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# BENSON'S COFIBRANTS, GORENSTEIN PROJECTIVES AND A RELATED CONJECTURE

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*Abstract* In this short article, we will be principally investigating two classes of modules over any given group ring – the class of Gorenstein projectives and the class of Benson's cofibrants. We begin by studying various properties of these two classes and studying some of these properties comparatively against each other. There is a conjecture made by Fotini Dembegioti and Olympia Talelli that these two classes should coincide over the integral group ring for any group. We make this conjecture over group rings over commutative rings of finite global dimension and prove it for some classes of groups while also proving other related results involving the two classes of modules mentioned.

*Keywords:* Benson's cofibrants; Gorenstein projectives; Kropholler's hierarchy; complete resolutions

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Secondary: 13D02

As mentioned in the abstract, our dealings in this article will focus around two classes of modules, namely the class of Gorenstein projectives and the class of Benson's cofibrants, which are conjectured to be coinciding all the time (see Conjecture 1.1).

Most of the arguments presented in this article are straightforward and easy to follow. There is a lot of background from the theory of cohomological invariants of groups, etc., without which one can struggle to appreciate the results, and this is why we have taken great care to expand on the background in the first section.

We start by providing a number of useful definitions and known results in §1. Then, in §2, we show that for any group $G$ and any commutative ring $R$, both the classes of Gorenstein projective $RG$-modules and the class of $RG$-modules $M$ such that $M \otimes_R B(G, R)$ (for the rest of this paper, we will be writing just $\otimes$ to mean $\otimes_R$, with $R$ being clear from the immediate context) is projective are good classes in the sense of Definition 3.5 of [5] (= Definition 2.1 of this paper) – the $RG$-module $B(G, R)$, which is given by all $G \to R$ functions that are only allowed to take finitely many values, plays a big role in the definition of Benson's cofibrants, we define this module in Definition 1.15. Then, in §3, we investigate the relations between the classes of modules generated by those two classes and show how, for groups satisfying certain properties or belonging to certain classes, we can derive information about the coincidence of these classes from the coincidence of the classes generated by them. When we say 'generation' here, we mean generation in the sense introduced in [5]. And then, after investigating some less important or less relevant questions on Gorenstein projectivity in §4, we prove some general results on the coincidence of Gorenstein projectives and Benson's cofibrants for some classes of groups in §5.

## 1. Preliminaries and definitions

We begin by stating the main conjecture by Dembegioti and Talelli that we will be dealing with in this article.

**Conjecture 1.1 (made over $\mathbb{Z}$ in [10]).** *For any group $G$ and any commutative ring $R$ of finite global dimension, the class of Gorenstein projective $RG$-modules coincides with the class of Benson's cofibrant $RG$-modules, i.e. the class $\{M \in Mod(RG) : M \otimes_R B(G, R) \text{is projective}\}$ (see Definition 1.15).*

All of our tensor products in this article will be over $R$.

We now look into the two classes of modules mentioned in the statement of Conjecture 1.1 and study some of their important properties.

To define Gorenstein projectives, we need to provide a definition of complete resolutions because Gorenstein projectives are defined to be precisely those modules that occur as kernels in complete resolutions.

**Definition 1.2.** Let $A$ be a ring and let $M$ be an $A$-module. A complete resolution admitted by $M$ is an infinite exact sequence of $A$-projective modules, $(F_*, d_*)$: $.. \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \to \cdots$ such that

(a) There exists a projective resolution of $M$, $(P_*, \delta_*) \twoheadrightarrow M$, such that for some $n \geq 0$, $(P_i, \delta_i)_{i \geq n} = (F_i, d_i)_{i \geq n}$. The smallest such $n$ is called the coincidence index of the complete resolution with respect to the given projective resolution.

(b) $\mathrm{Hom}_A(F_*, P)$ is acyclic for any $A$-projective module $P$.

An $A$-module is called Gorenstein projective iff it occurs as a kernel in a complete resolution.

When $A$ is a group ring $RG$, where $G$ is a group and $R$ is a commutative ring, a group $G$ is said to admit a complete resolution over $R$ if the trivial $RG$-module $R$ admits a complete resolution.

First, note that if we are working over a group algebra over a commutative ring $R$ of finite global dimension, then Gorenstein projectives over the group algebra are projective over $R$. This follows straightforwardly from the definition of complete resolutions. We record this in the following lemma. We will be making use of this fact in §3.

**Lemma 1.3.** *Let $G$ be a group and let $R$ be a commutative ring of finite global dimension and let $M$ be a kernel in an exact sequence of projective $RG$-modules that extends to infinity in both directions. Then, $M$ is $R$-projective.*

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**Proof.** Let the global dimension of $R$ be $r$. We have $M$ as a kernel in a doubly infinite acyclic complex of projectives, $(F_*, d_*)$. Let $M = Ker(d_t)$. Looking at the exact sequence $0 \to M \hookrightarrow F_t \to F_{t-1} \to \cdots \to F_{t-s} \twoheadrightarrow Im(d_{t-s}) \to 0$ where $s \geq r-1$, we see that $M$ is the $(s+1)$th syzygy of $Im(d_{t-s})$ and since $r$ is the global dimension of $R$, this implies that $M$ is $R$-projective. □

**Theorem 1.4 (Thm 2.5, Prop. 2.27 of [15]).** *The class of Gorenstein projective $R$-modules, for any ring $R$, contains the class of projectives and is closed under arbitrary direct sums and direct summands. Additionally, Gorenstein projectives are either projective or of infinite projective dimension.*

The following result provides a nice class of examples of Gorenstein projectives.

**Lemma 1.5 (Lemma 2.21 of [1]).** *Let $G$ be a group and $R$ a commutative ring of finite global dimension. Then, any $RG$-permutation module with finite stabilizers is Gorenstein projective.*

It is important to note that, when working over group algebras over rings of finite global dimension, whenever we have the trivial module admitting complete resolutions, all other modules admit complete resolutions. This result can be stated in terms of the Gorenstein projective dimension, and since we will be dealing a lot with the Gorenstein projective dimension in the coming sections, we provide here its definition and a very useful property.

**Definition 1.6.** Let $R$ be a ring and let $M$ be an $R$-module. The Gorenstein projective dimension of $M$ as an $R$-module, denoted $Gpd_R(M)$, is defined as $\min\{n \in \mathbb{N} : \exists$ an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ where each $G_i$ is a Gorenstein projective $R$-module$\}$. If $M$ does not admit a finite length resolution by Gorenstein projective $R$-modules, we say $Gpd_R(M)$ is not finite.

For any group $G$, the Gorenstein cohomological dimension of $G$ with respect to $R$ (in such cases, we usually require $R$ to be commutative), denoted $Gcd_R(G)$, is defined as $Gpd_{RG}(R)$.

Any given group $G$ achieves its highest Gorenstein cohomological dimension over the ring of integers among all commutative rings. This was proved in [14].

**Proposition 1.7 (Proposition 2.1 of [14]).** *Let $G$ be a group. Then, for any commutative ring $R$, $Gcd_R(G) \leq Gcd_{\mathbb{Z}}(G)$.*

**Theorem 1.8 (Theorem 2.20 of [15]).** *Let $R$ be a ring and let $M$ be an $R$-module such that $Gpd_R(M) < \infty$. The following are equivalent.*

*(a)* $Gpd_R(M) \leq n$.

*(b)* $\mathrm{Ext}_R^{>n}(M, N) = 0$ *for any $R$-module $N$ satisfying* $\mathrm{proj.\,dim}_R(N) < \infty$.

*(c)* $\mathrm{Ext}_R^{>n}(M, P) = 0$ *for any projective $P$.*

(d) *In any exact sequence* $0 \to K \to L_{n-1} \to \cdots \to L_1 \to L_0 \to M \to 0$, *where each* $L_i$ *is Gorenstein projective,* $K$ *is Gorenstein projective.*

It follows from Theorem 2.24 of [15] that we cannot have a situation where there is a short exact sequence with one module of infinite Gorenstein projective dimension and every other module of finite Gorenstein projective dimension. Using the Ext-formulation of Gorenstein projective dimension ($(a)$–$(c)$ equivalence in Theorem 1.8), we get the following result.

**Lemma 1.9.** *Let* $R$ *be a ring. Take a short exact sequence of* $R$*-modules* $0 \to A \to B \to C \to 0$. *Then* $Gpd_R(C) \leq \max\{Gpd_R(A)+1, Gpd_R(B)\}$.

**Proof.** If $Gpd_R(A)$ or $Gpd_R(B)$ is not finite, we have nothing to prove, so we can assume they are both finite. So, by Theorem 2.24 of [15], it follows that $Gpd_R(C)$ is finite.

The proof now is done in exactly the same way the very well-known analogous result is proved for projective dimension.

Let $Gpd_R(A) = m$ and $Gpd_R(B) = n$.

First, we deal with the case where $m+1 \geq n$. For any $R$-projective $P$, look at the long exact Ext-sequence: $\ldots \to \mathrm{Ext}_R^{m+1}(A, P) \to \mathrm{Ext}_R^{m+2}(C, P) \to \mathrm{Ext}_R^{m+2}(B, P) \to ..$; here, $\mathrm{Ext}_R^{m+1}(A, P) = 0$ by Theorem 1.8 as $Gpd_R(A) = m$ and $\mathrm{Ext}_R^{m+2}(B, P) = 0$ again by Theorem 1.8 as $Gpd_R(B) = n$ and $m+2 > m+1 \geq n$. Therefore, $\mathrm{Ext}_R^{m+2}(C, P) = 0$, and by Theorem 1.8, we have $Gpd_R(C) \leq m+1$.

