard axioms of ZF, BLT validates only Extensionality, Empty Set, and Union (though BLT is also consistent with Pairing and standard formulations of Infinity).

§5. The quasi-categoricity of BLT. We have seen that every complemented hierarchy satisfies BLT, so that every complemented hierarchy has well-

¹⁹Beyond the fact that classical logic guarantees the existence of at least one stage; see footnotes 8 and 16.

²⁰Since BLT_+ proves Pairing, BLT_+ extends NF_2 , the sub-theory of Quine’s NF whose axioms are Extensionality, Pairing, and Theorem 4.2. However, BLT_+ does not extend NF_O , the theory which adds to NF_2 the axiom that $\{x : a \in x\}$ exists for every a ; in particular, $\{x : \emptyset \in x\}$ does not exist; see the proof of Proposition 4.7.5 in Appendix A. For discussion of NF_2 and NF_O , see Forster (2001, Section 2).

ordered bevels. In fact, we can push this point further, by noting that BLT is quasi-categorical.²¹

Informally, we can spell out BLT’s quasi-categoricity as follows: *Any two complemented hierarchies are structurally identical for so far as they both run, but one may be taller than the other.* So, when we set up a complemented hierarchy, our only choice is how tall to make it.

In fact, there are at least two ways to explicate the informal idea of quasi-categoricity, and BLT is quasi-categorical on both explications.²² The first notion of quasi-categoricity should be familiar from Zermelo’s results for ZF, and uses the full semantics for second-order logic:

THEOREM 5.1. *Given full second-order logic:*

- (1) *The bevels of any model of BLT are well-ordered.*²³
- (2) *For any ordinal $\alpha > 0$, there is a model of BLT whose bevels form an α -sequence.*²⁴
- (3) *Given any two models of BLT, one is isomorphic to an initial segment of the other.*²⁵

Since this result involves semantic ascent, it is an *external* quasi-categoricity result. There is also an *internal* quasi-categoricity result for BLT, which is a theorem of the (second-order) object language, but this point requires a little more explanation.

In embracing Extensionality, BLT assumes that everything is a pure set. Here is an easy way to avoid making that assumption. Consider the following formula, which relativises BLT to a new primitive predicate, *Pure*:²⁶

$$\begin{aligned} \text{BLT}(\text{Pure}, \varepsilon) := & (\forall a : \text{Pure})(\forall b : \text{Pure})(\forall x(x \varepsilon a \leftrightarrow x \varepsilon b) \rightarrow a = b) \wedge \\ & (\forall a : \text{Pure})(\exists c : \text{Pure})((\forall x : \text{Pure})(x \varepsilon c \leftrightarrow x \notin a) \\ & \wedge (a \notin a \leftrightarrow c \varepsilon c)) \wedge \\ & \forall F(\forall a : \text{Pure})(a \notin a \rightarrow (\exists b : \text{Pure})(b \notin b \\ & \wedge \forall x(x \varepsilon b \leftrightarrow (F(x) \wedge x \varepsilon a)))) \wedge \\ & (\forall a : \text{Pure})(a \notin a \rightarrow (\exists s : \text{Bev})a \subseteq s) \wedge \\ & \forall x \forall y(y \varepsilon x \rightarrow (\text{Pure}(x) \wedge \text{Pure}(y))) \end{aligned}$$

The first four conjuncts say that the pure sets satisfy BLT;²⁷ the last says that, when we use ‘ ε ’, we restrict our attention to membership facts between pure

²¹This mirrors the discussion of LT’s quasi-categoricity; see Pt.1 Section 6.

²²Both ways make essential use of second-order logic, albeit in different ways.

²³That is, if $\mathcal{M} \models \text{BLT}$ then $\{s \in \mathcal{M} : \mathcal{M} \models \text{Bev}(s)\}$ is well-ordered by $\in^{\mathcal{M}}$.

²⁴That is, there is some $\mathcal{M} \models \text{BLT}$ such that $\{s \in \mathcal{M} : \mathcal{M} \models \text{Bev}(s)\}$ is isomorphic to α .

²⁵When \mathcal{A} and \mathcal{M} are models of BLT, say that \mathcal{A} is an *initial segment* of \mathcal{M} iff either $\mathcal{A} = \mathcal{M}$ or there is some s such that $\mathcal{M} \models \text{Bev}(s)$ and \mathcal{A} is isomorphic to the substructure of \mathcal{M} whose domain is $\{x \in \mathcal{M} : \mathcal{M} \models \mathbf{B}x \in s\}$.

²⁶Here, ‘ \subseteq ’ and ‘*Bev*’ should be defined in terms of ε rather than \in ; similarly for ‘*B*’ in the statement of Theorem 5.2.

²⁷With one insignificant caveat (see footnote 16): whereas classical logic guarantees that any model of BLT contains an empty set and a universal set, $\text{LT}(\text{Pure}, \varepsilon)$ allows that there may be no pure sets.

sets. This avoids the assumption that everything is a pure set. Moreover, I can use this formula to state our internal quasi-categoricity result (I have labelled the lines to facilitate its explanation).²⁸

THEOREM 5.2. *This is a deductive theorem of impredicative second-order logic:*

$$(\text{BLT}(Pure_1, \varepsilon_1) \wedge \text{BLT}(Pure_2, \varepsilon_2)) \rightarrow$$

$$\exists R(\forall v \forall y(R(v, y) \rightarrow (Pure_1(v) \wedge Pure_2(y))) \wedge \tag{1}$$

$$((\forall v : Pure_1) \exists y R(v, y) \vee (\forall y : Pure_2) \exists v R(v, y)) \wedge \tag{2}$$

$$\forall v \forall y \forall x \forall z ((R(v, y) \wedge R(x, z)) \rightarrow (v \varepsilon_1 x \leftrightarrow y \varepsilon_2 z)) \wedge \tag{3}$$

$$\forall v \forall y \forall z ((R(v, y) \wedge R(v, z)) \rightarrow y = z) \wedge \tag{4}$$

$$\forall v \forall x \forall y ((R(v, y) \wedge R(x, y)) \rightarrow v = x) \wedge \tag{5}$$

$$\forall v \forall x \forall y ((\mathbf{B}_1 x \subseteq_1 \mathbf{B}_1 v \wedge R(v, y)) \rightarrow \exists z R(x, z)) \wedge \tag{6}$$

$$\forall v \forall y \forall z ((\mathbf{B}_2 z \subseteq_2 \mathbf{B}_2 y \wedge R(v, y)) \rightarrow \exists x R(x, z)) \tag{7}$$

Intuitively, the point is this. Suppose two people are using their versions of BLT, subscripted with ‘1’ and ‘2’ respectively. Then there is some second-order entity, a relation R , which takes us between their sets (1), exhausting the sets of one or the other person (2); which preserves membership (3); which is functional (4) and injective (5); and whose domain is an initial segment of one (6) or the other’s (7) hierarchy. Otherwise put: BLT is (internally) quasi-categorical.

As a bonus, this internal quasi-categoricity result can be lifted into an internal *total*-categoricity result. To explain how, consider this abbreviation (where ‘ P ’ is a second-order function-variable):

$$\exists! x \Phi(x) := \exists P(\forall x \Phi(P(x)) \wedge (\forall y : \Phi) \exists! x P(x) = y).$$

This formalizes the idea that there is a bijection between the Φ s and the universe (see Pt.1 Section 6). Using this notation, we can state our internal total-categoricity result:

THEOREM 5.3. *This is a deductive theorem of impredicative second-order logic:*

$$(\text{BLT}(Pure_1, \varepsilon_1) \wedge \exists! x Pure_1(x) \wedge \text{BLT}(Pure_2, \varepsilon_2) \wedge \exists! x Pure_2(x)) \rightarrow$$

$$\exists R(\forall v \forall y(R(v, y) \rightarrow (Pure_1(v) \wedge Pure_2(y))) \wedge$$

$$(\forall v : Pure_1) \exists! y R(v, y) \wedge (\forall y : Pure_2) \exists! v R(v, y) \wedge$$

$$\forall v \forall y \forall x \forall z ((R(v, y) \wedge R(x, z)) \rightarrow (v \varepsilon_1 x \leftrightarrow y \varepsilon_2 z)))$$

Intuitively, if both BLT-like hierarchies are as large as the universe, then there is a structure-preserving *bijection* between them.

