The Bulletin of Symbolic Logic
Volume 23, Number 2, June 2017

# ELIMINATING DISJUNCTIONS BY DISJUNCTION ELIMINATION

DAVIDE RINALDI, PETER SCHUSTER, AND DANIEL WESSEL

**Abstract.** Completeness and other forms of Zorn's Lemma are sometimes invoked for semantic proofs of conservation in relatively elementary mathematical contexts in which the corresponding syntactical conservation would suffice. We now show how a fairly general syntactical conservation theorem that covers plenty of the semantic approaches follows from an utmost versatile criterion for conservation given by Scott in 1974.

To this end we work with multi-conclusion entailment relations as extending single-conclusion entailment relations. In a nutshell, the additional axioms with disjunctions in positive position can be eliminated by reducing them to the corresponding disjunction elimination rules, which in turn prove admissible in all known mathematical instances. In deduction terms this means to fold up branchings of proof trees by way of properties of the relevant mathematical structures.

Applications include the syntactical counterparts of the theorems or lemmas known under the names of Artin–Schreier, Krull–Lindenbaum, and Szpilrajn. Related work has been done before on individual instances, e.g., in locale theory, dynamical algebra, formal topology and proof analysis.

**§1. Introduction.** As is well-known, certain additional axioms in which disjunctions occur in positive position such as

$$P(x * y) \to P(x) \vee P(y) \qquad\qquad P(e) \to \bot$$
$$\top \to Q(z) \vee Q(\sim z) \qquad Q(z) \wedge Q(\sim z) \to \bot$$

are extremely useful in proof practice: they make possible quicker and slicker proofs in the special cases specified by the axioms. Examples include the characteristic axioms of integral domain, local ring, linear order, ordered field, and valuation ring. The use of such axioms, however, is said [67] to obstruct the extraction of computational content from classical proofs, e.g., by negative translation.

To reduce the general case to the special case, moreover, one needs to have at hand—in the terminology of Hilbert's Programme—the *ideal objects* characterised by the axioms, as there are prime ideals, prime filters, ultrafilters, complete theories, and linear orders. Yet the existence of these ideal objects is tied together—again in Hilbert's terms—with *transfinite methods* (Axiom of Choice, Well-Ordering Theorem, Ultrafilter Theorem, Zorn's Lemma, etc.) in the appropriate mathematical forms shaped by Artin–Schreier, Hahn–Banach, Krull–Lindenbaum, Szpilrajn, and others.

Received December 5, 2016.
2010 *Mathematics Subject Classification.* 03F03, 03E25, 03F65.
*Key words and phrases*. syntactical conservation, axiom of choice, Hilbert's Programme, disjunction elimination, entailment relation.


1079-8986/17/2302-0003
DOI:10.1017/bsl.2017.13

This method is related to *semantic* conservation proofs with adequate completeness theorems at hand: by suitably embedding any given model of the base theory $T$ into a model of the extended theory $T^*$ [71]. In fact, if $T^* \vdash \varphi$, then $T^* \vDash \varphi$ by soundness; whence $T \vDash \varphi$ by embedding, and thus $T \vdash \varphi$ by completeness. While completeness and embedding normally need transfinite methods, in some cases Boolean-valued models can be used for constructive arguments [15].

More often than not one can also put and prove a *syntactical* conservation theorem the proof of which contains a proof-theoretic conversion algorithm that works at least for what is known as Horn sequents [40] or definite Horn clauses [75]. This approach is not new and has already proved practicable in different but related contexts: for example, in point-free topology such as locale theory and formal topology [11, 12, 16, 18, 53, 54]; in constructive algebra, especially with dynamical methods [21,26,42,43,45,46,78]; and in proof theory, e.g., in proof analysis [57, 58].

Towards a considerable generalisation (Theorem 3.3) we now employ a method pointed out in [11]: that is, to work with Scott's entailment relations. More specifically, we invoke the extremely efficient 'sandwich criterion' Scott [69] has proved equivalent to syntactical conservation of a multi-conclusion entailment relation extending a single-conclusion entailment relation (Theorem 3.1 below). This criterion has turned out to hold in numerous cases including the ones we have abstracted before [63].

In a nutshell, applying Scott's criterion means to eliminate the additional axioms with disjunctions in positive position by reducing them to the corresponding disjunction elimination rules, which have proved admissible in all mathematical instances considered so far. In deduction terms this means to fold up branchings of proof trees by way of properties of the relevant mathematical structures.

Perhaps it in order to remember a saying by Scott [68, pp. 793–794]:

> *Un*fortunately, in my opinion, both because of the aim of Gentzen's own work and in the light of later applications, the Gentzen systems have been very much oriented toward proof-theoretic analyses—especially the problems of establishing the so-called *cut elimination theorem*. For me this was misleading. It took me a long time to realize that cut is *not* eliminable—except in very special circumstances. This is not to say that cut elimination is uninteresting or unimportant, but there does seem to be a simple and basic point to make with the aid of Gentzen's idea which may not be so generally appreciated.

**On method.** All but Section 5 can be expressed within Elementary Constructive Zermelo–Fraenkel Set Theory **ECST** [2, 3]. This is a fragment of Constructive Zermelo–Fraenkel Set Theory **CZF**, which is based on intuitionistic logic and does not contain the Axiom of Power Set, let alone the Axiom of Choice. To supply semantics (Section 5) sometimes requires to use classical logic, to speak of power sets or to invoke Zorn's Lemma; for simplicity's sake we refer to **ZFC** in any such case.

By a finite set we understand a set that can be written as $\{a_1, \dots, a_n\}$ for some $n \geqslant 0$.[1] In **ECST** the generic subsets of a set $T$ form a proper class, which as usual we write as $\mathscr{P}(T)$, whereas the finite subsets of a set $T$ do form a set, which we denote by $\mathscr{P}_\omega(T)$.

## §2. Relation.

**2.1. Consequence.** Let $S$ be a set and $\rhd \subseteq \mathscr{P}(S) \times S$. All but one of Tarski's axioms of consequence [74] can be put as

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \quad \frac{\forall b \in U(V \rhd b) \quad U \rhd a}{V \rhd a}\ (\mathrm{T}) \quad \frac{U \rhd a}{\exists U_0 \in \mathscr{P}_\omega(U)(U_0 \rhd a)}\ (\mathrm{A}),$$

where $U, V \subseteq S$, and $a \in S$. Since Sambin's [64][2] these axioms have also characterised a finitary covering or Stone covering in formal topology; see also [14, 55, 56].

The notion of consequence has allegedly been described first by Hertz [34–36]. We do not employ the one of Tarski's axioms by which he requires that $S$ be countable. This axiom aside, Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathscr{P}(S)$ correlated to a relation $\rhd$ as above, viz.

$$U^{\rhd} \equiv \{a \in S : U \rhd a\}.$$

Rather than with Tarski's notion, we henceforth work with its restriction to finite subsets, that is, the notion of a *single-conclusion entailment relation*. This is a relation $\rhd \subseteq \mathscr{P}_\omega(S) \times S$ that satisfies

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{V \rhd b \quad V', b \rhd a}{V, V' \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\ (\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$.

Our focus thus is on *finite* subsets of $S$, for which we reserve the letters $U, V, W, \dots$; we also sometimes write $a_1, \dots, a_n$ in place of $\{a_1, \dots, a_n\}$. Redefining

$$T^{\rhd} \equiv \{a \in S : \exists U \in \mathscr{P}_\omega(T)(U \rhd a)\} \tag{1}$$

for *arbitrary* subsets $T$ of $S$ gives back an algebraic closure operator on $\mathscr{P}(S)$. Hence the single-conclusion entailment relations correspond exactly to the relations satisfying Tarski's axioms above.

[1] For the sake of a slicker wording we thus deviate from the prevalent terminology of constructive mathematics and set theory [2, 3, 8, 9, 46, 52]: (1) to call 'subfinite' or 'finitely enumerable' a finite set in the sense above, i.e., a set $T$ for which there is a surjection from $\{1, \dots, n\}$ to $T$ for some $n \geqslant 0$; and (2) to reserve the term 'finite' to sets which are in bijection with $\{1, \dots, n\}$ for a necessarily unique $n \geqslant 0$. Also, finite sets in this stricter sense do not play a role in this paper.