Now, we deal with the case where $n > m+1$. Again, for any $R$-projective $P$, look at the long exact Ext-sequence:$\ldots \to \mathrm{Ext}_R^{n}(A, P) \to \mathrm{Ext}_R^{n+1}(C, P) \to \mathrm{Ext}_R^{n+1}(B, P) \to \ldots$ Here, by Theorem 1.8 and since $n > 1 + Gpd_R(A)$, $\mathrm{Ext}_R^{n}(A, P) = 0$, and again by Theorem 1.8 and since $Gpd_R(B) = n$, $\mathrm{Ext}_R^{n+1}(B, P) = 0$. Therefore, $\mathrm{Ext}_R^{n+1}(C, P) = 0$, and by Theorem 1.8, $Gpd_R(C) \leq n$.

Although the two cases dealt with above can be summed up in one argument, we dealt with them separately for clarity. □

**Remark 1.10.** We will be making use of Lemma 1.9 later in §2 to see why the class of Gorenstein projectives is a 'good' class in the sense of Definition 3.5 of [5]. However, it is noteworthy that instead of using Lemma 1.9, we can just use Theorem 2.24 of [15] in its place in §3.

It is easy to see that a module admits a complete resolution iff it has finite Gorenstein projective dimension – this is a very useful result. In our next result, we see, as touched upon before, that the trivial module admitting complete resolutions and all modules admitting complete resolutions are equivalent.

**Theorem 1.11 (Theorem 1.7 of [14]).** *Let* $G$ *be a group and let* $R$ *be a commutative ring of finite global dimension. Then,* $Gpd_{RG}(R) < \infty$ *iff* $Gpd_{RG}(M) < \infty$ *for all* $RG$*-modules* $M$.

On the topic of groups admitting complete resolutions, it is to be noted that there are infinitely many groups that do not admit complete resolutions over any commutative ring

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of finite global dimension-free abelian groups of infinite rank, for example. Before looking at some interesting examples of infinite groups that admit complete resolutions, we make the following definition of two large classes of groups.

**Definition 1.12 (see [18, 24]).** Let $\mathcal{X}$ be a class of groups. We define $H_0\mathcal{X} := \mathcal{X}$, and for any successor ordinal $\alpha$, a group $G$ is said to be in $H_\alpha\mathcal{X}$ iff there exists a finite dimensional contractible $CW$-complex on which $G$ acts by permuting the cells with cell stabilizers in $H_{\alpha-1}\mathcal{X}$. If $\alpha$ is a limit ordinal, then we define $H_\alpha\mathcal{X}$ as $\bigcup_{\beta<\alpha} H_\beta\mathcal{X}$. $G$ is said to be in $H\mathcal{X}$ iff $G$ is in $H_\alpha\mathcal{X}$ for some ordinal $\alpha$ (note that $\alpha$ may be chosen to be a successor ordinal here). We also use the notation $H_{<\alpha}\mathcal{X} := \bigcup_{\beta<\alpha} H_\beta\mathcal{X}$, for any ordinal $\alpha$.

A group $G$ is said to be of type $\Phi$ over a commutative ring $R$ if for any $RG$-module $M$, $\operatorname{proj.dim}_{RG}(M) < \infty$ iff $\operatorname{proj.dim}_{RH}(M) < \infty$ for all finite subgroups $H$ of $G$.

**Notation 1.13.** *Throughout this paper, $\mathcal{F}$ will denote the class of all finite groups.*

Taking $\mathcal{X} = \mathcal{F}$, the class of all finite groups, in the above definition, we get many groups in $H\mathcal{F}$ that do not admit complete resolutions. For example, the free abelian group of rank $t$ with $\aleph_0 \le t < \aleph_\omega$ where $\omega$ is the first infinite ordinal, does not admit complete resolutions but is in $H_2\mathcal{F} \setminus H_1\mathcal{F}$ (see Theorem 7.10 of [11]). In fact, if $t$ here is at least 1 but strictly smaller than $\aleph_0$, then the free abelian group of rank $t$ is in $H_1\mathcal{F} \setminus H_0\mathcal{F}$. The class $H_1\mathcal{F}$ is an interesting class here because all of its groups admit complete resolutions. This was proved in [8], but as we noted in [6], we can have a much shorter proof using the language of Gorenstein projectives:

**Theorem 1.14 (different proof in [8], also noted in Proposition 2.4 of [6]).** *If $G \in H_1\mathcal{F}$, then $G$ admits complete resolutions over any commutative ring. And, for any commutative ring $R$ of finite global dimension, all $RG$-modules admit complete resolutions.*

**Proof.** From the definition of $H_1\mathcal{F}$, it follows that there exists a finite dimensional contractible $CW$-complex $X$, say of dimension $n$, on which $G$ acts by permuting its cells with finite cell stabilizers. We look at the augmented cellular complex of this action: $0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to \mathbb{Z} \to 0$ where each $A_i$ is a direct sum of permutation $\mathbb{Z}G$-modules with finite stabilizers. Permutation modules with finite stabilizers are Gorenstein projective (see Lemma 1.5) and as Gorenstein projectives are closed under direct sums (see Lemma 1.4), we conclude that each $A_i$ is Gorenstein projective. Therefore, $Gpd_{\mathbb{Z}G}(\mathbb{Z}) \le n$, and it follows that $G$ admits complete resolutions over the ring of integers and by Proposition 1.7, it follows that $G$ admits complete resolutions over any ring. The second part follows from Theorem 1.11 over commutative rings of finite global dimension. □

If $G$ is of type $\Phi$ over a commutative ring $R$ of finite global dimension, then it also admits complete resolutions. This was shown in [19], but we can provide a very short proof using Benson's cofibrants and a result of Cornick and Kropholler. To show this short proof, we need to make use of an $RG$-module which we usually denote by $B(G, R)$, which is also crucial in defining the class of modules that we have been calling 'Benson's cofibrants'. We define these modules below and state a couple of important and useful results involving $B(G, R)$.

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**Definition 1.15.** For any group $G$ and any commutative ring $R$, the module $B(G, R)$ is defined to be the $RG$-module of all $G \to R$ functions that are only allowed to take finitely many values. The module structure is given by $g.f(h) = f(g^{-1}h)$ for all $g, h \in G$ and $f \in B(G, R)$. When $R = \mathbb{Z}$, $B(G, \mathbb{Z})$ is the module of all bounded functions from $G$ to $\mathbb{Z}$.

We define an $RG$-module $M$ to be Benson's cofibrant if $M \otimes_R B(G, R)$ is projective as an $RG$-module. For $R = \mathbb{Z}$, this definition was made by Benson in [3]. As mentioned earlier, throughout this article, we will drop the subscript '$R$' while writing this tensor product.

For the rest of this article, whenever we say 'cofibrant', we mean 'Benson's cofibrant' as in Definition 1.15.

**Lemma 1.16 (Lemma 3.4 of [3]).** *For any group $G$ and any commutative ring $R$, $B(G, R)$ is $R$-free and $RH$-free for every finite subgroup $H$ of $G$. Also, $B(G, R) = B(G, \mathbb{Z}) \otimes_{\mathbb{Z}} R$.*

Using the word 'cofibrant' without a model theoretic context might seem strange, so the following is a natural question to ask.

**Question 1.17.** *What is the motivation behind using the word 'cofibrant' in Definition 1.15?*

**Answer 1.18.** *In [4], Benson put a closed model category structure on the module category of $RG$-modules, for any group $G$ and any commutative Noetherian ring $R$, and defined cofibrations in the closed model category with the class of modules that we are calling 'Benson's cofibrants' as the cofibrant objects. He additionally showed, in Theorem 10.10 of [4], that the homotopy category of the closed model category, denoted $\mathrm{Ho.\,Mod}(RG)$, is equivalent to the 'stable module category' where the objects are the $RG$-modules and the arrows between modules $M$ and $N$ are given by $\widehat{\mathrm{Ext}}^0_{RG}(M, N)$ (See Remark 1.19 for more on $\widehat{\mathrm{Ext}}^*_{RG}(M, N)$.)*

*For a group of type $\Phi$ over $R$, now making the assumption that $R$ has finite global dimension in addition to being Noetherian, stable module categories for $RG$-modules were studied by Mazza and Symonds in [19], and their definition (Definition 3.2 of [19]) of the stable module category exactly coincided with Benson's definition above (to note that the definitions for Hom-sets coincide, compare Definition 3.2 of [19] with the definition of $\widehat{\mathrm{Ext}}^0_{RG}$ provided in Remark 1.19). The hypotheses that $R$ has finite global dimension and that $G$ is of type $\Phi$ over $R$ become important in [19] in a key result of the paper (Theorem 3.10 of [19]) where it is shown that the stable module category is equivalent, as a triangulated category, to several other known and well-behaved triangulated categories.*

*Because of the similarity in definition of the stable module category highlighted above, we can conclude that if one repeated the treatment in [4] of putting a closed model category structure on the module category of $RG$-modules, with $G$ a group of type $\Phi$ over $R$ where $R$ is commutative Noetherian and of finite global dimension, we will get*

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*the same conclusion regarding the homotopy category being equivalent to the Mazza–Symonds stable module category and the class of 'Benson's cofibrants' being the cofibrant objects.*

**Remark 1.19.** Following the treatment in §3 of [3], we denote by $\underline{\mathrm{Hom}}_{RG}(M, N)$ the quotient of $\mathrm{Hom}_{RG}(M, N)$ by the additive subgroup consisting of homomorphisms $M \to N$ that factor through an $RG$-projective. As noted in §3 of [3], there is a natural homomorphism $\underline{\mathrm{Hom}}_{RG}(M, N) \to \underline{\mathrm{Hom}}_{RG}(\Omega(M), \Omega(N))$. Now, for any integer $r$, $\widehat{\mathrm{Ext}}^r_{RG}(M, N)$ is defined as $\varinjlim_i \underline{\mathrm{Hom}}_{RG}(\Omega^{i+r}(M), \Omega^i(N))$.