²⁸Button and Walsh’s (2018, Chapter 11) proofs carry over straightforwardly to BLT.

§6. Ordinary set theory as a proper part of BLT. The Complemented Story provides two clauses for finding sets. Clause (*Lo*) tells us that, at each stage s and for any sets found before s , we find a set whose members are exactly those sets. But this is exactly what we would find according to the Basic Story (see Section 1), which deals with ordinary, uncomplemented hierarchies. Intuitively, then, we should be able to recover an ordinary hierarchy by considering a complemented hierarchy whilst ignoring any use of clause (*Hi*). This intuitive idea is exactly right; the aim of this section is to explain it carefully.

First, I must formalize the notion of a set which we find without ever using clause (*Hi*). I call such sets *hereditarily low*, or *helow* for short. So: helow sets are low, their members are low, the members of their members are low, etc. Here is the precise definition:

DEFINITION 6.1. Say that a is *helow*, or $Helo(a)$, iff there is some transitive $c \supseteq a$ such that $(\forall x \in c)x \notin x$.

To restrict our attention to the ordinary (uncomplemented) hierarchy, we then just restrict our attention to the helow sets. To implement this formally, for any formula ϕ , let ϕ^∇ be the formula which results by restricting all of ϕ 's quantifiers to helow sets. Using this notation, we can then prove results of this shape: *If some theory of uncomplemented hierarchies proves ϕ , then some suitable theory of complemented hierarchies proves ϕ^∇ .*

To state these results precisely, we need a suitable theory of uncomplemented hierarchies. That theory is LT, discussed in Pt.1. In a nutshell, LT stands to uncomplemented hierarchies exactly as BLT stands to complemented hierarchies. I will now briefly recap LT's key elements. To formalize the Basic Story, we define a predicate, *Lev*, to capture the notion of a *level* of an uncomplemented hierarchy (Pt.1 Definition 2.2); then LT is the theory whose axioms are Extensionality, Separation, and Stratification, which states that $\forall a(\exists s : Lev)a \subseteq s$ (see Pt.1 Section 2). It transpires that LT is quasi-categorical, and that every uncomplemented hierarchy satisfies LT, regardless of its height (see Pt.1 Sections 5–6). If we want to secure a tall uncomplemented hierarchy, we can consider the axioms Endless, Infinity and Unbounded (see Pt.1 Section 7); these are exactly like $Endless_\neq$, $Infinity_\neq$ and $Unbounded_\neq$ (see Section 3 of this part), except that they replace '*Bev*' with '*Lev*'. Let LT_+ stand for $LT + Endless$; it turns out that ZF is deductively equivalent to $LT + Infinity + Unbounded$; so LT , LT_+ , and ZF are three theories which axiomatize uncomplemented hierarchies, making successively stronger demands on the hierarchy's height. With this background in place, here is the result which intuitively states that the helow part of any complement hierarchy is an ordinary (uncomplemented) hierarchy (see Appendix C for the proof):

THEOREM 6.2. For any LT-sentence ϕ :

- (1) If $LT \vdash \phi$, then $BLT \vdash \phi^\nabla$
- (2) If $LT_+ \vdash \phi$, then $BLT_+ \vdash \phi^\nabla$
- (3) If $ZF \vdash \phi$, then $BLT_{ZF} \vdash \phi^\nabla$

§7. Definitional equivalence. Theorem 6.2.3 allows us to regard ZF as the result of restricting attention to the below-fragment of BLT_{ZF} 's universe of sets. But we also have a much deeper interpretative result, as follows (see Appendix D):²⁹

THEOREM 7.1. *ZF and BLT_{ZF} are definitionally equivalent, as are LT_+ and BLT_+ .*

As an immediate consequence, ZF and BLT_{ZF} are *equiconsistent*, as are LT_+ and BLT_+ . However, definitional equivalence is much stronger than mere equiconsistency.

Roughly, to say that two theories are definitionally equivalent is to say that each theory can define all the primitive expressions of the other, such that each theory can simulate the other perfectly, and where combining the two simulations gets you back exactly where you began.³⁰ So, in some purely formal sense, ZF and BLT_{ZF} can be regarded as notational variants; as wrapping the same deductive content in different notational packaging.

One might be tempted to go further, and suggest that Theorem 7.1 shows that there is *no* relevant difference between ZF and BLT_{ZF} . That, however, would require further argument.³¹ Precisely because definitional equivalence is a purely formal property, it ignores all non-formal matters, and these may be philosophically significant. There is more philosophical discussion to be had about the significance of Theorem 7.1, but that must wait for another time.

§8. Conway games and surreal numbers in BLT. Since ZF and BLT_{ZF} are definitionally equivalent, there is a sense in which each can do anything that the other can. Still, BLT_{ZF} can do some things more easily than ZF. This is neatly illustrated by considering John Conway's theory of games and surreal numbers.³²

Consider two-player games in which players move alternately, with no element of chance, where the game must end in a win or loss. (Think of chess, but without the possibility of stalemate.) Abstractly, such games can be thought of as specifications of permissible positions: to make a move in such a game is just to select a new position which is permissible given the current game state; and you lose when it is your turn to move but there is no permissible position. (Think of being checkmated: you must move to a position where your King is not in check, but no such move is available.) Crucially, any position in any such game can be considered as a game in its own right. (Imagine the version of chess which always starts with the pieces arranged as after the Queen's Gambit in regular chess.) So every game can

²⁹Forster conjectured that a result of this shape should hold.

³⁰For a more precise statement of what definitional equivalence requires, see Button and Walsh (2018, Chapter 5).

³¹Compare Pt.2 Section 9.

³²Joel David Hamkins suggested this application of BLT to me; many thanks to him, both for the initial suggestion, and for much subsequent correspondence.

be regarded, abstractly, as nothing other than a specification of which games each player can move to. Otherwise put, if we call the two game-players Low and High, then a game is just a specification of *low options*, i.e., games that Low can move to, and *high options*, i.e., games that High can move to.

The idea is very natural. However, as Conway remarked, formalizing it ‘in ZF destroys a lot of its symmetry.’ He therefore suggested that ‘the proper set theory in which to perform such a formalisation would be one with two kinds of membership’: a game would just be a set with ‘low-members’ (low options) and ‘high-members’ (high options).³³ However, we can easily implement this idea in BLT, using only *one* kind of membership. We start by saying that the games are the sets, and then stipulate:

DEFINITION 8.1 (BLT). If a is low, the set of a ’s *low options* is $La := \{x \in a : x \notin x\}$; the set of a ’s *high options* is $Ha := \{x \in a : x \in x\}$. If a is high, $La := \bar{La}$ and $Ha := \bar{Ha}$.

Intuitively, then, a and \bar{a} represent the same game. Moreover, there is a natural algebra on the games, given as follows (I explain the definitions below):³⁴

DEFINITION 8.2 (BLT). With $-$ as in Definition 4.4, define $+$ and \leq recursively:

$$\begin{aligned}
 a + c &:= \{x + c : x \in La\} \cup \{a + x : x \in Lc\} \cup \{\overline{y + c} : y \in Ha\} \\
 &\quad \cup \{\overline{a + y} : y \in Hc\} \\
 a \leq c &\text{ iff } (\forall y \in Hc) y \not\leq a \wedge (\forall x \in La) c \not\leq x
 \end{aligned}$$

We stipulate that $a \equiv c$ iff $a \leq c \leq a$, and define $a - c := a + (-c)$.