[2] This also is from where we have taken the symbol $\rhd$.

**2.2. Entailment.** Let $S$ be a set and $\vdash \subseteq \mathscr{P}_\omega(S) \times \mathscr{P}_\omega(S)$. Scott's [69] axioms of entailment can be put as

$$\frac{U \between W}{U \vdash W}\ (\mathrm{R}) \qquad \frac{V \vdash W, b \quad V', b \vdash W'}{V, V' \vdash W, W'}\ (\mathrm{T}) \qquad \frac{U \vdash W}{U, U' \vdash W, W'}\ (\mathrm{M})$$

for finite $U, V, W \subseteq S$ and $b \in S$, where $U \between W$ means that $U$ and $W$ have an element in common.[3] To be precise, any such $\vdash$ is a *multi-conclusion entailment relation*, where 'multi' includes 'empty'. The axioms are symmetric: that is, $\vdash$ satisfies the axioms if and only if so does the converse relation $\dashv$.

This fairly general notion of entailment has been introduced by Scott [68–70], building on Hertz's and Tarski's work (see above), and of course on Gentzen's sequent calculus [31,32]. Analogous concepts have been developed before by Lorenzen [47–50], who has even listed [48, pp. 84–85] counterparts of the axioms (R), (T), and (M) for single- and multi-conclusion entailment [22, 23].[4] The relevance of the notion of entailment relation to point-free topology and constructive algebra has been pointed out in [11]; this has been used e.g., in [16, 17, 20, 24, 25, 58, 62]. Consequence and entailment have caught interest from various angles [6, 29, 30, 37, 38, 59, 72, 77].

It is in order to point out a major virtue Scott's treatment of entailment [69] has in comparison with the ones of his predecessors: the base set $S$ may be any set whatsoever, and especially need not consist of formulas. The recommended reading of $U \vdash V$ still is as a sequent à la Gentzen, or rather as

$$\bigwedge_{b \in U} v(b) \to \bigvee_{c \in V} v(c),$$

where $v$ is a distinguished predicate on $S$, normally the one that is of primary interest in the given context. In more logical terms one may view the elements of $S$ as propositional variables and $v$ as a valuation. Although this can be made precise by semantics, as we recall in Section 5, let us stress that—apart from heuristics—it is by no means constituent for our syntactical considerations, i.e., for all but Section 5.

Just as for sequents, it is common to abbreviate $\emptyset \rhd a$ by $\rhd a$, and to use $\vdash V$ and $U \vdash$ as shorthands of $\emptyset \vdash V$ and $U \vdash \emptyset$, respectively. One occasionally even writes $\vdash$ in place of $\emptyset \vdash \emptyset$.

**2.3. Generation.** Let $\mathscr{E} \subseteq \mathscr{P}(X \times Y)$ be a class of relations between sets $X$ and $Y$. We order $\mathscr{E}$ by inclusion $\subseteq$, and call every $R \in \mathscr{E}$ an *$\mathscr{E}$-relation*. Let $R \subseteq X \times Y$, and $(x_i, y_i) \in X \times Y$ with $i \in I$. We say that *$R$ is the $\mathscr{E}$-relation generated by the axioms $x_i R y_i$ with $i \in I$* if $R$ is the least $\mathscr{E}$-relation to which $(x_i, y_i)$ belongs for every $i \in I$. Note that we thus do not incur circularity inasmuch as we suppose $R$ to be given.

We will apply this in the two cases in which $\mathscr{E}$ either consists of all single-conclusion entailment relations on a given set $S$, or else of all multi-conclusion entailment relations on $S$. In these cases the axioms rather are

[3] We have adopted this notation from Giovanni Sambin.

[4] Stefan Neuwirth has hinted us at this circumstance.

axiom schemes in the sense that every parameter is tacitly understood as ranging over its domain. For example, if $\circ$ is a binary operation on $S$, then by saying that $\rhd$ is generated by the axiom $a, b \rhd a \circ b$ we mean that $\rhd$ is generated by *all* axioms $a, b \rhd a \circ b$ with $a, b \in S$.

To actually construct an entailment relation generated by axioms, in a noncircular way, one can *inductively generate* it from the axioms by closing up with respect to (R), (T), and (M). This anyway is how we deal with our applications (Section 4), but is not always necessary for making proofs work. For example, for proving our main result (Theorem 3.3) it is enough to know that the entailment relations under consideration literally are the least entailment relations that satisfy the required axioms.

## §3. Conservation.

Let $\rhd$ and $\vdash$ stand for a single-conclusion and a multi-conclusion entailment relation, respectively.

**3.1. Back and forth.** Given $\vdash$, its *trace* $\rhd_\vdash$ is defined by

$$U \rhd_\vdash a \equiv U \vdash a$$

and in fact is a single-conclusion entailment relation.

Given $\rhd$ and $\vdash$, it makes sense to say that

1. $\vdash$ is an *extension* of $\rhd$ if $\rhd \subseteq \rhd_\vdash$;
2. $\vdash$ is *conservative* over $\rhd$ if $\rhd \supseteq \rhd_\vdash$.

By the very definitions, every $\vdash$ is a conservative extension of its trace $\rhd_\vdash$.

Given $\rhd$, there are $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ as follows [69]:[5]

$$U \vdash_\rhd^{\min} V \equiv \exists b \in V(U \rhd b)$$
$$U \vdash_\rhd^{\max} V \equiv \forall W \in \mathscr{P}_\omega(S) \forall c \in S \big(\forall b \in V(W, b \rhd c) \to W, U \rhd c\big).$$

These $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ indeed are multi-conclusion entailment relations, and $\vdash_\rhd^{\min} \subseteq \vdash_\rhd^{\max}$. As $\rhd$ is the trace of either relation, both $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ are conservative extensions of $\rhd$.

For later use we notice that $a_1, \dots, a_k \vdash_\rhd^{\max} b_1, \dots, b_\ell$ is tantamount to the validity of the rule

$$\frac{W, b_1 \rhd c \quad \dots \quad W, b_\ell \rhd c}{W, a_1, \dots, a_k \rhd c} \tag{2}$$

for all finite $W \subseteq S$ and $c \in S$.

**3.2. Unfolding sandwiches.** Given $\rhd$, the $\vdash$ which are extensions of $\rhd$ (respectively, which are conservative over $\rhd$) are closed upwards (respectively, closed downwards) with respect to $\subseteq$. Hence the $\vdash$ which are conservative extensions of $\rhd$ form an interval. This interval has endpoints $\vdash_\rhd^{\min}$ and $\vdash_\rhd^{\max}$ by the following 'sandwich criterion' for conservative extension by Scott [69]:

[5]This definition of $\vdash_\rhd^{\max}$ is not identical but equivalent to the one given in [69].

https://doi.org/10.1017/bsl.2017.13 Published online by Cambridge University Press

THEOREM 3.1. *A multi-conclusion entailment relation* $\vdash$ *is a conservative extension of the single-conclusion entailment relation* $\rhd$ *if and only if* $\vdash$ *lies between* $\vdash_{\rhd}^{\min}$ *and* $\vdash_{\rhd}^{\max}$, *which is to say that*

$$\rhd = \rhd_{\vdash} \iff \vdash_{\rhd}^{\min} \subseteq \vdash \subseteq \vdash_{\rhd}^{\max}.$$

Lorenzen [47, Satz 14, Satz 15] already had $\vdash_{\rhd}^{\min}$ and $\vdash_{\rhd}^{\max}$ as well as $\Rightarrow$ of Theorem 3.1 [22, 23].[6]

By proof inspection we could make Scott's criterion slightly more precise, as follows:

LEMMA 3.2.

1. $\vdash$ *is an extension of* $\rhd$ *if and only if* $\vdash_{\rhd}^{\min} \subseteq \vdash$.
2. *If* $\vdash$ *is an extension of* $\rhd$, *then* $\vdash$ *is conservative over* $\rhd$ *if and only if* $\vdash \subseteq \vdash_{\rhd}^{\max}$.

To have that conservation follows from $\vdash \subseteq \vdash_{\rhd}^{\max}$ it is not necessary that $\vdash$ be an extension of $\rhd$. In fact, if $U \vdash a$, then by $\vdash \subseteq \vdash_{\rhd}^{\max}$ we have $U \vdash_{\rhd}^{\max} a$, from which we get $U \rhd a$ in view of $a \rhd a$ by (R).

