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DUALITY OF (2, 3, 5)-DISTRIBUTIONS AND LAGRANGIAN CONE STRUCTURES

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Abstract. As was shown by a part of the authors, for a given $(2, 3, 5)$-distribution D on a five-dimensional manifold Y, there is, locally, a Lagrangian cone structure C on another five-dimensional manifold X which consists of abnormal or singular paths of (Y, D). We give a characterization of the class of Lagrangian cone structures corresponding to $(2, 3, 5)$-distributions. Thus, we complete the duality between $(2, 3, 5)$-distributions and Lagrangian cone structures via pseudo-product structures of type G_2. A local example of nonflat perturbations of the global model of flat Lagrangian cone structure which corresponds to $(2, 3, 5)$-distributions is given.

§1. Introduction

A distribution D on a five-dimensional manifold Y is called a $(2, 3, 5)$-*distribution* if there is a local section η_1, η_2 of D such that

$$\eta_1, \eta_2, [\eta_1, \eta_2], [\eta_1, [\eta_1, \eta_2]], [\eta_2, [\eta_1, \eta_2]]$$

form a local frame of the tangent bundle to Y, in other words, if D has the weak growth $(2, 3, 5)$, namely, if $\mathrm{rank}(\partial\mathcal{D}) = 3$ and $\mathrm{rank}(\partial^{(2)}\mathcal{D}) = 5$, where $\partial\mathcal{D} := [\mathcal{D}, \mathcal{D}]$ $(= \mathcal{D} + [\mathcal{D}, \mathcal{D}])$, the derived system, and $\partial^{(2)}\mathcal{D} := [\mathcal{D}, \partial\mathcal{D}]$ $(= \mathcal{D} + \partial\mathcal{D} + [\mathcal{D}, \partial\mathcal{D}])$ for the sheaf $\mathcal{D}$ of section germs to D.

The geometry and classification problem of $(2, 3, 5)$-distributions are studied after Cartan [13], related to the simple Lie group G_2, by many mathematicians [5, 9, 19, 26–30]. The $(2, 3, 5)$-distributions are related to many problems, for instance, to the problem of "rolling balls" [1, 6–8], to indefinite conformal metrics [19, 21], to nonlinear differential equations [22], and so on.

In [16–18], we studied the global duality of G_2-homogeneous (flat) $(2, 3, 5)$-distribution and a Lagrangian cone structure from Cayley's split Octonions and classified the related generic singularities. In [15], we associated locally with any given $(2, 3, 5)$-distribution D on a five-dimensional manifold Y, a Lagrangian cone structure C on another five-dimensional manifold X, which consists of abnormal or singular paths of (Y, D), in the sense of sub-Riemannian geometry or geometric control theory (see [2, 20]). Moreover, it was shown in [15] that the original space Y turns to be the totality of singular paths of the "Lagrangian cone structure" (X, C), when the cone field C is regarded as a control system on X.

In this paper, we give the characterization of the class of Lagrangian cone structures corresponding to $(2, 3, 5)$-distributions, and, thus, we complete the duality between $(2, 3, 5)$-distributions and Lagrangian cone structures (Theorem 3.1). The duality is actually

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understood via pseudo-product structure of G_2-type $E = K \oplus L$ on a six-dimensional manifold Z (Section 2), which is regarded both as the prolongation of (Y, D) and (X, C) in the sense of Bryant [9, 10], via the double fibration

$$(Y, D) \xleftarrow{\pi_Y} (Z, E) \xrightarrow{\pi_X} (X, C).$$

We realize, for the characterization of a class of Lagrangian cone structures, that the language of cone structures is actually lacking. We introduce, regarding the cone structures as control systems, the notions of linear approximations and osculating bundles of cone structures as well as the exact definition of nondegenerate Lagrangian cone structures (Definition 2.3).

We remark that our correspondence is purely local in nature. It is "spatially" local for (Z, E) while "spatially and directionally" local for (Y, D) and for (X, C). Moreover, the "directional locality" for the distribution (Y, D) is resolved by taking the linear hull; however, it is not the case for the cone structure (X, C). This fact makes our duality delicate.

It is clear that $(2, 3, 5)$-distributions form an open set, for Whitney C^∞-topology, in the space of all distributions of rank 2 on a five-dimensional manifold. In particular, a $(2, 3, 5)$-distribution remains a $(2, 3, 5)$-distribution by sufficiently small perturbations with compact supports. However, such a stability for cone structures corresponding to $(2, 3, 5)$-distributions via the duality is not clear. We give a local example of nonflat perturbations of the global model of flat Lagrangian cone structure [16], which corresponds to $(2, 3, 5)$-distributions (Example 4.3). The existence of nonflat *global* perturbations of Lagrangian cone structures, which correspond to $(2, 3, 5)$-distributions, is open. The classification of nondegenerate Lagrangian cone structures based on their symmetries is an interesting open problem, regarding our duality and the studies on G_2-contact structures [12, 19, 26]. The direct study on symmetries of nondegenerate Lagrangian cone structures should be desirable.

The cone structure was first given in [5] by a foliation on the space $P((\partial D)^\perp) \subset P(T^*Y)$ for the derived system ∂D of a $(2, 3, 5)$-distribution D, which is an essentially same foliation in the space $P(D) \subset P(TY)$ of [15]. See also [5, 14]. In fact, there exists the natural fiber-preserving diffeomorphism $P(D) \to P((\partial D)^\perp)$ which preserves also the foliation induced from singular paths of D. Moreover, the Lagrangian cone structure $C \subset TX$, which is contained in a contact structure $D' \subset TX$ on X, has the essentially same information with the Jacobi curves introduced in [3, 4]. In fact, each cone $C_x \subset D'_x, (x \in X)$ gives the (reduced) Jacobi curve associated with the singular path x of D in Lagrangian Grassmannian of D'_x by taking tangent planes to C_x.

In [29], it was shown that the Cartan tensor of any $(2, 3, 5)$-distribution is given by the fundamental invariant of Jacobi curves of singular paths and, in particular, the Cartan tensor is determined by the projective equivalence classes of the pointwise curves $P(C_x), x \in X$ of the corresponding Lagrangian cone structure (X, C). We give a short proof (Proposition 4.1), related to the study on G_2-contact structures [12, 19], that the $(2, 3, 5)$-distribution which corresponds to a cubic Lagrangian cone structure via our duality is necessarily flat, by using Zelenko's theorem [29] (Proposition 4.1). Since the degrees of cone structures are invariant under isomorphisms of cone structures and by Theorem 3.1 of the present paper, we see that any cone structure which corresponds to a flat $(2, 3, 5)$ structure must be cubic. Then we can say that, to check the flatness of a $(2, 3, 5)$-distribution is easier, if it is given by a corresponding Lagrangian cone structure. In fact, the condition

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$\partial(T_sC) \subset O_s^{(2)}C$ of Theorem 3.1 is checked by straightforward computations of differentials and then it is sufficient to see whether the degree of the cone is cubic or not. However, it is a difficult task, given a $(2, 3, 5)$-distribution, to get the corresponding Lagrangian cone structure concretely.