We can make these algebraic operations intuitive as follows. To take the *negative* of a game is to reverse the players’ roles (cf. Theorem 4.5). To *add* two games is to place them side-by-side, allowing a player to move in one game without affecting the other. But the *partial-order* requires slightly more explanation. Suppose High plays first on the game a ; then Low has a winning strategy *iff* whatever move High makes, i.e., for all $y \in Ha$, if Low plays first on y then High has no winning strategy. Similarly, suppose Low plays first on a ; then High has a winning strategy *iff* for all $x \in La$, if High plays first on x then Low has no winning strategy. So, if we gloss ‘ $\emptyset \leq z$ ’ as ‘Low has a winning strategy as second player on z ’ and gloss ‘ $z \leq \emptyset$ ’ as ‘High has a winning strategy as second player on z ’, this motivates two important special

³³Conway (1976, p. 66). Cox and Kaye (2012) take up this suggestion and offer an axiomatic theory with two kinds of membership; they prove it is definitionally equivalent with ZF. By Theorem 7.1, it is definitionally equivalent with BLT_{ZF}.

³⁴The well-ordering of bevels guarantees determinacy, and licenses induction and recursive definitions (see footnote 37, below). Definition 8.2 and 8.4 are BLT-implementations of Conway’s (1976, Chapters 0–1) definitions. (As defined, the sum of two low sets is always low; an arbitrary choice was required.) For Theorem 8.3, see Conway (1976, p. 78); for Theorem 8.5, see Conway (1976, Chapter 1). For an accessible presentation, see also Schleicher and Stoll (2006, Sections 2–4).

cases of the partial order:

$$\emptyset \leq a \text{ iff } (\forall y \in \text{Ha})y \not\leq \emptyset \qquad a \leq \emptyset \text{ iff } (\forall x \in \text{La})\emptyset \not\leq x.$$

The remainder of Definition 8.2 is then set up so that $a - b \leq \emptyset$ iff $a \leq b$. More generally, we have the following foundational result:

THEOREM 8.3 (BLT). *The sets form a partially-ordered abelian Group, with $\emptyset = 0$ and $+, -, \leq$ as in Definition 8.2, all modulo \equiv .*³⁵

We can obtain a totally-ordered Field by restricting our attention to surreals:

DEFINITION 8.4 (BLT). We specify that a is *surreal* iff: for all $x \in \text{La}$ and all $y \in \text{Ha}$, both x and y are surreal and $x \not\leq y$. We define multiplication on surreals thus:

$$a \cdot c := \{x \cdot c + a \cdot y - x \cdot y : (x \in \text{La} \wedge y \in \text{Lc}) \vee (x \in \text{Ha} \wedge y \in \text{Hc})\} \cup \{\overline{x \cdot c + a \cdot y - x \cdot y} : (x \in \text{La} \wedge y \in \text{Hc}) \vee (x \in \text{Ha} \wedge y \in \text{Lc})\}.$$

We say that a is a *surreal-ordinal* iff a is both helow and surreal.

THEOREM 8.5 (BLT). *The surreals form a totally-ordered Field, modulo \equiv .*

Summing up: Conway’s beautifully rich, nonstandard, theory of surreal numbers is available, essentially off-the-shelf, within BLT.

§9. Conclusion. The Complemented Story lays down a conception of *set* which rivals the (ordinary) cumulative notion, but which accepts Complementation and rejects Separation (see Section 1).

I have shown that any complemented hierarchy satisfies BLT (see Sections 2–3). So, given the characteristic results of BLT, the sets of any complemented hierarchy are arranged into well-ordered bevels, and constitute a boolean algebra (see Section 4). Moreover, BLT is quasi-categorical (see Section 5); so our only choice, in setting up a complemented hierarchy, is how tall to make it.

The theory BLT_{ZF} arises from BLT just by adding axioms which state that the complemented hierarchy *is* quite tall (see Section 4). And we can regard ZF as either a proper part of BLT_{ZF} (see Section 6), or as a notational variant (in a purely formal sense) of BLT_{ZF} (see Section 7). But both interpretations suggest that there is no obvious *a priori* reason to favour Separation over Complementation. And in some settings, such as the discussion of Conway games, using Complementation is extremely natural (see Section 8).

§Appendix A. Characteristics of BLT. The remainder of this paper consists of proofs of the results discussed in the main text. Many of the simpler proofs are similar to results for Pt.1; in such cases, I omit the proof and refer interested readers to the appropriate result from Pt.1.

³⁵To quotient by \equiv , define $[a] := \{b \equiv a : (\forall x \equiv a)\mathbf{B}b \subseteq \mathbf{B}x\}$; cf. Scott (1955) and Conway (1976, p. 65).

This first appendix deals with the results from Section 4. Initially, I will work in ECS, the subtheory of BLT whose only axioms are Extensionality, Complements and Separation_≠ (see Section 3). I start with some simple results and definitions:

LEMMA A.1 (ECS). *If $c \subseteq a \not\subseteq a$, then $c \not\subseteq c$; if $a \in a \subseteq c$, then $c \in c$.*

PROOF. If $c \subseteq a \not\subseteq a$, then $c \not\subseteq c = \{x \in a : x \in c\}$ by Separation_≠ and Extensionality. If $a \in a \subseteq c$, then $\bar{c} \subseteq \bar{a} \not\subseteq \bar{a}$ by Complements, so that $\bar{c} \not\subseteq \bar{c}$ as before, and $c \in c$ by Complements. ⊥

DEFINITION A.2. Say that a is *potent*_≠ iff $\forall x(\exists c(x \subseteq c \not\subseteq c \in a) \rightarrow x \in a)$. Say that a is *transitive*_≠ iff $(\forall x \not\subseteq x \in a)x \subseteq a$. Say that a is *complement-closed* iff $\forall x(x \in a \leftrightarrow \bar{x} \in a)$.

LEMMA A.3 (ECS). *If $\mathbf{P}a$ exists (see Definition 3.1), then:*

- (1) $(\forall x \not\subseteq x \in \mathbf{P}a)\exists c(x \subseteq c \not\subseteq c \in a)$.
- (2) $\mathbf{P}a$ is *potent*_≠.
- (3) $\mathbf{P}a$ is *complement-closed*.

PROOF. (1) Fix $x \not\subseteq x \in \mathbf{P}a$; so for some $c \not\subseteq c \in a$, either $x \subseteq c$ or $\bar{x} \subseteq c$. But $\bar{x} \in \bar{x}$ by Complements, so $\bar{x} \not\subseteq c$ by Lemma A.1.

(2) Fix $x \subseteq c \not\subseteq c \in \mathbf{P}a$; so $x \subseteq c \subseteq b \not\subseteq b \in a$ for some b by (1); hence $x \in \mathbf{P}a$.

(3) Fix $x \in \mathbf{P}a$. If $x \subseteq c$ for some $c \not\subseteq c \in a$, then $x = \bar{x} \subseteq c$ so that $\bar{x} \in \mathbf{P}a$; if $\bar{x} \subseteq c$ for some $c \not\subseteq c \in a$, then $\bar{x} \in \mathbf{P}a$ straightforwardly. ⊥

It follows that bevels (see Definition 3.2) have several important closure properties:

LEMMA A.4 (ECS). *Every bevel is transitive_≠, potent_≠, complement-closed, and non-self-membered.*

PROOF. Let s be a bevel, i.e., $s = \mathbf{P}h$ for some history h . So s is potent_≠ and complement-closed by Lemma A.3. For transitivity_≠, fix $a \not\subseteq a \in s = \mathbf{P}h$; so $a \subseteq c \not\subseteq c \in h$ for some c by Lemma A.3.1; and $c = \mathbf{P}(c \cap h)$ as h is a history; so $a \subseteq \mathbf{P}(c \cap h) \subseteq \mathbf{P}h = s$. To see that $s \not\subseteq s$, suppose $s \in s$ for reductio. Then $\bar{s} \not\subseteq \bar{s} \in s$ by Complements, so $\bar{s} \subseteq s$ by transitivity_≠, so $s = V$. Since $h \not\subseteq h$ by definition, and $h \in V = s = \mathbf{P}h$, by Lemma A.3.1 there is some c such that $h \subseteq c \not\subseteq c \in h$. Since h is a history, $c = \mathbf{P}(h \cap c) = \mathbf{P}h = V$, contradicting the fact that $c \not\subseteq c$. ⊥

From here, we can prove the well-ordering of the bevels, by proving a sequence of results like those from Pt.1 Section 3; I leave this to the reader:³⁶

LEMMA A.5 (ECS). *If there is an F , and all F s are non-self-membered and potent_≠, then there is an \in -minimal F . Formally: $\forall F((\exists x F(x) \wedge (\forall x : F)(x \not\subseteq x \wedge x \text{ is potent}_{\neq})) \rightarrow (\exists a : F)(\forall x : F)x \not\subseteq a)$*

³⁶For Lemma A.7, first note that if h is a history and $c \in h$, then $c = \mathbf{P}(c \cap h) \subseteq \mathbf{P}h \not\subseteq \mathbf{P}h$ by Lemma A.4, so $c \not\subseteq c$ by Lemma A.1. For Lemmas A.8 and A.9, reason about non-self-membered sets in the first instance, then deal with self-membered sets using Complements and complement-closure.