Remembering the recommended disjunctive reading of the conclusion $V$ of any sequent $U \vdash V$, in the light of Lemma 3.2 a possible interpretation of extension and conservation is as follows:

1. extension as that disjunctions introduced from $\rhd$ can be expressed as sequents of $\vdash$;
2. conservation as that disjunctions expressed as sequents of $\vdash$ can be eliminated in terms of $\rhd$.

**3.3. Adding axioms.** Let the single-conclusion entailment relation $\rhd$ be generated by axioms. Let the multi-conclusion entailment relation $\vdash$ be generated by the axioms of $\rhd$, of course with $\vdash$ in place of $\rhd$, and by *additional axioms* of the form

$$\varphi : \quad a_1, \dots, a_k \vdash b_1, \dots, b_\ell,$$

where $k, \ell \geqslant 0$. In any such situation we say that $\vdash$ *extends* $\rhd$, and list the additional axioms if needed. This is legitimate inasmuch as if $\vdash$ extends $\rhd$, then $\vdash$ is an extension of $\rhd$ in the sense of Section 3.1. By the *conservation criterion* of an axiom $\varphi$ as above we understand the rule

$$\frac{W, b_1 \rhd c \quad \dots \quad W, b_\ell \rhd c}{W, a_1, \dots, a_k \rhd c} \ (\mathrm{E}_\varphi),$$

where $W$ is a finite subset of $S$ and $c \in S$.

THEOREM 3.3. *If* $\vdash$ *extends* $\rhd$, *then* $\vdash$ *is conservative over* $\rhd$ *precisely when, for every additional axiom* $\varphi$ *of* $\vdash$, *the conservation criterion* $\mathrm{E}_\varphi$ *is satisfied for* $\rhd$.

PROOF. As $\vdash$ is an extension of $\rhd$, the former is conservative over the latter if and only if $\vdash \subseteq \vdash_{\rhd}^{\max}$ (Lemma 3.2). Now recall that $\vdash$ is the least multi-conclusion entailment relation that satisfies not only the axioms of $\rhd$ but also the additional axioms of $\vdash$; and that $\vdash_{\rhd}^{\max}$ as an extension of $\rhd$

[6] Stefan Neuwirth has hinted us at this circumstance.

https://doi.org/10.1017/bsl.2017.13 Published online by Cambridge University Press

already satisfies the former axioms. Hence $\vdash \subseteq \vdash_{\rhd}^{\max}$ is tantamount to $\vdash_{\rhd}^{\max}$ too satisfying all additional axioms, i.e.,

$$a_1, \dots, a_k \vdash_{\rhd}^{\max} b_1, \dots, b_\ell$$

for every additional axiom $\varphi$ as above. By definition of $\vdash_{\rhd}^{\max}$, this is equivalent to $\mathrm{E}_\varphi$ being valid for $\rhd$. ⊣

Given an axiom such as $\varphi$ above, let $\vdash_\varphi$ denote the multi-conclusion entailment relation that extends $\rhd$ with the single additional axiom $\varphi$. If $\vdash_\varphi$ is conservative over $\rhd$, we say—par abus de langage—that $\varphi$ is *conservative* over $\rhd$. The related special case of Theorem 3.3 reads as follows:

Corollary 3.4. *An axiom $\varphi$ is conservative over $\rhd$ if and only if the rule $\mathrm{E}_\varphi$ is valid for $\rhd$.*

By reduction to the corresponding rule we can thus eliminate from proof trees occurrences of an additional axiom, roughly as follows; note that the result of this conversion does not contain $\vdash$ at all:

$$\dfrac{\dfrac{W \rhd a_1 \quad \dots \quad W \rhd a_k \qquad \varphi : a_1, \dots, a_k \vdash b_1, \dots, b_l}{W \vdash b_1, \dots, b_l} \qquad W, b_1 \rhd c \quad \dots \quad W, b_l \rhd c}{W \vdash c}$$

$$\downarrow$$

$$\dfrac{W \rhd a_1 \quad \dots \quad W \rhd a_k \qquad \dfrac{W, b_1 \rhd c \quad \dots \quad W, b_l \rhd c}{W, a_1, \dots, a_k \rhd c}\,(\mathrm{E}_\varphi)}{W \rhd c}.$$

The following is an equivalent formulation of Theorem 3.3; of course also a direct proof is possible.

Corollary 3.5. *If $\vdash$ extends $\rhd$, then $\vdash$ is conservative over $\rhd$ precisely when every additional axiom $\varphi$ of $\vdash$ is conservative over $\rhd$.*

The next lemma will prove useful for modifying the base of an instance of conservation.

Lemma 3.6. *Let $\rhd$ be a single-conclusion entailment relation on $S$ that is generated by axioms. For any subset $D$ of $S$, let $\rhd'$ be generated by the axioms of $\rhd$, and by the extra axioms $\rhd d$ with $d \in D$.*

1. *We have $U \rhd' a$ if and only if $U, V \rhd a$ for a finite subset $V \subseteq D$.*
2. *If an axiom*

$$\varphi : \quad a_1, \dots, a_k \vdash b_1, \dots, b_\ell$$

*is conservative over $\rhd$, then it is conservative over $\rhd'$.*

## §4. Application.

**4.1. Szpilrajn's Theorem.** Our approach subsumes the existing syntactical treatment [58] of order extension, the semantics of which is that every (proper) quasiorder can be extended to a linear one [33, 73].

*4.1.1. Quasi-orders.* As a binary relation $\leqslant$ on a set $X$ is a quasiorder if $\leqslant$ is reflexive and transitive, the single-conclusion entailment relation $\rhd$

of *quasiorder* on $S = X \times X$ is generated by the corresponding axioms of *reflexivity* and *transitivity*:

$$\rho : \quad \rhd (a, a) \qquad\qquad \tau : \quad (a, b), (b, c) \rhd (a, c).$$

The multi-conclusion entailment relation $\vdash_\lambda$ of *linear quasiorder* extends $\rhd$ with the single additional axiom of *linearity*:

$$\lambda : \quad \vdash (a, b), (b, a).$$

The conservation criterion of $\lambda$ reads as follows:

$$\frac{W, (a, b) \rhd (r, s) \qquad W, (b, a) \rhd (r, s)}{W \rhd (r, s)} \ (\mathrm{E}_\lambda).$$

The closure operator corresponding to $\rhd$ assigns to a subset $T$ of $S$ its *reflexive-transitive closure* $T^*$. With this at hand, or following the proof of [58, Theorem 5.1], one readily verifies that $\mathrm{E}_\lambda$ is valid for $\rhd$; whence $\vdash_\lambda$ is conservative over $\rhd$ (Theorem 3.3). This can equally be seen by restricting to single-conclusion instances an alternative description of $\vdash_\lambda$ in terms of cycles [58, Section 7]. Reflexivity $\rho$ is necessary for conservation, by way of the special case $a = b = r = s$ of $\mathrm{E}_\lambda$.

*4.1.2. Bounded quasi-orders.* We say that a quasiorder $(X, \leqslant)$ with distinguished elements $0, 1$ is *bounded* if $0 \leqslant s$ and $r \leqslant 1$ for all $r, s \in X$; and that $\leqslant$ is *proper* if $1 \not\leqslant 0$. Accordingly, the single-conclusion entailment relation $\rhd'$ of *bounded quasiorder* on $S = X \times X$ is generated by the axioms $\rho$ and $\tau$ as above plus the following:

$$\beta_0 : \quad \rhd (0, s) \qquad\qquad \beta_1 : \quad \rhd (r, 1).$$

The multi-conclusion entailment relation $\vdash'$ of *linear proper bounded quasiorder* extends $\rhd'$ with the additional axioms of linearity $\lambda$ and *properness*:

$$\pi : \quad (1, 0) \vdash$$

The conservation criterion of $\pi$ reads as follows:

$$\frac{}{W, (1, 0) \rhd (r, s)} \ (\mathrm{E}_\pi).$$

By transitivity $\tau$ it is easy to see that $\mathrm{E}_\pi$ is valid for $\rhd'$. As $\mathrm{E}_\lambda$ is valid for $\rhd$ (Section 4.1.1), $\mathrm{E}_\lambda$ is valid for $\rhd'$ too (Corollary 3.4, Lemma 3.6). In all, $\vdash'$ is conservative over $\rhd'$ (Theorem 3.3).