In Section 2, we review the results given in the previous paper [15] with additional explanations. In particular, we give the exact definition of (nondegenerate) Lagrangian cone structures (Definition 2.3).

In Section 3, we complete the duality between $(2, 3, 5)$-distributions and nondegenerate Lagrangian cone structures with an additional condition via pseudo-product structures of type G_2.

We conclude this paper by several remarks related to the duality in Section 4.

All manifolds and mappings are supposed to be of class C^∞ unless otherwise stated.

§2. Pseudo-product structures of G_2-type

Let D be a $(2, 3, 5)$-distribution on a five-dimensional manifold Y. Let $Z := P(D) = (D - 0)/\mathbb{R}^\times$ be the space of tangential lines in D, $Z := \{(y, \ell) \mid y \in Y, \ell \subset D_y (\subset T_yY), \dim(\ell) = 1\}$. Then $\dim(Z) = 6$ and the projection $\pi_Y : Z \to Y$ is an $\mathbb{R}P^1$-bundle.

We define a subbundle $E \subset TZ$ of rank 2, *Cartan prolongation* of $D \subset TY$, by setting for each $(y, \ell) \in Z$, $\ell \subset D_y$, $E_{(y,\ell)} := \pi_{Y*}^{-1}(\ell)$ $(\subset T_{(y,\ell)}Z)$. Then E is a distribution with (weak) growth $(2, 3, 4, 5, 6)$: $\mathrm{rank}(E) = 2$, $\mathrm{rank}(\partial E) = 3$, $\mathrm{rank}(\partial^{(2)}E) = 4$, $\mathrm{rank}(\partial^{(3)}E) = 5$, $\mathrm{rank}(\partial^{(4)}E) = 6$.

Then we see that there exists an *intrinsic* decomposition

$$E = K \oplus L$$

of E with $L := \mathrm{Ker}(\pi_{Y*}) \subset E$ and a complementary line subbundle K of E, a *pseudo-product structure* in the sense of Tanaka [24, 25].

We will explain this in terms of "geometric control theory" [2, 20].

A *control system* $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \to M$ on a manifold M is given by a locally trivial fibration $\pi_{\mathcal{U}} : \mathcal{U} \to M$ over M and a map $F : \mathcal{U} \to TM$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & TM \\ & \pi_{\mathcal{U}} \searrow \quad \swarrow \pi_{TM} & \\ & M & \end{array}$$

Any section $s : M \to \mathcal{U}$ defines a vector field $F \circ s : M \to TM$ over M. Via a local triviality on M, a control system is given by a family of vector fields $f_u(x) = F(x, u)$ over M, $(x, u) \in \mathcal{U}$, $x \in M$.

A distribution $D \subset TM$ is regarded as a control system $\mathbb{D} : D \hookrightarrow TM \longrightarrow M$, by the inclusion.

Two control systems $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \xrightarrow{\pi_{TM}} M$ and $\mathbb{C}' : \mathcal{U}' \xrightarrow{F'} TM' \xrightarrow{\pi_{TM'}} M'$ are called *isomorphic* if the diagram

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{F} & TM & \xrightarrow{\pi_{TM}} & M \\ \psi \downarrow & & \varphi_* \downarrow & & \downarrow \varphi \\ \mathcal{U}' & \xrightarrow{F'} & TM' & \xrightarrow{\pi_{TM'}} & M' \end{array}$$

commutes for some diffeomorphisms ψ and φ. Here, φ_* is the differential of φ.

The pair (ψ, φ) of diffeomorphisms is called an *isomorphism* of the control systems $\mathbb{C}$ and $\mathbb{C}'$.

Given a control system $\mathbb{C}: \mathcal{U} \xrightarrow{F} TM \to M$, an L^∞ (measurable, essentially bounded) map $c: [a, b] \to \mathcal{U}$ is called an *admissible control* if the curve

$$\gamma := \pi_{\mathcal{U}} \circ c : [a, b] \to M$$

satisfies the differential equation

$$\dot{\gamma}(t) = F(c(t)) \quad (\text{a.e. } t \in [a, b]).$$

Then the Lipschitz curve γ is called a *trajectory*. If we write $c(t) = (x(t), u(t))$, then $x(t) = \gamma(t)$ and

$$\dot{x}(t) = F(x(t), u(t)), \quad (\text{a.e. } t \in [a, b]).$$

We use the term "path" for a smooth (C^∞) immersive trajectory regarded up to parametrization.

The totality $\mathcal{C}$ of admissible controls $c: [a, b] \to \mathcal{U}$ with a given initial point $q_0 \in M$ is a Banach manifold. The *endpoint mapping* $\mathrm{End}: \mathcal{C} \to M$ is defined by

$$\mathrm{End}(c) := \pi_{\mathcal{U}} \circ c(b).$$

An admissible control $c: [a, b] \to \mathcal{U}$ with the initial point $\pi_{\mathcal{U}}(c(a)) = q_0$ is called *singular* or *abnormal*, if $c \in \mathcal{C}$ is a singular point of End, namely if the differential $\mathrm{End}_*: T_c\mathcal{C} \to T_{\mathrm{End(c)}}M$ is not surjective. If c is a singular control, then the trajectory $\gamma = \pi_{\mathcal{U}} \circ c$ is called a *singular trajectory* or an *abnormal extremal*.

Let $D \subset TY$ be a $(2, 3, 5)$-distribution. Then, it can be shown that for any point y of Y and for any direction $\ell \subset D_y$, there exists uniquely a *singular D-path* (an immersed abnormal extremal for D) through y with the given direction ℓ. Thus, the singular D-paths form another five-dimensional manifold X.

Let $Z = P(D) = (D - 0)/\mathbb{R}^\times$ be the space of tangential lines in D, $\dim(Z) = 6$. Then Z is naturally foliated by the liftings of singular D-paths, and we have locally double fibrations:

$$Y \xleftarrow{\pi_Y} Z \xrightarrow{\pi_X} X.$$

If we put $L = \mathrm{Ker}(\pi_{Y*})$, $K = \mathrm{Ker}(\pi_{X*})$, then we have a decomposition $E = K \oplus L$ by subbundles of rank 1.