LEMMA A.6 (ECS). *If some bevel is F , then there is an \in -minimal bevel which is F . Formally: $\forall F((\exists s : Bev)F(s) \rightarrow (\exists s : Bev)(F(s) \wedge (\forall r : Bev)(F(r) \rightarrow r \notin s)))$*

LEMMA A.7 (ECS). *Every member of a history is a bevel.*

LEMMA A.8 (ECS). $s = \mathbf{P}\{r \in s : Bev(r)\}$, for any bevel s .

LEMMA A.9 (ECS). *All bevels are comparable, i.e., $(\forall s : Bev)(\forall t : Bev)(s \in t \vee s = t \vee t \in s)$*

Combining Lemmas A.6 and A.9, ECS proves that the bevels are well-ordered by \in ; this is Theorem 4.1. This licenses our use of the \mathbf{B} -operator (see Definition 4.3). Here are some simple results about that operator, which can be proved by tweaking the proof of Pt.1 Lemma 3.12:

LEMMA A.10 (BLT). *For any sets a, c , and any bevels r, s :*

- (1) $\mathbf{B}a$ exists
- (2) $a \notin \mathbf{B}a$
- (3) $r \subseteq s$ iff $s \notin r$
- (4) $s = \mathbf{B}s$
- (5) if $c \subseteq a \notin a$ or $a \in a \subseteq c$, then $\mathbf{B}c \subseteq \mathbf{B}a$
- (6) if $c \in a \notin a$ or $c \notin a \in a$, then $\mathbf{B}c \in \mathbf{B}a$

Moreover, we can now show that sets are closed under arbitrary pairwise intersection:

LEMMA A.11 (BLT). *For any sets a and c , the set $a \cap c = \{x : x \in a \wedge x \in c\}$ exists.*

PROOF. First suppose that either $a \notin a$ or $c \notin c$ (or both); without loss of generality, suppose $a \notin a$; now $a \cap c = \{x \in a : x \in c\}$ exists by Separation $_{\notin}$. Next suppose that both $a \in a$ and $c \in c$. So both $\bar{a} \notin \bar{a}$ and $\bar{c} \notin \bar{c}$ by Complements. Let s be the maximum of $\mathbf{B}\bar{a}$ and $\mathbf{B}\bar{c}$. Since s is potent $_{\notin}$, both $\bar{a} \subseteq s$ and $\bar{c} \subseteq s$, so $\bar{a} \cup \bar{c} = \{x \in s : x \in \bar{a} \vee x \in \bar{c}\}$ exists by Separation $_{\notin}$. Now $a \cap c = \overline{\bar{a} \cup \bar{c}}$ exists by Complements. \dashv

This immediately entails that the sets form a boolean algebra, which is Theorem 4.2. Our next result shows that the universe is contra-automorphic:³⁷

THEOREM 4.5 (BLT). $\forall a \forall b(a \in b \leftrightarrow -a \notin -b)$

PROOF. Recall that negative is given as in Definition 4.4 by

$$-a := \overline{\{-x : x \in a\}}, \text{ if } a \notin a \quad -a := \{-x : x \notin a\}, \text{ if } a \in a.$$

Fix a bevel s and for induction suppose that, for any $x, y \in s$:

³⁷Theorem 4.1 licenses recursive definitions. If we are using second-order logic, such definitions yield a second-order entity. If we are using first-order logic, then (as usual) we define a term by considering a strictly increasing sequence of first-order ‘bounded approximations’ (specifying the behavior of the term over the last few bevels manually, if there is a last bevel).

- (1) $\neg x$ is well-defined and $\mathbf{B}x = \mathbf{B}(\neg x)$, and
- (2) $x = y$ iff $\neg x = \neg y$.

It suffices to show that both properties hold of a, b when $\mathbf{B}a = \mathbf{B}b = s$.

Concerning (1). Suppose $a \notin a$. If $x \in a$, then $\mathbf{B}(\neg x) = \mathbf{B}x \in \mathbf{B}a$ by induction assumption (1) and Lemma A.10.6. Using Separation_≠, let $c \notin c = \{v \in \mathbf{B}a : (\exists x \in a)v = \neg x\} = \{\neg x : x \in a\}$. Moreover, $\mathbf{B}c = \mathbf{B}a$, by the well-ordering of bevels and since $\mathbf{B}(\neg x) = \mathbf{B}x \in \mathbf{B}a$ for all $x \in a$. Now $\bar{c} \in \bar{c} = \neg a$ by Complements; so $\mathbf{B}a = \mathbf{B}c = \mathbf{B}\bar{c} = \mathbf{B}(\neg a)$. The case when $a \in a$ is similar, defining $c \notin c = \{v \in \mathbf{B}a : (\exists x \notin a)v = \neg x\} = \{\neg x : x \notin a\} = \neg a$.

Concerning (2). If $a \in a \leftrightarrow b \in b$, then $a = b$ iff $\neg a = \neg b$ by induction assumption (2). Without loss of generality, suppose that $a \in a$ and $b \notin b$; in establishing (1), we found that $\neg a \notin \neg a$ and $\neg b \in \neg b$; so $a \neq b$ and $\neg a \neq \neg b$. ⊥

I ended Section 4 by stating some simple facts about extensions of BLT. I will prove the distinctively boolean facts, leaving the remainder to the reader:

PROPOSITION 4.7 (Fragment).

- (2) BLT proves Union, i.e., $\forall a(\bigcup a$ exists)
- (5) BLT contradicts Powersets, i.e., it proves $\exists a \neg \exists b \forall x(x \in b \leftrightarrow x \subseteq a)$
- (6) BLT proves Foundation-restricted-to-high-sets, i.e., $(\forall a \in a)(\exists x \in a)a \cap x = \emptyset$.
- (7) BLT₊ contradicts unrestricted Foundation, i.e., it proves $(\exists a \neq \emptyset)(\forall x \in a)a \cap x \neq \emptyset$.

PROOF. (2) If $a \in a$, then $\bigcup a = \overline{\{x \in \bar{a} : (\forall y \in a)x \notin y\}}$, which exists by Separation_≠ and Complements. If $a \notin a$, then using Separation_≠ let $a_0 = \{x \in a : x \notin x\}$ and let $a_1 = \{x \in a : x \in x\}$. I will show that $\bigcup a_0$ and $\bigcup a_1$ exist, so that, using Complements and Lemma A.11:

$$\bigcup a = \bigcup a_0 \cup \bigcup a_1 = \overline{\bigcup a_0 \cap \bigcup a_1}.$$

Clearly $\bigcup a_0$ exists by Separation_≠ on $\mathbf{B}a$. If $a_1 = \emptyset$ then $\bigcup a_1 = \emptyset$; otherwise, $\bigcup a_1 = \overline{\bigcap \{\bar{x} : x \in a_1\}}$, which exists by Complements and Separation_≠ on $\mathbf{B}a$.

(5) If there is only one bevel, then the only sets are \emptyset and $V = \{\emptyset, V\}$, so that $\wp\emptyset = \{\emptyset\}$ does not exist. Otherwise, we find $\overline{\{\emptyset\}}$ at the second bevel, and if $\wp\{\emptyset\}$ existed it would be $\{x : \emptyset \notin x\}$. So suppose for reductio that $a = \{x : \emptyset \notin x\}$. Then $\emptyset \notin \emptyset$, so $\emptyset \in a$, so $a \notin a$. Now $\bar{a} \in \bar{a} = \{x : \emptyset \in x\}$ by Complements, so that $\emptyset \in \bar{a}$, contradicting that $\emptyset \in a$.

