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ON NEW MINIMAL EXCLUDANTS OF OVERPARTITIONS RELATED TO SOME q-SERIES OF RAMANUJAN

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Abstract

Inspired by work of Andrews and Newman ['Partitions and the minimal excludant', *Ann. Comb.* **23** (2019), 249–254] on the minimal excludant or 'mex' of partitions, we define four new classes of minimal excludants for overpartitions and establish relations to certain functions due to Ramanujan.

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1. Introduction

Given an integer partition π, the minimal excludant of π is defined to be the smallest positive integer that is not a part of π. This partition statistic seems to have been first considered by Grabner and Knopfmacher [12] in 2006, who call it the least gap. They obtained the result that the sum of the minimal excludants over all partitions of n is equal to the number of partitions of n into distinct parts with two colours. In 2019, Andrews and Newman introduced the terminology 'minimal excludant' or 'mex' of an integer partition function and initiated the study of the connections of the mex to other partition theoretic objects and statistics. This statistic was also reported earlier in 2011 by Andrews [3], where he relates the minimal excludant, then called the smallest part of a partition that is not a summand, to the Frobenius symbol representation of partitions.

In the first of two papers, Andrews and Newman [5] considered $\sigma\,\mathrm{mex}(n)$, the sum of $\mathrm{mex}(\pi)$ taken over all partitions π of n. Among other results, they rediscovered the result of Grabner and Knopfmacher. Aricheta and Donato [8] extended this concept to overpartitions. To recall, an overpartition is a partition in which the first occurrence of a number may be overlined [9]. Aricheta and Donato define the minimal excludant or mex of an overpartition π, denoted by $\overline{\mathrm{mex}}(\pi)$, to be the smallest positive integer that is not a part of the nonoverlined parts of π. For a positive integer n, they define $\sigma\,\overline{\mathrm{mex}}(n)$ to be the sum of the minimal excludants over all overpartitions π of n and prove that $\sigma\,\overline{\text{mex}}(n)$ equals the number of partitions of n into distinct parts using three colours [8, Theorem 1.1].

In the second paper on the minimal excludant statistic, Andrews and Newman [6] defined an extended function of their minimal excludant and explored relations of this extended mex function to other well-studied partition statistics such as the rank and crank. The connection between the mex and the crank was independently made by Hopkins and Sellers [14] in the same year. In [10], we studied generalised versions of these relations by calculating the generating function of the general case of the extended minimal excludant of Andrews and Newman. At the end of that paper [10, Section 4], we defined a new minimal excludant for overpartitions which, to our surprise, was related to a function of Ramanujan. In this paper, we continue our study on minimal excludants of overpartitions and their relations to two fifth-order mock theta functions and some other q-series of Ramanujan.

Let L, m, n be nonnegative integers. Throughout the paper, we use the following standard notation [2]:

$$(a)_L = (a;q)_L := \prod_{k=0}^{L-1}(1-aq^k); \quad (a)_\infty = (a;q)_\infty := \lim_{L\to\infty}(a)_L \quad \text{where } |q|<1.$$

We define the q-binomial coefficient by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \begin{cases} \dfrac{(q)_m}{(q)_n(q)_{m-n}} & \text{for } m \geq n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\overline{P}(q) = (-q)_\infty/(q)_\infty$ and $\overline{P}_o(q) = (-q;q^2)_\infty/(q;q^2)_\infty$ denote the generating functions of overpartitions and overpartitions into odd parts, respectively.

We define four new classes of minimal excludants of overpartitions with an example for each. The first of these minimal excludants appears in our previous paper [10, Section 4].

We believe that there are more interesting classes of minimal excludants for overpartitions to be discovered, and further relations to other mock theta functions and interesting q-series. While not all definitions lead to interesting results, our results suggest that a systematic complete study is warranted.

1.1. Four overpartition mex statistics

DEFINITION 1.1. For an overpartition π, let $\text{omex}(\pi)$ be the smallest positive integer that is not a part (overlined or nonoverlined) of π.

REMARK 1.2. The minimal excludant in Definition 1.1 differs from that considered by Aricheta and Donato because their mex is the smallest part missing from the nonoverlined parts of the partition.