Note that, $\widehat{\mathrm{Ext}}^r_{RG}(M, N)$ can also be defined using satellite functors as shown in [20]. It is shown in §4 of [20] that there is a natural isomorphism between the complete cohomology functors defined using satellite functors (as done in [20]) and those defined as per the definition above.

Coming back to the question of complete resolutions, we show with the aid of the following result why groups of type $\Phi$ admit complete resolutions.

**Theorem 1.20 (Theorem 3.5 of [8]).** *For any group $G$ and any commutative ring $R$, if $M$ is an $RG$-module such that $M \otimes B(G, R)$ has finite projective dimension as an $RG$-module, then $M$ admits complete resolutions.*

**Corollary 1.21.** *If $G$ is a group of type $\Phi$ over a commutative ring $R$, then the trivial module admits complete resolutions. If, in addition, $R$ is of finite global dimension, then all $RG$-modules admit complete resolutions.*

**Proof.** From Lemma 1.16 and the definition of type $\Phi$, it follows that $B(G, R)$ is of finite projective dimension and then by Theorem 1.20, it follows that the trivial module admits complete resolutions. If $R$ has finite global dimension, then from Theorem 1.11, it follows that all $RG$-modules admit complete resolutions. □

We do not know of any groups that admit complete resolutions over a ring of finite global dimension but are not of type $\Phi$ over that ring or are not in $H_1\mathfrak{F}$. The following is conjectured regarding the classes $H_1\mathfrak{F}$ and groups of type $\Phi$.

**Conjecture 1.22 (Conjecture $A$ in [24], Conjecture 43.1 in [7]).** *For any group $G$, the following are equivalent.*

*(a) $G$ is of type $\Phi$ over $\mathbb{Z}$.*

*(b) $G$ admits a finite-dimensional model for its classifying space of proper actions.*

*(c) $G \in H_1\mathfrak{F}$.*

*(d) $G$ admits complete resolutions over $\mathbb{Z}$.*

$(c) \Rightarrow (a)$ is very easy to check and it has been proved in [19]. We checked $(a) \Rightarrow (d)$ and $(c) \Rightarrow (d)$ although it must be noted that they were already known in the literature.

We end this section with some more examples of groups in these classes. We have already mentioned some free abelian examples of $H_1\mathfrak{F}$-groups. Some other examples of

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groups that are type $\Phi$ (over the ring of integers) are groups of finite virtual cohomological dimension like discrete subgroups of Lie groups with finitely many components, for example, groups acting on trees with finite stabilizers like $\mathbb{Q}/\mathbb{Z}$. See §3 in [23] for a reference.

## 2. Both classes in Conjecture 1.1 are good classes

For the rest of this article for a group $G$ and a commutative ring $R$, we shall denote by $\mathrm{GProj}(RG)$ the class of Gorenstein projective $RG$-modules and by $\mathrm{CoF}(RG)$ the class of $RG$-modules $M$ such that $M \otimes B(G, R)$ is projective. Our first result here is that $\mathrm{GProj}(RG)$, for all groups $G$ and all rings $R$, is a 'good' class. We recall the definition of a good class first.

**Definition 2.1 (see Definition 3.5 of [5]).** Let $R$ be a ring. Let $\mathfrak{T}$ be a class of $R$-modules. An $R$-module $M$ is generated in zero steps from $\mathfrak{T}$ iff it is in $\mathfrak{T}$ and in $n$ steps iff there is an exact sequence $0 \to M_2 \to M_1 \to M \to 0$, where $M_i$ is generated from $\mathfrak{T}$ in $a_i$ steps, and $a_1 + a_2 \leq n - 1$. The class of all $R$-modules generated in finitely many steps from $\mathfrak{T}$ is denoted $\langle \mathfrak{T} \rangle$.

The $\mathfrak{T}$-dimension of an $R$-module $M$, denoted $\mathfrak{T}\text{-}\dim(M)$, is defined to be the smallest integer $n$ such that there exists an exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to M \to 0$, where each $T_i \in \mathfrak{T}$, for some $n$. The class of all $R$-modules with finite $\mathfrak{T}$-dimension is denoted $[\mathfrak{T}]$.

A class of $R$-modules is called good iff $[\mathfrak{T}] = \langle \mathfrak{T} \rangle$.

We proved the following characterization result for good classes in [5].

**Proposition 2.2 (part of Proposition 7.2 of [5]).** *Let $R$ be a ring and let $\mathfrak{T}$ be a class of $R$-modules. Then, the following two statements are equivalent.*

*(a) For any short exact sequence of $R$-modules $0 \to A \to B \to C \to 0$, if $A, B \in [\mathfrak{T}]$, then $C \in [\mathfrak{T}]$.*

*(b) For any exact sequence of $R$-modules $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C \to 0$, for any $n > 1$, if each $C_i \in [\mathfrak{T}]$, then $C \in [\mathfrak{T}]$. Note that this is equivalent to saying $[[\mathfrak{T}]] = [\mathfrak{T}]$.*

*(c) $\mathfrak{T}$ is a good class.*

**Proof.** $(a) \Rightarrow (b)$: We shall proceed by induction on $n$. If $n = 2$, $(b)$ holds true, by $(a)$. Let the statement of $(b)$ hold true for $n = k$, this is our induction hypothesis. Now let $0 \to C_{k+1} \to C_k \to \cdots \to C_1 \to C \to 0$ be an exact sequence where each $C_i$ is in $[\mathfrak{T}]$. We split this into two exact sequences.

(S0) $0 \to C_{k+1} \to C_k \to \mathrm{Im}(C_k \to C_{k-1}) \to 0$.

(S1) $0 \to \mathrm{Im}(C_k \to C_{k-1}) \hookrightarrow C_{k-1} \to \cdots \to C_1 \to C \to 0$.

From $(a)$, it follows that in $(S0)$, $\mathrm{Im}(C_k \to C_{k-1}) \in [\mathfrak{T}]$. Therefore, in $(S1)$, every module other than $C$ is in $[\mathfrak{T}]$. So, by our induction hypothesis, $C \in [\mathfrak{T}]$.

$(b) \Rightarrow (c)$: First note that we always have $[\mathfrak{T}] \subseteq \langle \mathfrak{T} \rangle$, i.e. for any $R$-module $M$, if the $\mathfrak{T}\text{-}\dim(M) < \infty$, then $M \in \langle \mathfrak{T} \rangle$. This follows from an easy induction – if

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$\mathfrak{T}\text{-}\dim(M) = 0$, then $M \in \mathfrak{T} \subseteq \langle\mathfrak{T}\rangle$. Assume as an induction hypothesis that for all $M$ satisfying $\mathfrak{T}\text{-}\dim(M) \leq n$, $M \in \langle\mathfrak{T}\rangle$. Now, let $\mathfrak{T}\text{-}\dim(M) = n+1$; then there exists an exact sequence $0 \to T_{n+1} \to T_n \to \cdots \to T_0 \to M \to 0$ where each $T_i \in \mathfrak{T}$. It follows that $\mathfrak{T}\text{-}\dim(\mathrm{Ker}(T_0 \to M)) \leq n$ and therefore $\mathrm{Ker}(T_0 \to M) \in \langle\mathfrak{T}\rangle$ by our induction hypothesis, and from the exact sequence $0 \to \mathrm{Ker}(T_0 \to M) \to T_0 \to M \to 0$, since all the modules except $M$ are in $\langle\mathfrak{T}\rangle$, we get from Definition 2.1 that $M \in \langle\mathfrak{T}\rangle$.

Now, we need to show $\langle\mathfrak{T}\rangle \subseteq [\mathfrak{T}]$ assuming $(b)$ to be true. For any $M \in \langle\mathfrak{T}\rangle$, denote by $\alpha_{\mathfrak{T}}(M)$ the least number of steps required to generate $M$ from $\mathfrak{T}$. We make the induction hypothesis that for all $M$ such that $\alpha_{\mathfrak{T}}(M) \leq n$, $M \in [\mathfrak{T}]$. The base case is obvious as if $\alpha_{\mathfrak{T}}(M) = 0$, then $M \in \mathfrak{T} \subseteq [\mathfrak{T}]$. Now assume $\alpha_{\mathfrak{T}}(M) = n+1$, then there is an exact sequence $0 \to M_2 \to M_1 \to M \to 0$ where $\alpha_{\mathfrak{T}}(M_i) \leq n$, for $i = 1, 2$. By the induction hypothesis, $M_1, M_2 \in [\mathfrak{T}]$ and by $(b)$, we have $M \in [\mathfrak{T}]$.

$(c) \Rightarrow (a)$: Let $0 \to A \to B \to C \to 0$ be an exact sequence where $A, B \in [\mathfrak{T}]$. From $(c)$, it follows that $A, B \in \langle\mathfrak{T}\rangle$, and therefore $C \in \langle\mathfrak{T}\rangle = [\mathfrak{T}]$ by Definition 2.1. □

**Lemma 2.3.** *Let $G$ be a group and let $R$ be a commutative ring. Then,* $\mathrm{GProj}(RG)$ *is a good class.*

**Proof.** We saw in Lemma 1.9 that if we have a short exact sequence of $RG$-modules $0 \to A \to B \to C \to 0$ where $A, B \in [\mathrm{GProj}(RG)]$, i.e. $Gpd_{RG}(A), Gpd_{RG}(B) < \infty$, then $Gpd_{RG}(C) < \infty$. Now it follows from Proposition 2.2 that $\mathrm{GProj}(RG)$ is a good class. □

Before we show that $\mathrm{CoF}(RG)$ is a good class for all $G$ and $R$, note that since $B(G, R)$ is $R$-free, if $P_* \twoheadrightarrow M$ is an $RG$-projective resolution of an $RG$-module $M$, then $P_* \otimes B(G, R) \twoheadrightarrow M \otimes B(G, R)$ is an $RG$-projective resolution of $M \otimes B(G, R)$.