(6) If $a \in a$ then $\bar{a} \in a$ by Complements, and $a \cap \bar{a} = \emptyset$.

(7) We find $\{V\}$ at the second bevel, and $\{V\} \cap V \neq \emptyset$. ⊥

§Appendix B. The set-theoretic equivalence of BST and BLT. I now want to prove Theorem 3.3, which states that BLT and BST say exactly the same things about sets. (This mirrors Pt.1 Section 4.)

To show that BST says no more about sets than BLT does, I define a translation $*$: BST \rightarrow BLT, whose non-trivial actions are as follows:³⁸

$$Lo(x) := x \notin x \quad Hi(x) := x \in x$$

$$(\mathbf{s} < \mathbf{t})^* := \mathbf{s} \in \mathbf{t} \quad (x \preceq \mathbf{s})^* := (x \subseteq \mathbf{s} \vee \bar{x} \subseteq \mathbf{s}) \quad (\forall \mathbf{s}\phi)^* := (\forall \mathbf{s} : Bev)(\phi^*).$$

After translation, I treat all first-order variables as being of the same sort. Fairly trivially, for any BLT-sentence ϕ , if $BST \vdash \phi$ then $BST^* \vdash \phi$. The left-to-right half of Theorem 3.3 now follows as $*$ is an interpretation:

LEMMA B.1 (BLT). BST^* holds.

PROOF. Extensionality* is Extensionality. Order* holds by Lemma A.4; Staging* holds by Stratification \notin and Complements; and Cases* is trivial. Next, by Lemma A.4 and Lemma A.8, we can simplify $(x \prec \mathbf{s})^*$ to $x \in \mathbf{s}$. So, using Lemmas A.1 and A.4, we can simplify Priority*_{Lo} thus:

$$(\forall s \in Bev)(\forall a \notin a)((a \subseteq s \vee \bar{a} \subseteq s) \rightarrow (\forall x \in a)x \in s)$$

i.e., $(\forall s \in Bev)(\forall a \subseteq s)(\forall x \in a)x \in s$

which is trivial; then Priority*_{Hi} holds similarly, by Complements. A similar simplification allows us to obtain Specification*_{Lo} via Separation \notin ; then Specification*_{Hi} holds similarly, by Complements.³⁹ \dashv

To obtain the right-to-left half of Theorem 3.3, I will work in BST. I start by defining *slices*, which will go proxy for stages, and will turn out to be bevels, and then stating a few elementary results (for proofs, tweak those of Pt.1 Section 4):

DEFINITION B.2 (BST). For each \mathbf{s} , let $\check{\mathbf{s}} = \{x : x \prec \mathbf{s}\}$. Say that a is a slice iff $a = \check{\mathbf{s}}$ for some stage \mathbf{s} .

LEMMA B.3 (BST). $\forall F(\forall a : Lo)(\exists b : Lo)\forall x(x \in b \leftrightarrow (F(x) \wedge x \in a))$

LEMMA B.4 (BST). $\forall \mathbf{s}(\forall a : Lo)(a \preceq \mathbf{s} \leftrightarrow (\forall x \in a)x \prec \mathbf{s})$

LEMMA B.5 (BST). For any \mathbf{s} :

- (1) $\check{\mathbf{s}}$ exists and is low
- (2) $\forall \mathbf{r}(\forall a : Lo)(a \preceq \mathbf{r} \leq \mathbf{s} \rightarrow a \preceq \mathbf{s})$
- (3) $(\forall a : Lo)(a \subseteq \check{\mathbf{s}} \leftrightarrow a \preceq \mathbf{s})$

We must now part company slightly with the strategy of Pt.1 Section 4, to handle low and high sets, and their relation to (non-)self-membership:

LEMMA B.6 (BST). If some slice is F , then there is an \in -minimal slice which is F .

³⁸So the other clauses are: $(\neg\phi)^* := \neg\phi^*$; $(\phi \wedge \psi)^* := (\phi^* \wedge \psi^*)$; $(\forall x\phi)^* := \forall x\phi^*$; $(\forall F\phi)^* := \forall F\phi^*$; and $\alpha^* := \alpha$ for all atomic formulas α which are not of the forms mentioned in the main text.

³⁹Note that the $*$ -translation of any BST-Comprehension instance is a BLT-Comprehension instance.

PROOF. Every slice is low, by Lemma B.5.1. Subsets of low sets are low, by a result like Lemma A.1. From this, and Lemma B.5, it follows that $\forall \check{s} \forall x ((\exists c : Lo)x \subseteq c \in \check{s} \rightarrow x \in \check{s})$. The result now follows, reasoning as in Pt.1 Lemma 3.5. \dashv

LEMMA B.7 (BST). *a is low iff $a \notin a$; and a is high iff $a \in a$.*

PROOF. Suppose for reductio that $a \in a$ is low. Using Staging and Lemma B.6, let \check{s} be an \in -minimal slice such that $\exists \mathbf{t}(a \preceq \mathbf{t} \wedge \check{\mathbf{t}} = \check{\mathbf{s}})$; let \mathbf{t} witness this. Since $a \in a \preceq \mathbf{t}$ and a is low, $a \preceq \mathbf{r} < \mathbf{t}$ for some \mathbf{r} by Priority $_{Lo}$; so $\check{\mathbf{r}} \in \check{\mathbf{t}} = \check{\mathbf{s}}$ by Lemma B.5, contradicting $\check{\mathbf{s}}$'s minimality. Discharging the reductio: if a is low, then $a \notin a$. Similarly: if a is high, then $a \in a$. The biconditionals follow by Cases. \dashv

LEMMA B.8 (BST). *\bar{a} exists; and $a \notin a \leftrightarrow \bar{a} \in \bar{a}$; and $\forall \mathbf{s}(a \preceq \mathbf{s} \leftrightarrow \bar{a} \preceq \mathbf{s})$.*

PROOF. Using Staging, let $a \preceq \mathbf{s}$. If $a \notin a$, then a is low by Lemma B.7, so $(\forall x \in a)x \prec \mathbf{s}$ by Priority $_{Lo}$, so that by Specification $_{Hi}$ and Extensionality $\{x : x \notin a\} = \bar{a} \preceq \mathbf{s}$ exists and is high, i.e., $\bar{a} \in \bar{a}$ by Lemma B.7. If $a \in a$, reason similarly using Priority $_{Hi}$ and Specification $_{Lo}$. \dashv

Note that $BST \vdash ECS$ by Lemmas B.3, B.7, and B.8. So Lemmas A.1–A.9 hold verbatim within BST. We can now complete our reasoning about slices, by resuming the proof-strategy of Pt.1 Section 4; at this point, I leave the remaining details to the reader:

LEMMA B.9 (BST). *$\check{\mathbf{s}} \notin \check{\mathbf{s}}$; and $\check{\mathbf{s}}$ is transitive $_{\neq}$; and $\check{\mathbf{s}} = \mathbf{P}\{\check{\mathbf{r}} : \check{\mathbf{r}} \in \check{\mathbf{s}}\}$.*

LEMMA B.10 (BST). *All slices are comparable, i.e., $\forall \check{\mathbf{s}} \forall \check{\mathbf{t}}(\check{\mathbf{s}} \in \check{\mathbf{t}} \vee \check{\mathbf{s}} = \check{\mathbf{t}} \vee \check{\mathbf{t}} \in \check{\mathbf{s}})$.*

LEMMA B.11 (BST). *s is a bevel iff s is a slice.*

It follows that BST proves Stratification $_{\neq}$, delivering Theorem 3.3.