EXAMPLE 1.3. From the two overpartitions

$$\pi_1 = \overline{5} + \overline{4} + 4 + 2 + 1, \quad \pi_2 = \overline{10} + 8 + 5 + \overline{3} + 2 + \overline{1},$$

we have $\mathrm{omex}(\pi_1) = 3$ and $\mathrm{omex}(\pi_2) = 4$.

Let $\overline{m}(n)$ denote the number of overpartitions π of n having the property that no positive integer less than $\mathrm{omex}(\pi)$ is overlined. As an example, the eight overpartitions of 3 are

$$3, \overline{3}, 2 + 1, \overline{2} + 1, 2 + \overline{1}, \overline{2} + \overline{1}, 1 + 1 + 1, \overline{1} + 1 + 1.$$

The overpartitions $\pi \in \{3, \overline{3}, 2 + 1, 1 + 1 + 1\}$ have the stated property and hence, $\overline{m}(3) = 4$. Define

$$\overline{M}(q) = \sum_{n=0}^{\infty} \overline{m}(n) q^n.$$

DEFINITION 1.4. For an overpartition π into odd parts, let $\mathrm{omoex}(\pi)$ be the smallest positive odd integer which is not a part (overlined or nonoverlined) of π.

EXAMPLE 1.5. From the overpartitions

$$\pi_3 = \overline{7} + \overline{7} + 3 + 1, \quad \pi_4 = \overline{7} + 7 + \overline{5} + 3 + 1,$$

we have, $\mathrm{omoex}(\pi_3) = 5$ and $\mathrm{omoex}(\pi_4) = 9$.

Let $\overline{m}_o(n)$ denote the number of overpartitions π of n into odd parts having the property that no positive integer less than $\mathrm{omoex}(\pi)$ is overlined. For example, for $n = 3$, the overpartitions $\pi \in \{3, \overline{3}, 1 + 1 + 1\}$ have the stated property and, hence, $\overline{m}_o(3) = 3$. Define

$$\overline{M}_o(q) = \sum_{n=0}^{\infty} \overline{m}_o(n) q^n.$$

For our other two classes of overpartition mexes, we define an ordering on the parts of an overpartition π, where every nonoverlined part is smaller than its overlined counterpart, that is,

$$1 < \overline{1} < 2 < \overline{2} < 3 < \overline{3} < \cdots.$$

In the two definitions that follow, we use a $\sim$ symbol to denote the minimal excludant taken over overpartitions with this ordering on the parts.

DEFINITION 1.6. For an overpartition π where we take into consideration the ordering on the parts, let $\widetilde{\mathrm{omex}}(\pi)$ be the smallest overlined positive integer which is not a part of π.

EXAMPLE 1.7. From the overpartitions

$$\pi_5 = \overline{5} + \overline{3} + 2 + 1, \quad \pi_6 = \overline{7} + 7 + \overline{5} + 3 + \overline{2} + \overline{1},$$

we have $\widetilde{\mathrm{omex}}(\pi_5) = \overline{1}$ and $\widetilde{\mathrm{omex}}(\pi_6) = \overline{3}$.

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Let $\widetilde{m}(n)$ denote the number of overpartitions π of n having the property that all overlined and nonoverlined parts smaller than $\widetilde{\overline{\text{omex}}}(\pi)$ occur as parts. For example, for $n = 3$, the overpartitions $\pi \in \{2 + 1, \overline{2} + 1, 1 + 1 + 1\}$ have the stated property and, hence, $\widetilde{m}(3) = 3$. Define

$$\widetilde{M}(q) = \sum_{n=0}^{\infty} \widetilde{m}(n)q^n.$$

Definition 1.8. For an overpartition π into odd parts where we once again take into consideration the ordering on the parts, let $\widetilde{\overline{\text{omoex}}}(\pi)$ be the smallest overlined positive odd integer which is not a part of π.

Example 1.9. From the overpartitions

$$\pi_7 = \overline{5} + \overline{3} + 3 + \overline{1}, \quad \pi_8 = 11 + \overline{7} + \overline{5} + \overline{3} + 3 + \overline{1},$$

we have $\widetilde{\overline{\text{omoex}}}(\pi_7) = \overline{7}$ and $\widetilde{\overline{\text{omoex}}}(\pi_8) = \overline{9}$.