*4.1.3. Discussion.* A proof-theoretic analysis of order relations is carried out in [58], with sequent calculi **GPO** and **GLO** which correspond to the theories of quasiorder and linear quasiorder, respectively. It is shown that a single-conclusion sequent derivable in **GLO** is derivable already in **GPO** [58, Theorems 5.1].

This conservation result is then carried over to nondegenerate nontrivial quasiorders [58, Theorem 5.2]. While *nondegenerate* means $1 \not\leqslant 0$, i.e., what we have called ‘proper’, a quasiorder $\leqslant$ with distinguished elements 0 and 1 is said *nontrivial* if $0 \leqslant 1$. In terms of the single-conclusion entailment relation of quasiorder, *nontriviality* means to add the axiom

$$\rhd(0,1)$$

to reflexivity $\rho$ and transitivity $\tau$. With this add-on, conservation of linearity $\lambda$ carries over (Lemma 3.6), whereas the conservation of properness $\pi$ depends on the presence of $\beta_0$ and $\beta_1$.

The conservation of linearity for quasiorders is an instance of the Universal Krull–Lindenbaum principle, to which we now turn our attention.

**4.2. Universal Krull–Lindenbaum.** In the sequel $\rhd$ and $\vdash$ always stand for a single-conclusion and a multi-conclusion entailment relation, respectively.

*4.2.1. Universal Krull.* Let $S$ come with a (partial) binary operation $* : S \times S \to S$ and with a distinguished element $e \in S$. Given $\rhd$, let $\vdash$ extend $\rhd$ with additional axioms

$$\mu : \quad a * b \vdash a, b \qquad\qquad \pi : \quad e \vdash .$$

In this situation Theorem 3.3 reads as follows:

COROLLARY 4.1. $\vdash$ *is a conservative extension of* $\rhd$ *precisely when the following rules hold*:

$$\frac{W, a \rhd c \qquad W, b \rhd c}{W, a * b \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, e \rhd c}\,(\mathrm{E}_\pi).$$

The first and second conservation criterion above have occurred [63] as '$\rhd$ satisfies *Encoding*' and as '$e$ is *convincing* for $\rhd$', respectively. It is noteworthy that the axiom of *contraction*

$$a * a \rhd a$$

is necessary for conservation, by the special case $a = b = c$ of $\mathrm{E}_\mu$.

Corollary 4.1 can be compared with *disjunction elimination* and *ex falso quodlibet*, especially if $S$ is a bounded distributive lattice such as an intuitionistic Lindenbaum algebra, with $\vee$ as $*$ and $\bot$ as $e$. In these cases $\rhd$ is the single-conclusion entailment relation of filter or theory, see 4.2.4.

*4.2.2. Universal Lindenbaum.* Let $S$ come with a (partial) unary operation $\sim$. Given $\rhd$, let $\vdash$ extend $\rhd$ with additional axioms

$$\eta : \quad \vdash a, \sim a \qquad \nu : \quad a, \sim a \vdash .$$

In this situation Theorem 3.3 reads as follows:

COROLLARY 4.2. $\vdash$ *is a conservative extension of* $\rhd$ *precisely when*

$$\frac{W, a \rhd c \qquad W, \sim a \rhd c}{W \rhd c}\,(\mathrm{E}_\eta) \qquad \frac{}{W, a, \sim a \rhd c}\,(\mathrm{E}_\nu).$$

Corollary 4.2 can be compared with *excluded middle* and *noncontradiction*, especially if $S$ is a Boolean algebra such as a Lindenbaum algebra in classical logic, with complement or negation $\neg$ as $\sim$. In these cases $\rhd$ again is the entailment relation of filter or theory, see 4.2.4. Further applications of Corollary 4.2 include Artin and Schreier's theorem, see 4.2.6 below.

*4.2.3. Order extension revisited.* Some proof-theoretic variants of Szpilrajn's theorem may also be viewed as instances of Universal Krull–Lindenbaum. In order to see this, consider once more the single-conclusion entailment relation $\rhd$ of bounded quasiorder on $S = X \times X$. The swap operation

$$\sim(a, b) = (b, a)$$

on $S$ fits Universal Lindenbaum in parts, giving rise to Section 4.1.1 above. In fact, the axiom $\eta$ becomes linearity $\lambda$ and thus $\mathrm{E}_\eta$ can be proved valid, whereas $\mathrm{E}_\nu$ does not hold in general.

We consider next, on the same $S = X \times X$, the *partial* binary operation $\circ$ of composition defined by

$$(a, b) \circ (b, d) = (a, d).$$

With $\circ$ in place of $*$, the axiom $\mu$ becomes *cotransivitity*

$$\kappa : \quad (a, d) \vdash (a, b), (b, d),$$

the conservation criterion of which reads

$$\frac{W, (a, b) \rhd (r, s) \qquad W, (b, d) \rhd (r, s)}{W, (a, d) \rhd (r, s)} \, (\mathrm{E}_\kappa)$$

and can be proved by transitivity $\tau$ only. By reflexivity $\rho$, (conservation of) linearity $\lambda$ is a special case of (conservation of) cotransitivity $\kappa$. With $(1, 0)$ as $e$, Section 4.1.2 above is an instance of Universal Krull as a whole.

An interesting variant is given by the *total* binary operation

$$(a, b) * (c, d) = (a, d)$$

with additional axiom

$$(a, d) \vdash (a, b), (c, d)$$

and conservation criterion

$$\frac{W, (a, b) \rhd (r, s) \qquad W, (c, d) \rhd (r, s)}{W, (a, d) \rhd (r, s)} .$$

This rule can be shown valid for the single-conclusion entailment relation of quasiorder, over which by reflexivity $\rho$ it yields conservation of *strong linearity* [51]:

$$\vdash (a, b), (c, a).$$

*4.2.4. Distributive lattices.* Krull's Lemma for distributive lattices says that every proper filter (respectively, proper ideal) can be extended to a proper prime filter (respectively, proper prime ideal). To view the corresponding conservation statement, let $L$ be a bounded lattice with meet $\wedge$ and join $\vee$, and with bottom and top element $0$ and $1$, respectively. The entailment relation $\rhd$ of *filter* on $S = L$ is generated by the axioms

$$\rhd 1 \qquad a, b \rhd a \wedge b \qquad a \rhd a \vee b.$$

The multi-conclusion entailment relation $\vdash$ of *proper prime filter* extends $\rhd$ with additional axioms

$$\mu : \quad a \vee b \vdash a, b \qquad \pi : \quad 0 \vdash$$

https://doi.org/10.1017/bsl.2017.13 Published online by Cambridge University Press

for which the conservation criteria read as follows:

$$\frac{W, a \rhd c \qquad W, b \rhd c}{W, a \vee b \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, 0 \rhd c}\,(\mathrm{E}_\pi).$$

The closure operator corresponding to $\rhd$ assigns to a subset $T$ of $S$ the filter generated by $T$. Thus 0 is convincing, while the validity of $\mathrm{E}_\mu$, i.e., Encoding for $\rhd$, follows from $L$ being distributive [63, 4.2]. Therefore, with $\vee$ as $*$ and 0 as $e$, Universal Krull implies that the multi-conclusion entailment relation of proper prime filter on a distributive lattice is a conservative extension of the single-conclusion entailment relation of filter.

Dually, the single-conclusion entailment relation $\rhd'$ of *ideal* on $S = L$ is generated by the axioms

$$\rhd' 0 \qquad a, b \rhd' a \vee b \qquad a \rhd' a \wedge b.$$

This $\rhd'$ extends to the multi-conclusion entailment relation $\vdash'$ of *proper prime ideal* by adding the following axioms:

$$\mu' : \quad a \wedge b \vdash' a, b \qquad \pi' : \quad 1 \vdash' .$$

The closure operator corresponding to $\rhd'$ assigns to a subset $T$ of $S$ the ideal generated by $T$. Reasoning dually to the case of filters shows that the multi-conclusion entailment relation of proper prime ideal is a conservative extension of the single-conclusion entailment relation of ideal.