§Appendix C. Helow sets. In this appendix I prove Theorem 6.2, which shows how to recover ordinary, uncomplemented hierarchies via helow sets (see Definition 6.1). For readability, I refer to non-self-membered sets as *low*, and self-membered sets as *high* (cf. Lemma B.7). Note that every helow set is low, since all its members are low (i.e. non-self-membered). Now:

DEFINITION C.1 (BLT). If a is low, let $a_{\nabla} := \{x \in a : x \text{ is helow}\}$; by Separation $_{\neq}$, a_{∇} exists and is low.

LEMMA C.2 (BLT). *a is helow iff every member of a is helow.*

PROOF. *Left-to-right.* Where c witnesses that a is helow, if $x \in a$, then $x \in c$ and hence $x \subseteq c$, so c also witnesses that x is helow. *Right-to-left.* Let every member of a be helow. Every member of a is low, so a itself is low; hence $a \subseteq (\mathbf{B}a)_{\nabla}$. Now $(\mathbf{B}a)_{\nabla}$ witnesses that a is helow: if $x \in c \in (\mathbf{B}a)_{\nabla}$ then c is helow so x is helow (by left-to-right), so $x \in (\mathbf{B}a)_{\nabla}$ as $\mathbf{B}a$ is transitive $_{\neq}$. \dashv

I can now begin to show that $\nabla : LT \rightarrow BLT$, which simply restricts all quantifiers to helow sets (see Section 6), is an interpretation of LT:

LEMMA C.3 (BLT). *Both Extensionality[∇] and Separation[∇] hold.*

PROOF. For Extensionality[∇], fix below a and b and suppose that $(\forall x : Helo)(x \in a \leftrightarrow x \in b)$; then $\forall x(x \in a \leftrightarrow x \in b)$ by Lemma C.2, so $a = b$ by Extensionality. Similarly, repeated use of Lemma C.2 shows that Separation[∇] follows from Separation_≠. ⊢

The next task is to connect bevels with levels[∇]. (See Pt.1 Definitions 2.1–3.1 for the definitions of *potent*, \mathbb{Q} , *Hist* and *Lev*.)

LEMMA C.4 (BLT). *For any bevels r, s :*

- (1) s_∇ is below, potent and transitive
- (2) $r \in s$ iff $r_\nabla \in s_\nabla$
- (3) $s = \mathbf{B}(s_\nabla)$
- (4) $s_\nabla = \mathbb{Q}h = \mathbb{Q}^\nabla(h)$, where $h = \{r_\nabla \in s_\nabla : Bev(r)\}$.
- (5) s_∇ is a level[∇]

PROOF. (1) By Lemma C.2, s_∇ is below; then s_∇ is potent and transitive as s is potent_≠ and transitive_≠.

(2) *Left-to-right.* By (1). *Right-to-left.* Let $r_\nabla \in s_\nabla$. So $r \neq s$, since $r_\nabla \notin r_\nabla$. Similarly, $s_\nabla \notin r_\nabla$, since s_∇ is transitive; so $s \notin r$ by *left-to-right*. So $r \in s$, by Lemma A.9.

(3) Induction on bevels, using (2).

(4) By (1) and Lemma C.2, h is below. If $a \in \mathbb{Q}h$, then $a \in s_\nabla$ as s_∇ is potent by (1). Conversely, if $a \in s_\nabla$, then $a \subseteq r \in s$ for some bevel r by Lemma A.8, and $a \subseteq r_\nabla \in s_\nabla$ by (2) and Lemma C.2, so $a \in \mathbb{Q}h$. So $s_\nabla = \mathbb{Q}h$. Repeated use of Lemma C.2, as in Lemma C.3, now yields that $\mathbb{Q}h = \mathbb{Q}^\nabla(h)$.

(5) With h as in (4), since $s = \mathbb{Q}^\nabla(h)$ it suffices to show that *Hist*[∇](h). If $r_\nabla \in h$, then $r_\nabla \cap h = \{q_\nabla \in r_\nabla : Bev(q)\}$, by (1); so $r_\nabla = \mathbb{Q}^\nabla(r_\nabla \cap h)$ by (4). ⊢

LEMMA C.5 (BLT). *The levels[∇] are the bevels_∇, i.e.: Lev[∇](a) iff $(\exists s : Bev)a = s_\nabla$.*

PROOF. By Lemma C.4, if s is a bevel then both *Lev*[∇](s_∇) and $\mathbf{B}(s_\nabla) = s$. To complete the proof, it suffices to note that if p and q are distinct levels[∇], then $\mathbf{B}p \neq \mathbf{B}q$; this follows from Lemma A.10.6 and the fact that the levels[∇] are well-ordered by \in . (The well-ordering of levels[∇] is Pt.1 Theorem 3.10[∇], which holds via Lemma C.3.) ⊢

COROLLARY C.6 (BLT). *Stratification[∇] holds; Endless_≠ proves Endless[∇]; Infinity_≠ proves Infinity[∇]; and Unbounded_≠ proves Unbounded[∇].*

Recalling that LT+Infinity+Unbounded is equivalent to ZF (see Section 6), Lemmas C.3 and C.6 yield Theorem 6.2.

§Appendix D. Definitional equivalence. In this appendix, I prove the definitional equivalence discussed in Section 7.⁴⁰

⁴⁰Recall: both LT and BLT (and their extensions) are formulated as second-order theories. I continue to frame my discussion in second-order terms in this appendix. However, the theories

D.1. Interpreting BLT_{ZF} in ZF . I first define an interpretation, I , to simulate (extensions of) BLT within (extensions of) LT. The key idea is to use \emptyset as a flag to indicate whether to treat a set as low or high. To allow \emptyset to play this role, I define a bijection $\sigma : V \rightarrow V \setminus \{\emptyset\}$:⁴¹

$$\sigma(a) := \begin{cases} \{a\} & \text{if } a \text{ is a Zermelo number,} \\ a & \text{otherwise,} \end{cases}$$

where the Zermelo numbers are $0 = \emptyset$ and $n + 1 = \{n\}$. I then interpret membership thus:

$$x \in^I a \text{ iff } (\sigma(x) \in a \leftrightarrow \emptyset \notin a).$$

Since $\sigma(a) \notin a$ for all a , it follows that $a \notin^I a$ iff $\emptyset \notin a$ (i.e., a is treated as low), and $a \in^I a$ iff $\emptyset \in a$ (i.e. a is treated as high). I will now prove a sequence of results which establish that I is an interpretation of BLT. The first few are straightforward:

LEMMA D.1 (LT_+). *Where $a \subseteq^I b$ abbreviates $(\forall x \in^I a)x \in^I b$:*

- (1) *If $\emptyset \notin a$ and $\emptyset \notin b$, then: $a \subseteq b$ iff $a \subseteq^I b$.*
- (2) *If $\emptyset \in a$ and $\emptyset \in b$, then: $a \supseteq b$ iff $a \subseteq^I b$.*

PROOF. (1) Since σ is a bijection $V \rightarrow V \setminus \{\emptyset\}$, $a \subseteq b$ iff $\forall x(\sigma(x) \in a \rightarrow \sigma(x) \in b)$ iff $a \subseteq^I b$.

(2) Similarly, $a \supseteq b$ iff $\forall x(\sigma(x) \notin a \rightarrow \sigma(x) \notin b)$ iff $a \subseteq^I b$. ⊢

LEMMA D.2 (LT_+). *Extensionality^I holds.*

PROOF. Suppose $\forall x(x \in^I a \leftrightarrow x \in^I b)$. If $a \notin^I a$ but $b \in^I b$, then $\forall x(\sigma(x) \in a \leftrightarrow \sigma(x) \notin b)$, so that $a \cup b = V$, which is impossible. Generalising, $a \in^I a$ iff $b \in^I b$. Now apply Lemma D.1. ⊢

LEMMA D.3 (LT_+). *Separation^I holds.*

PROOF. Fix F and $a \notin^I a$, i.e., $\emptyset \notin a$. Using Separation, let $b = \{\sigma(x) \in a : F(x)\}$. Since $\emptyset \notin b$ we have $\forall x(x \in^I b \leftrightarrow (F(x) \wedge x \in^I a))$. ⊢

The interpretation of complementation is obvious: $\bar{a}^I = a \cup \{\emptyset\}$ if $a \notin^I a$, and $\bar{a}^I = a \setminus \{\emptyset\}$ if $a \in^I a$. The next result follows trivially:

LEMMA D.4 (LT_+). *$\forall a \forall x(x \in^I a \leftrightarrow x \notin^I \bar{a}^I)$, and Complements^I holds.*

The only intricate part of this interpretation concerns the treatment of bevels. Within LT_+ , we can define the von Neumann ordinals, and recursively define the following:

$$W_\gamma = \{\sigma(x) : (\exists \beta < \gamma)x \subseteq W_\beta \cup \{\emptyset\}\}.$$

can easily be reformulated as first-order formulations, and the definitional equivalences hold for these first-orderisations (only the quasi-categoricity results of Section 5 require second-order resources).