Let $\widetilde{m}_o(n)$ denote the number of overpartitions of n into odd parts having the property that all overlined and nonoverlined parts smaller than $\widetilde{\overline{\text{omoex}}}(\pi)$ occur as parts. For example, the only overpartition of 3 that has the stated property is $1 + 1 + 1$ and, hence, $\widetilde{m}_o(3) = 1$. Define

$$\widetilde{M}_o(q) = \sum_{n=0}^{\infty} \widetilde{m}_o(n)q^n.$$

1.2. Two arithmetic mex functions. Finally, analogous to the arithmetic function

$$\sigma\text{mex}(n) = \sum_{\pi \vdash n} \text{mex}(\pi)$$

over partitions π of n considered by Andrews and Newman, we study the following two analogous sums of our first two minimal excludants of overpartitions introduced in Definitions 1.1 and 1.4. To that end, let us consider the sum

$$\sigma\text{omex}(n) = \sum_{\pi \vdash n} \text{omex}(\pi)$$

taken over all overpartitions π of n. Define $\overline{M}(z, q)$ to be the double series in which the coefficient of $z^m q^n$ is the number of overpartitions π of n with $\text{omex}(\pi) = m$ and let

$$\sigma\overline{M}(q) = \sum_{n \geq 0} \sigma\text{omex}(n)q^n.$$

Again, we consider the sum

$$\sigma\text{omoex}(n) = \sum_{\pi \vdash n} \text{omoex}(\pi)$$

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taken over all overpartitions π of n into odd parts. Define $\overline{M}_o(z, q)$ to be the double series in which the coefficient of $z^m q^n$ is the number of overpartitions π of n into odd parts with $\mathrm{omoex}(\pi) = m$ and let

$$\sigma\overline{M}_o(q) = \sum_{n\geq 0} \sigma\mathrm{omoex}(n) q^n.$$

2. Main results

In this section, we present the statements of our results. Theorem 2.1 below is also stated and proved in our paper [10]. We have included the statement of the result here for the sake of completeness.

THEOREM 2.1 [10, Theorem 30]. *We have* $\overline{M}(q) = \overline{P}(q)(2 - R(q))$, *where*

$$R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} q^n (q)_{n-1} = (q)_\infty + 2(q)_\infty \sum_{n=1}^{\infty} \frac{q^n}{(q)_n(1+q^n)}.$$

REMARK 2.2. The first two q-series representations of $R(q)$ are due to Ramanujan [16] and the last representation is due to Gupta [13, (1.11)]. Andrews [1] studied $R(q)$ in connection with identities from Ramanujan's 'Lost' Notebook. Conjectures made by Andrews in his paper on the distribution of the coefficients of $R(q)$ were proved by Andrews *et al.* [4].

THEOREM 2.3. *We have* $\overline{M}_o(q) = \overline{P}_o(q)(1 - F(-q))$, *where*

$$F(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q; q^2)_n}.$$

REMARK 2.4. Consider partitions into odd parts, with the property that if k occurs as a part, then all positive odd numbers less than k also occur. Then, $F(q)$ is the generating function for the number of such partitions where the largest part is congruent to 3 modulo 4 minus the number of such partitions, where the largest part is congruent to 1 modulo 4. See [4, Section 5] for a treatment of $F(q)$, which is a companion function to $R(q)$.

THEOREM 2.5. *We have* $\widetilde{M}(q) = \overline{P}(q)(f_0(q) - 1)$, *where*

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}$$

is a fifth-order mock theta function of Ramanujan.

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THEOREM 2.6. *We have* $\widetilde{M}_o(q) = q\overline{P}_o(q)F_1(-q)$, *where*

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}}$$

is a fifth-order mock theta function of Ramanujan.

REMARK 2.7. Ramanujan's mock theta functions are examples of mock modular forms. The functions of $\widetilde{M}(q)$ and $\widetilde{M}_o(q)$ are mixed mock modular forms and possess interesting properties. See [15] for a treatment of mixed mock modular forms.

In the next theorem, we relate our sum of mex function $\sigma\overline{M}(q)$ to a q-series which comprises a sum of tails formed by discarding the initial terms of a certain q-series.