While this approach fits Universal Krull, if $L$ is a Boolean algebra with complement $-$, then we may instead add the axioms

$$\eta : \quad \vdash a, -a \qquad \nu : \quad a, -a \vdash$$

to both $\rhd$ and $\rhd'$. Now Universal Lindenbaum applies, giving rise to conservation over $\rhd$ and $\rhd'$ of the multi-conclusion entailment relations of *proper complete filter* and *proper complete ideal*, respectively. This conservation corresponds to Lindenbaum's Lemma for Boolean algebras, which says that every proper filter (respectively, proper ideal) can be extended to a proper complete filter (respectively, proper complete ideal).

*4.2.5. Commutative rings.* The original form of Krull's Lemma, for commutative rings [41], says that every proper filter can be extended to a proper prime filter, which can be carried over from ideals to filters. In order to display the corresponding conservation results, let $\rhd$ be the single-conclusion entailment relation of *radical ideal* (or *reduced ring*) on a commutative ring $S$ which is generated by the axioms of *ideal* (or *zero*)

$$\rhd 0 \qquad a, b \rhd a + b \qquad a \rhd ab$$

together with the characteristic axiom of *radical ideal*

$$a^2 \rhd a\,.$$

The corresponding closure operator assigns to every subset $T$ of $S$ the radical of the ideal generated by $T$. The following rules are valid (see e.g., [63, Lemma 19] for a proof of the first one):

$$\frac{W, a \rhd c \qquad W, b \rhd c}{W, ab \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, 1 \rhd c}\,(\mathrm{E}_\pi).$$

https://doi.org/10.1017/bsl.2017.13 Published online by Cambridge University Press

By Universal Krull, with multiplication as $*$ and 1 as $e$, the following axioms of *prime ideal* (or *integral domain*) are conservative over $\rhd$:

$$\mu : \quad ab \vdash a, b \qquad\qquad \pi : \quad 1 \vdash .$$

The five axioms for $\vdash$ stem from [11], and conservation of $\vdash$ over $\rhd$ is essentially known from dynamical algebra [26]. As the relevant case of contraction, the characteristic axiom of radical ideal $a^2 \rhd a$ is necessary for conservation.

Dually, the single-conclusion entailment relation of *filter* (or *unit*) on a commutative ring $S$ is generated by the following axioms:

$$\rhd 1 \qquad a, b \rhd ab \qquad ab \rhd a.$$

The corresponding closure operator assigns to every subset $T$ of $S$ the filter generated by $T$. The following rules hold (see e.g., [63, Lemma 20] for a proof of the first one):

$$\frac{W, a \rhd c \qquad W, b \rhd c}{W, a + b \rhd c}\,(\mathrm{E}_\mu) \qquad\qquad \frac{}{W, 0 \rhd c}\,(\mathrm{E}_\pi).$$

By Universal Krull, now with addition $+$ in place of $*$ and 0 as $e$, the axioms of *prime filter* (or *local ring*) are conservative over $\rhd$:

$$\mu : \quad a + b \vdash a, b \qquad\qquad \pi : \quad 0 \vdash .$$

This again is essentially known from dynamical algebra [42].

*4.2.6. Ordered fields.* Artin and Schreier's Theorem [5], saying that every proper quadratic preorder on a field can be extended to a total order, was used to solve Hilbert's 17th Problem in the affirmative [4]. Towards the corresponding conservation result in terms of entailment relations, let the single-conclusion entailment relation $\rhd$ of *quadratic preorder* on a field $S$ of char $\neq 2$ be generated by the following axioms:

$$\rhd a^2 \qquad a, b \rhd a + b \qquad a, b \rhd ab.$$

The corresponding closure operator assigns to every subset $T$ of $S$ the quadratic preorder generated by $T$. The following rules are valid for $\rhd$ (see [63, Lemma 24] for a proof of the first one):

$$\frac{W, a \rhd c \quad W, b \rhd c}{W, a + b \rhd c}\,(\mathrm{E}_\mu) \qquad \frac{}{W, -1 \rhd c}\,(\mathrm{E}_\pi).$$

By Universal Krull, with addition $+$ as $*$ and $-1$ as $e$, the following axioms are conservative over $\rhd$:

$$\mu : \quad a + b \vdash a, b \qquad \pi : \quad -1 \vdash .$$

Equivalently so are the axioms of *total order* on $S \setminus \{0\}$:

$$\eta : \quad \vdash a, -a \qquad \nu : \quad a, -a \vdash .$$

We thus not only have an instance of Universal Krull but also one of Universal Lindenbaum, with minus $-$ in place of $\sim$. A related set of axioms [11] already contains $\mu$ and $\nu$.

Once more, this conservation statement is essentially known from dynamical algebra [43]. There is vast literature on computational and continuous aspects of the Artin–Schreier Theorem, Hilbert's 17th Problem and related results, see e.g., [7, 10, 27, 28, 60].

*4.2.7. Valuation rings.* Let $R$ be a subring of a field $K$, and let $R[U]$ denote the subring of $K$ containing $R$ that is generated by $U \subseteq K$. Take the single-conclusion entailment relation $\rhd$ on $S = K$ that is generated by the axioms of *subring of* $K$ *containing* $R$

$$\rhd r \quad (r \in R) \qquad a, b \rhd a + b \qquad a, b \rhd ab$$

together with the following axiom of *integral closure*:

$$s_1, \dots, s_n \rhd a \quad (a^n + s_1 a^{n-1} + \cdots + s_n = 0,\ n \geqslant 1).$$

The corresponding closure operator assigns to every subset $T$ of $S$ the integral closure $\overline{R[T]}$ of $R[T]$; and the following rule can be shown valid [63, Lemma 23]:

$$\frac{W, a \rhd c \qquad W, b \rhd c}{W, ab \rhd c}\,(\mathrm{E}_\mu).$$

By Corollary 3.4, the additional axiom

$$\mu : \quad ab \vdash a, b$$

is conservative over $\rhd$. As there is no convincing element for $\rhd$, this is a partial instance of Universal Krull, with multiplication in place of $*$.

Not only $a^2 \rhd a$ as an instance of contraction, but even the full axiom of integral closure is necessary for conservation, which can be seen along the lines of the usual proof that a valuation ring is integrally closed. For example, if $c^2 + s_1 c + s_2 = 0$, then $s_1, s_2 \rhd c$ follows from the instance $a = -c, b = c + s_1$ and $W = \{ s_1, s_2 \}$ of $\mathrm{E}_\mu$.

Up to this point everything equally works for subrings of a ring $K$ rather than a field $K$. For a field $K$, however, an alternative generation [11, 24] of $\vdash$ makes use of the axiom of *valuation*:

$$\eta : \quad \vdash a, a^{-1} \quad (a \neq 0).$$

Given the axioms of $\rhd$, this $\eta$ is equivalent to $\mu$ as above whenever the field $K$ is *discrete* in the sense that the characteristic axiom of a field holds in the form

$$a = 0 \vee \exists b\, (ab = 1).$$

Hence $\eta$ too is conservative over $\rhd$ for discrete fields $K$, and we have a partial instance of Universal Lindenbaum as well. This approach has been followed in the context of Kronecker's Theorem [44] and Dedekind's Prague Theorem [24], and more generally to study valuations in a point-free way [19].

*4.2.8. Ordered vector spaces.* Let $E$ be a vector space over the field $\mathbb{Q}$ of rationals. Let the single-conclusion entailment relation $\rhd$ on $S = E$ be generated by the following axioms:

$$\rhd 0 \qquad a, b \rhd a + b \qquad n \cdot a \rhd a \quad (n \in \mathbb{N}, n \geqslant 1).$$

https://doi.org/10.1017/bsl.2017.13 Published online by Cambridge University Press

The corresponding closure operator assigns to every $T \subseteq S$ the *positive cone* generated by $T$, and

$$\frac{W, a \rhd c \qquad W, b \rhd c}{W, a + b \rhd c}\ (\mathrm{E}_\mu)$$

is a valid rule for $\rhd$. By Theorem 3.3, the multi-conclusion entailment relation $\vdash$ extending $\rhd$ with the additional axiom

$$\mu : \quad a + b \vdash a, b$$

is conservative over $\rhd$. The axiom $n \cdot a \rhd a$ of $\rhd$ follows from $\mathrm{E}_\mu$ by induction and thus is necessary for conservation. A different set of axioms [11] for $\vdash$ includes $\mu$ as above and the following axiom:

$$\nu : \quad a, -a \vdash .$$

From this one can go to the point-free treatments [11, 12, 16, 18] of the Hahn–Banach theorem in succession to [53, 54]. Unlike for many of the ring-theoretic applications mentioned above, it is not clear whether one can make computational use of the Hahn–Banach theorem itself [16].