⁴¹Many thanks to Randall Holmes for discussion of this construction (and other constructions); the proof in this section is much more self-contained than it would have been, had it not been for his input. Thanks also to Thomas Forster, for encouraging me to consider the question of definitional equivalence. The proof-strategy is similar to Löwe (2006).

Now LT_+ proves that W_γ exists for each γ , and that these are the bevels^I:

LEMMA D.5 (LT_+). *Bev^I(s) iff $s = W_\gamma$ for some γ .*

PROOF. Lemmas D.2–D.4 show that LT_+ proves ECS^I . Hence LT_+ proves Theorem 4.1^I, i.e., that the bevels^I are well-ordered by \in^I . For induction on γ , suppose that if $\beta < \gamma$ then W_β is the β^{th} bevel^I. Let s be the γ^{th} bevel^I. By Lemma D.1:

$$\begin{aligned} W_\gamma &= \{\sigma(x) : (\exists \beta < \gamma)x \subseteq W_\beta \cup \{\emptyset\}\} \\ &= \{\sigma(x) : (\exists \beta < \gamma)(x \subseteq^I W_\beta \vee \bar{x}^I \subseteq^I W_\beta)\} \\ &= \{\sigma(x) : (\exists W_\beta \notin^I W_\beta \in^I s)(x \subseteq^I W_\beta \vee \bar{x}^I \subseteq^I W_\beta)\} \\ &= (\mathbf{P}\{w \in s : Bev(w)\})^I. \end{aligned}$$

So $W_\gamma = s$ by Lemma A.8^I. By induction, the bevels^I are the W_γ s. ⊖

I can now prove the crucial proposition:

LEMMA D.6 (LT_+). *Stratification^I_≠ holds.*

PROOF. By Lemma D.5, it suffices to show that $(\forall a \notin^I a)\exists \gamma a \subseteq^I W_\gamma$. Since the levels are well-ordered by \in (Pt.1 Theorem 3.10), we can write V_γ for the γ^{th} level. I claim: if $a \notin^I a \subseteq V_\gamma$, then $a \subseteq W_\gamma$. For induction, suppose this holds for all ordinals $\beta < \gamma$. Fix $a \notin^I a \subseteq V_\gamma$. If $\gamma = 0$, then $a = \emptyset \subseteq^I W_0 = \emptyset$. Otherwise, fix $x \in^I a$, i.e., $\sigma(x) \in a \subseteq V_\gamma$; now $x \subseteq V_\beta$ for some $\beta < \gamma$, by Pt.1 Lemma 3.12, so that $x \subseteq W_\beta \cup \{\emptyset\}$ by the induction hypothesis; so $\sigma(x) \in W_\gamma$, i.e., $x \in^I W_\gamma$. Generalising, $a \subseteq^I W_\gamma$. ⊖

LEMMA D.7. $LT_+ \vdash BLT_+^I$ and $ZF \vdash BLT_{ZF}^I$.

PROOF. Lemmas D.2–D.6 establish that $LT_+ \vdash BLT^I$. And $LT_+ \vdash$ Endless^I_≠, using Endless and our explicitly defined bevels^I, the W_γ s. Evidently, Infinity yields Infinity^I_≠. For Unbounded^I_≠, fix P and $a \notin^I a$; by Unbounded, the set $c = \{\sigma(P(x)) : \sigma(x) \in a\}$ exists; by construction, $\emptyset \notin c$ and $(\forall x \in^I a)P(x) \in^I c$. The result follows, since ZF is equivalent to $LT +$ Infinity + Unbounded (see Section 6). ⊖

D.2. Interpreting ZF in BLT_{ZF} . I now switch to working in BLT_+ . Using σ —i.e., using verbatim the same definitions of ‘Zermelo number’ and of σ in BLT_+ as we used in LT_+ —consider this function:

$$\eta(a) = \begin{cases} \{\sigma(\eta(x)) : x \in a\} & \text{if } a \notin a, \\ \{\sigma(\eta(x)) : x \notin a\} \cup \{\emptyset\} & \text{if } a \in a. \end{cases}$$

I will prove that η is a bijection $V \rightarrow Helo$. I then define a translation, J , by stipulating:

$$x \in^J a \text{ iff } \eta(x) \in \eta(a).$$

It will follow that J is an interpretation of LT_+ in BLT_+ .

LEMMA D.8 (BLT_+). *If $\eta(a) = \eta(b)$, then $a = b$.*

PROOF. Let $\eta(a) = \eta(b)$, so that $a \notin a \leftrightarrow b \notin b$. For induction, suppose that $\eta(x) = \eta(y) \rightarrow x = y$ for all x, y with $\mathbf{B}x, \mathbf{B}y \in \mathbf{B}a \cup \mathbf{B}b$. If $a \notin a$ and $b \notin b$, then $\{\sigma(\eta(x)) : x \in a\} = \{\sigma(\eta(x)) : x \in b\}$, so that $a = b$ by the induction hypothesis and the injectivity of σ . The case when $a \in a$ is similar. \dashv

LEMMA D.9 (BLT₊). $\eta(a)$ is helow, for any a .

PROOF. For induction, suppose that $\eta(x)$ is helow for all x with $\mathbf{B}x \in \mathbf{B}a$. Suppose $a \notin a$; since $\sigma(\eta(x))$ is helow iff $\eta(x)$ is helow, every member of $\eta(a)$ is helow; so $\eta(a)$ is helow by Lemma C.2. The case when $a \in a$ is similar. \dashv

LEMMA D.10 (BLT₊). If a is helow, then $a = \eta(c)$ for some c .

PROOF. By Lemma D.8, η^{-1} is functional. For induction, suppose that for all helow $z \in \mathbf{B}a$, we have that $\eta^{-1}(z)$ is defined and $\mathbf{B}(\eta^{-1}(z)) \subseteq \mathbf{B}z$.

If $\emptyset \notin a$, let $c \notin c = \{\eta^{-1}(\sigma^{-1}(x)) \in \mathbf{B}a : x \in a\}$ using Separation_≠. Fix $x \in a$; then $\sigma^{-1}(x) \in \mathbf{B}a$ and $\sigma^{-1}(x)$ is helow, recalling that a is helow and using Lemma C.2. Now $\mathbf{B}(\eta^{-1}(\sigma^{-1}(x))) \subseteq \mathbf{B}(\sigma^{-1}(x)) \in \mathbf{B}a$ by the induction hypothesis, i.e., $\eta^{-1}(\sigma^{-1}(x)) \in \mathbf{B}a$. So $c = \{\eta^{-1}(\sigma^{-1}(x)) : x \in a\}$, so that $a = \eta(c)$ and $\mathbf{B}c \subseteq \mathbf{B}a$.

If $\emptyset \in a$, then instead let $c = \{\eta^{-1}(\sigma^{-1}(x)) : \emptyset \neq x \in a\}$; now $a = \eta(\bar{c})$. \dashv

LEMMA D.11. $\text{BLT}_+ \vdash \text{LT}_+^J$ and $\text{BLT}_{ZF} \vdash \text{ZF}^J$.