**Lemma 2.4.** *For any group $G$ and any commutative ring $R$,* $[\mathrm{CoF}(RG)]$ *and the class of all $RG$-modules $M$ satisfying* $\mathrm{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$ *coincide.*

**Proof.** Let $M \in [\mathrm{CoF}(RG)]$. Then, there exists an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to M \to 0$$

where each $C_i$ is a cofibrant module, i.e. $C_i \otimes B(G, R)$ is projective for all $i$. Since $B(G, R)$ is $R$-free, we can tensor the exact sequence by $B(G, R)$ to get

$$0 \to C_n \otimes B(G,R) \to \cdots \to C_1 \otimes B(G,R) \to C_0 \otimes B(G,R) \to M \otimes B(G,R) \to 0$$

Now, as each term in the exact sequence, other than $M \otimes B(G, R)$, is projective, we can say that $\mathrm{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$. Thus, $[\mathrm{CoF}(RG)]$ is a subclass of the class of all $RG$-modules $M$ satisfying $\mathrm{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$.

Now let $M$ be an $RG$-module satisfying $\mathrm{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$. Let $(P_i, d_i)_{i \geq 0}$ be a projective resolution of $M$, and, for any positive integer $r$, let $K_r(M)$ denote the $r$th kernel in this resolution. Let $\mathrm{proj.dim}_{RG}(M \otimes B(G, R)) = t$. Then, $K_t(M) \otimes B(G, R)$ is projective. By definition of $K_t(M)$, we have the following exact sequence

$$0 \to K_t(M) \to P_{t-1} \to P_{t-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

$P_i \otimes B(G, R)$ is projective for all $i$ (as $B(G, R)$ is $R$-free), so each $P_i$ is cofibrant. And, $K_t(M) \otimes B(G, R)$ is projective as well, so $K_t(M)$ is cofibrant. Thus, $M$ is in $[\mathrm{CoF}(RG)]$.

Therefore, the class of all $RG$-modules $M$ satisfying $\operatorname{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$ is a subclass of $[\mathrm{CoF}(RG)]$. □

**Lemma 2.5.** *For any group $G$ and any commutative ring $R$, $\mathrm{CoF}(RG)$ is a good class. In particular, we claim $[[\mathrm{CoF}(RG)]] = [\mathrm{CoF}(RG)]$.*

**Proof.** It is obvious that $[\mathrm{CoF}(RG)]$ is a subclass of $[[\mathrm{CoF}(RG)]]$.

Now let $M$ be a module in $[[\mathrm{CoF}(RG)]]$. There exists an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to M \to 0$$

where each $C_i$ is in $[\mathrm{CoF}(RG)]$. We can tensor the above exact sequence by $B(G, R)$, which is $R$-free, to get the following exact sequence

$$0 \to C_n \otimes B(G,R) \to C_{n-1} \otimes B(G,R) \to \cdots \to C_0 \otimes B(G,R) \to M \otimes B(G,R) \to 0.$$

By Lemma 2.4, $\operatorname{proj.dim}_{RG}(C_i \otimes B(G, R)) < \infty$ for $i = 0, 1, \ldots, n$. Therefore, $\operatorname{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$, and by Lemma 2.4 again, $M$ is in $[\mathrm{CoF}(RG)]$. Thus, the classes $[[\mathrm{CoF}(RG)]]$ and $[\mathrm{CoF}(RG)]$ coincinde. Therefore, $\mathrm{CoF}(RG)$ is a good class by Proposition 2.2. □

## 3. Relations between the classes generated by Gorenstein projectives and the cofibrants

We start this section with a technical result.

**Proposition 3.1.** *Let $G$ be a group and $R$ be a commutative ring. Let $\mathcal{T}$ be a class of $RG$-modules satisfying the following conditions:*

*(a) $\mathcal{T}$ is closed under tensoring with $R$-free modules.*

*(b) For any $RG$-module $M$, there exists a surjective map of $RG$-modules $\phi : T_M \to M$ for some $T_M \in \mathcal{T}$.*

*(c) For any $RG$-module $M$, if $\mathcal{T}$-$dim(M) \leq n$, then in any exact sequence $0 \to K_n \to T_{n-1} \to T_{n-2} \to \cdots \to T_1 \to T_0 \to M \to 0$ where each $T_i \in \mathcal{T}$, $K_n \in \mathcal{T}$.*

*(d) For any short exact sequence $0 \to A \to B \to C \to 0$, where $B \in \mathcal{T}$ and $\mathcal{T}\text{-}\dim(A) = k > 0$, $\mathcal{T}\text{-}\dim(C) = k + 1$.*

*For any $R$-free $RG$-module $F$, let $\mathcal{X}_{F,\mathcal{T}} := \{M \in Mod(RG) : M \otimes F \in \mathcal{T}\}$, and let $\mathcal{S}$ denote the class of all $RG$-modules that occur as kernels of $(-\infty, \infty)$-exact sequences of modules in $\mathcal{T}$ (this means the exact sequences are extending to infinity in both directions). Then, $\mathcal{S} \cap [\mathcal{X}_{F,\mathcal{T}}] = \mathcal{S} \cap \mathcal{X}_{F,\mathcal{T}}$ if $F\mathcal{T}D(RG) < \infty$. Here, $F\mathcal{T}D(RG) := sup\{\mathcal{T}\text{-}\dim(M) : M \in Mod(RG)$ satisfying $\mathcal{T}\text{-}\dim(M) < \infty\}$.*

**Proof.** All tensor products are being taken over $R$.

First we prove that $[\mathcal{X}_{F,\mathcal{T}}] = \{M \in Mod(RG) : \mathcal{T}\text{-}dim(M \otimes F) < \infty\}$. To show this, we start with an $RG$-module $M$ satisfying $\mathcal{T}\text{-}dim(M \otimes F) = r < \infty$. We know from condition $(b)$ of our hypothesis, that there exists a surjective $RG$-linear map $\phi_0 : T_0 \to M$ for

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some $T_0 \in \mathcal{T}$. Similarly, there exists an $RG$-surjective map $\phi_1 : T_1 \to Ker(\phi_0)$. Going on like this, we get an exact sequence $0 \to Ker(\phi_{r-1}) \hookrightarrow T_{r-1} \to T_{r-2} \to \cdots \to T_1 \to T_0 \to M \to 0$, where each $T_i \in \mathcal{T}$. When we tensor this exact sequence by $F$ which is $R$-free, we get an exact sequence $0 \to Ker(\phi_{r-1}) \otimes F \to T_{r-1} \otimes F \to \cdots \to T_0 \otimes F \to M \otimes F \to 0$, where each $T_i \otimes F \in \mathcal{T}$ as $\mathcal{T}$ is closed under tensoring with $R$-free modules by condition $(a)$ of our hypothesis. Now, as $\mathcal{T}\text{-}dim(M \otimes F) = r$, by condition $(c)$ of our hypothesis, $\text{Ker}(\phi_{r-1}) \otimes F \in \mathcal{T}$, i.e. $\text{Ker}(\phi_{r-1}) \in \mathcal{X}_{F,\mathcal{T}}$. Thus, $M \in [\mathcal{X}_{F,\mathcal{T}}]$. Now if we take $M \in [\mathcal{X}_{F,\mathcal{T}}]$, then we have an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ for some $n$ and some $X_0, \ldots, X_n$ where each $X_i \in \mathcal{X}_{F,\mathcal{T}}$. Tensoring this sequence by the $R$-free $F$, we get a finite length resolution of $M \otimes F$ by modules in $\mathcal{T}$ as each $X_i \otimes F \in \mathcal{T}$. Thus, $\mathcal{T}\text{-dim}(M \otimes F) < \infty$.

We are now in a position to complete our proof of the proposition. Note that since $\mathcal{X}_{F,\mathcal{T}}$ is a subclass of $[\mathcal{X}_{F,\mathcal{T}}]$, we have $\mathcal{S} \cap \mathcal{X}_{F,\mathcal{T}}$ to be a subclass of $\mathcal{S} \cap [\mathcal{X}_{F,\mathcal{T}}]$. To prove the other direction, we start with an $RG$-module $M_0 \in \mathcal{S} \cap [\mathcal{X}_{F,\mathcal{T}}]$. From our first paragraph, it follows that $\mathcal{T}\text{-}dim(M_0 \otimes F) < \infty$. We need to show $M_0 \otimes F \in \mathcal{T}$, so we start with the assumption that it is not the case and that $\mathcal{T}\text{-}dim(M_0 \otimes F) = r > 0$. By definition of $\mathcal{S}$, there exists an exact sequence $\ldots \to T_1 \to T_0 \to T_{-1} \to T_{-2} \to \ldots$ where each $T_i \in \mathcal{T}$ and $M_0 = \text{Ker}(T_0 \to T_{-1})$. For $i \neq 0$, let $M_i := \text{Ker}(T_i \to T_{i-1})$. We can tensor the short exact sequence $0 \to M_0 \hookrightarrow T_0 \twoheadrightarrow M_{-1} \to 0$ by $F$, which is $R$-free, to get the short exact sequence $0 \to M_0 \otimes F \hookrightarrow T_0 \otimes F \twoheadrightarrow M_{-1} \otimes F \to 0$ where $T_0 \otimes F \in \mathcal{T}$ since $\mathcal{T}$ is closed under tensoring with $R$-free modules. By $(d)$, we can say that $\mathcal{T}\text{-dim}(M_{-1} \otimes F) = r + 1$. Similarly, we get that $\mathcal{T}\text{-dim}(M_{-F\mathcal{T}D(RG)} \otimes F) = r + F\mathcal{T}D(RG) > F\mathcal{T}D(RG)$ which is not possible. So, $M_0 \otimes F \in \mathcal{T}$, i.e. $M_0 \in \mathcal{X}_{F,\mathcal{T}}$. Thus, $\mathcal{S} \cap [\mathcal{X}_{F,\mathcal{T}}]$ is a subclass of $\mathcal{S} \cap \mathcal{X}_{F,\mathcal{T}}$, and we are done. □

The proposition above is a general result. We can use it to show that when $fin.dim(RG) < \infty$, the class generated by Benson's cofibrants coincides with the class generated by Gorenstein projectives if and only if the class of Benson's cofibrants coincides with the class of Gorenstein projectives. To see why that is true, we state this following result, which we have already mentioned in the first section (see Theorem 1.20), from the literature using the language of generation.