§5. **Semantics.** Now we place ourselves within **ZFC**. As before let $S$ be a set, and write $U, V$ for finite subsets of $S$.

**5.1. Lindenbaum's Lemma and completeness theorems.** According to [69], a (multi-conclusion) entailment relation $\vDash$ on $S$ is *complete* if for each $a \in S$ either $\vDash a$ or $a \vDash$, and *consistent* if for no $a \in S$ both $\vDash a$ and $a \vDash$. Note that [69] if $\vDash$ is inconsistent, then $\vDash$ holds in the sense that $\emptyset \vDash \emptyset$. Conversely, if $\vDash$ holds, then $\vDash$ is inconsistent unless $S = \emptyset$.

The complete consistent entailment relations $\vDash$ are just the *valuations* on $S$, i.e., predicates $v \in 2^S$. More precisely, if $\vDash$ corresponds to $v$, then

$$U \vDash V \iff \left( \bigwedge_{b \in U} v(b) \Rightarrow \bigvee_{c \in V} v(c) \right). \tag{3}$$

The following [69] surely is one of the most general versions of *Lindenbaum's Lemma*:

THEOREM 5.1. *Each entailment relation $\vdash$ on a set $S$ equals the intersection of all complete consistent entailment relations $\vDash$ on $S$ with $\vDash \supseteq \vdash$.*

For an arbitrary subset $P$ of $S$ we set

$$U \Vdash_P V \iff (P \supseteq U \Rightarrow V \between P).$$

As the valuations $v$ on $S$ are just the subsets $P$ of $S$, in view of (3) the complete consistent entailment relations $\vDash$ are precisely the relations of the form $\Vdash_P$ where $P$ is a—not necessarily finite—subset of $S$.

Now let $\vdash$ denote an arbitrary entailment relation. If $\vdash \subseteq \Vdash_P$, then $P$ is said to be an *ideal element* [25] of $S$ or a *model* [17] of $\vdash$, for short $P \in \mathrm{Mod}(\vdash)$. Hence Lindenbaum's Lemma (Theorem 5.1) is tantamount to the following *Completeness Theorem* [25]:

THEOREM 5.2. *Every entailment relation* $\vdash$ *on a set* $S$ has enough models *in the sense that*

$$U \vdash V \iff \forall P \in \mathrm{Mod}(\vdash)(U \Vdash_P V) \tag{4}$$

*for all finite subsets* $U$ *and* $V$ *of* $S$.

The models of the converse relation $\dashv$ are exactly the complements of the models of $\vdash$. For example, the prime filters of a distributive lattice or commutative ring are exactly the complements of the prime ideals.

*Constructive semantics.* We hasten to add that a constructive semantics is possible, to be expressed in **ECST**.[7] If $S$ is a distributive lattice, then a natural choice of an entailment relation is

$$U \vdash V \equiv \bigwedge U \leqslant \bigvee V$$

with (R) and (M) being automatic but (T) equivalent to distributivity [69]. In this case the models of $\vdash$ and $\dashv$ are nothing but the prime filters and prime ideals, respectively, of $S$.

Conversely, in [11, Theorem 3] the following seminal theorem has been proved, which now is already called 'fundamental theorem of entailment relations' [46, XI, Theorem 5.3]:[8]

THEOREM 5.3. *For every entailment relation* $\vdash$ *on a set* $S$ *there is a distributive lattice* $D$ *with a map* $i : S \to D$ *satisfying*

$$U \vdash V \iff \bigwedge i(U) \leqslant \bigvee i(V) \tag{5}$$

*such that if* $L$ *is an arbitrary distributive lattice and* $j : S \to L$ *is a map satisfying* $\Rightarrow$ *of* (5) *with* $j$ *in place of* $i$, *then there is a unique lattice homomorphism* $f : D \to L$ *such that* $f \circ i = j$.

This can also be seen as the constructive essence of Theorem 5.2. In fact, (4) follows from (5) in **ZFC**, where every distributive lattice has enough prime filters by the adequate variant of Krull's Lemma.

**5.2. Extension and conservation, semantically.** Back to **ZFC**, let $\rhd$ and $\vdash$ be a single-conclusion and multi-conclusion entailment relation, respectively. The models of $\rhd$ are exactly the subsets of $S$ which are closed under the associated algebraic closure operator (1). The corresponding counterpart of Theorem 5.2 is trivial with $\rhd$ in place of $\vdash$ and for singleton $V$: the closure of $U$ equals the intersection of the closed supersets of $U$.

The next statements largely rely on completeness (Theorem 5.2).

[7] For the question whether in **CZF** the models of a given entailment relation form at least a 'set-generated class' we refer to [1, 39, 76]. When consulting this literature we suggest to observe that an entailment relation $R$ on $S$ is a specific sort of 'set of rules' on $S$, and the models of $R$ are exactly the subsets of $S$ that are '$R$–closed'.

[8] The anonymous referee has kindly indicated this name to us, as well as the fact that both this theorem and the analogous one for single-conclusion entailment relations, with semilattices in place of lattices, were already known to Lorenzen [48].

LEMMA 5.4.

1. $\vdash$ *is an extension of* $\rhd$ *if and only if, for every finite* $U \subseteq S$,

$$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq U\} \supseteq U^{\rhd} .$$

2. *If* $\vdash$ *is an extension of* $\rhd$, *then* $\vdash$ *is conservative over* $\rhd$ *if and only if, for every finite* $U \subseteq S$,

$$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq U\} \subseteq U^{\rhd} .$$

In fact, by Theorem 5.2 for singleton $V$ we have that, for every finite $U \subseteq S$,

$$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq U\} = U^{\rhd\vdash} .$$

PROPOSITION 5.5.

1. $\vdash$ *is an extension of* $\rhd$ *if and only if every model of* $\vdash$ *is a model of* $\rhd$.
2. *If* $\vdash$ *is an extension of* $\rhd$, *then* $\vdash$ *is conservative over* $\rhd$ *if and only if, for every model* $I$ *of* $\rhd$,

$$\bigcap\{P \in \mathrm{Mod}(\vdash) : P \supseteq I\} = I .$$

More often than not the characterisation of conservation from Proposition 5.5.2 occurs in its contrapositive form, viz.

> *for every* $I \in \mathrm{Mod}(\rhd)$ *and* $a \in S$, *if* $I \not\ni a$, *then there is* $P \in \mathrm{Mod}(\vdash)$ *with* $P \supseteq I$ *and* $P \not\ni a$.

While this perfectly fits Zorn's Lemma, to prove the original form of Proposition 5.5.2 (and Theorems 5.1 and 5.2) it is perhaps more natural [13, 63, 65, 66] to use Raoult's Open Induction [61].

In the situation of 4.1.1 the semantics of Proposition 5.5.2 reads as

> *every quasiorder* $R$ *on a set* $X$ *equals the intersection of all linear quasiorders on* $X$ *that contain* $R$,

the contrapositive of which is known as Szpilrajn's Theorem [33, 58, 73]:

> *every proper quasiorder* $R$ *on a set* $X$ *can be extended to a proper linear quasiorder on* $X$ *that contains* $R$.

More generally, if $\vdash$ is to $\rhd$ as in 4.2.1, then the semantics of Proposition 5.5.2 is the *Universal Krull–Lindenbaum Theorem* (*UKL*) [63, Theorem 14, Corollary 15], the crucial hypothesis of which is just Encoding recalled in 4.2.1. This UKL has been abstracted from Krull's [41] and Lindenbaum's [74, p. 394] results in commutative algebra and formal logic, respectively.