We denote, for any distribution E, by $\mathcal{E}$ the sheaf of local sections to E. We set

$$\partial\mathcal{E} := [\mathcal{E}, \mathcal{E}] = \mathcal{E} + [\mathcal{E}, \mathcal{E}], \qquad \partial^{(2)}\mathcal{E} := [\mathcal{E}, \partial\mathcal{E}] = \mathcal{E} + \partial\mathcal{E} + [\mathcal{E}, \partial\mathcal{E}]$$

and so on. If $\partial\mathcal{E}$ is generated by a local section of a distribution, then we denote it by ∂E.

DEFINITION 2.1. A distribution (Z, E) of rank 2 on a six-dimensional manifold Z with a decomposition $E = K \oplus L$ by subbundles K, L of rank 1 is called a pseudo-product structures of G_2-type if E has small growth $(2, 3, 4, 5, 6)$ and, moreover, satisfies that

$$[\mathcal{K}, \mathcal{L}] = \partial\mathcal{E}, \qquad [\mathcal{K}, \partial\mathcal{E}] = \partial^{(2)}\mathcal{E}, [\mathcal{L}, \partial\mathcal{E}] = \partial\mathcal{E},$$
$$[\mathcal{K}, \partial^{(2)}\mathcal{E}] = \partial^{(3)}\mathcal{E}, \qquad [\mathcal{L}, \partial^{(2)}\mathcal{E}] = \partial^{(2)}\mathcal{E}, \qquad [\mathcal{K}, \partial^{(3)}\mathcal{E}] = \partial^{(3)}\mathcal{E}, \qquad [\mathcal{L}, \partial^{(3)}\mathcal{E}] = \partial^{(4)}\mathcal{E}.$$

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Then, by taking the gradation of the filtration

$$\mathcal{E} \subset \partial\mathcal{E} \subset \partial^{(2)}\mathcal{E} \subset \partial^{(3)}\mathcal{E} \subset \partial^{(4)}\mathcal{E},$$

we have, at each point $z \in Z$, the symbol algebra

$$\mathfrak{m} = \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \langle e_6\rangle \oplus \langle e_5\rangle \oplus \langle e_4\rangle \oplus \langle e_3\rangle \oplus \langle e_1, e_2\rangle,$$
$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4, \qquad [e_2, e_3] = 0, \qquad [e_1, e_4] = e_5,$$
$$[e_2, e_4] = 0, \qquad [e_1, e_5] = 0, \qquad [e_2, e_5] = e_6,$$

with the decomposition $\mathfrak{g}_{-1} = \mathfrak{k} \oplus \mathfrak{l} = \langle e_1\rangle \oplus \langle e_2\rangle$.

Then we have the following.

THEOREM 2.2. *There exists a natural bijective correspondence of local isomorphism classes between* $(2, 3, 5)$*-distributions and pseudo-product structures of* G_2*-type.*

Proof. First, let us make sure that the prolongation E of a $(2, 3, 5)$-distribution D on a five-dimensional manifold Y has small growth $(2, 3, 4, 5, 6)$.

Let η_1, η_2 be a local frame of D. Then, setting

$$\eta_3 := [\eta_1, \eta_2], \qquad \eta_4 := [\eta_1, \eta_3], \qquad \eta_5 := [\eta_2, \eta_3],$$

we have a local frame $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ of TY. For each $y \in Y$, directions in D_y are, locally, parametrized via $\eta_1(y) + t\eta_2(y)$ $(t \in \mathbb{R})$. Then, for any system of local coordinates $y = (y_1, y_2, y_3, y_4, y_5)$ of Y centered at base point of Y, (y, t) form a system of local coordinates of Z such that π_Y is expressed by $(y, t) \mapsto y$. We regard $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ as vector fields over Z. Then

$$\zeta_1 := \eta_1 + t\eta_2 \qquad \zeta_2 := \frac{\partial}{\partial t}$$

form a local frame of E, and $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \zeta_2$ of TZ.

Since $[\zeta_1, \zeta_2] = [\eta_1 + t\eta_2, \zeta_2] = -\eta_2$, we have

$$\partial\mathcal{E} = \langle\zeta_1, \zeta_2, \eta_2\rangle = \langle\eta_1, \eta_2, \zeta_2\rangle,$$

which is of rank 3. Here, $\langle\zeta_1, \zeta_2, \eta_2\rangle$ means the distribution generated by ζ_1, ζ_2, η_2. Since $[\zeta_1, \eta_2] = [\eta_1 + t\eta_2, \eta_2] = \eta_3$ and $[\zeta_2, \eta_2] = 0$, we have

$$\partial^{(2)}\mathcal{E} = \langle\eta_1, \eta_2, \eta_3, \zeta_2\rangle,$$

which is of rank 4. Since $[\zeta_1, \eta_3] = [\eta_1 + t\eta_2, \eta_3] = \eta_4 + t\eta_5$ and $[\zeta_2, \eta_3] = 0$, we have

$$\partial^{(3)}\mathcal{E} = \langle\eta_1, \eta_2, \eta_3, \eta_4 + t\eta_5, \zeta_2\rangle,$$

which is of rank 5. Since $[\zeta_2, \eta_4 + t\eta_5] = \eta_5$, we have $\partial^{(4)}\mathcal{E} = TZ$. Therefore, E has small growth $(2, 3, 4, 5, 6)$.

Note that $\mathcal{L}$ is generated by ζ_2. Moreover, there exists a generator of $\mathcal{K}$ of form $\zeta_1 + e(y, t)\zeta_2$. In fact, the function $e(y, t)$ is uniquely determined by the condition $[\mathcal{K}, \partial^{(3)}\mathcal{E}] = \partial^{(3)}\mathcal{E}$, which is equivalent to the condition

$$e\eta_5 + [\eta_1, \eta_4] + t[\eta_1, \eta_5] + t[\eta_2, \eta_4] + t^2[\eta_1, \eta_5] \equiv 0, \quad \text{mod. } \partial^{(3)}\mathcal{E}.$$

Then other remaining conditions that $E = K \oplus L$ is a pseudo-product structure of type G_2 follow.