**Theorem 3.2.** *For any group $G$ and any commutative ring $R$,*
$[\text{CoF}(RG)] = \langle \text{CoF}(RG) \rangle \subseteq \langle \text{GProj}(RG) \rangle = [\text{GProj}(RG)]$.

**Remark 3.3.** Note that in [8] although the statement of the result was exactly what we have stated above, the proof showed that cofibrants admit complete resolutions of index 0, i.e. they are Gorenstein projective. So we know $\text{CoF}(RG) \subseteq \text{GProj}(RG)$ for all $G$ and $R$ and the challenge resides in proving the other direction.

We can actually determine whether the class of modules generated by $\text{CoF}(RG)$ contains every module just by looking at the projective dimension of $B(G, R)$ as long as the ring $R$ is of finite global dimension. It follows from Theorem 3.2 that if the class of modules generated by the cofibrants includes everything, then the same is true of the class of modules generated by the Gorenstein projectives. The reverse can be shown to be true if we assume a related conjecture to be true. We first state this conjecture below.

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**Conjecture 3.4.** *For any group $G$ and any commutative ring $R$ of finite global dimension, $Gcd_R(G) = \text{proj.}\dim_{RG}(B(G, R))$.*

In Remark 3.7, we deal with some progress that has been achieved on Conjecture 3.4.

**Proposition 3.5.** *For any group $G$ and any commutative ring $R$ which has finite global dimension, the following implications hold with the following statements: $(a) \iff (b) \implies (c)$, and Conjecture 3.4 $\implies ((c) \implies (b))$.*

*(a)* $\langle \text{CoF}(RG)\rangle$ *contains all $RG$-modules.*

*(b)* $\text{proj.}\dim_{RG} B(G, R) < \infty$.

*(c)* $\langle \text{GProj}(RG)\rangle$ *contains all $RG$-modules.*

**Proof.** $(a) \implies (b)$: If $\langle \text{CoF}(RG)\rangle$ contains all $RG$-modules, then it contains the trivial module, and since $\text{CoF}(RG)$ is a good class, we have $\langle \text{CoF}(RG)\rangle = [\text{CoF}(RG)] = \{M \in Mod\text{-}RG : \text{proj.}\dim_{RG}(M \otimes B(G, R)) < \infty\}$ (by Lemmas 2.4 and 2.5), and so $R \otimes B(G, R) = B(G, R)$ must have finite projective dimension.

$(b) \implies (a)$ : Let $t$ be the global dimension of $R$. Let $M$ be an $RG$-module. Let $(P_i, d_i)_{i\geq 0}$ be a projective resolution admitted by $M$ (denote by $K_r(M)$ the $r$th kernel in this projective resolution, for any positive integer $r$). As $B(G, R)$ is $R$-free, $(P_i \otimes B(G, R), d_i \otimes id)_{i\geq 0}$ is a projective resolution admitted by $M \otimes B(G, R)$ and $K_r(M) \otimes B(G, R)$ the $r$th kernel in this projective resolution, for any positive integer $r$. $K_t(M)$ is $R$-projective for any $RG$-module $M$ as $t$ is the global dimension of $R$. As $\text{proj.}\dim_{RG}(B(G, R)) < \infty$, we have $\text{proj.}\dim_{RG}(K_t(M) \otimes B(G, R)) < \infty$ and since $(K_t(M) \otimes B(G, R))$ is the $t$th kernel in the projective resolution $P_* \otimes B(G, R) \twoheadrightarrow M \otimes B(G, R)$, we have $\text{proj.}\dim_{RG}(M \otimes B(G, R)) < \infty$. So, $M$ is in $[\text{CoF}(RG)]$, by Lemma 2.4. Thus, $[\text{CoF}(RG)]$ contains all $RG$-modules, and, as $\text{CoF}(RG)$ is a good class (by Lemma 2.5), $\langle \text{CoF}(RG)\rangle$ contains all $RG$-modules.

$(a) \implies (c)$: obvious from Theorem 3.2.

Conjecture 3.4 $\implies ((c) \implies (b))$: note that if $\langle \text{GProj}(RG)\rangle$ is the class of all $RG$-modules, then $Gpd_{RG}(R) = Gcd_R(G) < \infty$, and therefore by Conjecture 3.4, $\text{proj.}\dim_{RG}(B(G, R)) < \infty$. □

**Remark 3.6.** In the proof of Proposition 3.5, only the part $(b) \implies (a)$ makes use of the fact that $R$ is of finite global dimension.

**Remark 3.7.** In [6], we proved the following result regarding Conjecture 3.4: for any group $G$ and any commutative ring $R$ of finite global dimension, $\text{proj.}\dim_{RG}(B(G, R)) = Gcd_R(G)$ if $\text{proj.}\dim_{RG}(B(G, R))$ is finite. To do this, we need to invoke an invariant called the generalized cohomological dimension of a group that was first defined over the integers in [16], denoted $\underline{cd}_R(G)$. $\underline{cd}_R(G)$ is defined to be the supremum over all integers $n$ such that there is an $R$-free $RG$-module $M$ and an $RG$-free module $F$ such that $\text{Ext}^n_{RG}(M, F) \neq 0$.

First note that, when $\text{proj.}\dim_{RG}(B(G, R)) = n < \infty$, $\Omega^n(R)$ is Gorenstein projective by Remark 3.3 and thus, $Gcd_R(G) \leq n$. We also have, under the same hypothesis, $\text{proj.}\dim_{RG}(B(G, R)) \leq \underline{cd}_R(G)$ (see Lemma 3.8). And finally, it follows from the proof

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of Theorem 2.5 of [2] that if $Gcd_R(G) < \infty$, then $Gcd_R(G) = \underline{cd}_R(G)$. It is noteworthy that here we do not need $R$ to be Noetherian because as we see in the proof of Theorem 2.5 of [2], showing $Gcd_R(G) = \underline{cd}_R(G)$ when it is already known that $Gcd_R(G)$ is finite does not require using $silp(RG) = spli(RG)$ which is the only place where $R$ is required to be Noetherian. Recall that for any ring $A$, $silp(A)$ and $spli(A)$ denote, respectively, the supremum over the injective dimensions of $A$-projectives and the supremum over the projective dimensions of $A$-injectives.

**Lemma 3.8 (Lemma 1.19 of [6]).** *Let $G$ be a group and let $R$ be a commutative ring. If* $\operatorname{proj.dim}_{RG}(B(G, R))$ *is finite, then* $\underline{cd}_R(G) \geq \operatorname{proj.dim}_{RG}(B(G, R))$.

**Proof.** We can assume that $\underline{cd}_R(G)$ is finite. Recall that for any commutative ring $R$ and any group $G$, $\underline{cd}_R(G) := sup\{i \in \mathbb{Z} : \operatorname{Ext}^i_{RG}(M, F) \neq 0$ for some $R$-free $M$ and some $RG$-free $F\}$. Let us start with the assumption that $\operatorname{proj.dim}_{RG}(B(G, R)) = k > \underline{cd}_R(G)$. There must exist some $RG$-module $M$ such that $\operatorname{Ext}^k_{RG}(B(G, R), M) \neq 0$ because otherwise $\operatorname{proj.dim}_{RG}(B(G, R)) \leq k - 1$. Now take $F_M$, the $RG$-free module on $M$, and form the short exact sequence $0 \to \Omega(M) \to F_M \to M \to 0$. We now look at the associated long exact Ext-sequence associated with this short exact sequence and get $.. \to \operatorname{Ext}^k_{RG}(B(G, R), \Omega(M)) \to \operatorname{Ext}^k_{RG}(B(G, R), F_M) \to \operatorname{Ext}^k_{RG}(B(G, R), M) \to \operatorname{Ext}^{k+1}_{RG}(B(G, R), \Omega(M)) \to \dots$ Here, $\operatorname{Ext}^k_{RG}(B(G, R), F_M) = 0$ because $k > \underline{cd}_R(G)$ and $B(G, R)$ is $R$-free and $F_M$ is $RG$-free. Also, $\operatorname{Ext}^{k+1}_{RG}(B(G, R), \Omega(M)) = 0$ because $\operatorname{proj.dim}_{RG}(B(G, R)) = k$. So, $\operatorname{Ext}^k_{RG}(B(G, R), M) = 0$ which gives us a contradiction. □

The following now follows directly from Proposition 3.5 and the more technical Proposition 3.1.