§**6. Acknowledgments.** The research summarised in this communication was carried out within the project "Abstract Mathematics for Actual Computation: Hilbert's Program in the 21st Century" funded by the John Templeton Foundation, within which Rinaldi worked as Research Fellow at the University of Leeds. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

Further studies had been undertaken when Schuster was visiting the Munich Center for Mathematical Philosophy upon kind invitation by Hannes Leitgeb and with a research fellowship "Erneuter Aufenthalt" by the Alexander-von-Humboldt Foundation. Stefan Neuwirth has kindly explained Lorenzen's work. The authors are most grateful to Henri Lombardi, Giuseppe Rosolini and Helmut Schwichtenberg for their interest and suggestions. Schuster and Wessel would like to thank Sara Negri for the enlightening discussions during her "Cooperint Azione 3" visiting fellowship provided by the University of Verona in 2016; and Thomas Streicher for the helpful advice he gave during his INdAM-GNSAGA visiting professorship at the University of Padua in 2016. Rinaldi and Schuster owe special thanks to Giovanni Sambin, who has raised the issue of conservativity of spatiality already in 2010. Last but not least, all authors are indebted to the anonymous referee for a really constructive critique.

## REFERENCES

[1] P. Aczel, H. Ishihara, T. Nemoto, and Y. Sangu, *Generalized geometric theories and set-generated classes*. ***Mathematical Structures in Computer Science***, vol. 25 (2015), no. 7, pp. 1466–1483.

[2] P. Aczel and M. Rathjen, *Notes on constructive set theory*, Technical report, Institut Mittag–Leffler, 2000/01, Report No. 40.

[3] ———, *Constructive set theory*, Book draft, 2010.

[4] E. Artin, *Über die Zerlegung definiter Funktionen in Quadrate*. ***Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg***, vol. 5 (1927), no. 1, pp. 100–115.

[5] E. Artin and O. Schreier, *Algebraische Konstruktion reeller Körper*. ***Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg***, vol. 5 (1927), no. 1, pp. 85–99.

[6] A. Avron, *Simple consequence relations*. ***Information and Computation***, vol. 92 (1991), pp. 105–139.

[7] S. Basu, R. Pollack, and M.-F. Roy, ***Algorithms in Real Algebraic Geometry***, Springer, Berlin, 2003.

[8] E. Bishop, ***Foundations of Constructive Analysis***, McGraw-Hill, New York, 1967.

[9] E. Bishop and D. Bridges, ***Constructive Analysis***, Springer, Berlin and Heidelberg, 1985.

[10] J. Bochnak, M. Coste, and M.-F. Roy, ***Real Algebraic Geometry***, Springer-Verlag, Berlin, 1998.

[11] J. Cederquist and T. Coquand, *Entailment relations and distributive lattices*, ***Logic Colloquium '98.*** (S. R. Buss, P. Hájek, and P. Pudlák, editors), Lecture Notes Logic, vol. 13, A. K. Peters, Natick, MA, 2000, pp. 127–139.

[12] J. Cederquist, T. Coquand, and S. Negri, *The Hahn-Banach theorem in type theory*, ***Twenty-five Years of Constructive Type Theory (Venice, 1995)*** (G. Sambin and J. M. Smith, editors), Oxford Logic Guides, vol. 36, Oxford University Press, New York, 1998, pp. 57–72.

[13] F. Ciraulo, D. Rinaldi, and P. Schuster, *Lindenbaum's lemma via open induction*, ***Advances in Proof Theory*** (R. Kahle, T. Strahm, and T. Studer, editors), Progress in Computer Science and Applied Logic, vol. 28, Birkhäuser, Basel, 2016, pp. 65–77.

[14] F. Ciraulo and G. Sambin, *Finitary formal topologies and Stone's representation theorem*. ***Theoretical Computer Science***, vol. 405 (2008), no. 1–2, pp. 11–23.

[15] T. Coquand, *Two applications of Boolean models*. ***Archive for Mathematical Logic***, vol. 37 (1998), no. 3, pp. 143–147.

[16] ———, *A direct proof of the localic Hahn-Banach theorem*, 2000. Manuscript, available from author's webpage, http://www.cse.chalmers.se/~coquand/formal.html.

[17] ———, *Lewis Carroll, Gentzen and entailment relations*, 2000. Manuscript, available from the author's website, http://www.cse.chalmers.se/~coquand/formal.html.

[18] ———, *Geometric Hahn-Banach theorem*. ***Mathematical Proceedings of the Cambridge Philosophical Society***, vol. 140 (2006), pp. 313–315.

[19] ———, *Space of valuations*. ***Annals of Pure and Applied Logic***, vol. 157 (2009), pp. 97–109.

[20] T. Coquand and H. Lombardi, *Hidden constructions in abstract algebra (3): Krull dimension of distributive lattices and commutative rings*, ***Commutative Ring Theory and Applications*** (M. Fontana, S.-E. Kabbaj, and S. Wiegand, editors), Lecture Notes in Pure and Applied Mathematics, vol. 231, Marcel Dekker, New York, 2002, pp. 477–499.

[21] ———, *A logical approach to abstract algebra*. ***Mathematical Structures in Computer Science***, vol. 16 (2006), pp. 885–900.

[22] T. Coquand, H. Lombardi, and S. Neuwirth, *Lorenzen's theory of divisibility*, preprint, 2016.

[23] ———, *Lattice-ordered groups generated by ordered groups and regular systems of ideals*, preprint, 2017, https://arxiv.org/pdf/1701.05115.pdf.

[24] T. Coquand and H. Persson, *Valuations and Dedekind's Prague theorem*. ***Journal of Pure and Applied Algebra***, vol. 155 (2001), no. 2–3, pp. 121–129.

[25] T. Coquand and G.-Q. Zhang, *Sequents, frames, and completeness*, ***Computer science logic 2000*** (P. G. Clote and H. Scwichtenberg, editors), Lecture Notes in Computer Science, vol. 1862, Springer, Berlin, 2000, pp. 277–291.

[26] M. Coste, H. Lombardi, and M.-F. Roy, *Dynamical method in algebra: Effective Nullstellensätze*. ***Annals of Pure and Applied Logic***, vol. 111 (2001), no. 3, pp. 203–256.

[27] C. N. Delzell, *Kreisel's Unwinding of Artin's Proof*, ***Kreiseliana.*** (P. Odifreddi, editor), A K Peters, Wellesley, MA, 1996, pp. 113–246.

[28] C. N. Delzell, L. González-Vega, and H. Lombardi, *A continuous and rational solution to Hilbert's 17th problem and several cases of the Positivstellensatz*, ***Computational Algebraic Geometry*** (F. Eyssette and A. Galligo, editors), Birkhäuser, Boston, MA, 1993, pp. 61–75.

[29] K. Došen, *On passing from singular to plural consequences*, ***Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa*** (E. Orlowska, editor), Studies in Fuzziness and Soft Computing, vol. 24, Physica, Heidelberg, 1999, pp. 533–547.

[30] D. M. Gabbay, ***Semantical Investigations in Heyting's Intuitionistic Logic***, Synthese Library, vol. 148, D. Reidel Publishing Co., Dordrecht-Boston, MA, 1981.

[31] G. Gentzen, *Untersuchungen über das logische Schließen I*. ***Mathematische Zeitschrift***, vol. 39 (1934), pp. 176–210.

[32] ———, *Untersuchungen über das logische Schließen II*. ***Mathematische Zeitschrift***, vol. 39 (1934), pp. 405–431.

[33] B. Hansson, *Choice structures and preference relations*. ***Synthese***, vol. 18 (1968), no. 4, pp. 443–458.

[34] P. Hertz, *Über Axiomensysteme für beliebige Satzsysteme. I. Teil. Sätze ersten Grades*. ***Mathematische Annalen***, vol. 87 (1922), no. 3, pp. 246–269.

[35] ———, *Über Axiomensysteme für beliebige Satzsysteme. II. Teil. Sätze höheren Grades*. ***Mathematische Annalen***, vol. 89 (1923), no. 1, pp. 76–102.

[36] ———, *Über Axiomensysteme für beliebige Satzsysteme*. ***Mathematische Annalen***, vol. 101 (1929), no. 1, pp. 457–514.

[37] L. Humberstone, *On a conservative extension argument of Dana Scott*. ***Logic Journal of the IGPL***, vol. 19 (2011), pp. 241–288.