Conversely, suppose $E = K \oplus L$ is a pseudo-product structure of type G_2. Then L is the Cauchy characteristic of ∂E (see [11]). Let Y be the leaf space of L, which is a locally defined five-dimensional manifold. Moreover, Z has a system of local coordinates (y, t) centered at the base point such that π_Y is given by $(y, t) \mapsto y$. Let D be the reduction of ∂E by L. Take a local frame η_1, η_2 of D such that, regarded as vector fields over Z, η_1 generates the quotient bundle $(\partial E)/E$. Moreover, $\zeta_1 = \eta_1 + \varphi(y, t)\eta_2$ and $\zeta_2 = \partial/\partial t$ generate K and L, respectively, for some function $\varphi(y, t)$ with $\varphi(0, 0) = 0$. Since

$$[\zeta_1, \zeta_2] = [\eta_1 + \varphi\eta_2, \zeta_2] = -(\partial\varphi/\partial t)\eta_2,$$

we have that $\partial\varphi/\partial t \neq 0$. Set $\zeta_3 := \eta_2$. Then

$$[\zeta_1, \zeta_3] = [\eta_1, \eta_2] + \eta_2(\varphi)\eta_2 \equiv [\eta_1, \eta_2] \mod. \partial\mathcal{E}.$$

Therefore, $\eta_1, \eta_2, [\eta_1, \eta_2]$ are linearly independent pointwise on Y. We set $\zeta_4 := \eta_3 = [\eta_1, \eta_2]$ as a vector field over Z. Then

$$[\zeta_1, \zeta_4] = [\eta_1, \eta_3] + \varphi[\eta_2, \eta_3] - \eta_3(\varphi)\eta_2 \equiv [\eta_1, \eta_3] + \varphi[\eta_2, \eta_3] \mod. \partial^{(2)}\mathcal{E},$$

and $[\zeta_2, \zeta_4] = [\partial/\partial t, \eta_3] = 0$. Set $\eta_4 = [\eta_1, \eta_3]$, $\eta_5 = [\eta_2, \eta_3]$ and $\zeta_5 = \eta_4 + \varphi\eta_5$. Then $\eta_4(0) \in (\partial^{(3)}E)_0 \setminus (\partial^{(2)}E)_0$. Then we have that $[\zeta_2, \zeta_5](0) \notin (\partial^{(3)}E)_0$, while $[\zeta_2, \zeta_5] = (\partial\varphi/\partial t)\eta_5(0)$. Therefore, $\eta_5(0) \notin (\partial^{(3)}E)_0$. Therefore, $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ are linearly independent pointwise. Thus, we see that D is a $(2, 3, 5)$-distribution.

These correspondences induce the bijection between local isomorphism classes of $(2, 3, 5)$-distributions and pseudo-product structures of G_2-type on a 5-manifold. □

Note that the original $(2, 3, 5)$-distribution D is obtained as the linear hull of the cone field ("bowtie") induced from K:

$$D_y = \text{linear hull}\left(\bigcup_{z \in \pi_Y^{-1}(y)} \pi_{Y*}(K_z) \subset T_yY\right).$$

Also, the $(2, 3, 5)$-distribution D is obtained as the reduction of ∂E by Cauchy characteristic $L = \text{Ker}(\pi_{Y*})$.

On the other hand, we obtain a cone field $C \subset TX$ on X by setting, for each $x \in X$,

$$(\triangle) \qquad C_x := \bigcup_{z \in \pi_X^{-1}(x)} \pi_{X*}(L_z) \subset T_xX.$$

Now, to make sure, we formulate exactly the notion of Lagrangian cone structures (see [12]).

DEFINITION 2.3. (1) Let X be a manifold of dimension m. A subset $C \subset TX$ is called a *cone structure* if there is an $\mathbb{R}^\times$-invariant subset $\underline{C} \subset \mathbb{R}^m$, a model cone, such that, for any $x \in X$, there exist an open neighborhood U of x and a local triviality $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ of $\pi : TX \to X$ over U satisfying $\Phi(\pi^{-1}(U) \cap C) = U \times \underline{C}$.

(2) Suppose that the model cone $\underline{C}$ is nonsingular away from the origin in $\mathbb{R}^m$. Then $P(C)$ is a submanifold of $P(TX)$. For each section $s : X \to P(C)$ for the projection $P(C) \to X$,

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we have the subbundle $T_sC \subset TX$ by taking tangent planes of C_x along the direction $s(x)$ at every point $x \in X$. We call the distribution T_sC the *linear approximation* of C along s.

(3) A cone structure $C \subset TX$ is called a *Lagrangian cone structure* if there exists a contact structure D' on X such that $C \subset D'$ and, for any section $s : X \to P(C)$, T_sC is a Lagrangian subbundle of D'. The last condition is equivalent to that, for any $x \in X$, $C_x \setminus \{0\}$ is a Lagrangian submanifold of the linear symplectic manifold D'_x or, equivalently, $P(C_x)$ is a Legendrian submanifold of the contact manifold $P(D'_x)$ induced from the conformal symplectic vector space D'_x.

(4) Let $\dim(X) = 5$. A Lagrangian cone structure $C \subset TX$ for a contact structure $D' \subset TX$ is called *nondegenerate* if the spatial projective curve segment $P(C_x) \subset P(D'_x) \cong P^3$ is nondegenerate, i.e. the first, second, and third derivatives of a parametrization of $P(C_x)$ are linearly independent.

From condition (4), for each direction field s of C, we define *osculating bundles* $O^{(2)}_s C \subset TX$ of rank 3 and $O^{(3)}_s C \subset TX$ of rank 4, generated by osculating planes O_2 and three-dimensional osculating spaces O_3 to $P(C_x)$ with direction s. Then the contact structure D' coincides with $O^{(3)}_s C$ which is independent of s.

Because distributions are regarded as cone structures of special type, the notion of Lagrangian cone structures is a natural generalization for that of Lagrangian subbundle of the tangent bundle over a contact manifold.

LEMMA 2.4. *In our case, the above $C \subset TX$ defined as ($\triangle$) corresponding to a $(2, 3, 5)$-distribution $D \subset TY$ is a nondegenerate Lagrangian cone structure in the sense of Definition 2.3.*

Proof. By the condition $[\mathcal{K}, \partial\mathcal{E}] = \partial^{(2)}\mathcal{E}$, C satisfies conditions (1) and (2) of Definition 2.3. By the condition $[\mathcal{K}, \partial^{(3)}\mathcal{E}] = \partial^{(3)}\mathcal{E}$, K is the Cauchy characteristic of $\partial^{(3)}\mathcal{E}$. Then the distribution $D' \subset TX$ induced from $\partial^{(3)}\mathcal{E}$ is a contact structure by the condition $[\mathcal{L}, \partial^{(3)}\mathcal{E}] = \partial^{(4)}\mathcal{E}$. Moreover, $\partial^{(2)}\mathcal{E}$ projects to tangent spaces to C_x along $\pi_X^{-1}(x)$. For any section $s : X \to L$, $s(x) \neq 0$, we have that the linear approximation T_sC is a Lagrangian subbundle of D' by the condition $[\mathcal{L}, \partial^{(3)}\mathcal{E}] = \partial^{(3)}\mathcal{E}$. Therefore, C satisfies also condition (3) of Definition 2.3. Thus, (X, C) is a Lagrangian cone structure. Moreover, by the condition $[\mathcal{K}, \partial^{(2)}\mathcal{E}] = \partial^{(3)}\mathcal{E}$, condition (4) of Definition 2.3 is satisfied. Therefore, (X, C) is a nondegenerate Lagrangian cone structure. ∎

Now, we regard the cone field $C \subset TX$ as a control system over X:

$$\mathbb{C} : L \xrightarrow{\pi_{X*}|_L} TX \to X,$$

for the subbundle L of TZ. Then we have shown in [15] the following theorem.