**Theorem 3.9.** *Let $G$ be a group and let $R$ be a commutative ring of finite global dimension. If* $\operatorname{proj.dim}_{RG}(B(G, R)) < \infty$, *then* $\operatorname{CoF}(RG) = \operatorname{GProj}(RG)$.

**Proof.** From Proposition 3.5, it follows that $\langle \operatorname{CoF}(RG) \rangle$ and $\langle \operatorname{GProj}(RG) \rangle$ coincide with the class of all modules. Therefore, $Gcd_R(G) < \infty$. Now, we know from Theorem $C$ of [9] that $fin.dim(RG) \leq silp(RG)$, where $silp(RG)$, following the notation mentioned in Remark 3.7, is the supremum over the injective dimensions of all projective $RG$-modules (note that although the statement of Theorem $C$ in [9] has the hypothesis that $G$ is an $H\mathfrak{F}$-group, for the proof of this inequality that assumption was not used). Also, from Corollary 1.6 of [14], we know that $silp(RG) \leq Gcd_R(G)$+global dimension of $R$. (We are dealing with weak Gorenstein projective modules here which are formally defined at the beginning of §4, see Remark 3.10 for a slightly general remark) Thus, we have in this case that $fin.dim(RG) < \infty$, and therefore by Proposition 3.1, $\operatorname{CoF}(RG) = \operatorname{GProj}(RG)$. □

**Remark 3.10.** It can be noted that, in the setup used by Theorem 3.9, if $silp(RG)$ is finite, then all weak Gorenstein projective $RG$-modules, i.e. modules occurring as kernels in infinite acylic complexes of projectives satisfying condition $(a)$ of Definition 1.2 but not necessarily condition $(b)$ of Definition 1.2, are necessarily Gorenstein projective.

Although Theorem 3.9 implies that groups of type $\Phi$ over any commutative ring of finite global dimension satisfy Conjecture 1.1 – this is because it follows from Lemma 1.16 and

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the definition of type $\Phi$ groups that if $G$ is of type $\Phi$ over $R$ then $B(G, R)$ is of finite projective dimension – we provide a simpler proof below.

**Theorem 3.11.** *Let $G$ be a group of type $\Phi$ over a commutative ring $R$ of finite global dimension. Then,* $\mathrm{CoF}(RG) = \mathrm{GProj}(RG)$.

**Proof.** We only need to show $\mathrm{GProj}(RG) \subseteq \mathrm{CoF}(RG)$ in light of Remark 3.3. Now, if $M \in \mathrm{GProj}(RG)$, then $M \otimes B(G, R)$, since $B(G, R)$ is $R$-free, occurs as a kernel in a doubly infinite acyclic complex of projectives. Over type $\Phi$ groups, doubly infinite acyclic complexes of projectives are totally acyclic (note that, for any ring $A$, an acyclic complex of projective $A$-modules, $C_*$, is called *totally acyclic* iff $\mathrm{Hom}_A(C_*, P)$ is acyclic for any $A$-projective $P$) – this result is proved in Lemma 3.14 of [19], see Remark 3.12. So, $M \otimes B(G, R)$ is Gorenstein projective. Now, $M$ is $R$-projective by Lemma 1.3 and over any finite subgroup $H$ of $G$, $B(G, R)$ is $RH$-free by Lemma 1.16, so $M \otimes B(G, R)$ is $RH$-projective. As $G$ is of type $\Phi$, this means $M \otimes B(G, R)$ has finite projective dimension as an $RG$-module. Now, since $M \otimes B(G, R)$ is Gorenstien projective, it follows from Theorem 1.4 that $M \otimes B(G, R)$ is projective. Therefore, $M \in \mathrm{CoF}(RG)$. □

**Remark 3.12.** In Lemma 3.14 of [19], the base ring $R$ is Noetherian but the Noetherian assumption is only used (this is stated explicitly in Remark 3.15 of [19]) to get $silp(RG) < \infty$ where $R$ is a commutative Noetherian ring of finite global dimension and $G$ is of type $\Phi$ over $R$. Actually, we do not need the Noetherian assumption here because, for just commutative $R$ of finite global dimension, we already have $B(G, R)$ to be of finite projective dimension as an $RG$-module and from Remark 3.7, we get $Gcd_R(G) < \infty$ and, then using Corollary 1.6 of [14], that $silp(RG) < \infty$.

**Remark 3.13.** As we can see, it is actually quite easy to prove that groups of type $\Phi$ satisfy Conjecture 1.1. We have provided two different proofs already and in §5, we get a slightly stronger result as a corollary from a technical result for type $\Phi$ groups again. Since for any commutative ring $R$ of finite global dimension and any group $G$, $B(G, R)$ being of finite projective dimension as an $RG$-module implies that $G$ must admit complete resolutions over $R$, and since Conjecture 1.22 when stated over $R$ claims that a group admitting complete resolutions and being of type $\Phi$ are equivalent conditions, it is plausible that $B(G, R)$ being of finite projective dimension and $G$ being of type $\Phi$ are equivalent conditions as well, but if it is not, then Theorem 3.9 proves Conjecture 1.1 for a larger class of groups than the class of groups of type $\Phi$.

We now look at some conjectured properties of Gorenstein projectivity and how those conjectures relate to Conjecture 1.1.

## 4. Two other questions on Gorenstein projectivity

In this section, we introduce a class of modules called the *weak Gorenstein projectives*. Recall that Gorenstein projectives are defined as modules that arise as kernels in complete resolutions and if, for any ring $A$, $(F_i, d_i)_{i\in\mathbb{Z}}$ is a complete resolution of projective $A$-modules, we require by definition that $Hom_A(F_*, P)$ be acyclic for all projective $A$-modules $P$. We define $F_*$ to be a *weak complete resolution* if it satisfies the same definition

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as complete resolutions except we do not put the condition that $Hom_A(F_*, P)$ is acyclic for all projective $P$. *Weak Gorenstein projectives* are defined as modules that occur as kernels in weak complete resolutions. For any ring $A$, we shall be denoting the class of all weak Gorenstein projective $A$-modules by $\mathrm{WGProj}(A)$. Weak Gorenstein projectives share the following crucial property with Gorenstein projectives.

**Lemma 4.1.** *Let $R$ be a commutative ring of finite global dimension and let $G$ be a group. Then, weak Gorenstein projectives over $RG$ are $R$-projective. Also,* $\mathrm{WGProj}(RG)$ *is closed under direct sums.*

**Proof.** The proof of the first part follows from Lemma 1.3. The proof for weak Gorenstein projectives being closed under direct sums is obvious as a direct sum of weak complete resolutions is still a weak complete resolution. □

The following conjecture was made in [10] by Dembegioti and Talelli over the ring of integers, we make it here over rings of finite global dimension.

**Conjecture 4.2.** *For any group $G$ and a commutative ring $R$ of finite global dimension, an $RG$-module $M$ admits a complete resolution iff it admits a weak complete resolution.*

We wish to focus on the following related conjecture regarding the class of weak Gorenstein projectives.

**Conjecture 4.3.** *For any group $G$ and a commutative ring $R$ of finite global dimension, the class of weak Gorenstein projective $RG$-modules coincides with the class of Gorenstein projective $RG$-modules.*

Before going further in this section, we wish to state another question on Gorenstein projectivity which will be important in our study of groups in the next section where we will deal with a situation where a slight variation of Conjecture 1.1 is satisfied at the level of finitely generated subgroups.

**Conjecture 4.4.** *For any group $G$ and any commutative ring $R$ of finite global dimension, if $M$ is Gorenstein projective as an $RG$-module then it is also Gorenstein projective as an $RH$-module for any subgroup $H$ of $G$, i.e. Gorenstein projectivity is closed upon restriction to subgroups.*

The following is an immediate observation.

**Lemma 4.5.** *If Conjecture 4.3 is satisfied for a subgroup-closed class of groups $\mathfrak{X}$ and a commutative ring $R$ of finite global dimension, then $\mathfrak{X}$-groups satisfy Conjecture 4.4 over $R$.*

**Proof.** Take $G \in \mathfrak{X}$. Let $M$ be a Gorenstein projective $RG$-module that occurs as a kernel in the complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ of $RG$-projective modules. Upon restriction to a subgroup $H$ of $G$, $M$ occurs as a kernel in the same weak complete resolution upon restriction to $H$ (it is still a weak complete resolution because projectivity is closed

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under restriction to subgroups). This means, upon restriction to $H$, $M$ is weak Gorenstein projective, but since Conjecture 4.3 is satisfied for $H$ and $R$ (this is because $\mathfrak{X}$ is subgroup-closed), we have that $M$ is Gorenstein projective upon restriction to $H$. □

If it is known for some group $G$ and some commutative ring $R$ that the class of weak Gorenstein projective $RG$-modules coincides with the class of Benson's cofibrants, then Conjecture 4.2, in fact a slightl y stronger version of it, follows.

**Lemma 4.6.** *Let $G$ be a group and let $R$ be a commutative ring. If* $\mathrm{WGProj}(RG) = \mathrm{CoF}(RG)$*, then, over $RG$, every weak complete resolution is a complete resolution.*

**Proof.** This follows from Corollary $D$ of [10] where they have dealt with the exactly similar situation in different language over $\mathbb{Z}$. The same proof works for any commutative ring $R$. □

We now note the following straightforward result involving $B(G, R)$ that helps us to show why the statement of Conjecture 1.1 over a commutative ring $R$ implies Conjecture 4.4.