[38] ———, *Dana Scott's work with generalized consequence relations*, ***Universal Logic: An Anthology - From Paul Hertz to Dov Gabbay*** (J.-Y. Béziau), Studies in Universal Logic, Birkhäuser, Basel, 2012, pp. 263–279.

[39] H. Ishihara and T. Nemoto, *Non-deterministic inductive definitions and fullness*, ***Concepts of Proof in Mathematics, Philosophy, and Computer Science*** (D. Probst and P. Schuster, editors), Ontos Mathematical Logic, vol. 6, Walter de Gruyter, Berlin, 2016, pp. 163–170.

[40] P. T. Johnstone, ***Sketches of an Elephant: A Topos Theory Compendium, vol. 1***, Oxford Logic Guides, vol. 43, The Clarendon Press Oxford University Press, New York, 2002.

[41] W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*. ***Annals of Mathematics***, vol. 101 (1929), pp. 729–744.

[42] H. Lombardi, *Le contenu constructif d'un principe local-global avec une application à la structure d'un module projectif de type fini*, ***Publications Mathématiques de Besançon. Algèbre et Théorie des Nombres***, 1997, Fascicule 94–95 & 95–96.

[43] ———, *Relecture constructive de la théorie d'Artin-Schreier*. ***Annals of Pure and Applied Logic***, vol. 91 (1998), pp. 59–92.

[44] ———, *Hidden constructions in abstract algebra. I. Integral dependance*. ***Journal of Pure and Applied Algebra***, vol. 167 (2002), pp. 259–267.

[45] ———, *Algèbre dynamique, espaces topologiques sans points et programme de Hilbert*. ***Annals of Pure and Applied Logic***, vol. 137 (2006), pp. 256–290.

[46] H. Lombardi and C. Quitté, ***Commutative Algebra: Constructive Methods: Finite Projective Modules***, Springer, Netherlands, Dordrecht, 2015.

[47] P. Lorenzen, *Über halbgeordnete Gruppen*. ***Mathematische Zeitschrift***, vol. 52 (1950), no. 1, pp. 483–526.

[48] ———, *Algebraische und logistische Untersuchungen über freie Verbände*. ***The Journal of Symbolic Logic***, vol. 16 (1951), no. 2, pp. 81–106.

[49] ———, *Teilbarkeitstheorie in Bereichen*. ***Mathematische Zeitschrift***, vol. 55 (1952), no. 3, pp. 269–275.

[50] ———, *Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen*. ***Mathematische Zeitschrift***, vol. 58 (1953), no. 1, pp. 15–24.

[51] M. McKubre-Jordens, *Material implications over minimal logic* (*joint work with Hannes Diener*), Conference Presentation, May 2016, Mathematics for Computation, Benediktinerabtei Niederaltaich, Germany, 8–13 May 2016.

[52] R. Mines, F. Richman, and W. Ruitenburg, ***A Course in Constructive Algebra***, Universitext, Springer, New York, 1988.

[53] C. J. Mulvey and J. Wick-Pelletier, *The dual locale of a seminormed space*. ***Cahiers de topologie et géométrie différentielle catégoriques***, vol. 23 (1982), no. 1, pp. 73–92.

[54] ———, *A globalization of the Hahn-Banach theorem*. ***Advances in Mathematics***, vol. 89 (1991), pp. 1–59.

[55] S. Negri, *Stone bases alias the constructive content of Stone representation*, ***Logic and Algebra.*** (A. Ursini and P. Aglianò, editors), Lecture Notes in Pure and Applied Mathematics, vol. 180, Marcel Dekker, New York, 1996, pp. 617–636.

[56] S. Negri, *Continuous domains as formal spaces*. ***Mathematical Structures in Computer Science***, vol. 12 (2002), no. 1, pp. 19–52.

[57] S. Negri and J. von Plato, ***Proof Analysis***, Cambridge University Press, Cambridge, 2011.

[58] S. Negri, J. von Plato, and T. Coquand, *Proof-theoretical analysis of order relations*. ***Archive for Mathematical Logic***, vol. 43 (2004), pp. 297–309.

[59] G. Payette and P. K. Schotch, *Remarks on the Scott–Lindenbaum theorem*. ***Studia Logica***, vol. 102 (2014), no. 5, pp. 1003–1020.

[60] A. Prestel and C. N. Delzell, ***Positive Polynomials. From Hilbert's 17th Problem to Real Algebra***, Springer-Verlag, Berlin, 2001.

[61] J.-C. Raoult, *Proving open properties by induction*. ***Information Processing Letters***, vol. 29 (1988), no. 1, pp. 19–23.

[62] D. Rinaldi, ***Formal Methods in the Theories of Rings and Domains***, Doctoral dissertation, Universität München, 2014.

[63] D. Rinaldi and P. Schuster, *A universal Krull-Lindenbaum theorem*. ***Journal of Pure and Applied Algebra***, vol. 220 (2016), pp. 3207–3232.

[64] G. Sambin, *Intuitionistic formal spaces—a first communication*, ***Mathematical Logic and its Applications*** (D. Skordev, editor), Plenum, New York, 1987, pp. 187–204.

[65] P. Schuster, *Induction in algebra: A first case study*, ***27th Annual ACM/IEEE Symposium on Logic in Computer Science***, IEEE Computer Society Publications, 2012, Proceedings, LICS 2012, Dubrovnik, Croatia, pp. 581–585.

[66] ———, *Induction in algebra: A first case study*. ***Logical Methods in Computer Science***, vol. 9 (2013), no. 3, p. 20.

[67] H. Schwichtenberg and C. Senjak, *Minimal from classical proofs*. ***Annals of Pure and Applied Logic***, vol. 164 (2013), pp. 740–748.

[68] D. Scott, *On engendering an illusion of understanding*. ***Journal of Philosophy***, vol. 68 (1971), pp. 787–807.

[69] D. Scott, *Completeness and axiomatizability in many-valued logic*, ***Proceedings of the Tarski Symposium*** (L. Henkin, J. Addison, C. C. Chang, W. Craig, D. Scott, and R. Vaught, editors), American Mathematical Society, Providence, RI, 1974, pp. 411–435.

[70] D. S. Scott, *Background to formalization*, ***Truth, Syntax and Modality*** (H. Leblanc, editor), Studies in Logic and the Foundations of Mathematics, vol. 68, North-Holland, Amsterdam, 1973, pp. 244–273.

[71] J. R. Shoenfield, ***Mathematical Logic***, Addison-Wesley, Reading, MA, 1967.

[72] D. J. Shoesmith and T. J. Smiley, ***Multiple-Conclusion Logic***, Cambridge University Press, Cambridge, 1978.

[73] E. Szpilrajn, *Sur l'extension de l'ordre partiel*. ***Fundamenta Mathematicae***, vol. 16 (1930), pp. 368–389.

[74] A. Tarski, *Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I*. ***Monatshefte für Mathematik und Physik***, vol. 37 (1930), pp. 361–404.

[75] A. S. Troelstra and H. Schwichtenberg, ***Basic Proof Theory***, second ed., Cambridge University Press, Cambridge, 2000.

[76] B. van den Berg, *Non-deterministic inductive definitions*. ***Archive for Mathematical Logic***, vol. 52 (2013), no. 1–2, pp. 113–135.

[77] R. Wójcicki, ***Theory of Logical Calculi. Basic Theory of Consequence Operations***, Synthese Library, vol. 199, Kluwer Academic Publishers Group, Dordrecht, 1988.

[78] I. Yengui, ***Constructive Commutative Algebra. Projective Modules Over Polynomial Rings and Dynamical Gröbner Bases***, Lecture Notes in Mathematics, vol. 2138, Springer International Publishing, Switzerland, 2015.

DIPARTIMENTO DI INFORMATICA
UNIVERSITÀ DEGLI STUDI DI VERONA
STRADA LE GRAZIE, 15
37134 VERONA, ITALY
*E-mail*: davide.rinaldi@univr.it
*E-mail*: peter.schuster@univr.it

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI TRENTO
VIA SOMMARIVE, 14
38123 POVO (TN), ITALY
*E-mail*: daniel.wessel@unitn.it

https://doi.org/10.1017/bsl.2017.13 Published online by Cambridge University Press