THEOREM 2.5. (Duality theorem [15]) *Singular paths of the control system*

$$\mathbb{C} : L \xrightarrow{\pi_{X*}|_L} TX \to X$$

are given by π_X-images of π_Y-fibers.

Therefore, for any $x \in X$ and for any direction $\ell \subset C_x$, there exists uniquely a singular $\mathbb{C}$-path passing through x with the direction ℓ at x.

Thus, the original space Y is identified with the space of singular paths for (X, C), while X is the space of singular paths for (Y, D).

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We recall the local characterization of singular controls.

For a control system $\mathbb{C}: \mathcal{U} \xrightarrow{F} TM \to M$ on a manifold M, we consider the fiber product $\mathcal{U} \times_M T^*M$ and define the *Hamiltonian function* $H: \mathcal{U} \times_M T^*M \to \mathbb{R}$ of the control system $F: \mathcal{U} \to TM$ by

$$H(x,p,u) := \langle p, F(x,u) \rangle, \qquad ((x,u),(x,p)) \in \mathcal{U} \times_M T^*M.$$

A singular control $(x(t), u(t))$ is characterized by the liftability to an *abnormal bi-extremal* $(x(t), p(t), u(t))$ satisfying the constrained Hamiltonian equation

$$\begin{cases} \dot{x}_i(t) = \dfrac{\partial H}{\partial p_i}(x(t), p(t), u(t)), \quad (1 \leqslant i \leqslant m) \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial x_i}(x(t), p(t), u(t)), \quad (1 \leqslant i \leqslant m) \\ \dfrac{\partial H}{\partial u_j}(x(t), p(t), u(t)) = 0, \quad (1 \leqslant j \leqslant r), p(t) \neq 0. \end{cases}$$

Let $E \subset TZ$ be a distribution on a manifold Z regarded as a control system. A singular path $x(t)$ for $E \subset TZ$ is called *regular singular* if it is associated with an abnormal bi-extremal $(x(t), p(t), u(t))$ such that $p(t) \in (\partial E)^\perp \setminus (\partial^{(2)} E)^\perp \subset T^*Z$. A singular path $x(t)$ for $E \subset TZ$ is called *totally irregular singular* if any associated abnormal bi-extremal $(x(t), p(t), u(t))$ satisfies that $p(t) \in (\partial^{(2)} E)^\perp \subset T^*Z$.

From the pseudo-product structure on $E \subset TZ$, we have the following.

THEOREM 2.6. (Asymmetry theorem [15]) *A singular path for $E \hookrightarrow TZ \to Z$ is either a π_Y-fiber or a π_X-fiber. Each π_Y-fiber is regular singular, while each π_X-fiber is totally irregular singular.*

§3. Complete duality

The description of the duality on $(2,3,5)$-distributions (Y, D) and nondegenerate Lagrangian cone structures (X, C) via (Z, E) which is given in Section 2 should be completed by answering the question: *What kinds of nondegenerate Lagrangian cone structures do they correspond to $(2,3,5)$-distributions?*

Then we have the following.

THEOREM 3.1. *There exist natural bijective correspondences of isomorphism classes:*

$$\{(2,3,5)\text{-}\textit{distributions } (Y,D)\}/\cong$$

$$\longleftrightarrow \left\{ \begin{array}{c} \textit{pseudo-product structures of } G_2\textit{-type } (Z,E): \\ (2,3,4,5,6)\textit{-distributions } E \textit{ with a decomposition} \\ E = K \oplus L, \operatorname{rank}(K) = \operatorname{rank}(L) = 1, \\ [\mathcal{K}, \mathcal{L}] = \partial\mathcal{E} \ (:= [\mathcal{E}, \mathcal{E}] = \mathcal{E} + [\mathcal{E}, \mathcal{E}]), \\ [\mathcal{K}, \partial\mathcal{E}] = \partial^{(2)}\mathcal{E}, [\mathcal{L}, \partial\mathcal{E}] = \partial\mathcal{E}, \\ [\mathcal{K}, \partial^{(2)}\mathcal{E}] = \partial^{(3)}\mathcal{E}, [\mathcal{L}, \partial^{(2)}\mathcal{E}] = \partial^{(2)}\mathcal{E}, \\ [\mathcal{K}, \partial^{(3)}\mathcal{E}] = \partial^{(3)}\mathcal{E}, [\mathcal{L}, \partial^{(3)}\mathcal{E}] = \partial^{(4)}\mathcal{E}. \end{array} \right\} / \cong$$

$$\longleftrightarrow \left\{ \begin{array}{c} \textit{nondegenerate Lagrangian cone structures } (X,C) \\ \textit{on five-dimensional manifolds } X \textit{ with the condition} \\ \partial(T_sC) \subset O_s^{(2)}C, \ \textit{for any direction field } s \textit{ of } C. \end{array} \right\} / \cong$$

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Proof of Theorem 3.1. Let X be a five-dimensional manifold and $C \subset TX$ a nondegenerate Lagrangian cone structure (Definition 2.3). Then $Z = P(C) := (C \setminus \text{(zero section)})/\mathbb{R}^{\times}$ is a six-dimensional manifold and $\pi_X : Z \to X$ is a C^∞-fibration with projective curves $P(C_x) \subset P(T_x X) \cong P^4$ as fibers.

By the nondegeneracy condition, we have that the first, second, and third derivatives are linearly independent everywhere on $P(C_x)$, for any $x \in X$.

Then we define a subbundle $E \subset TZ$ of rank 2 by setting

$$E_{(x,\ell)} := (\pi_X)_*^{-1}(\ell),$$

for each $(x, \ell) \in Z$ as the prolongation of the cone structure $C \in TX$. We set $K = \mathrm{Ker}((\pi_X)_*)$.