**Lemma 4.7.** *For any group $G$ and any commutative ring $R$, $B(H, R)$ is a summand of $Res^G_H B(G, R)$, where $H$ is any subgroup of $G$.*

**Proof.** This result has been proved in [22] for $R = \mathbb{Z}$. It follows over any ring commutative $R$ because $B(G, R) = B(G, \mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any group $G$. □

**Lemma 4.8.** *Let $G$ be a group and $R$ a commutative ring. Then, if $M$ is cofibrant as an $RG$-module, $M$ is cofibrant as an $RH$-module for any subgroup $H \leq G$ as well.*

**Proof.** Let $M$ be cofibrant as an $RG$-module. So, $M \otimes B(G, R)$ is projective as an $RG$-module. This implies that $Res^G_H(M \otimes B(G, R))$ is projective as an $RH$-module for any subgroup $H$ of $G$. By Lemma 4.7, $B(H, R)$ is a direct summand of $Res^G_H B(G, R)$, and so $Res^G_H M \otimes B(H, R)$ is a summand of $Res^G_H(M \otimes B(G, R))$ which is projective. So, $Res^G_H M$ is cofibrant as an $RH$-module. □

**Corollary 4.9.** *Let $R$ be a commutative ring. If a group $G$ satisfies Conjecture 1.1, then Conjecture 4.4 holds for $G$ as well.*

**Proof.** When Conjecture 1.1 is satisfied for $G$, the class of Gorenstein projective $RG$-modules coincides with the class of $RG$-modules $M$ such that $M \otimes B(G, R)$ is projective. Thus, by Lemma 4.8 and the fact that cofibrant $RH$-modules are Gorenstein projective for any subgroup $H \leq G$, our result follows. □

## 5. Groups satisfying a variation of Conjecture 1.1 locally

We start this section by recalling the definition of $LH\mathfrak{X}$-groups, where $\mathfrak{X}$ is a class of groups.

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**Definition 5.1.** We define a group $G$ to be in $LH\mathfrak{X}$ iff all finitely generated subgroups of $G$ are in $H\mathfrak{X}$. Similarly, for any class of groups $\mathfrak{X}$, $L\mathfrak{X}$ or the class of groups that are locally in $\mathfrak{X}$, is defined to be the class of groups all of whose finitely generated subgroups are in $\mathfrak{X}$.

It is important to note that it follows from Theorem 1.1 of [17] that if $\alpha$ is a countable ordinal, then $LH_\alpha\mathfrak{F} \subset LH\mathfrak{F}$. This implies that the class $LH\mathfrak{F}$ contains groups that are not contained in $H_n\mathfrak{F}$ for any integer $n$.

Before going forward, we need to prove a technical result which is a close variant of Theorem B of [10]. Before we state and prove this result, we need to state two lemmas that will be crucial in its proof and in the proofs of some other results in this section.

**Lemma 5.2 (Lemma 5.6 of [3]).** *Let $R$ be a commutative ring and let $G = \bigcup_{\alpha<\gamma} G_\alpha$ where $(G_\alpha)_{\alpha<\gamma}$ is an ascending chain of subgroups of $G$ for some ordinal $\gamma$. Then for any $RG$-module $M$ that is projective over each $G_\alpha$, $\operatorname{proj.dim}_{RG}(M) \leq 1$.*

**Lemma 5.3 (Lemma 2.5 of [17]).** *Every countable group admits an action on a tree with finitely generated vertex and edge stabilizers.*

**Proposition 5.4 (done over $\mathbb{Z}$ and $\mathfrak{F}$ in Theorem B of [10], almost the same proof works here).** *Let $G$ be a group and let $R$ be a commutative ring of finite global dimension. Let $\mathfrak{X}$ be a class of groups and let $B_{G,\mathfrak{X}}$ be an $R$-free $RG$-module that restricts to a projective module on every subgroup of $G$ that is in $\mathfrak{X}$. Now if $M$ is a weak Gorenstein projective $RG$-module, then $M \otimes B_{G,\mathfrak{X}}$ is projective over every subgroup of $G$ that is in $LH\mathfrak{X}$.*

**Proof.** We first show the result is true over all $H\mathfrak{X}$-subgroups of $G$, i.e. over all $H_\alpha\mathfrak{X}$-subgroups of $G$ where $\alpha$ is some ordinal. We shall proceed by transfinite induction on $\alpha$. When $\alpha = 0$, then $B_{G,\mathfrak{X}}$ is projective over all $H_0\mathfrak{X}$ subgroups of $G$ by definition and since $M$ is $R$-projective by Lemma 4.1, we have that $M \otimes B_{G,\mathfrak{X}}$ is projective over all $H_0\mathfrak{X}$ subgroups. Now assume that the result is true over $H_\beta\mathfrak{X}$-subgroups of $G$ for all $\beta < \alpha$ – this is our induction hypothesis. Now if $H$ is a subgroup of $G$ in $H_\alpha\mathfrak{X}$, then $H$ acts on, say, an $n$-dimensional contractible cell complex with $H_{<\alpha}\mathfrak{X}$-stabilizers and the trivial $RH$-module admits a length $n$ resolution by direct sums of permutation modules with stabilizers in $H_\beta\mathfrak{X}$ for some $\beta < \alpha$, i.e. modules that are direct sums of modules of the form $Ind_F^H$(trivial module), for some $F \in H_{<\alpha}\mathfrak{X}$. For any $M \in \mathrm{WGProj}(RG)$, we can tensor this resolution by $M \otimes B_{G,\mathfrak{X}}$ and since by the induction hypothesis, $M \otimes B_{G,\mathfrak{X}}$ is projective over all $H_{<\alpha}\mathfrak{X}$-subgroups of $G$, what we get after tensoring is a finite length resolution of $M \otimes B_{G,\mathfrak{X}}$ by modules that are projective over $RH$. (This is because, for any $F \in H_{<\alpha}\mathfrak{X}$, when we tensor $Ind_F^H$(trivial module) with $M \otimes B_{G,\mathfrak{X}}$ as an $RH$-module, we get $Ind_F^H Res_F^H(M \otimes B_{G,\mathfrak{X}})$ and it follows from the induction hypothesis that $Ind_F^H Res_F^H(M \otimes B_{G,\mathfrak{X}})$ is projective over $RH$.) So, $\operatorname{proj.dim}_{RH}(M \otimes B_{G,\mathfrak{X}}) \leq n$. Now note that this is true for all weak Gorenstein projectives $M$. Now since $B_{G,\mathfrak{X}}$ is $R$-free, if $M$ is a kernel in a weak complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ of $RH$-modules (note that weak Gorenstein projectivity is closed under restriction to subgroups) then $M \otimes B_{G,\mathfrak{X}}$ occurs as a kernel in the weak complete resolution $(F_i \otimes B_{G,\mathfrak{X}}, d_i \otimes id)_{i\in\mathbb{Z}}$ and all the kernels in

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this weak complete resolution have finite projective dimension over $RH$, bounded by $n$, as we have just showed, so by Proposition 3.1, $M \otimes B_{G,\mathfrak{X}}$ is projective over $RH$.

Now we show the statement of the proposition is true for $LH\mathfrak{X}$-subgroups of $G$. Let $H$ be an $LH\mathfrak{X}$ subgroup of $G$. Then, if $H$ is countable, then by Lemma 5.3, $H$ is in $H\mathfrak{F}$. Let $H$ be an uncountable $LH\mathfrak{X}$-subgroup of $G$. Assume as induction hypothesis that the statement of the proposition holds true for all $LH\mathfrak{X}$-subgroups of $G$ of cardinality strictly smaller than the cardinality of $H$. Now as $H$ is uncountable, it can be written as $\bigcup_{\gamma<\alpha} H_\gamma$ where $(H_\gamma)_{\gamma<\alpha}$ is a strictly ascending chain of subgroups of $H$ and $\alpha$ is some ordinal, where each $H_\gamma$ is strictly smaller than $H$ in cardinality. By our induction hypothesis, $M \otimes B_{G,\mathfrak{X}}$ is projective over $RH_\gamma$ for each $\gamma < \alpha$, therefore by Lemma 5.2, $\operatorname{proj.dim}_{RH}(M \otimes B_{G,\mathfrak{X}}) \leq 1$. Again, if $M$, as a weak Gorenstein projective $RH$-module, occurs as the $r$th kernel in a complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ and if the $(r-1)$th kernel, which is also a weak Gorenstein projective $RH$-module, is denoted by $K_{-1}(M)$, then we have a short exact sequence $0 \to M \otimes B_{G,\mathfrak{X}} \hookrightarrow F_{r-1} \otimes B_{G,\mathfrak{X}} \twoheadrightarrow K_{-1}(M) \otimes B_{G,\mathfrak{X}} \to 0$. So, if $\operatorname{proj.dim}_{RH}(M \otimes B_{G,\mathfrak{X}}) = 1$, then $\operatorname{proj.dim}_{RH}(K_{-1}(M) \otimes B_{G,\mathfrak{X}}) = 2$ which is not possible because since $K_{-1}(M)$ is weak Gorenstein projective, the above claim for weak Gorenstein projectives shows that $\operatorname{proj.dim}_{RH}(K_{-1}(M) \otimes B_{G,\mathfrak{X}}) \leq 1$. Thus, $\operatorname{proj.dim}_{RH}(M \otimes B_{G,\mathfrak{X}}) = 0$. □

**Corollary 5.5.** *Let $G$ be a group in $LH\mathfrak{F}$ or of type $\Phi$ over a commutative ring $R$ of finite global dimension. Then, the class of weak Gorenstein projective $RG$-modules, the class of Gorenstein projective $RG$-modules and the class of Benson's cofibrant $RG$-modules all coincide.*

**Proof.** For the case where $G$ is in $LH\mathfrak{F}$, it follows directly from Proposition 5.4, with $\mathfrak{X} = \mathfrak{F}$ and $B_{G,\mathfrak{X}} = B(G, R)$, that weak Gorenstein projective $RG$-modules are Benson's cofibrant. Exactly this was handled and proved in Theorem $B$ and subsequently in Corollary C of [10] by Dembegioti and Talelli.