THEOREM 2.8. *We have*

$$\sigma\overline{M}(q) = \overline{P}(q)\Big(R(q) - 2(q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q)_n(1+q^n)} G_n(q)\Big),$$

where $G_n(q)$ *is the* q*-series tail given by*

$$G_n(q) = \sum_{i=n+1}^{\infty} \frac{q^i}{1+q^i}.$$

THEOREM 2.9. *We have*

$$\sigma\overline{M}_o(q) = \overline{P}_o(q)\Big(1 + q\sum_{n=1}^{\infty}(-1)^n(q^2;q^2)_n q^n H_n(q^2)\Big),$$

where the partial sum

$$H_n(q) = \sum_{i=1}^{n} \frac{q^i}{1-q^i}$$

is the q*-analogue of the harmonic number* $H_n = \sum_{i=1}^{n} 1/i$.

REMARK 2.10. The q-harmonic series $H_n(q)$ are partial sums of the generating function of the divisor function given by

$$\sum_{i=1}^{\infty} d(i)q^i = \sum_{i=1}^{\infty} \frac{q^i}{1-q^i} = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}q^{i(i+1)/2}}{(1-q^i)(q;q)_i}.$$

Interesting formulae for harmonic and q-harmonic numbers H_n and $H_n(q)$ were re-established by Andrews and Uchimura in [7] using differentiation of classical hypergeometric series. One such formula relevant to this discussion is the finite analogue of the generating function of the divisor function,

$$H_n(q) = \sum_{i=1}^{n} \frac{q^i}{1-q^i} = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{1-q^i} q^{i(i+1)/2} \begin{bmatrix} n \\ i \end{bmatrix}_q.$$

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3. Proofs of the main results

In this section, we provide proofs of our results. We follow the notation of Gasper and Rahman [11]. The unilateral basic hypergeometric series ${}_r\phi_{r-1}$ with base q and argument z is defined by

$$ {}_r\phi_{r-1}\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_{r-1}\end{matrix};q,z\right) := \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_{r-1};q)_k} z^k, \quad |z|<1. $$

We also need the q-binomial theorem:

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix};q,z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1, \tag{3.1} $$

and Heine's ${}_2\phi_1$ transformation: for $|z|<1$ and $|b|<1$,

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/b,z \\ az \end{matrix};q,b\right). \tag{3.2} $$

Finally, we will also make use of the following simple identity [10, (3)] obtained from a ${}_1\phi_1$ summation of Gasper and Rahman:

$$ \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(-zq;q)_n} = 1+z. \tag{3.3} $$

Proof of Theorem 2.3. By standard combinatorial arguments,

$$ \begin{aligned} \overline{M}_o(q) &= \sum_{n=0}^{\infty} \overline{m}_o(n) q^n = \sum_{n=1}^{\infty} \frac{q^{1+3+\cdots+(2n-3)} \prod_{m=n+1}^{\infty}(1+q^{2m-1})}{\prod_{\substack{m=1\\ m\neq n}}^{\infty}(1-q^{2m-1})} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{(n-1)^2}(1-q^{2n-1})}{(-q;q^2)_n} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \left[\sum_{n=1}^{\infty} \frac{q^{(n-1)^2}}{(-q;q^2)_n} - \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n} \right] \\ &= \overline{P}_o(q)(1-F(-q)), \end{aligned} $$

where the last line follows by replacing $q \mapsto q^2$ and substituting $z=q$ in (3.3). □

Proof of Theorem 2.5. By standard combinatorial arguments,

$$ \begin{aligned} \widetilde{M}(q) &= \sum_{n=1}^{\infty} \widetilde{m}(n) q^n = \sum_{n=1}^{\infty} \frac{q^{1+1+2+2+\cdots+(n-1)+(n-1)+n} \prod_{m=n+1}^{\infty}(1+q^m)}{\prod_{m=1}^{\infty}(1-q^m)} \\ &= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_n} = \overline{P}(q)[f_0(q)-1]. \end{aligned} $$ □

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PROOF OF THEOREM 2.6. By standard combinatorial arguments,

$$\begin{aligned}\widetilde{M}_{\mathrm{o}}(q) &= \sum_{n=1}^{\infty} \widetilde{m}_o(n) q^n = \sum_{n=1}^{\infty} \frac{q^{1+1+3+3+\cdots+(2n-3)+(2n-3)+(2n-1)} \prod_{m=n+1}^{\infty}(1+q^{2m-1})}{\prod_{m=1}^{\infty}(1-q^{2m-1})} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n^2-2n+1}}{(-q;q^2)_n} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}}{(-q;q^2)_{n+1}} \\ &= q\overline{P}_{\mathrm{o}}(q) F_1(-q).\end{aligned}$$