Let $x = (x_1, x_2, x_3, x_4, x_5)$ be a system of local coordinates of X and x, θ that of Z such that $\pi_X : Z \to X$ is given by $(x, \theta) \mapsto x$ and E is generated by $\zeta_1 = \frac{\partial}{\partial \theta}$ and a vector field $\zeta_2(x, \theta)$ of form

$$\zeta_2(x, \theta) = \frac{\partial}{\partial x_1} + A\frac{\partial}{\partial x_2} + B\frac{\partial}{\partial x_3} + S\frac{\partial}{\partial x_4} + T\frac{\partial}{\partial x_5},$$

where A, B, S, T are function germs of x, θ. The projective curve $C_x \subset P(T_x X)$ is given by

$$\theta \mapsto [1 : A(x, \theta) : B(x, \theta) : S(x, \theta) : T(x, \theta)]$$

in homogeneous coordinates, for each $x \in X$.

We have, on Z,

$$[\zeta_1, \zeta_2](x, \theta) = \frac{\partial \zeta_2}{\partial \theta}(x, \theta) =: \zeta_3$$

and

$$[\zeta_1, \zeta_3](x, \theta) = \frac{\partial^2 \zeta_2}{\partial \theta^2}(x, \theta) =: \zeta_4.$$

In local coordinates,

$$\zeta_3 = A_\theta \frac{\partial}{\partial x_2} + B_\theta \frac{\partial}{\partial x_3} + S_\theta \frac{\partial}{\partial x_4} + T_\theta \frac{\partial}{\partial x_5}, \qquad \zeta_4 = A_{\theta\theta} \frac{\partial}{\partial x_2} + B_{\theta\theta} \frac{\partial}{\partial x_3} + S_{\theta\theta} \frac{\partial}{\partial x_4} + T_{\theta\theta} \frac{\partial}{\partial x_5},$$

and

$$\zeta_5 = A_{\theta\theta\theta} \frac{\partial}{\partial x_2} + B_{\theta\theta\theta} \frac{\partial}{\partial x_3} + S_{\theta\theta\theta} \frac{\partial}{\partial x_4} + T_{\theta\theta\theta} \frac{\partial}{\partial x_5}.$$

Any direction field s of C is given by $x \mapsto (x, \theta(x))$ for some functions $\theta(x)$ of x and the linear approximation $T_s C$ of C along the direction field s is generated by $\zeta_2(x, \theta(x))$, $\frac{\partial \zeta_2}{\partial \theta}(x, \theta(x))$. Moreover, the osculating bundles $O_s^{(2)}C$ and $O_s^{(3)}C$ are generated by $\zeta_2(x, \theta(x))$, $\frac{\partial \zeta_2}{\partial \theta}(x, \theta(x))$, $\frac{\partial^2 \zeta_2}{\partial \theta^2}(x, \theta(x))$ and by $\zeta_2(x, \theta(x))$, $\frac{\partial \zeta_2}{\partial \theta}(x, \theta(x))$, $\frac{\partial^2 \zeta_2}{\partial \theta^2}(x, \theta(x))$, $\frac{\partial^3 \zeta_2}{\partial \theta^3}$ $(x, \theta(x))$, respectively.

By the condition $\partial(T_s C) \subset O_s^{(2)}C$, we have that $[\zeta_2, \zeta_3] \equiv 0, \mathrm{mod.}\ \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle$. Then there exists uniquely a function $U(x, \theta)$ such that $\widetilde{\zeta}_2 = \zeta_2 + U\zeta_1$ is the Cauchy characteristic vector field of ∂E so that $[\widetilde{\zeta}_2, \zeta_3] \equiv 0, \langle \mathrm{mod.}\ \zeta_1, \zeta_2, \zeta_3 \rangle$.

Taking the subbundle $L \subset E$ generated by $\widetilde{\zeta}_2$, we have a pseudo-product structure $E = K \oplus L$ on Z satisfying the conditions

$$[\mathcal{K}, \mathcal{L}] = \partial \mathcal{E}, \qquad [\mathcal{K}, \partial \mathcal{E}] = \partial^{(2)} \mathcal{E}, \qquad [\mathcal{L}, \partial \mathcal{E}] = \partial \mathcal{E}, \qquad [\mathcal{K}, \partial^{(2)} \mathcal{E}] = \partial^{(3)} \mathcal{E}.$$

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By Jacobi identity, $[\widetilde{\zeta}_2, [\zeta_1, \zeta_3]] + [\zeta_1, [\zeta_3, \widetilde{\zeta}_2] + [\zeta_3, [\widetilde{\zeta}_2, \zeta_1]] = 0$, we have that

$$[\widetilde{\zeta}_2, \zeta_4] \equiv [\zeta_1, [\widetilde{\zeta}_2, \zeta_3]] \equiv 0, \quad \text{mod. } \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle.$$

Therefore, the condition $[\mathcal{L}, \partial^{(2)}\mathcal{E}] = \partial^{(2)}\mathcal{E}$ is satisfied. Since $O_s^{(3)}C \subset TX$ is independent of s and is a contact structure on X, we have that $[\mathcal{K}, \partial^{(3)}\mathcal{E}] = \partial^{(3)}\mathcal{E}$ and that $[\mathcal{L}, \partial^{(3)}\mathcal{E}]$ generates the total tangent bundle TZ. Thus, the last condition $[\mathcal{L}, \partial^{(3)}\mathcal{E}] = \partial^{(4)}\mathcal{E}$ holds.

Consequently, if C is a nondegenerate Lagrangian cone structure with the condition that $\partial(T_sC) \subset O_s^{(2)}C$ for any direction field s of C, then $E = K \oplus L$ is a pseudo-product structure of G_2-type.

This completes the proof of Theorem 3.1. ∎

REMARK 3.2. The cone structure $C \subset TX$ is regarded as the control system over X,

$$\mathbb{C} : L \to TX \to X, \qquad L \ni ((x, \ell), v) \mapsto (x, v) \mapsto x,$$

with 2-control parameters. In local coordinates, the control system $\mathbb{C}$ is given by

$$F(x; r, \theta) := r\left(\frac{\partial}{\partial x_1} + A(x, \theta)\frac{\partial}{\partial x_2} + B(x, \theta)\frac{\partial}{\partial x_3} + S(x, \theta)\frac{\partial}{\partial x_4} + T(x, \theta)\frac{\partial}{\partial x_5}\right),$$

with the control parameters r, θ.

§4. $(2, 3, 5)$-distributions and cubic Lagrangian cone structures

Let us denote by G_2' the automorphism group of the split octonion algebra $\mathbb{O}'$. Then for a Borel group subgroup B and parabolic subgroups P_1, P_2 containing B of G_2', we have a double fibration

$$Y = G_2'/P_1 \xleftarrow{\pi_Y} Z = G_2'/B \xrightarrow{\pi_X} X = G_2'/P_2,$$

a $(2, 3, 5)$-distribution $D \subset TY$ on Y, a pseudo-product structure of type G_2 as $E = K \oplus L \subset TZ$ on Z, and a nondegenerate Lagrangian cubic cone structure $C \subset TX$ (see [16]). It is known also that Y is diffeomorphic to $S^3 \times S^2$ (resp. Z to $S^3 \times S^3$, X to $S^2 \times S^3$). On each of three places, there exists Cartan's parabolic geometry as a natural nonflat geometry modeled on the homogeneous space. On Y, it is the geometry of $(2, 3, 5)$-distributions. On Z, it is the geometry of pseudo-product structures of type G_2. On X, it is G_2-contact structures [12, 19]. Moreover, any G_2-contact structure is accompanied with and is recovered from a nondegenerate Lagrangian cubic cone structure.