PROOF. By Lemmas D.8–D.10, $\eta : V \rightarrow \text{Helo}$ is a bijection; now use Theorem 6.2. \dashv

D.3. The interpretations are inverse. It remains to show that I and J are mutually inverse, in the sense required for definitional equivalence.⁴² The key lies in their treatments of the Zermelo numbers. Working informally, let z_n be the n^{th} Zermelo number, and let v_n be defined similarly, but starting from V rather than \emptyset , i.e.:

$$z_n = \underbrace{\{\dots\{\emptyset\}\dots\}}_{n \text{ times}} \quad v_n = \underbrace{\{\dots\{V\}\dots\}}_{n \text{ times}}$$

We can now consider two sequences:

$$\begin{array}{cccccccc} z_0, & z_1, & z_2, & z_3, & \dots, & z_{2n}, & z_{2n+1}, & \dots \\ z_0, & v_0, & z_1, & v_1, & \dots, & z_n, & v_n, & \dots \end{array}$$

Intuitively, I treats the former sequence as the latter, and J treats the latter as the former. The proof that I and J are mutually inverse simply builds on this intuitive thought.

Here are two facts which make the intuitive thought precise:

LEMMA D.12 (LT₊). $\forall x \ x \notin^I \emptyset$, and $\forall x \ x \in^I \{\emptyset\}$, and $\forall x (x \in^I z_{n+2} \leftrightarrow x = z_n)$ for all n .

LEMMA D.13 (BLT₊). $\eta(z_n) = z_{2n}$ and $\eta(v_n) = z_{2n+1}$, for all n .

⁴²Via Friedman and Visser (2014, Corollary 5.5), to establish Theorem 7.1 we could instead verify that I and ∇ (from Section C) are bi-interpretations.

The proofs of both facts are trivial. Using the second fact, though, I can build up to the proof in BLT_+ that $x \in a$ iff $(x \in^I a)^J$:

LEMMA D.14 (BLT_+). *The function σ^J , i.e., the J -interpretation of LT 's definition of σ , maps $z_n \mapsto v_n \mapsto z_{n+1}$, and $x \mapsto x$ otherwise.*

PROOF. Note that $z_{2n} \in z_{2n+1} \in z_{2n+2}$, with these membership facts unique. So $\eta(z_n) \in \eta(v_n) \in \eta(z_{n+1})$, by Lemma D.13, i.e., $z_n \in^J v_n \in^J z_{n+1}$. \dashv

LEMMA D.15 (BLT_+). $\eta(\sigma^J(a)) = \sigma(\eta(a))$, for all a .

PROOF. By Lemmas D.13–D.14, we have $\eta(\sigma^J(z_n)) = \eta(v_n) = z_{2n+1} = \sigma(z_{2n}) = \sigma(\eta(z_n))$ and $\eta(\sigma^J(v_n)) = \eta(z_{n+1}) = z_{2n+2} = \sigma(z_{2n+1}) = \sigma(\eta(v_n))$. Now suppose $a \neq z_n$ and $a \neq v_n$ for any n , so that $\sigma^J(a) = a$ and hence $\eta(\sigma^J(a)) = \eta(a)$; moreover, $\eta(a) \neq z_n$ for any n by Lemma D.13; so $\eta(\sigma^J(a)) = \eta(a) = \sigma(\eta(a))$. \dashv

LEMMA D.16 (BLT_+). $\eta(\sigma^J(x)) \in \eta(a) \leftrightarrow a \notin a$ iff $x \in a$

PROOF. If $a \notin a$ then $\eta(a) = \{\eta(\sigma^J(x)) : x \in a\}$ by Lemma D.15. If $a \in a$ then $\eta(a) = \{\eta(\sigma^J(x)) : x \notin a\} \cup \{\emptyset\}$, and note that $\emptyset \neq \eta(\sigma^J(x)) = \sigma(\eta(x))$ for all x . \dashv

LEMMA D.17 (BLT_+). $x \in a$ iff $(x \in^I a)^J$

PROOF. Using Lemma D.16 and the fact that $a \notin a$ iff $\eta(\emptyset) = \emptyset \notin \eta(a)$, note the following chain of equivalent formulas:

- (1) $x \in a$
- (2) $\eta(\sigma^J(x)) \in \eta(a) \leftrightarrow \eta(\emptyset) \notin \eta(a)$
- (3) $(\sigma(x) \in a \leftrightarrow \emptyset \notin a)^J$
- (4) $(x \in^I a)^J$ \dashv

It remains to show in LT_+ that $x \in a$ iff $(x \in^J a)^I$. Working in BLT_+ , define μ as a map sending $z_{n+1} \mapsto v_n \mapsto z_n$ and $x \mapsto x$ otherwise; by Lemma D.14, if $x \neq \emptyset$ then $\mu^{-1}(x) = \sigma^J(x)$. We then have two quick results:

LEMMA D.18 (BLT_+). $\eta(x) \in \eta(a)$ iff $(x = \emptyset \wedge a \in a) \vee (x \neq \emptyset \wedge (\mu(x) \in a \leftrightarrow a \notin a))$

PROOF. If $x = \emptyset$, then $\eta(\emptyset) = \emptyset \in \eta(a)$ iff $a \in a$. If $x \neq \emptyset$; use Lemma D.16. \dashv

LEMMA D.19 (LT_+). If $x \neq \emptyset$, then $\sigma(\mu^I(x)) = x$.

PROOF. By Lemma D.12, μ^I maps $z_{n+2} \mapsto z_{n+1} \mapsto z_n$, and $x \mapsto x$ otherwise. \dashv

LEMMA D.20 (LT_+). $x \in a$ iff $(x \in^J a)^I$

PROOF. Using Lemmas D.19 and D.18^I, note the following chain of equivalent formulas:

- (1) $x \in a$
- (2) $(\emptyset = x \wedge x \in a) \vee (\emptyset \neq x \wedge x \in a)$
- (3) $(\emptyset = x \wedge x \in a) \vee (\emptyset \neq x \wedge \sigma(\mu^I(x)) \in a)$
- (4) $(\emptyset = x \wedge a \in^I a) \vee (\emptyset \neq x \wedge (\mu^I(x) \in^I a \leftrightarrow \emptyset \notin a))$

- (5) $(\emptyset = x \wedge a \in^I a) \vee (\emptyset \neq x \wedge (\mu^I(x) \in^I a \leftrightarrow a \notin^I a))$
 (6) $((\emptyset = x \wedge a \in a) \vee (\emptyset \neq x \wedge (\mu(x) \in a \leftrightarrow a \notin a)))^I$
 (7) $(\eta(x) \in \eta(a))^I$
 (8) $(x \in^J a)^I$ −

Theorem 7.1 now follows from Lemmas D.7, D.11, D.17, and D.20.

D.4. Finitary cases of definitional equivalences. The base theories, LT and BLT, are not definitionally equivalent. To see this, consider:

$$\begin{aligned} \text{lt}(1) &:= 1 & \text{blt}(1) &:= 2 \\ \text{lt}(n+1) &:= 2^{\text{lt}(n)} & \text{blt}(n+1) &:= 2^{\text{blt}(n)+1}. \end{aligned}$$

Any model of LT with n levels has $\text{lt}(n)$ sets, and any model of BLT with n levels has $\text{blt}(n)$ sets. In particular, there is a model of LT with four sets, but no model of BLT has four sets. So LT and BLT are not definitionally equivalent.

There is, though, a nice definitional equivalence when we insist that there are infinitely many sets but that every (low) set is finite. Concretely: let LT_{fin} be $\text{LT}_+ + \neg\text{Infinity}$, and let BLT_{fin} be $\text{BLT}_+ + \neg\text{Infinity}_{\neq}$. Our earlier results immediately entail that LT_{fin} and BLT_{fin} are definitionally equivalent. Moreover, as noted in Pt.1 Section 7, LT_{fin} is equivalent to ZF_{fin} . Finally, ZF_{fin} and PA are definitionally equivalent.⁴³ So:

LEMMA D.21. PA, ZF_{fin} , LT_{fin} , and BLT_{fin} are definitionally equivalent.

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⁴³Kaye and Wong (2007, Theorems 3.3, 6.5, and 6.6). ZF_{fin} is the theory with all of ZF's axioms except that: (i) Zermelo's axiom of infinity is replaced with its negation and (ii) it has a new axiom, $\forall a(\exists t \supseteq a)(t \text{ is transitive})$.

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