□

PROOF OF THEOREM 2.8. In the q-series transformations, we use Heine's ${}_2\phi_1$ transformation and the following identity due to Gupta [13, (1.15)], valid for any $c \in \mathbb{R}$ and $|t| < 1$:

$$\sum_{n=0} c^n((t)_n - (t)_\infty) = (t)_\infty \sum_{n=1}^{\infty} \frac{t^n}{(q)_n(1-cq^n)}. \tag{3.4}$$

Proceeding with the proof of our theorem,

$$\begin{aligned}\overline{M}(z,q) &= \sum_{n=1}^{\infty} \frac{z^n q^{1+2+\cdots+(n-1)} \prod_{m=n+1}^{\infty}(1+q^m)}{\prod_{\substack{m=1\\ m\neq n}}^{\infty}(1-q^m)} = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=1}^{\infty} \frac{z^n q^{n(n-1)/2}(1-q^n)}{(-q)_n} \\ &= \overline{P}(q)\Big[\sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(-q)_n} - \sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(-q)_n}\Big].\end{aligned}$$

Thus,

$$\sigma\overline{M}(q) = \frac{\partial}{\partial z}\Big|_{z=1} \overline{M}(z,q), = \overline{P}(q)[A(q) - B(q)],$$

where

$$\begin{aligned}A(q) &= \frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(-q)_n} = \frac{\partial}{\partial z}\Big|_{z=1} \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix}-1/\tau, q \\ -q\end{matrix}; q, z\tau\right) \\ &= \frac{\partial}{\partial z}\Big|_{z=1} \lim_{\tau\to 0} \frac{(q)_\infty(-z)_\infty}{(-q)_\infty(z\tau)_\infty} {}_2\phi_1\left(\begin{matrix}-1, z\tau \\ -z\end{matrix}; q, q\right) \\ &\quad \text{(using (3.2) with } (a,b,c,z) \mapsto (-1/\tau, q, -q, z\tau)) \\ &= \frac{\partial}{\partial z}\Big|_{z=1} \frac{(q)_\infty}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)_n(-zq^n)_\infty q^n}{(q)_n} = 2(q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q)_n} \sum_{i=n}^{\infty} \frac{q^i}{1+q^i} \\ &= 2(q)_\infty \sum_{n=1}^{\infty} \frac{q^n}{(q)_n} \sum_{i=n}^{\infty} \frac{q^i(-q)_{i-1}}{(-q)_i} + 2(q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{1+q^i}\end{aligned}$$

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$$= 2(q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{(q)_n} \sum_{i=0}^{\infty} \frac{q^i(-q)_{i+n-1}}{(-q)_{i+n}} + 2(q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{1+q^i}$$

$$= 2(q)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}(-q)_{n-1}}{(q)_n(-q)_n} {}_2\phi_1\left(\begin{matrix} q, -q^n \\ -q^{n+1} \end{matrix}; q, q\right) + 2(q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{1+q^i}$$

$$= 2(q^2)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{(q)_n} \sum_{i=0}^{\infty} \frac{(-1)^i (q)_i q^{ni}}{(q^2)_i} + 2(q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{1+q^i}$$

(using (3.2) with $(a, b, c, z) \mapsto (q, -q^n, -q^{n+1}, q)$ in the first double sum)

$$= 2(q^2)_\infty \sum_{i=0}^{\infty} \frac{(-1)^i (q)_i}{(q^2)_i} \sum_{n=1}^{\infty} \frac{q^{(i+2)n}}{(q)_n} + 2(q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{1+q^i}$$

$$= 2(q^2)_\infty \sum_{i=0}^{\infty} \frac{(-1)^i (q)_i}{(q^2)_i} \left[\frac{1}{(q^{i+2})_\infty} - 1\right] + 2(q)_\infty \sum_{i=0}^{\infty} \frac{q^i}{1+q^i}$$