Sato [23] has suggested to the first author that any G_2-contact structure corresponding to a $(2, 3, 5)$-distribution should be flat, from the exact comparison of curvatures for associated Cartan connections on pseudo-product G_2-structure and on G_2-contact structures [24, 27]. Here we would like to provide alternative proof for the fact. In fact, we have the following.

PROPOSITION 4.1. *Any $(2, 3, 5)$-distribution (Y, D) which corresponds to a cubic cone structure (X, C) must be flat. Any Lagrangian cone structure which corresponds to a flat $(2, 3, 5)$-distributions must be cubic.*

Proof of Proposition 4.1. For each $x \in X$, the cone $C_x \subset D'_x (\subset T_xX)$ gives the (reduced) "Jacobi curve" in the sense of Agrachev and Zelenko [3–5, 29]. Then, in [29], it is proved that "Cartan tensor" of D is recovered by a projective invariant, the fundamental invariant, a kind of cross ratio, of $P(C_x)$ pointwise. In fact, for the cone $C_x \subset D_x \cong \mathbb{R}^4$, there is

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associated a curve $P(C_x)$ in Grassmannian $\mathrm{Gr}(2, \mathbb{R}^4)$, and the fundamental invariants is calculated from $P(C_x)$ in a projective invariant way.

Suppose a $(2,3,5)$-distribution D corresponds to a cubic cone structure $C \subset D' \subset TX$. Then the cone structure is nondegenerate. Since all nondegenerate cubic cones are projectively equivalent pointwise, the Cartan tensor of D coincides with the flat $(2,3,5)$-distribution. Therefore, D must be flat.

Suppose a Lagrangian cone structure (X, C) corresponds to a flat $(2,3,5)$-distribution (Y, D). The flat model (Y_0, D_0) has the standard cubic dual (X_0, C_0) as in [16]. Since $(Y, D) \equiv (Y_0, D_0)$, we see $(X, C) \equiv (X_0, C_0)$ by Theorem 3.1. Then C is cubic because the degree is invariant under the isomorphism of cone structures. □

EXAMPLE 4.2. (Cubic Lagrangian cone structures not corresponding to $(2,3,5)$-distributions) Consider a cubic cone structure C on $(\mathbb{R}^5, 0)$ around the direction $\theta = 0$,

$$F(x; r, \theta) = r\bigg(\frac{\partial}{\partial x_1} + \theta\frac{\partial}{\partial x_2} + (\theta^2 + a)\frac{\partial}{\partial x_3} + (\theta^3 - 3\theta a)\frac{\partial}{\partial x_4} + \{x_3\theta - 2x_2(\theta^2 + a) + x_1(\theta^3 - 3\theta a)\}\frac{\partial}{\partial x_5}\bigg),$$

defined by a C^∞ function $a(x_1)$ with $a(0) = 0$.

Then C is a nondegenerate Lagrangian cone structure for the contact structure D': $dx_5 - x_3 dx_2 + 2x_2 dx_3 - x_1 dx_4 = 0$. Moreover, C satisfies the condition $\partial(T_s C) \subset O_s^{(2)} C$ for any $s : X \to L \setminus \{0\}$, to correspond to a $(2,3,5)$-distribution, if and only if $a \not\equiv 0$. The case $a \equiv 0$ corresponds to the G_2-homogeneous flat case [16].

The following gives examples of nondegenerate Lagrangian noncubic cone structures which correspond to $(2,3,5)$-distributions and shows the necessity of the additional condition $\partial(T_s C) \subset O_s^{(2)} C$ of Theorem 3.1.

EXAMPLE 4.3. (Noncubic Lagrangian cone structures corresponding to $(2,3,5)$-distributions) Consider a cone structure on $(\mathbb{R}^5, 0)$ around the direction $\theta = 0$,

$$F(x; r, \theta) = r\bigg(\frac{\partial}{\partial x_1} + \theta\frac{\partial}{\partial x_2} + (\theta^2 + b)\frac{\partial}{\partial x_3} + (\theta^3 + c)\frac{\partial}{\partial x_4} + \{x_3\theta - 2x_2(\theta^2 + b) + x_1(\theta^3 + c)\}\frac{\partial}{\partial x_5}\bigg),$$

where $b = b(\theta), c = c(\theta)$, with $\mathrm{ord}_0\, b(\theta) \geqslant 3, \mathrm{ord}_0\, c(\theta) \geqslant 4$.

Then F is a nondegenerate Lagrangian cone structure for the contact structure D': $dx_5 - x_3 dx_2 + 2x_2 dx_3 - x_1 dx_4 = 0$. Moreover, F satisfies the condition $\partial(T_s C) \subset O_s^{(2)} C$, for any direction field s, to correspond to a $(2,3,5)$-distribution, if and only if $c_\theta = 3\theta b_\theta - 3b$.

If $b_{\theta\theta\theta\theta} \neq 0$, for example, if $b = \theta^4, c = \frac{9}{5}\theta^5$, then C is not cubic. Therefore, the corresponding $(2,3,5)$-distribution is never flat.

Here we present the computation of the prolongation (Z, E) from the above example of cone structures. The bundle E is generated by

$$\begin{cases} \zeta_1 = \dfrac{\partial}{\partial \theta}, \\ \zeta_2 = \dfrac{\partial}{\partial x_1} + \theta\dfrac{\partial}{\partial x_2} + (\theta^2 + b)\dfrac{\partial}{\partial x_3} + (\theta^3 + c)\dfrac{\partial}{\partial x_4} + \{x_3\theta - 2x_2(\theta^2 + b) + x_1(\theta^3 + c)\}\dfrac{\partial}{\partial x_5} \end{cases}$$