Now, let us assume that $G$ is of type $\Phi$ over a commutative ring $R$ of finite global dimension. Let $M$ be a weak Gorenstein projective $RG$-module which occurs as a kernel of a weak complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ of projective $RG$-modules. Then $M \otimes B(G, R)$ is projective as an $RF$-module for all finite subgroups $F$ of $G$ (this is because $M$ is $R$-projective and $B(G, R)$ is $RF$-free for all finite $F \leq G$). Since $G$ is of type $\Phi$, this implies that $\operatorname{proj.dim}_{RG}(M \otimes B(G, R)) < \infty$. But $fin.dim(RG) < \infty$ as $G$ is of type $\Phi$ (see Lemma 2.4 of [19]) and $M$ can be any kernel in $F_*$; so by Proposition 3.1, $M \otimes B(G, R)$ is projective as an $RG$-module. By Remark 3.3, it now follows that $M \in \operatorname{GProj}(RG)$. Thus, we have $\operatorname{WGProj}(RG) \subseteq \operatorname{CoF}(RG) \subseteq \operatorname{GProj}(RG) \subseteq \operatorname{WGProj}(RG)$. So, $\operatorname{WGProj}(RG) = \operatorname{CoF}(RG) = \operatorname{GProj}(RG)$. This gives us a new proof of Theorem 3.11. □

Although it is usually not much convenient, we can replace the class of finite subgroups in the definition of type $\Phi$ and have in its place subgroups belonging to some other arbitrary class.

**Definition 5.6.** A group $G$ is said to be of type $\Phi$-$\mathfrak{X}$, for a class of groups $\mathfrak{X}$, over a commutative ring $R$ if the following conditions are equivalent for all $RG$-modules $M$.

(a) $\operatorname{proj.dim}_{RG}(M) < \infty$.

(b) $\operatorname{proj.dim}_{RH}(M) < \infty$ for all subgroups $H \leq G$ such that $H \in \mathfrak{X}$.

Now, from Proposition 5.4, the following two corollaries follow.

**Corollary 5.7.** *Let $G$ be a group and let $R$ be a commutative ring of finite global dimension. Let $\mathfrak{X}$ be a class of groups, and let $B_{G,\mathfrak{X}}$ be an $R$-free $RG$-module that restricts to a projective module on every $\mathfrak{X}$-subgroup of $G$. Then, for any weak Gorenstein projective $RG$-module $M$, $M \otimes B_{G,\mathfrak{X}}$ is projective over every subgroup of $G$ that is of type $\Phi$-$LH\mathfrak{X}$ over $R$.*

**Proof.** Let $M \in \operatorname{WGProj}(RG)$. From Proposition 5.4, we get that $M \otimes B_{G,\mathfrak{X}}$ is projective over $LH\mathfrak{X}$-subgroups of $G$, and then from Definition 5.6 it follows that $M \otimes B_{G,\mathfrak{X}}$ has finite projective dimension over all $\Phi$-$LH\mathfrak{X}$-subgroups of $G$. Let us fix $K$ to be a $\Phi$-$LH\mathfrak{X}$-subgroup of $G$. Note that as $\operatorname{WGProj}(RG)$ is closed under restriction to subgroups, $M$, restricted as an $RK$-module, is in $\operatorname{WGProj}(RK)$ and as $B_{G,\mathfrak{X}}$ is $R$-free, $M \otimes B_{G,\mathfrak{X}} \in \operatorname{WGProj}(RK)$ (here, we are restricting both $M$ and $B_{G,\mathfrak{X}}$ as $RK$-modules, and we will be considering $B_{G,\mathfrak{X}}$ as an $RK$-module for the rest of this proof).

We claim that $sup\{\operatorname{proj.dim}_{RK}(N \otimes B_{G,\mathfrak{X}}) : N \in \operatorname{WGProj}(RK)\}$ is finite. This is easy to see because otherwise for every integer $n > 0$, we will have an $N_n \in \operatorname{WGProj}(RK)$ such that $\operatorname{proj.dim}_{RK}(N_n \otimes B_{G,\mathfrak{X}}) > n$, and then $\operatorname{proj.dim}_{RK}(\bigoplus_n N_n \otimes B_{G,\mathfrak{X}})$ will not be finite which will be a contradiction as $\operatorname{WGProj}(RK)$ is closed under direct sums. Now, from Proposition 3.1, it follows that $M \otimes B_{G,\mathfrak{X}}$ is actually projective over $K$. □

**Corollary 5.8.** *Let $G$ be a group of type $\Phi$-$LH\mathfrak{F}$ over a commutative ring $R$ of finite global dimension. Then, the class of weak Gorenstein projective $RG$-modules, the class of Gorenstein projective $RG$-modules and the class of Benson's cofibrant $RG$-modules all coincide.*

One of the main arguments in the proof of Proposition 5.4 is moving from the module $M \otimes B_{G,\mathfrak{X}}$ being projective over all $H\mathfrak{X}$-subgroups to proving that it must be so over all subgroups that are locally $H\mathfrak{X}$. The proof of Proposition 5.4, however, allows us to prove the following.

**Theorem 5.9.** *Let $G$ be a group and let $R$ be a commutative ring. Let $\mathfrak{X}$ be a class of groups. Assume that for any weak Gorenstein projective $RG$-module $M$, $M \otimes B(G, R)$ is projective over every $\mathfrak{X}$-subgroup of $G$. Then, $M \otimes B(G, R)$ is projective over every $LH\mathfrak{X}$-subgroup of $G$. Note that from this it follows that $M \otimes B(G, R)$ is projective over every $L\mathfrak{X}$-subgroup and over every $H\mathfrak{X}$-subgroup of $G$.*

**Corollary 5.10.** *Let $R$ be a commutative ring of finite global dimension. If a group $G$ is in $LH\mathfrak{F}_{\phi,R}$, where $\mathfrak{F}_{\phi,R}$ is the class of all groups of type $\Phi$ over $R$, or locally $\Phi$-$LH\mathfrak{F}$ over $R$, then, the class of weak Gorenstein projectives, Benson's cofibrants and Gorenstein projectives all coincide.*

**Proof.** The proof is obvious from Theorem 5.9 and Corollary 5.7 which shows that the weak Gorenstein projectives form a subclass of Benson's cofibrants and from the

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logic in the proof of Corollary 5.5 it follows that we have $\mathrm{WGProj}(RG) \subseteq \mathrm{CoF}(RG) \subseteq \mathrm{GProj}(RG) \subseteq \mathrm{WGProj}(RG)$. □

**Remark 5.11.** I do not know of any examples of groups that are locally $\Phi$-$LH\mathfrak{F}$ over, say, $\mathbb{Z}$, but are not of type $\Phi$-$LH\mathfrak{F}$ over the same. The following question becomes crucial in investigating locally $\Phi$-$LH\mathfrak{F}$ should imply $\Phi$-$LH\mathfrak{F}$: is a module over a group of finite projective dimension if it is of finite projective dimension over any finitely generated subgroup of the given group?

For any commutative ring $R$ of finite global dimension, taking as base class $\mathfrak{F}_{\phi,R}$, the class of all groups of type $\Phi$ over $R$, we get that Kropholler's class $H\mathfrak{F}_{\phi,R}$ is much larger than $\mathfrak{F}_{\phi,R}$ because $H\mathfrak{F}_{\phi,R}$ contains $H\mathfrak{F}$ as $\mathfrak{F}_{\phi,R}$ contains $\mathfrak{F}$ and there are examples of free abelian groups of infinite rank that are in $H\mathfrak{F}$ which do not admit complete resolutions as we have discussed before and therefore cannot be in $\mathfrak{F}_{\phi,R}$. Free abelian groups of infinite rank are also locally type $\Phi$ but not type $\Phi$ themselves over any ring of finite global dimension. However, if $(a) \Leftrightarrow (c)$ holds in Conjecture 1.22, then $H\mathfrak{F}_{\phi,\mathbb{Z}} = H(H_1\mathfrak{F}) = H\mathfrak{F}$, and $LH\mathfrak{F}_{\phi,\mathbb{Z}} = LH\mathfrak{F}$, and that would mean, for $R = \mathbb{Z}$, our result (Corollary 5.10) for $LH\mathfrak{F}_{\phi,\mathbb{Z}}$-groups does not give us any groups for which Conjecture 1.1 is not already known to be true.

We end this section with the following corollary which shows that, for a given commutative ring $R$ of finite global dimension, Conjecture 4.4 is satisfied by $LH\mathfrak{F}_{\phi,R}$-groups and groups that are either fully or locally $\Phi$-$LH\mathfrak{F}$.

**Corollary 5.12.** *Fix a commutative ring $R$ of finite global dimension. Using the notation of Corollary 5.10, Gorenstein projectivity is closed under*

*(a) subgroups that are in $LH\mathfrak{F}_{\phi,R}$.*

*(b) subgroups that are either locally or fully $\Phi$-$LH\mathfrak{F}$ (or even $\Phi$-$LH\mathfrak{F}_{\phi,R}$) over $R$.*

**Proof.** If $M$ is a Gorenstein projective $RG$-module, then for any subgroup $H$ of $G$, $Res^G_H(M)$ is a weak Gorenstein projective $RH$-module, and if $H$ is a subgroup of $G$ satisfying $(a)$ or $(b)$ of the hypothesis of Corollary 5.12, then $Res^G_H(M)$ is a Gorenstein projective $RH$-module by Corollary 5.10. □

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