(using (3.1) with $(a, z) \mapsto (0, q^{i+2})$ in the first double sum)

$$= 2\sum_{n=0}^{\infty}(-1)^n (q)_n (1 - (q^{n+2})_\infty) + 2(q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{1+q^n}$$

$$= 2\sum_{n=0}^{\infty}(-1)^n (q)_n (1 - (q^{n+2})_\infty) + 2(q)_\infty \sum_{r=0}^{\infty} \frac{(-1)^r q^{r+1}}{1-q^{r+1}} + (q)_\infty$$

$$= 2\sum_{n=0}^{\infty}(-1)^n (q)_n (1 - (q^{n+2})_\infty) + 2\sum_{n=0}^{\infty}(-1)^n q^{n+1} (q)_n (q^{n+2})_\infty + (q)_\infty$$

$$= 2\sum_{n=0}^{\infty}(-1)^n ((q)_n - (q)_\infty) + (q)_\infty = 2(q)_\infty \sum_{n=1}^{\infty} \frac{q^n}{(q)_n(1+q^n)} + (q)_\infty$$

(using (3.4) with $(c, t) \mapsto (-1, q)$)

$$= R(q),$$

and

$$B(q) = \frac{\partial}{\partial z}\bigg|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(-q)_n}$$

$$= \frac{\partial}{\partial z}\bigg|_{z=1} \lim_{\tau\to 0} \sum_{n=0}^{\infty} \frac{(-q/\tau)_n z^n \tau^n}{(-q)_n}$$

$$= \frac{\partial}{\partial z}\bigg|_{z=1} \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix} -q/\tau, q \\ -q \end{matrix}; q, z\tau\right)$$

$$= \frac{\partial}{\partial z}\bigg|_{z=1} \lim_{\tau\to 0} \frac{(q)_\infty (-zq)_\infty}{(-q)_\infty (z\tau)_\infty} {}_2\phi_1\left(\begin{matrix} -1, z\tau \\ -z\tau \end{matrix}; q, q\right)$$

(using (3.2) with $(a, b, c, z) \mapsto (-q/\tau, q, -q, z\tau)$ in the first double sum)

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$$= \frac{\partial}{\partial z}\Big|_{z=1} \frac{(q)_\infty}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)_n(-zq^{n+1})_\infty q^n}{(q)_n}$$
$$= \frac{(q)_\infty}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)_n(-q^{n+1})_\infty q^n}{(q)_n} \sum_{i=n+1}^{\infty} \frac{q^i}{1+q^i}$$
$$= 2(q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q)_n(1+q^n)} \sum_{i=n+1}^{\infty} \frac{q^i}{1+q^i}.$$

This gives the desired result. □

PROOF OF THEOREM 2.9. We have

$$\overline{M}_o(z,q) = \sum_{n=1}^{\infty} \frac{z^n q^{1+3+\cdots+(2n-3)} \prod_{m=n+1}^{\infty}(1+q^{2m-1})}{\prod_{\substack{m=1\\ m\neq n}}^{\infty}(1-q^{2m-1})} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{z^n q^{(n-1)^2}(1-q^{2n-1})}{(-q;q^2)_n}$$
$$= \overline{P}_o(q)\Big[\sum_{n=1}^{\infty} \frac{z^n q^{(n-1)^2}}{(-q;q^2)_n} - \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(-q;q^2)_n}\Big].$$

Thus,

$$\sigma\overline{M}_o(q) = \frac{\partial}{\partial z}\Big|_{z=1} \overline{M}_o(z,q) = \overline{P}_o(q)[C(q) - D(q)],$$

where

$$C(q) = \frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=1}^{\infty} \frac{z^n q^{(n-1)^2}}{(-q;q^2)_n} = \frac{\partial}{\partial z}\Big|_{z=1} z \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(-q;q^2)_{n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q^2)_{n+1}} + z\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(-q;q^2)_{n+1}}$$
$$= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(-q;q^2)_{n+1}}$$

(using (3.3) with $(z,q) \mapsto (q,q^2)$ in the first sum)

$$= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{n^2}(q^2;q^2)_n}{(q^2;q^2)_n(-q;q^2)_{n+1}}$$
$$= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{z^n q^{n^2}(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n(q^{2n+2};q^2)_\infty}$$
$$= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{(2n+2)m}}{(q^2;q^2)_m}$$

(using (3.1) with $(a,z,q) \mapsto (-q,q^{2n+2},q^2)$)