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on the space Z with coordinates $\theta, x_1, x_2, x_3, x_4, x_5$. Then we have over Z,

$$\zeta_3 := [\zeta_1, \zeta_2] = \frac{\partial}{\partial x_2} + (2\theta + b_\theta)\frac{\partial}{\partial x_3} + (3\theta^2 + c_\theta)\frac{\partial}{\partial x_4} + \{x_3 - 2x_2(2\theta + b_\theta) + x_1(3\theta^2 + c_\theta)\}\frac{\partial}{\partial x_5},$$

$$\zeta_4 := [\zeta_1, \zeta_3] = (2 + b_{\theta\theta})\frac{\partial}{\partial x_3} + (6\theta + c_{\theta\theta})\frac{\partial}{\partial x_4} + \{-2x_2(2 + b_{\theta\theta}) + x_1(6\theta + c_{\theta\theta})\}\frac{\partial}{\partial x_5},$$

$$[\zeta_2, \zeta_3] = (c_\theta - 3\theta b_\theta + 3b)\frac{\partial}{\partial x_5} = 0,$$

$$\zeta_5 = [\zeta_1, \zeta_4] = b_{\theta\theta\theta}\frac{\partial}{\partial x_3} + (6 + c_{\theta\theta\theta})\frac{\partial}{\partial x_4} + \{-2x_2 b_{\theta\theta\theta} + x_1(6 + c_{\theta\theta\theta})\}\frac{\partial}{\partial x_5},$$

$$[\zeta_2, \zeta_4] = (c_{\theta\theta} - 3\theta b_{\theta\theta})\frac{\partial}{\partial x_5} = 0,$$

$$[\zeta_1, \zeta_5] = b_{\theta\theta\theta\theta}\frac{\partial}{\partial x_3} + c_{\theta\theta\theta\theta}\frac{\partial}{\partial x_4} + (-2x_2 b_{\theta\theta\theta\theta} + x_1 c_{\theta\theta\theta\theta})\frac{\partial}{\partial x_5},$$

$$\zeta_6 := [\zeta_2, \zeta_5] = 3(2 + a_{\theta\theta})\frac{\partial}{\partial x_5}.$$

We have that $\partial\mathcal{E} = \langle\zeta_1, \zeta_2, \zeta_3\rangle$, $\partial^{(2)}\mathcal{E} = \langle\zeta_1, \zeta_2, \zeta_3, \zeta_4\rangle$, $\partial^{(3)}\mathcal{E} = \langle\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\rangle$, and $\partial^{(4)}\mathcal{E} = \langle\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\rangle$. Then E has the pseudo-product structure of G_2-type given by $\mathcal{K} = \langle\zeta_1\rangle$, $\mathcal{L} = \langle\zeta_2\rangle$ and it descends to a nonflat $(2, 3, 5)$-distribution.

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References

[1] A. A. Agrachev, *Rolling balls and octonions*, Proc. Steklov Math. Inst. **258** (2007), 13–22.
[2] A. A. Agrachev and Y. L. Sachkov, "*Control theory from the geometric viewpoint*", in *Control Theory and Optimization, II*, Encyclopaedia of Mathematical Sciences **87**, Springer, Berlin, 2004.
[3] A. A. Agrachev and I. Zelenko, *Geometry of Jacobi curves, I*, J. Dyn. Control Syst. **8**(1) (2002), 93–140.
[4] A. A. Agrachev and I. Zelenko, *Geometry of Jacobi curves, II*, J. Dyn. Control Syst. **8**(2) (2002), 167–215.
[5] A. A. Agrachev and I. Zelenko, "*Nurowski's conformal structures for* $(2, 5)$*-distributions via dynamics of abnormal extremals*", in *Proceedings of RIMS Symposium on Developments of Cartan Geometry and Related Mathematical Problems*, RIMS Kokyuroku **1502**, 2006, 204–218.
[6] D. An and P. Nurowski, *Twistor space for rolling bodies*, Comm. Math. Phys. **326**(2) (2014), 393–414.
[7] J. C. Baez and J. Huerta, *G_2 and the rolling ball*, Trans. Amer. Math. Soc. **366**(10) (2014), 5257–5293.
[8] G. Bor and R. Montgomery, *G_2 and the rolling distributions*, Enseign. Math. **55** (2009), 157–196.
[9] R. L. Bryant, *Élie Cartan and geometric duality*, A lecture given at the Institut d'Élie Cartan on 19 June 1998.
[10] R. L. Bryant and L. Hsu, *Rigidity of integral curves of rank 2 distributions*, Invent. Math. **114**(1) (1993), 435–461.
[11] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths, *Exterior Differential Systems*, Springer, New York, 1991.
[12] A. Čap and J. Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2009.
[13] E. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. Éc. Norm. Supér. (3) **27** (1910), 109–192.

[14] B. Doubrov and I. Zelenko, *Equivalence of variational problems of higher order*, Differential Geom. Appl. **29** (2011), 255–270.
[15] G. Ishikawa, Y. Kitagawa and W. Yukuno, *Duality of singular paths for* $(2, 3, 5)$*-distributions*, J. Dyn. Control Syst. **21** (2015), 155–171.
[16] G. Ishikawa, Y. Machida and M. Takahashi, *Singularities of tangent surfaces in Cartan's split* G_2*-geometry*, Asian J. Math. **20**(2) (2016), 353–382.
[17] Y. Kitagawa, *The Infinitesimal automorphisms of a homogeneous subriemannian contact manifold*, Thesis, Nara Women's University, 2005.
[18] Y. Kitagawa, *The duality of abnormal extremals on subriemannian Cartan structures*, RIMS Kôkyûroku Bessatsu **B55** (2016), 111–132.
[19] T. Leistner, P. Nurowski and K. Sagerschung, *New relations between* G_2*-geometries in* 5 *and* 7, Int. J. Maths **28**(13) (2017), 1750094.
[20] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Mathematical Surveys and Monographs **91**, American Mathematical Society, Providence, RI, 2002.
[21] P. Nurowski, *Differential equations and conformal structures*, J. Geom. Phys. **55**(1) (2005), 19–49.
[22] M. Randall, *Flat (2, 3, 5)-distributions and Chazy's equations*, Symmetry, Integrability and Geometry: Methods and Applications **12** (2016), 029, 28 pages.
[23] H. Sato, private communication, January 2018.
[24] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.
[25] N. Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Math. J. **14** (1985), 277–351.
[26] D. The, *Exceptionally simple PDE*, Differential Geom. Appl. **56** (2018), 13–41.
[27] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Adv. Stud. Pure Math. **22** (1993), 413–494.
[28] K. Yamaguchi, "G_2*-geometry of overdetermined systems of second order*", in *Analysis and Geometry in Several Complex Variables*, Birkhäuser, Boston, 1999, 289–314.
[29] I. Zelenko, *Fundamental form and the Cartan tensor of* $(2, 5)$*-distributions coincide*, J. Dyn. Control Syst. **12**(2) (2006), 247–276.
[30] M. Zhitomirskii, *Exact normal form for* $(2, 5)$ *distributions*, RIMS Kokyuroku **1502** (2006), 16–28.

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