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$$
\begin{aligned}
&= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{z^n q^{n^2+2mn}}{(q^2;q^2)_n} \\
&= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}(-zq^{2m+1};q^2)_\infty}{(q^2;q^2)_m} \\
&\qquad \text{(using [2, (2.2.6)] with } (z,q) \mapsto (zq^{2m+1}, q^2)) \\
&= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \frac{(q^2;q^2)_\infty(-zq;q^2)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}}{(q^2;q^2)_m(-zq;q^2)_m} \\
&= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} \frac{(q^2;q^2)_\infty(-zq;q^2)_\infty}{(-q;q^2)_\infty} {}_2\phi_1\left(\begin{matrix}-q,0\\ -zq\end{matrix}; q^2, q^2\right) \\
&= 1 + z\frac{\partial}{\partial z}\Big|_{z=1} {}_2\phi_1\left(\begin{matrix}z,q^2\\ 0\end{matrix}; q^2, -q\right) \\
&\qquad \text{(using (3.2) with } (a,b,c,z,q) \mapsto (-q,0,-zq,q^2,q^2)) \\
&= 1 - \sum_{n=1}^{\infty} (-1)^n (q^2;q^2)_{n-1} q^n.
\end{aligned}
$$

Then,

$$
\begin{aligned}
C(-q) &= 1 - \sum_{n=1}^{\infty} (q^2;q^2)_{n-1} q^n = 1 - q\lim_{c\to 0} {}_2\phi_1\left(\begin{matrix}q^2,q^2\\ c\end{matrix}; q^2, q\right) \\
&= 1 - q\lim_{c\to 0} \frac{(c/q^2;q^2)_\infty(q^3;q^2)_\infty}{(c;q^2)_\infty(q;q^2)_\infty} {}_2\phi_1\left(\begin{matrix}q^5/c,q^2\\ q^3\end{matrix}; q^2, c/q^2\right) \\
&\qquad \text{(using (3.2) with } (a,b,c,z,q) \mapsto (q^2,q^2,c,q,q^2)) \\
&= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(q;q^2)_{n+1}} = 1 + F(q),
\end{aligned}
$$

which gives $C(q) = 1 + F(-q)$ and

$$
\begin{aligned}
D(q) &= \frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=1}^{\infty} \frac{z^n q^{n^2}}{(-q;q^2)_n} = \frac{\partial}{\partial z}\Big|_{z=1} z\sum_{n=0}^{\infty} \frac{z^n q^{(n+1)^2}}{(-q;q^2)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(-q;q^2)_{n+1}} + z\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{(n+1)^2}}{(-q;q^2)_{n+1}} \\
&= F(-q) + z\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{(n+1)^2}}{(-q;q^2)_{n+1}} \\
&= F(-q) + z\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=0}^{\infty} \frac{z^n q^{(n+1)^2}(q^2;q^2)_n}{(q^2;q^2)_n(-q;q^2)_{n+1}}
\end{aligned}
$$

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$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{z^n q^{(n+1)^2}(-q^{2n+3};q^2)_\infty}{(q^2;q^2)_n(q^{2n+2};q^2)_\infty}$$

$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} \frac{q(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{z^n q^{n^2+2n}}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{(2n+2)m}}{(q^2;q^2)_m}$$

(using (3.1) with $(a,z,q) \mapsto (-q, q^{2n+2}, q^2)$)

$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} \frac{q(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{z^n q^{n^2+2mn+2n}}{(q^2;q^2)_n}$$

$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} \frac{q(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}(-zq^{2m+3};q^2)_\infty}{(q^2;q^2)_m}$$

(using [2, (2.2.6)] with $(z,q) \mapsto (zq^{2m+3}, q^2)$)

$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} \frac{q(q^2;q^2)_\infty(-zq^3;q^2)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}}{(q^2;q^2)_m(-zq^3;q^2)_m}$$

$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} \frac{q(q^2;q^2)_\infty(-zq^3;q^2)_\infty}{(-q;q^2)_\infty} {}_2\phi_1\left(\begin{matrix}-q, 0\\ -zq^3\end{matrix}; q^2, q^2\right)$$

$$= F(-q) + z\frac{\partial}{\partial z}\bigg|_{z=1} q\, {}_2\phi_1\left(\begin{matrix}zq^2, q^2\\ 0\end{matrix}; q^2, -q\right)$$

(using (3.2) with $(a,b,c,z,q) \mapsto (-q, 0, -zq^3, q^2, q^2)$)

$$= F(-q) - q\sum_{n=1}^{\infty}(-1)^n(q^2;q^2)_n q^n \sum_{i=1}^{n} \frac{q^{2i}}{1-q^{2i}}.$$

This gives the desired result. □

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